Observability by Viability Kernels

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Outline

1. Statement of the problem
2. Connection to Viability kernels
3. Using single-valuedness results
Outline

1 Statement of the problem

2 Connection to Viability kernels

3 Using single-valuedness results
A new framework for output

Consider the system

\[ \dot{z}(t) = f(t, z(t)), \quad t \in [t_0, t_f], \quad \text{ODE} \]
\[ (t, z(t)) \in \Theta, \quad t \in [t_0, t_f], \quad \text{output} \]

where

- \( f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous function, and \( n, t_0, t_f > 0 \).
- \( \Theta \subset \mathbb{R}_+ \times \mathbb{R}^n \), which we call the output domain

For instance, for standard output equation

\[ \theta(t) = h(t, z(t)), \quad t \in [0, t_f] \]

one gets

\[ \Theta = \{(t, z) \in \mathbb{R}_+ \times \mathbb{R}^n \mid \theta(t) = h(t, z)\} \]

To describe partial or uncertain information

\[ \Theta = \{(t, z) \in \mathbb{R}_+ \times \mathbb{R}^n \mid |\theta(t) - h(t, z)| \leq \epsilon\} \]
A new framework for output

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\dot{z}(t) &= f(t, z(t)), \quad t \in [t_0, t_f], \quad \text{ODE} \\
(t, z(t)) &\in \Theta, \quad t \in [t_0, t_f], \quad \text{output}
\end{align*}
\]

where

- \( f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n \) is a continuous function, and \( n, t_0, t_f > 0 \).
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where

- \( f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a \textit{continuous} function, and \( n, t_0, t_f > 0 \).
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To describe \textit{partial} or \textit{uncertain} information

\[ \Theta = \{(t, z) \in \mathbb{R}_+ \times \mathbb{R}^n \mid |\theta(t) - h(t, z)| \leq \epsilon\} \]
Definitions

Let \( t_0 \in [0, t_f) \), \( z_0, z_1, z_2 \in \mathbb{R}^n \) and \( \Sigma \subset \mathbb{R}^n \). We say that:

- \( z_0 \) generates output \( \Theta \) on the horizon \([t_0, t_f]\), if the system with output has a solution \( \bar{z} \) which satisfies \( \bar{z}(t_0) = z_0 \). Notation:
  \[
  z_0 \leadsto \Theta \text{ on } [t_0, t_f]
  \]

- \( z_1 \) and \( z_2 \) are indistinguishable if both generate \( \Theta \)

The system with output is:

- \( \Sigma \)-observable: no distinct indistinguishable states are in \( \Sigma \)
- loc. observable around \( z_0 \): \( W \)-observable for \( W \in \mathcal{N}(z_0) \)
- observable from near \( \bar{t} \) : there exists \( I \in \mathcal{N}(\bar{t}) \) such that for all \( t_0 \in I \), the system is observable on \([t_0, t_f]\)
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Statement of the problem

Definitions

Let \( t_0 \in [0, t_f) \), \( z_0, z_1, z_2 \in \mathbb{R}^n \) and \( \Sigma \subset \mathbb{R}^n \). We say that :

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Let $t_0 \in [0, t_f)$, $z_0, z_1, z_2 \in \mathbb{R}^n$ and $\Sigma \subset \mathbb{R}^n$. We say that:

- $z_0$ **generates** output $\Theta$ on the horizon $[t_0, t_f]$, if the system with output has a solution $\bar{z}$ which satisfies $\bar{z}(t_0) = z_0$. Notation:

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- $z_1$ and $z_2$ are **indistinguishable** if both generate $\Theta$

- The system with output is:
  - **$\Sigma$–observable**: no distinct indistinguishable states are in $\Sigma$
  - **loc. observable around $z_0$**: $W$–observable for $W \in \mathcal{N}(z_0)$
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The Observability Kernel

For each \( t \in [0 \ t_f) \), define the multifunction

\[
\mathcal{J}(t) \doteq \{ z \in \Sigma \mid z \sim \Theta \text{ on } [t, t_f] \}
\]

Then, two immediate results:

- System is \( \Sigma \)-observable on \([t_0 \ t_f]\) iff \( \text{card}(\mathcal{J}(t_0)) \leq 1 \)
- Let \( \mathcal{K} \) be the viability kernel of \( \Theta \) under system

\[
\begin{align*}
\dot{t} &= 1 \\
\dot{z} &= f(t, z)
\end{align*}
\]

(context of viability theory Aubin, Saint-Pierre, Bonneuil etc)

then, for all \( t \in [0 \ t_f) \)

\[
\mathcal{J}(t) = \{ z \in \Sigma \mid (t, z) \in \mathcal{K} \}
\]

We call subset \( \mathcal{K} \) the observability kernel of the system.
The Observability Kernel

For each $t \in [0 \ t_f)$, define the multifunction

$$\mathcal{J}(t) \doteq \{ z \in \Sigma \mid z \leadsto \Theta \text{ on } [t, t_f] \}$$

Then, two immediate results:

- **System is $\Sigma$–observable on** $[t_0 \ t_f]$ if and only if $\text{card}(\mathcal{J}(t_0)) \leq 1$
- Let $\mathcal{K}$ be the viability kernel of $\Theta$ under system

  $$\begin{align*}
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The Observability Kernel

For each $t \in [0 \ t_f)$, define the multifunction

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Then, two immediate results:

- **System is $\Sigma$–observable on** $[t_0 \ t_f]$ **iff** $\text{card}(J(t_0)) \leq 1$
- Let $\mathcal{K}$ be the **viability kernel** of $\Theta$ under system

  $$\begin{align*}
  \dot{t} &= 1 \\
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  then, for all $t \in [0 \ t_f)$

  $$J(t) = \{ z \in \Sigma \mid (t, z) \in \mathcal{K} \}$$

We call subset $\mathcal{K}$ the **observability kernel** of the system.
A graphical illustration

The system is observable:
- on both $[0 \ t_f]$ and on all the horizons $[t \ t_f]$ for $t > t_4$
- from near $t_3$
- loc. on $[t_2 \ t_f]$ around $x_0$

But it is unobservable on $[t_1 \ t_f]$
The linear autonomous case

Consider the standard l. a. system

$$\dot{z} = Az, \quad \theta = Cz$$

where $A \in \mathcal{L}(\mathbb{R}^n), C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^q)$ and $\theta \in L^2(t_0, t_f, \mathbb{R}^q)$

Then the associated observability kernel $\mathcal{K}$ is given by

$$\mathcal{K} = z_0 + \ker(Q)$$

where $Q \doteq [C \ CA \ldots CA^{n-1}]'$ and $z_0$ is such that

$$C\exp(A(t - t_0))z_0 = \theta(t) \text{ for all } t \in [t_0 \ t_f]$$

If such $z_0$ does not exist then $\mathcal{K} = \emptyset$. Therefore the system is observable iff $\text{rank}(Q) = n$ or $z_0$ does not exist.
An example from Lorentz equation

\[
\begin{align*}
\dot{z}_1 &= \sigma z_1 - \sigma z_2, \\
\dot{z}_2 &= -rz_1 + z_2 + tz_1
\end{align*}
\]

with output:

\[(z_1, z_2) \in [-30, 30]^2 \text{ and } bt = z_1 z_2\]

The system is \textit{observable} on \([t_1, 50]\), for all \(t_1 \in \{0\} \cup [t_0, 50]\)
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Observability and Convexity

Consider the following implication,

\[(P) \quad z_1 \sim \Theta \text{ on } [t_1, t_f] \quad z_2 \sim \Theta \text{ on } [t_2, t_f] \implies \frac{z_1 + z_2}{2} \sim \Theta \text{ on } \left[\frac{t_1 + t_2}{2}, t_f\right]\]

Theorem (using a result by Nikodem et al.)

Assume \( \Sigma \) convex and that \((P)\) be satisfied for all \( z_1, z_2 \) in \( \Sigma \). Let \( I \subset \text{dom}(\mathcal{J}) \) be an open interval and \( \bar{t} \in I \) such that the system is \( \Sigma \)-observable on \( [\bar{t}, t_f] \). Then:

- The system is \( \Sigma \)-observable on \( [t_0, t_f] \) for all \( t_0 \in \mathcal{J} \)
- There exist an additive function \( \xi : \mathcal{J} \to \mathbb{R}^n \) and \( y_0 \in \mathbb{R}^n \) such that for each \( t_0 \in \mathcal{J} \), \( \xi(t_0) + y_0 \sim \Theta \text{ on } [t_0, t_f] \)
Using single-valuedness results

Observability and Monotonicity

Consider the following statement,

\[(Q)\]

\[
\begin{align*}
z_1 & \rightsquigarrow \Theta \text{ on } [\pi_1(y_1) \, t_f] \\
z_2 & \rightsquigarrow \Theta \text{ on } [\pi_1(y_2) \, t_f] \\
\implies (y_2 - y_1, z_2 - z_1) & \geq 0,
\end{align*}
\]

for \(z_1, z_2\) and \(y_1, y_2\) such that \(\pi_1(y_i) \in [0, t_f]\) for \(i = 1\) or \(2\)

\(\pi_1\) : the first projection

**Theorem (using a result by Zarantonello)**

Assume \(\text{int}(\mathcal{K}) \neq \emptyset\) and \((Q)\) holds for all \(z_1, z_2\) in \(\Sigma\). Then System is \(\Sigma\)–observable on \([t_0 \, t_f]\) for a. e. \(t_0\) in \(\text{dom}(\mathcal{J})\).
A Characterization

\[ z_1 \sim \Theta \text{ on } [\pi_1(y_1), t_f] \text{ and } z_2 \sim \Theta \text{ on } [\pi_1(y_2), t_f], \]

\[ \langle y_2 - y_1, z_2 - z_1 \rangle \geq -\langle y_2 - y_1, \tau(y_2) - \tau(y_1) \rangle, \]

for couples \((z_1, z_2), (y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^n\), such that \(\pi_1(y_i) \in [0, t_f]\) for \(i = 1\) or \(2\) and a function \(\tau : \mathbb{R}^n \to \mathbb{R}^n\).

**Theorem (using a result by Levy and Poliquin)**

Let \(\bar{t} \in [0, t_f]\), \(\bar{y} = (\bar{t}, 0, \ldots, 0)\)' and \(\bar{z} \sim \Theta \text{ on } [\bar{t}, t_f]\). Then the system is continuously observable from near \(\bar{t}\) iff:

- There exist \(U \in \mathcal{N}(\bar{y}), V \in \mathcal{N}(\bar{z})\), and \(\tau : U \to V\) continuous, such that \((R)\) holds for all \((y_i, z_i) \in U \times V\)

- If \((t_q)_q\) converges near \(\bar{t}\), there exists \((z_q)_q\) which converges near \(\bar{z}\) such that \(z_q \sim \Theta \text{ on } [t_q, t_f]\), for all \(q\).