Inverse Magnetization Problems for Thin Plates

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Paleomagnetism

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- Until recently, nearly all paleomagnetic techniques were only analyzing bulk samples (several centimeters in diameter).
- In fact, the vast majority of magnetometers in use in the Geosciences infer the net magnetic moment of a rock sample from a set of field measurements taken at some distance.
- The development of scanning magnetic microscopes (superconductive coils) can extend paleomagnetic measurements to submillimeter scales.
- Typical scanning magnetic microscopes map a single component of the field, measured in a planar grid, at fixed distance above a planar sample whose section is three orders of magnitude smaller than its horizontal dimension. Thus, assuming planar magnetization distribution is an accurate model for the sample.
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A full characterization of silent sources was apparently not given before. In this talk, we use tools from harmonic analysis to achieve this.

A generalization of the classical Helmholtz-Hodge decomposition, that we call the Hardy-Hodge decomposition, is a key tool for characterizing silent sources.
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$$B = \mu_0 (H + M),$$

(1)
Constitutive Relations

Given a quasi-static $\mathbb{R}^3$-valued magnetization $\mathbf{M}$, the magnetic-flux density $\mathbf{B}$ and the magnetic field $\mathbf{H}$ satisfy

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where $\mu_0 = 4\pi \times 10^{-7} \text{Hm}^{-1}$ is the vacuum permeability.
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$$\Delta \phi = \nabla \cdot \mathbf{M}$$

(2)
As $1/(4\pi|\mathbf{r}|)$ is a fundamental solution of $-\Delta$, where $\mathbf{r}$ is the position vector in $\mathbb{R}^3$, we infer since $\phi$ is zero at infinity that

$$
\phi(\mathbf{r}) = -\frac{1}{4\pi} \iiint \frac{\nabla \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{r}'.
$$  \hspace{1cm} (3)

Integrating by parts we get

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whenever $\mathbf{M}$ is a distribution for which (4) is well-defined for all $\mathbf{r}$ not in the support of $\mathbf{M}$. 
We single out the third component of $r \in \mathbb{R}^3$ by writing $r = (x, z)$, where $x \in \mathbb{R}^2$. 
Thin plate Magnetizations

- We single out the third component of $\mathbf{r} \in \mathbb{R}^3$ by writing $\mathbf{r} = (x, z)$, where $x \in \mathbb{R}^2$.
- We assume that the support of the magnetization is contained in the $z = 0$ plane, that is $\mathbf{M}$ is a distribution of the form

$$\phi(x, z) = \mathbf{m}(x) \otimes \delta_0(z) =: (m_T(x), m_3(x)) \otimes \delta_0(z), \quad (5)$$
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where \( m_T = (m_1, m_2) \) and \( m_3 \) are distributions on \( \mathbb{R}^2 \) corresponding, respectively, to the tangential and normal components of \( m \).
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- By Fubini’s rule

$$\phi(x, z) = \frac{1}{4\pi} \int \int \left( \frac{m_T(x') \cdot (x - x')}{(|x - x'|^2 + z^2)^{3/2}} + \frac{m_3(x')z}{(|x - x'|^2 + z^2)^{3/2}} \right) dx', \quad (6)$$

for all $(x, z)$ such that either $z \neq 0$ or $x \notin \text{supp. } m$. 
Letting $z > 0$ for definiteness, equation (6) means that

$$\phi(x, z) = \frac{1}{2} (H_z \ast m^T(x) + P_z \ast m_3(x)) \quad (7)$$

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Thin plate potentials as convolutions

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$$H_z(x) := \frac{1}{2\pi} \frac{x}{(|x|^2 + z^2)^{3/2}}$$  \hfill (9)

is another kernel that we now analyze.
For $f \in L^p(\mathbb{R}^2)$, $p \in (1, \infty)$, the Riesz transforms of $f$, denoted by $R_1(f)$ and $R_2(f)$, are defined by

$$R_j(f)(x) := \lim_{\epsilon \to 0} \frac{1}{2\pi} \int \int_{\mathbb{R}^2 \setminus B(x, \epsilon)} f(x') \frac{(x_j - x_j')}{|x - x'|^3} \, dx', \quad j = 1, 2,$$

which exists a.e. and continuously maps $L^p(\mathbb{R}^2)$ into itself.

If $f_1, f_2 \in L^p(\mathbb{R}^2)$, it can be shown that

$$(f_1, f_2)^\ast H = Pz^\ast (R_1(f_1) + R_2(f_2)).$$

(11) We shall generalize this to more general distributions.
Riesz transforms

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- Poisson and Riesz transforms are defined on $W^{-\infty, p}$ by duality:

$$\langle R_j(m), f \rangle := -\langle m, R_j(f) \rangle, \quad \langle P_z \ast m, f \rangle := \langle m, P_z \ast f \rangle,$$

$$m \in W^{-\infty, p}, \quad f \in W^{\infty, q}.$$
Thin plate potentials as Poisson integrals

For \( m_1, m_2 \in W^{-\infty,p} \), it is still true that

\[
H_z \ast (m_1, m_2) = P_z \ast (R_1(m_1) + R_2(m_2)),
\]

(12)

More generally, for \( z \neq 0 \)

\[
\phi(x, z) = \frac{1}{2} P_{|z|} \ast (R_1(m_1) + R_2(m_2) + m_3),
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For $m_1, m_2 \in W^{-\infty, p}$, it is still true that

$$H_z * (m_1, m_2) = P_z * (R_1(m_1) + R_2(m_2)),$$

hence if $m \in (W^{-\infty, p})^3$, we have for $z > 0$:

$$\phi(x, z) = \frac{1}{2} (H_z * m^T(x) + P_z * m_3(x))$$

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**Theorem**

*Let \( m = (m_T, m_3) = (m_1, m_2, m_3) \in (W^{-\infty, p})^3. \) Then \( \phi(m)(x, z) \) is harmonic for \( z \neq 0. \)*
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**Theorem**

Let \( \mathbf{m} = (\mathbf{m}_T, m_3) = (m_1, m_2, m_3) \in (W^{-\infty,p})^3 \). Then \( \phi(m)(\mathbf{x}, z) \) is harmonic for \( z \neq 0 \). At such points it has the following representation in terms of the Riesz and Poisson transforms:

\[
\Lambda(m)(\mathbf{x}, z) = \frac{1}{2} P_{|z|} \ast \left( R_1(m_1) + R_2(m_2) + \frac{z}{|z|} m_3 \right)(\mathbf{x}).
\] (14)

Moreover, the limiting relation

\[
\lim_{z \to 0 \pm} \Lambda(m)(\mathbf{x}, z) = \frac{1}{2} \left( R_1(m_1)(\mathbf{x}) + R_2(m_2)(\mathbf{x}) \pm m_3(\mathbf{x}) \right)
\] (15)

holds in the distributional sense.
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**Theorem**

Let \( \mathbf{m} = (m_T, m_3) = (m_1, m_2, m_3) \in (W^{-\infty}, p)^3 \). Then \( \phi(\mathbf{m})(\mathbf{x}, z) \) is harmonic for \( z \neq 0 \). At such points it has the following representation in terms of the Riesz and Poisson transforms:

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Equivalent and silent sources

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- A magnetization is *silent from above* (resp. *below*) if it is equivalent from above (resp. below) to the null magnetization.

Since the Poisson transform is injective, Theorem 1 implies that \( m \) is silent from above if and only if \( R_1(m_1) + R_2(m_2) + m_3 = 0 \) and silent from below if and only if \( R_1(m_1) + R_2(m_2) - m_3 = 0 \). Hence, \( m \) is silent if and only if \( R_1(m_1) + R_2(m_2) = 0 \) and \( m_3 = 0 \).
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- Hence, \( \mathbf{m} \) is silent if and only if
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Hardy spaces of harmonic gradients

To understand better the role of the expression

\[ R_1(m_1)(x) + R_2(m_2)(x) \pm m_3(x), \]

we introduce Hardy spaces of harmonic gradients in the upper and lower half-space respectively:

\[ H^+: = \{ (R_1(f), R_2(f), f) : f \in W_{-\infty}, p \} \]

\[ H^-: = \{ (-R_1(f), -R_2(f), f) : f \in W_{-\infty}, p \} \]

We also let

\[ S: = \{ (s_1, s_2, 0) : s_1, s_2 \in W_{-\infty}, p, \nabla \cdot (s_1, s_2) = 0 \} \]
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\[m = (m_1, m_2, m_3) = P_{H^+}(m) + P_{H^-}(m) + P_S(m),\]  

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$$P_{H^+}(m) = \left( R_1(m^+), R_2(m^+), m^+ \right), \quad 2m^+ := -\sum_{j=1}^{2} R_j(m_j) + m_3$$
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P_{H^-}(m) = \left( -R_1(m^-), -R_2(m^-), m^- \right), \quad 2m^- := \sum_{j=1}^2 R_j(m_j) + m_3
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$$m = (m_1, m_2, m_3) = P_{H^+}(m) + P_{H^-}(m) + P_S(m),$$

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$$P_{H^+}(m) = \left( R_1(m^+), R_2(m^+), m^+ \right), \quad 2m^+ := -\Sigma_{j=1}^{2} R_j(m_j) + m_3$$

$$P_{H^-}(m) = \left( -R_1(m^-), -R_2(m^-), m^- \right), \quad 2m^- := \Sigma_{j=1}^{2} R_j(m_j) + m_3$$

$$P_S(m) = \left( -R_2(d), R_1(d), 0 \right), \quad d := R_2(m_1) - R_1(m_2).$$
Remark

- Each \((R_1(f), R_2(f), f) \in H^+\) is the trace on \(\{z = 0\}\) of a harmonic gradient in the upper half-space, namely 
  \(P_z \ast (R_1(f), R_2(f), f)\)

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Hence the decomposition \((W_{-\infty}, p)_3 = H^+ \oplus H^- \oplus S\) generalizes the classical decomposition of \(L^p(\mathbb{R})\) into a direct sum of holomorphic Hardy spaces.

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- If \( \text{supp} \mathbf{m} \neq \mathbb{R}^2 \), then \( \mathbf{m} \) is silent from above if and only if it is silent from below.
Equivalent magnetizations with compact support

Theorem

Let $m \in (W^{-\infty}, p)^3$ be supported on a compact set $K \subset \mathbb{R}^2$. 
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Theorem

Let $m \in (W^{-\infty,p})^3$ be supported on a compact set $K \subset \mathbb{R}^2$. The magnetizations supported on $K$ which are equivalent to $m$ (either from above or below) are all sums $m + s$, where $s \in S$ is supported on $K$. Such magnetizations are in fact equivalent to $m$ from above and below.
**Theorem**

*Let* \( m \in (L^2(\mathbb{R}^2))^3 \) *be supported on a compact, Lipschitz-smooth and finitely connected set* \( K \subset \mathbb{R}^2 \), *with interior* \( \Omega \).
Theorem

Let \( \mathbf{m} \in (L^2(\mathbb{R}^2))^3 \) be supported on a compact, Lipschitz-smooth and finitely connected set \( K \subset \mathbb{R}^2 \), with interior \( \Omega \).
Write \( P_S(\mathbf{m}) = (\mathbf{s}, 0) \) for the divergence-free component in the Hardy-Hodge decomposition of \( \mathbf{m} \).
Digression on the $L^2$ case

**Theorem**

Let $m \in (L^2(\mathbb{R}^2))^3$ be supported on a compact, Lipschitz-smooth and finitely connected set $K \subset \mathbb{R}^2$, with interior $\Omega$.

Write $P_S(m) = (s, 0)$ for the divergence-free component in the Hardy-Hodge decomposition of $m$.

The magnetization $m_K \in (L^2(\mathbb{R}^2))^3$ which is equivalent to $m$, supported on $K$, and has minimum $L^2$ norm under these constraints is

$$m_K = P_{H^+}(m) + P_{H^-}(m) + (h, 0), \quad (16)$$

where $h$ is the concatenation $v \lor s|_{\mathbb{R}^2 \setminus K}$ with $v$ the unique integrable harmonic field on $\Omega$, with normal component $v_n = (s|_{\Omega})_n$ on $\partial K$. 
The next figure shows the silent magnetization \( \mathbf{m}(x, y) = (\psi(x)\psi'(y), -\psi'(x)\psi(y), 0) \) where 
\[ \psi(t) := \frac{1}{2}(1 - \cos(2\pi t)) \]
for \( t \in [0, 1] \) and zero otherwise.

Parts A and B show the magnetization 
\( \mathbf{m}_1(x, y) = (\psi(x)\psi'(y), 0, 0) \) and resulting vertical component of the field measured at height \( z = 0.1 \) mm.

Parts C and D show the magnetization 
\( \mathbf{m}_2(x, y) = (0, -\psi'(x)\psi(y), 0) \) and resulting vertical component.

Parts E and F illustrate the silent source magnetization 
\( \mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2 \) and resulting null vertical component of the magnetic field measured at height \( z = 0.1 \) mm. In this case, \( \mathbf{m}_1 \) and \( -\mathbf{m}_2 \) are equivalent magnetizations.

Each image corresponds to an area of 1 mm \( \times \) 1 mm.
A compactly supported silent source

Figure:
We call \( m \) \textit{unidirectional} if \( m = Qu \) for some fixed \( u \in \mathbb{R}^3 \) and some scalar valued distribution \( Q \).
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Bidirectional magnetizations are common models for unidirectional magnetizations later corrupted by some superimposed field.
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- For $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ with $u_3 \neq 0$, any magnetization in $(W^{-\infty}, p)^3$ is equivalent from above to a unidirectional magnetization of the form $Q(x)\mathbf{u}$. 
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For $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ with $u_3 \neq 0$, any magnetization in $(W^{-\infty}, p)^3$ is equivalent from above to a unidirectional magnetization of the form $Q(x)u$.

A compactly supported unidirectional magnetization is equivalent from above (or below) to no other compactly supported unidirectional magnetization.
A proof

We prove the existence of an equivalent unidirectional magnetization from above.

By Theorem 3, \(Q\mathbf{u}\) is equivalent to \(\mathbf{m}\) from above iff

\[
    u_1 R_1(Q) + u_2 R_2(Q) + u_3 Q = R_1(m_1) + R_2(m_2) + m_3 =: h. \quad (17)
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  \[ \hat{Q}(\kappa) = \frac{\hat{h}(\kappa)}{u_3 - i u_T \cdot \kappa / |\kappa|}. \]
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- $1/(u_3 - i \mathbf{u} \cdot \kappa / |\kappa|)$ is smooth away from the origin, bounded, and homogeneous of degree 0, hence is a multiplier of $W^{\infty,q}$ by Hörmander’s theorem and since multiplier transformations commute with derivations.
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- By duality, (17) is solvable with $Q \in W^{-\infty,p}$ when $m \in (W^{-\infty,p})^3$. 
Compactly supported bidirectional silent sources

Theorem

Suppose $m(x) = Q(x)u + R(x)v$ where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are nonzero vectors in $\mathbb{R}^3$ while $Q$, $R$ are distributions with compact support. If $u_3$ or $v_3$ is nonzero, then $m$ is silent iff $m = 0$. If $u_3 = v_3 = 0$, then $m$ is silent iff $m^T(x) = Q(x)(u_1, u_2) + R(x)(v_1, v_2)$ is divergence free.
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1. If \( u_3 \) or \( v_3 \) is nonzero, then \( m \) is silent iff \( m = 0 \).
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is divergence free.
Inversion of experimental magnetic data from a synthetic sample measured with MIT SQUID microscope.

(A) Optical photograph of the synthetic sample comprised of a piece of paper with Vanderbilt University’s ‘Star V’ logo printed on it. The paper was glued to a nonmagnetic quartz disc to ensure flatness and facilitate scanning by the instrument. The sample was magnetized in the minus $z$ direction by applying a field pulse of 900 mT prior to mapping.

(B) Map of the $z$ component of the remanent magnetic field produced by the sample. The sample-to-sensor distance was approximately 0.27 mm.

(C) Estimated magnetization distribution obtained by inversion of magnetic data in the Fourier domain using Wiener deconvolution.
The Vanderbilt star

Figure:
Generalizations

- The previous theory carries over to magnetizations with components in $BMO^{-\infty}$, the space of finite sums of partial derivatives of any order of $BMO$ functions.
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- $BMO^{-\infty}$ is a quotient space of distributions by the constants, dual to the space $\mathcal{H}^{\infty,1}$ of functions lying in the real Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ together with all their derivatives.
Generalizations

- The previous theory carries over to magnetizations with components in $BMO^{-\infty}$, the space of finite sums of partial derivatives of any order of $BMO$ functions.

- $BMO^{-\infty}$ is a quotient space of distributions by the constants, dual to the space $h^{\infty,1}$ of functions lying in the real Hardy space $h^1(R^2)$ together with all their derivatives.

- In this case, nonzero silent unidirectional magnetizations exist: they are “ridge” distributions of the form $m(x) = uh(x \cdot v)$, where $v \in R^2$ is orthogonal to $(u_1, u_2)$ and $h \in BMO^{-\infty}(R)$. 
Another unidirectional example of retrieval

Inversion of the magnetic field produced by a simulated piecewise-continuous magnetization, comprised of rectangular slabs uniformly magnetized. The bottom part of the letter ‘I’ is magnetized in the antipodal direction, equivalent to a negative magnetization.

(A) Intensity plot of the synthetic magnetization distribution.
(B) Simulated map of the z component of the magnetic field at a sample-to-sensor distance of 0.15 mm. The map was calculated on a 128 x 128 square grid of positions. Gaussian white noise was added to the map to simulate instrument noise, yielding a signal-to-noise ratio of 100:1 or 40 dB.
(C) Estimated magnetization distribution obtained by inversion in the Fourier domain. The estimated distribution has 128 x 128 elements. Notice the ridge artifacts along the magnetization direction.
(D) Solution obtained by means of an improved Wiener deconvolution algorithm, with only a minor impact on accuracy.