Relaxation and discretization of control problems in the coefficients with a nonlinear functional in the gradient

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Model problem:

\[ \alpha, \beta, \mu > 0, \; \Omega \subset \mathbb{R}^N \text{ open, bounded}, \; f \in H^{-1}(\Omega) \]

\[ F_i: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \text{ Carathédory functions, } i = 1, 2, \]

\[ |F_i(x, s, \xi)| \leq C(1 + |s|^2 + |\xi|^2) \]

\[ \begin{align*}
\inf \left( \int_\omega F_1(x, u, \nabla u) dx + \int_{\Omega \setminus \omega} F_2(x, u, \nabla u) dx \right) \\
-\text{div} \left( (\alpha \chi_\omega + \beta \chi_{\Omega \setminus \omega}) \nabla u \right) = f \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega \\
\omega \subset \Omega \text{ measurable, } \; |\omega| \leq \mu
\end{align*} \]

F. Murat. This problem has not a solution in general.

It is interesting to work with a relaxed formulation.
If the functional to minimize is
\[
\int_\omega G_1(x, u) \, dx + \int_{\Omega \setminus \omega} G_2(x, u) \, dx + \int_\Omega h(x, u)(\alpha \chi_\omega + \beta \chi_{\Omega \setminus \omega}) |\nabla u|^2 \, dx,
\]
A relaxation of (CP) is given by
\[
\text{(RCP)} \quad \inf \int_\Omega (\theta G_1(x, u) + (1 - \theta)G_2(x, u) + h(x, u)M\nabla u\nabla u) \, dx
\]
\[
\begin{cases}
-\text{div}(M\nabla u) = f \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega \\
M \in \mathcal{K}(\theta), \quad \int_\Omega \theta \leq \mu
\end{cases}
\]
\mathcal{K}(\theta) \text{ set of matrices constructed via homogenization using } \alpha \text{ with proportion } \theta \text{ and } \beta \text{ with proportion } 1 - \theta.
L. Tartar.  K. Lurie, A. Cherkaev characterize $\mathcal{K}(p), \ 0 \leq p \leq 1$

Define $\lambda(p) = \left(\frac{p}{\alpha} + \frac{1-p}{\beta}\right)^{-1}$, \quad $\Lambda(p) = p\alpha + (1-p)\beta$

If $N \geq 2$, $\mathcal{K}(p)$ is the set of symmetric matrices with eigenvalues satisfying

$$\lambda(p) \leq \lambda_1 \leq \cdots \leq \lambda_N \leq \Lambda(p)$$

$$\sum_{i=1}^{N} \frac{1}{\lambda_i - \alpha} \leq \frac{N-1}{\Lambda(p) - \alpha} + \frac{1}{\lambda(p) - \alpha}$$

$$\sum_{i=1}^{N} \frac{1}{\beta - \lambda_i} \leq \frac{N-1}{\beta - \Lambda(p)} + \frac{1}{\beta - \lambda(p)}$$

For our purpose it is enough to know $\mathcal{K}(p)\xi, \xi \in \mathbb{R}^N$

$$\mathcal{K}(p)\xi = \begin{cases} B \left( \frac{\lambda(p) + \Lambda(p)}{2}, \frac{\Lambda(p) - \lambda(p)}{2} \right) |\xi| & \text{if } N \geq 2 \\ \lambda(p)\xi & \text{if } N = 1 \end{cases}$$
In general (JCD, J. Couce-Calvo, J.D. Martín-Gómez) the relaxed control problem has the form

\[
\inf \int_{\Omega} H(x, u, \nabla u, M\nabla u, \theta)dx
\]

\[
\begin{cases}
-\text{div}(M\nabla u) = f \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega \\
M \in \mathcal{K}(\theta), \quad \int_{\Omega} \theta dx \leq \mu,
\end{cases}
\]

or equivalently

\[
\inf \int_{\Omega} H(x, u, \nabla u, \sigma, \theta)dx
\]

\[
-\text{div}\sigma = f \text{ in } \Omega, \quad u \in H^1_0(\Omega), \quad \sigma \in K(\theta)\nabla u, \quad \int_{\Omega} \theta dx \leq \mu.
\]

Related results: Allaire, Bellido, Grabovski, Gutiérrez, Maestre, Munch, Pedregal, Tartar,…
Remark: If \((u_n, \omega_n)\) are solution of
\[-\text{div} \left( (\alpha\chi_{\omega_n} + \beta\chi_{\Omega \setminus \omega_n})\nabla u_n \right) = f \text{ in } \Omega,\ u_n = 0 \text{ on } \partial \Omega,\]
then for a subsequence we have
\[u_n \rightharpoonup u \text{ in } H^1_0(\Omega),\ \theta_n = \chi_{\omega_n} \rightharpoonup \theta \text{ in } L^\infty(\Omega)\]
\[\sigma_n = (\alpha\chi_{\omega_n} + \beta\chi_{\Omega \setminus \omega_n})\nabla u_n \rightharpoonup \sigma \text{ in } L^2(\Omega)^N\]
\[\text{div}\sigma_n \rightharpoonup \text{div}\sigma \text{ in } H^{-1}(\Omega).\]
We write \((u_n, \sigma_n, \theta_n) \rrightarrow (u, \sigma, \theta)\).
$$\overline{\mathcal{F}}(u, \sigma, \theta) = \int_{\Omega} H(x, u, \nabla u, \sigma, \theta) dx$$

is the lower semicontinuous envelope for the $\tau$-convergence of $\mathcal{F}$ given by

$$\mathcal{F}(u, \sigma, \theta) = \int_{\omega} F_1(x, u, \nabla u) dx + \int_{\Omega \setminus \omega} F_2(x, u, \nabla u) dx$$

if $\theta = \chi_\omega$, $\sigma = (\alpha \chi_\omega + \beta \chi_{\Omega \setminus \omega}) \nabla u$,

$$\mathcal{F}(u, \sigma, \theta) = +\infty \text{ otherwise.}$$
If \( N = 1 \),

\[
H(x, s, \xi, \eta, p) = \begin{cases} 
  pF_1 \left( x, s, \frac{\eta}{\alpha} \right) + (1 - p)F_2 \left( x, s, \frac{\eta}{\alpha} \right) & \text{if } \eta = \lambda(p)\xi \\
  +\infty & \text{otherwise}
\end{cases}
\]

If \( N > 1 \), we have

- \( \text{Dom}(H) = \{(x, s, \xi, \eta, p): \eta \in \mathcal{K}(p)\xi\} \).
- \( |H(x, s, \xi, \eta, p)| \leq C(1 + |s|^2 + |\xi|^2 + |\eta|^2) \)
- \( H \) satisfies the following convexity property

\[
H(x, s, \gamma \xi_1 + (1 - \gamma)\xi_2, \gamma \eta_1 + (1 - \gamma)\eta_2, \gamma p_1 + (1 - \gamma)p_2) \\
\leq \gamma H(x, s, \xi_1, \eta_1, p_1) + (1 - \gamma)H(x, \xi_2, \eta_2, p_2) \\
\text{if } \gamma \in [0, 1], \ (\xi_2 - \xi_1) \cdot (\eta_2 - \eta_1) = 0.
\]

- \( H(x, s, \xi, \eta, p) = pF_1 \left( x, s, \frac{\beta \xi - \eta}{(\beta - \alpha)p} \right) + (1 - p)F_1 \left( x, s, \frac{\eta - \alpha \xi}{(\beta - \alpha)(1 - p)} \right) \)
  \( \text{if } \eta \in \partial \mathcal{K}(p)\xi \)
If \( F_i(x, s, \xi), \ i = 1, 2, \) are convex in \( \xi, \) we have
\[
H(x, s, \xi, \eta, p) \geq pF_1 \left( x, s, \frac{\beta \xi - \eta}{(\beta - \alpha)p} \right) + (1 - p)F_1 \left( x, s, \frac{\eta - \alpha \xi}{(\beta - \alpha)(1 - p)} \right).
\]

Cases where \( H \) is known
\[
F_2(x, s, \xi) = r(x, s)|\xi|^2
\]
\[
Q(x, s, \xi) = F_1(x, s, \xi) - \frac{\alpha}{\beta} r(x, s)|\xi|^2 \quad \text{convex in } \xi.
\]
\[
\Rightarrow H(x, s, \xi, \eta, p) = \frac{h(x, s)}{\beta} \xi \cdot \eta + pQ \left( x, s, \frac{\eta - \beta \xi}{p(\beta - \alpha)} \right)
\]

It contains some cases proved by Bellido, Pedregal, Grabovsky, \ldots

\( \forall (x, s, \xi), \ \exists \xi \in \mathbb{R}^N \) such that the applications \( t \rightarrow F_i(x, s, \xi + t \zeta) \) are linear.
\[
\Rightarrow H(x, s, \xi, \eta, p) = pF_1 \left( x, s, \frac{\beta \xi - \eta}{(\beta - \alpha)p} \right) + (1 - p)F_1 \left( x, s, \frac{\eta - \alpha \xi}{(\beta - \alpha)(1 - p)} \right)
\]
A discretization using an upper approximation of $H$

Take $\bar{H}(x, s, \xi, \eta, p)$, with

\[
\bar{H}(x, s, \xi, \eta, p) \geq H(x, s, \xi, \eta, p)
\]

\[
\bar{H}(x, s, \xi, \alpha \xi, 1) = F_1(x, s, \xi), \quad \bar{H}(x, s, \xi, \beta \xi, 0) = F_2(x, s, \xi).
\]

For $h > 0$, we consider a triangulation $\mathcal{T}_h = \{T_{i,h}\}_{i=1}^{n_h}$ of $\Omega$

\[
\Omega = \bigcup_{i=1}^{n_h} T_{i,h}, \quad T_{i,h} \text{ measurable, } |T_{i,h}| > 0, \quad \text{diam}(T_{i,h}) \leq h
\]

\[
|T_{i,h} \cap T_{j,h}| = 0, \text{ if } i \neq j,
\]

and a sequence of closed subspaces $V_h \subset H_0^1(\Omega)$
Discretized problem

\[
\min \int_{\Omega} \bar{H}(x, u_h, \nabla u_h, M_h \nabla u_h, \theta_h) \, dx
\]

\[0 \leq \theta_h \leq 1, \int_{\Omega} \theta_h \, dx \leq \mu^i, \quad M_h \in \mathcal{K}(\theta_h) \text{ a.e. in } \Omega\]

\[u_h \in V_h, \quad \int_{\Omega} M_h \nabla u_h \cdot \nabla v_h \, dx = \int_{\Omega} f v_h \, dx, \quad \forall v_h \in V_h\]

\[M_h, \theta_h \text{ constants in the elements } T_{i,h}\]
Assumptions on \( V_h \)

i) \[
\lim_{h \to 0} \min_{v_h \in V_h} \| v - v_h \|_{H^1_0(\Omega)} = 0, \quad \forall v \in H^1_0(\Omega),
\]

ii) \[
\lim_{h \to 0} \min_{v_h \in V_h} \| w_h \phi - v_h \|_{H^1_0(\Omega)} = 0,
\]

\( \forall w_h \in V_h \) bounded in \( H^1_0(\Omega) \), \( \forall \phi \in C^\infty_c(\Omega) \)

iii) \[
\lim_{h \to 0} \int_\Omega H(x, u_h, \nabla u_h, \sigma_h, \theta_h) dx \geq \int_\Omega H(x, u, \nabla u, \sigma, \theta) dx
\]

\( \forall u_h \rightharpoonup u \) in \( H^1_0(\Omega) \), \( u_h \in V_h \), \( \forall \sigma_h \rightharpoonup \sigma \) in \( L^2(\Omega)^N \)

\( \forall \theta_h \rightharpoonup^* \theta \) in \( L^\infty(\Omega) \), \( 0 \leq \theta_h \leq 1 \) a.e. in \( \Omega \)

\[
\lim \max_{h \to 0} \frac{1}{\| v_h \|_{H^1_0(\Omega)}} \int_\Omega (\sigma_h - \sigma) \cdot \nabla v_h dx = 0.
\]
Properties i), ii), iii) are satisfied for $V_h = H^1_0(\Omega)$.

If $V_h$ is a usual space of finite elements, it satisfies i), ii).

In the examples where we know $H$, every sequence $V_h$ satisfies iii).

**Theorem:** The discrete problem has a solution $(u_h, M_h, \theta_h)$

Up to a subsequence

$$
\begin{cases}
  u_h \rightharpoonup u \text{ in } H^1_0(\Omega) \\
  M_h \nabla u_h \rightharpoonup M \nabla u \text{ in } L^2(\Omega)^N \\
  \theta_h \rightharpoonup^* \theta \text{ in } L^\infty(\Omega)
\end{cases}
$$

$$
\inf_{\Omega} \int H(x, u, \nabla u, M \nabla u, \theta) dx
$$

$(u, M, \theta)$ solution of

$$
\begin{cases}
  - \text{div}(M \nabla u) = f \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega \\
  0 \leq \theta \leq 1, \ M \in K(\theta) \text{ a.e.}, \ \int \theta dx \leq \mu
\end{cases}
$$

$$
\lim_{h \to 0} \int_{\Omega} H(x, u_h, \nabla u_h, M_h \nabla u_h, \theta_h) dx = \int_{\Omega} H(x, u, \nabla u, M \nabla u, \theta) dx.
$$
Example 1:
\[
\overline{H}(x, s, \xi, A\xi, 1) = F_1(x, s, \xi), \quad \overline{H}(x, s, \xi, B\xi, 0) = F_2(x, s, \xi)
\]
\[
\overline{H}(x, s, \xi, \eta, p) = +\infty \text{ otherwise.}
\]
In this case, we are solving a discrete version of the original (unrelaxed) problem, i.e.
\[
\inf\left(\int_{\omega} F_1(x, u_h, \nabla u_h) dx + \int_{\Omega \setminus \omega} F_2(x, u_h, \nabla u_h) dx\right)
\]
\[
\begin{aligned}
\left\{ \begin{array}{l}
u_h \in V_h \\
\int_{\Omega} (\alpha \chi_{\omega_h} + \beta \chi_{\Omega \setminus \omega_h}) \nabla u_h \cdot \nabla v_h dx = \int_{\Omega} f v_h dx, \forall v_h \in V_h \\
\omega_h \text{ a union of elements of } \mathcal{T}_h, \quad |\omega_h| \leq \mu.
\end{array} \right.
\end{aligned}
\]
Example 2: \( (N \geq 2) \) Since we know the values of \( \bar{H} \) in the boundary of its domain, we can take

\[
\bar{H}(x, s, \xi, \eta, p) = p F_1 \left( x, s, \frac{\beta \xi - \eta}{p(\beta - \alpha)} \right) + (1 - p) F_2 \left( x, s, \frac{\eta - \alpha \xi}{(1 - p)(\beta - \alpha)} \right)
\]

if \( \eta \in \partial \mathcal{K}(p) \xi \)

\[
\bar{H}(x, s, \xi, \eta, p) = +\infty \text{ elsewhere.}
\]

Clearly, when we know the function \( H \) we can just take \( \bar{H} = H \).
A lower approximation of \( H \)

We consider \( H(x, s, \xi, \eta, p) \) with

\[
H(x, s, \xi, \eta, p) \leq H(x, s, \xi, \eta, p)
\]

For \( h > 0 \), consider \( \mathcal{J}_h = \{T_{i,h}\}_{i=1}^{n_h} \) as above and closed subspaces \( V_h \subset H^1_0(\Omega) \) satisfying properties i) and ii) as above and

\[
\lim_{h \to 0} \int_{\Omega} H(x, u_h, \nabla u_h, \sigma_h, \theta_h) \, dx \geq \int_{\Omega} H(x, u, \nabla u, \sigma, \theta) \, dx
\]

\( \forall u_h \rightharpoonup u \) in \( H^1_0(\Omega) \), \( u_h \in V_h \), \( \forall \sigma_h \rightharpoonup \sigma \) in \( L^2(\Omega)^N \)

\( \forall \theta_h \rightharpoonup^* \theta \) in \( L^\infty(\Omega) \), \( 0 \leq \theta_h \leq 1 \) a.e. in \( \Omega \)

\[
\lim \max_{h \to 0} \frac{1}{\|v_h\|_{H^1_0(\Omega)}} \int_{\Omega} (\sigma_h - \sigma) \cdot \nabla v_h \, dx = 0.
\]
Discretized problem \( \mathcal{C} = \text{co}\{(M, p): M \in \mathcal{K}(p)\} \)

\[
\min_{\Omega} \int \frac{H(x, u_h, \nabla u_h, M_h \nabla u_h, \theta_h)}{\Omega} dx
\]

\[0 \leq \theta_h \leq 1, \quad \int_{\Omega} \theta_h dx \leq \mu^i, \quad (\theta_h, M_h) \in \mathcal{C} \text{ a.e. in } \Omega\]

\[u_h \in V^h, \quad \int_{\Omega} M_h \nabla u_h \cdot \nabla v_h dx = \int_{\Omega} f v_h dx, \quad \forall v_h \in V^h\]

\[M_h, \theta_h \text{ constants in the elements } T_{i,h}\]
Theorem: The discrete problem has a solution \((u_h, M_h, \theta_h)\)

Up to a subsequence

\[
\begin{align*}
&u_h \rightharpoonup u \text{ in } H^1_0(\Omega) \\
&M_h \nabla u_h \rightharpoonup M \nabla u \text{ in } L^2(\Omega)^N \\
&\theta_h \rightharpoonup^* \theta \text{ in } L^\infty(\Omega)
\end{align*}
\]

\[
\inf_{u} \int_{\Omega} H(x, u, \nabla u, M \nabla u, \theta) \, dx
\]

\((u, M, \theta)\) solution of

\[
\begin{align*}
&-\text{div}(M \nabla u) = f \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega \\
&0 \leq \theta \leq 1, \ M \in K(\theta) \text{ a.e., } \int \theta \, dx \leq \mu
\end{align*}
\]

\[
\lim_{h \to 0} \int_{\Omega} H(x, u_h, \nabla u_h, M_h \nabla u_h, \theta_h) \, dx = \int_{\Omega} H(x, u, \nabla u, M \nabla u, \theta) \, dx
\]
Remark: Looking for the optimality conditions, we hope that the solution \((\hat{u}, \hat{M}, \hat{\theta})\) satisfies

\[
\hat{M} \nabla \hat{u} \in \partial \mathcal{K}(\hat{\theta}) \nabla \hat{u} \text{ a.e. in } \Omega.
\]

Thus, we need to take \(H\) satisfying

\[
H(x, s, \xi, \eta, p) = H(x, s, \xi, \eta, p) \text{ if } \eta \in \partial \mathcal{K}(p)\xi.
\]

Example: If \(F_1(x, s, \xi), F_2(x, s, \xi)\) are convex in \(\xi\) take

\[
H(x, s, \xi, \eta, p) = H(x, s, \xi, \eta, p)
\]

\[
= p F \left(x, s, \frac{\beta \xi - \eta}{p(\beta - \alpha)} \right) + (1 - p) F \left(x, s, \frac{\eta - \alpha \xi}{(1 - p)(\beta - \alpha)} \right)
\]
We have shown that we can solve numerically the control problem discretizing the unrelaxed or the relaxed problem.

What is better?

Control problem (CP) \( F_1, F_2 \in W^{1,\infty} \)

\[
\inf \left( \int_{\omega} F_1 \left( x, u, \frac{du}{dx} \right) dx + \int_{(0,1)\setminus\omega} F_2 \left( x, u, \frac{du}{dx} \right) dx \right)
\]

\[
\begin{cases}
- \frac{d}{dx} \left( (\alpha \chi_\omega + \beta \chi_{(0,1)\setminus\omega}) \frac{du}{dx} \right) = f \text{ in } (0,1) \\
\left| \omega \right| \leq \mu \\
u(0) = u(1) = 0
\end{cases}
\]

Relaxed formulation (RCP)

\[
\min \int_{0}^{1} \left( \theta F_1 \left( x, u, \frac{\lambda(\theta) du}{\alpha} \right) + (1 - \theta) F_2 \left( x, u, \frac{\lambda(\theta) du}{\beta} \right) \right) dx
\]

\[
\begin{cases}
- \frac{d}{dx} \left( \lambda(\theta) \frac{du}{dx} \right) = f \text{ in } (0,1) \\
\left( \int_{0}^{1} \theta \right) dx \leq \mu \\
u(0) = u(1) = 0
\end{cases}
\]
Discretization

We take a partition $\mathcal{P}_r$ and refinement $\mathcal{Q}_h$ of respective diameters $r$ and $h$

$$V_h = \{ u \in C_0^0([0,1]): u \text{ is affine in each interval of } \mathcal{Q}^h \}$$

We consider the discretized problems

$$\min \left( \int_\omega F_1 \left( x, u, \frac{du}{dx} \right) dx + \int_{(0,1) \setminus \omega} F_1 \left( x, u, \frac{du}{dx} \right) dx \right)$$

(DCP)

$$\begin{cases} 
  u \in V_h \\
  \int_0^1 \left( \alpha \chi_\omega + \beta \chi_{(0,1) \setminus \omega} \right) \frac{du}{dx} \frac{dv}{dx} dx = \int_0^1 f \, vdx, \; \forall v \in V_h \\
  \omega \text{ is a union of intervals of } \mathcal{P}_r, \quad |\omega| \leq \mu. 
\end{cases}$$
\[
\min \int_{\omega} \left( \theta F_1 \left( x, u, \frac{\lambda(\theta)}{\alpha} du \right) + (1 - \theta) F_2 \left( x, u, \frac{\lambda(\theta)}{\beta} du \right) \right) dx
\]

\[(DRP)\]
\[
\left\{ \begin{array}{l}
\int_{0}^{1} \lambda(\theta) \frac{du}{dx} dv \, dx = \int_{0}^{1} f \, v \, dx, \, \forall v \in V_h \\
u \in V_h
\end{array} \right.
\]

\(\theta\) is constant in the intervals of \(\mathcal{P}_r, \int_{0}^{1} \theta \, dx \leq \mu\)

**Theorem.** Taking \(r = h\) and \(f \in W^{1,l}(0,1)\) we have

\[
|\min(RCP) - \min(DCP)| \leq c h^{l+1}
\]

Taking \(f \in L^\infty(0,1), r = \sqrt{h}\) and assuming that there exists an optimal control in \(BV(0,1)\), we have

\[
|\min(RCP) - \min(DRP)| \leq c h.
\]