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Semigroup inequalities, stochastic domination, Hardy's inequality, and strong ergodicity

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Abstract

For the classical L^p-spaces of signed measures on $\mathbb{N}$, we devise a framework in which bounds for a sub-Markovian semigroup of interest can be obtained, up to a constant factor, from bounds for another tractable semigroup that dominates stochastically the first one. The main tools are the Hardy inequality, the definition of related auxiliary L^p spaces suited to take advantage of the domination, and the proof that the norms are equivalent to the classical ones if the reference measure is quasi-geometrically decreasing. We illustrate the results using birth-death and single-birth processes.

KEYWORDS: semigroup inequalities, stochastic order, Hardy's inequality, strong ergodicity, exponential stability, spectral gap, birth-death and single-birth processes

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1 Introduction

The long-time behavior of sub-Markovian semigroups of signed measures is often studied in the classical L^p-spaces of the densities with respect to a reference measure. In this setting, we establish a framework in which bounds for a semigroup of interest can be deduced from bounds obtained for another semigroup that *stochastically dominates* the first semigroup.

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Usually, semigroups are given through their generators, and the semigroup of interest arises in the study of a particular problem and lacks structure and nice features. The dominating semigroup and reference measure can then be chosen adequately, for instance so as to satisfy exponential stability bounds. Powerful tools to prove such bounds are, *e.g.*, the Dirichlet form when the reference measure is invariant, or the spectral decomposition when the reference measure is self-adjoint, together with spectral gap estimates.

Such results on bound transfers are mainly found in the literature in cases when the semigroup of interest is *term-wise dominated* by another one [1, 9, 14, 17, 23]. This very strong assumption unsurprisingly yields results, but is seldom true, in particular it cannot be satisfied between different semigroups of probability measures.

Hereafter in this paper, *domination* between measures or semigroups refers to *stochastic domination*. This is a natural probabilistic notion, easily expressed also in the dual functional space perspective. It is powerfully related to *coupling* methods, and sample-path intuition may help find a convenient dominating semigroup for a complicated semigroup of interest.

We introduce a new family of auxiliary L^p spaces related to stochastic domination and the Hardy inequality. Under the mild assumption that the reference measure is quasi-geometrically decreasing (has exponential tails), we show that the norms of the classical and auxiliary L^p spaces are equivalent for $p > 1$. This enables the transfer of bounds from the dominating semigroup to the dominated one, up to a controlled constant factor.

Such ideas initially appeared in Graham [10], and yielded global exponential stability results, first for sub-Markovian semigroups, then for a non-linear dynamical system by adequate comparisons with the former. Even though the main sub-Markovian semigroup of interest was self-adjoint in a Hilbert space with very strong norm, the goals of [10] required bounds for weaker scalar products, applicable to much more general initial conditions.

We have a wider scope in the present paper, and the novel use of the Hardy inequality yields clearer arguments and nicer auxiliary L^p spaces. We consider sub-Markovian semigroups of bounded signed measures on $\mathbb{N} = \{0, 1, \dots\}$, and expect these techniques have wider applicability. Such semigroups may be rendered Markovian by adjoining a cemetery or absorbing state to $\mathbb{N}$, assumed to be the least state and denoted by -1.

The signed measure spaces are in duality with functional spaces, and the results apply to semigroups in both kinds of spaces. The methods and results can be readily extended to time-dependent "generators", or flows of linear time-inhomogeneous equations instead of semigroups, as long as the controls we use can be taken uniformly in time.

We illustrate the results with birth-death processes and single-birth processes. For the former, various spectral gap criteria and bounds exist, as in Callaert [3, 4], van Doorn [6, 7, 8] and Chen [5], We discuss a striking result of Liggett [16, Cor. 3.8], [5, Theorem 5.5 p. 93], showing that the assumption on quasi-geometrical decrease is nearly optimal.

Stochastic monotonicity properties of birth-death processes were used in van Doorn's monograph [6] for different purposes than ours.

2 The framework and main result

2.1 Signed measures and sub-Markovian semigroups

For μ in the space $\mathscr{M} = \mathscr{M}(\mathbb{N})$ of signed measures, we denote by $|\mu|$ its total variation measure and by μ^+ and μ^- its positive and negative parts, so that $|\mu| = \mu^+ + \mu^-$ and $\mu = \mu^+ - \mu^-$, and the duality bracket between $\mathscr{M}$ and the functional space $L^\infty = L^\infty(\mathbb{N})$ by

$$\langle \mu, f \rangle = \int f d\mu = \sum_{k \in \mathbb{N}} \mu(k) f(k) \,, \qquad \mu \in \mathscr{M},\, f \in L^\infty \,,$$

for which dual spaces, adjoints, etc., will be denoted classically using asterisks. The space $\mathscr{M}$ is Banach for the strong dual norm, which is the total variation norm

$$\|\mu\|_{\mathrm{TV}} = |\mu|(\mathbb{N}) = \sum_{k \in \mathbb{N}} |\mu(k)| = \sup_{\|f\|_\infty \le 1} \langle \mu, f \rangle \,, \qquad \mu \in \mathscr{M} \,.$$

We consider sub-Markovian semigroups: positivity-preserving contraction semigroups $(T_t)_{t \ge 0}$ on L^∞, satisfying $T_{t+s} = T_t T_s$ and $\|T_t f\|_\infty \le \|f\|_\infty$ and $T_t f \ge 0$ for all $t, s \ge 0$ and $f \ge 0$ in L^∞. The infinitesimal generators of such semigroups act on dense subspaces. The adjoint semigroup $(T_t^*)_{t \ge 0}$ given by $\langle T_t^* \mu, f \rangle = \langle \mu, T_t f \rangle$ for μ in $\mathscr{M}$ and f in L^∞ is also positivity-preserving and contractive on $\mathscr{M}$.

The decomposition of signed measures in their positive and negative parts allows to restrict our attention to probability measures (and even Dirac masses) as initial data for the adjoint sub-Markovian semigroups, which then evolve in the subset of sub-probability measures. Markovian semigroups preserve the set of probability measures.

2.2 Some classic Banach spaces

For $\alpha > 0$ in $\mathscr{M}$ and conjugate exponents $1 < p, q < \infty$ (satisfying $\frac{1}{p} + \frac{1}{q} = 1$) we have

$$|\langle \mu, f \rangle| = \left| \sum_{k \geq 0} \frac{\mu(k)}{\alpha(k)} f(k) \alpha(k) \right| \leq \left(\sum_{k \geq 0} \left| \frac{\mu(k)}{\alpha(k)} \right|^p \alpha(k) \right)^{1/p} \left(\sum_{k \geq 0} |f(k)|^q \alpha(k) \right)^{1/q}$$

by Hölder's inequality, and thus the functional space

$$L^q(\alpha) = \left\{ f : \|f\|_{L^q(\alpha)}^q = \langle |f|^q, \alpha \rangle = \sum_{k \geq 0} |f(k)|^q \alpha(k) < \infty \right\}$$

is in duality with the Banach space for signed measures

$$\mathscr{M}^p(\alpha) = \left\{ \mu \in \mathscr{M} : \|\mu\|_{\mathscr{M}^p(\alpha)}^p = \sum_{k \geq 0} \left| \frac{\mu(k)}{\alpha(k)} \right|^p \alpha(k) = \sum_{k \geq 0} |\mu(k)|^p \alpha(k)^{1-p} < \infty \right\}.$$

We have

$$\|\mu\|_{\mathscr{M}^p(\alpha)}^p = \| \, |\mu| \, \|_{\mathscr{M}^p(\alpha)}^p = \|\mu^+\|_{\mathscr{M}^p(\alpha)}^p + \|\mu^-\|_{\mathscr{M}^p(\alpha)}^p$$

and Hölder's inequality implies that for $1 \leq a \leq b < \infty$ we have dense continuous injections

$$\mathscr{M}^b(\alpha) \subset \mathscr{M}^a(\alpha) \subset \mathscr{M}^1(\alpha) = \mathscr{M}, \qquad \|\alpha\|_{\mathrm{TV}}^{-1/a} \|\mu\|_{\mathscr{M}^a(\alpha)} \leq \|\alpha\|_{\mathrm{TV}}^{-1/b} \|\mu\|_{\mathscr{M}^b(\alpha)}.$$

We state the following elementary fact as a lemma for further reference.

Lemma 1. *For $p > 1$, the $\mathscr{M}^p(\alpha)$ norm dominates the $\mathscr{M}^p(\beta)$ norm if and only if $\alpha = O(\beta)$, and these two norms are equivalent if and only if $\alpha = \Theta(\beta)$.*

The spaces $L^p(\alpha)$ are a classic choice for the study of stability bounds for semigroups, as in Saloff-Coste [21] and Roberts and Rosenthal [19]. When the semigroup has α as an invariant measure or is self-adjoint (or reversible) with respect to α, then the Hilbert space $L^2(\alpha)$ is a natural setting, in which Dirichlet forms, the resolution of the identity (see Rudin [20]) or other spectral decompositions such as Karlin and McGregor's [12, 13] for birth and death processes, see also [3, 4, 6, 7], may yield exponential stability bounds through spectral gap estimates. The book of Chen [5] gives many such stability bounds.

2.3 The Hardy inequality and some related Banach spaces

We write Hardy's inequality as follows: for any measure λ and function f on $\mathbb{R}$ and $p > 1$,

$$\int \left| \frac{\int_{[x,\infty[} f(y)\,\lambda(dy)}{\lambda[x,\infty[} \right|^p \lambda(dx) \le \left(\frac{p}{p-1}\right)^p \int |f(x)|^p \lambda(dx)\,. \tag{1}$$

This is obtained considering the image of λ and f by $x \mapsto -x$ from the inequality

$$\int \left| \frac{\int_{]-\infty,x]} f(y)\,\lambda(dy)}{\lambda]-\infty,x]} \right|^p \lambda(dx) \le \left(\frac{p}{p-1}\right)^p \int |f(x)|^p \lambda(dx)$$

itself derived from Hardy's classical result [11, Theorem 330] for the Lebesgue measure on $\mathbb{R}_+$ by Sinnamon [22, Theorem 1.1] using the non-increasing rearrangement f^* of f with respect to λ. A related alternative derivation, more amenable to probabilists, uses that λ is the image of the Lebesgue measure by $G(x) = \inf\{\, y \in \mathbb{R} : x \le \lambda]-\infty,y]\,\}$, the left-continuous inverse of the cumulative distribution function.

We consider $\alpha > 0$ in $\mathscr{M}$ and for $p \ge 1$ the Banach spaces for signed measures

$$\mathscr{B}^p(\alpha) = \left\{ \mu \in \mathscr{M} : \|\mu\|^p_{\mathscr{B}^p(\alpha)} = \sum_{k\ge 0} \left(\frac{|\mu|[k,\infty[}{\alpha[k,\infty[} \right)^p \alpha(k) < \infty \right\}$$

where $|\mu|[k,\infty[= \sum_{i\ge k} |\mu(i)| = |\mu(k)| + |\mu(k+1)| + \cdots$.

Theorem 1. *Let $\alpha > 0$ be in $\mathscr{M}$, $C(\alpha) = \sup_{k\ge 0} \frac{\alpha[k,\infty[}{\alpha(k)} \in]1,\infty]$ and $p > 1$. Then*

$$\|\cdot\|_{\mathscr{B}^p(\alpha)} \le \frac{p}{p-1} \|\cdot\|_{\mathscr{M}^p(\alpha)}\,, \qquad \|\cdot\|_{\mathscr{M}^p(\alpha)} \le C(\alpha) \|\cdot\|_{\mathscr{B}^p(\alpha)}\,,$$

and if $C(\alpha) < \infty$ then the $\mathscr{M}^p(\alpha)$ and $\mathscr{B}^p(\alpha)$ norms are equivalent.

Proof. The Hardy inequality (1) with $\lambda = \alpha = \sum_{k\in\mathbb{N}} \alpha(k)\delta_k$ and $f = \frac{d|\mu|}{d\alpha}$ yields

$$\sum_{k\ge 0} \left(\frac{|\mu|[k,\infty[}{\alpha[k,\infty[} \right)^p \alpha(k) \le \left(\frac{p}{p-1}\right)^p \sum_{k\ge 0} \left| \frac{\mu(k)}{\alpha(k)} \right|^p \alpha(k)$$

and clearly

$$\sum_{k\ge 0} \left(\frac{|\mu|[k,\infty[}{\alpha[k,\infty[} \right)^p \alpha(k) \ge \sum_{k\ge 0} \left(\frac{\alpha(k)}{\alpha[k,\infty[} \right)^p \left| \frac{\mu(k)}{\alpha(k)} \right|^p \alpha(k) \ge \frac{1}{C(\alpha)^p} \sum_{k\ge 0} \left| \frac{\mu(k)}{\alpha(k)} \right|^p \alpha(k)\,.$$

□

When $C(\alpha) < \infty$ we say that α is *quasi-geometrically decreasing*, or has an *exponential tail*. We will explain in the study of birth-death processes in Section 3 below how a result of Liggett [16, Cor. 3.8] (also [5, Theorem 5.5 p. 93]) shows that this is a rather weak assumption for our purposes. We recall a classical characterization [15].

Lemma 2. *The measure α is quasi-geometrically decreasing if and only if there are constants $m \geq 1$ and $r < 1$ and $R < \infty$ such that, for all $k \in \mathbb{N}$,*

$$\alpha(k+m) \leq r\alpha(k), \qquad \alpha(k+1) \leq R\alpha(k).$$

Then $C(\alpha) \leq \frac{1}{1-r}\frac{1-R^m}{1-R}$ for $R \neq 1$, $C(\alpha) \leq \frac{m}{1-r}$ for $R = 1$, and $\alpha(k) = O(r^{k/m})$.

Proof. The sufficiency and upper bound follow from

$$\alpha[k,\infty[= \sum_{i\geq 0}\sum_{j=0}^{m-1}\alpha(k+im+j) \leq \alpha(k)\sum_{i\geq 0} r^i \sum_{j=0}^{m-1} R^j .$$

The necessity follows from the fact that if $C(\alpha) < \infty$, then for any $n \in \mathbb{N}$,

$$C(\alpha)\alpha(k) \geq \alpha(k) + \cdots + \alpha(k+n) \geq \left(\frac{n}{C(\alpha)} + 1\right)\alpha(k+n), \quad C(\alpha)\alpha(k) \geq \alpha(k+1),$$

so that we may take m in $\mathbb{N}$ large enough that $\frac{C(\alpha)^2}{m+C(\alpha)} \leq r < 1$ and $R = C(\alpha)$. $\square$

For $\mu > 0$ we have

$$\|\mu\|^p_{\mathscr{B}^p(\alpha)} = \sum_{k\geq 0}\left(\frac{|\mu|[k,\infty[}{\mu(k)}\right)^p \left(\frac{\alpha(k)}{\alpha[k,\infty[}\right)^p \left|\frac{\mu(k)}{\alpha(k)}\right|^p \alpha(k)$$

and the second inequality in Theorem 2 is asymptotically saturated by $\alpha(k) = a^k$ and $\mu(k) = m^k$ for $0 < m < a < 1$ as m goes to 0, since $C(\alpha) = \frac{1}{1-a}$ and $\|\mu\|_{\mathscr{B}^p(\alpha)} = \frac{1-a}{1-m}\|\mu\|_{\mathscr{M}^p(\alpha)} < \infty$. Hence, this inequality is optimal.

Since $f \in L^p(\mathbb{R}_+, dx) \mapsto \frac{1}{x}\int_0^x f(y)\,dy \in L^p(\mathbb{R}_+, dx)$ is one-to-one but not onto, it is hopeless to try to find such reverse Hardy inequalities for completely general integrands. Results such as those in [2, (26)], [18, Theorem 4] and [15] have very restrictive assumptions, such as non-decreasing integrands or $p \leq 1$, and cannot be used for our purposes.

2.4 Stochastic domination and inequality transfer

In $\mathscr{M}$ we write $\mu \le^d \nu$ or $\nu \ge^d \mu$ and say that μ *is dominated by* ν or that ν *dominates* μ if

$$|\mu|[k,\infty[\, \le |\nu|[k,\infty[\,, \qquad \forall k \in \mathbb{N}. \tag{2}$$

For probability measures this is the notion of stochastic domination, for sub-probability measures of stochastic domination on the extended state-space $\mathbb{N} \cup \{-1\}$. It is much weaker than term-wise domination.

If S and T are operators on L^∞ such that $S^*\mu \le^d T^*\mu$ for all $\mu \in \mathscr{M}$, we say that S is dominated by T or S^* is dominated by T^*, and denote it by $S \le^d T$ or $S^* \le^d T^*$, etc. If the operators are positivity-preserving, it is enough to check this for Dirac masses μ, and equivalently $Sf \le Tf$ for all positive increasing f (it suffices to take the $\mathbb{1}_{[k,\infty[}$).

We extend these notions to operators on $L^q(\alpha)$ for $\alpha > 0$ in $\mathscr{M}$ and $q > 1$, which have adjoint operator on $\mathscr{M}^p(\alpha)$ for the conjugate exponent $p > 1$.

We have introduced this custom-made framework for the following theorem, which is the main result of the paper. Its deceptively short proof involves all the above ideas and tools.

Theorem 2. *Let $\alpha > 0$ in $\mathscr{M}$ be such that $C(\alpha) = \sup_{k\ge 0} \frac{\alpha[k,\infty[}{\alpha(k)} < \infty$ and conjugate exponents $1 < p,q < \infty$. If S and T are operators on $L^q(\alpha)$ such that $S \le^d T$ then*

$$\|S^*\mu\|_{\mathscr{M}^p(\alpha)} \le \frac{p}{p-1} C(\alpha) \|T^*\mu\|_{\mathscr{M}^p(\alpha)}, \qquad \mu \in \mathscr{M}^p(\alpha).$$

Proof. We have

$$\|S^*\mu\|_{\mathscr{M}^p(\alpha)} \le C(\alpha)\|S^*\mu\|_{\mathscr{B}^p(\alpha)} \le C(\alpha)\|T^*\mu\|_{\mathscr{B}^p(\alpha)} \le C(\alpha)\frac{p}{p-1}\|T^*\mu\|_{\mathscr{M}^p(\alpha)}$$

using Theorem 1, and the definition of domination and of $\mathscr{B}^p(\alpha)$. □

This result will be applied to semigroups such that $S_t \le^d T_t$ for all $t \ge 0$, yielding that the generators also satisfy the inequality. In this situation, if $(T_t^*)_{t\ge 0}$ is exponentially stable then so is $(S_t^*)_{t\ge 0}$, with the same exponent: for $p > 1$ and μ in $\mathscr{M}^p(\alpha)$ and $\gamma > 0$ and $K_\mu < \infty$,

$$\|T_t^*\mu\|_{\mathscr{M}^p(\alpha)} \le K_\mu e^{-\gamma t} \Rightarrow \|S_t^*\mu\|_{\mathscr{M}^p(\alpha)} \le \frac{p}{p-1} C(\alpha) K_\mu e^{-\gamma t}, \qquad t \ge 0.$$

This can be used for proofs of strong ergodicity in the sense of Chen [5].

2.5 Intrinsic formulations versus identifications

A natural identification between $\mathscr{M}$ and the summable sequence space ℓ^1, and between L^∞ and the bounded sequence space ℓ^∞, is obtained by identifying a signed measure and the sequence of its atoms, and a function with the sequence of its values.

Thus, a semigroup $(T_t)_{t\geq 0}$ may be identified with its sub-stochastic matrix indexed by $\mathbb{N}\times\mathbb{N}$ in the canonical basis, and its generator with a matrix with positive terms off the diagonal and negative row sums (in the wide sense), sometimes called a Q-matrix. The row sum is null for Markovian generators.

In matrix notation we further identify $\mu \in \mathscr{M}$ to a row vector and $f \in L^\infty$ to a column vector, and adjunction may be replaced by multiplication to the left of the matrices, so that

$$\langle \mu, f\rangle = \mu f\,, \qquad T_t^*\mu = \mu T_t\,, \qquad \langle T_t^*\mu, f\rangle = \langle \mu, T_t f\rangle = \mu T_t f\,.$$

These practical notations will be used in the sequel.

Intrinsic notations helped clarify the above arguments, and other interesting identifications exist. For instance, in the study of self-adjoint operators on $L^2(\alpha)$, one often identifies $\mu \in \mathscr{M}^p(\alpha)$ and its density $\frac{d\mu}{d\alpha} \in L^2(\alpha)$, and the duality bracket between measures and functions with the $L^2(\alpha)$ scalar product, see [21, 19].

3 Some applications using birth and death processes

3.1 Preliminaries

Karlin and McGregor [12, 13] studied irreducible sub-Markovian birth and death processes on $\mathbb{N}$, with birth rates $\lambda_n = A(n,n+1) > 0$ in states $n \geq 0$, death rates $\mu_n = A(n,n-1) > 0$ in states $n \geq 1$, and a killing rate $\mu_0 \geq 0$ in state 0. When $\mu_0 > 0$ the process may be rendered Markovian by adding a cemetery or absorbing state -1. The cases $\mu_0 > 0$ and $\mu_0 = 0$ may be related by a duality procedure.

The infinitesimal generator on $\mathbb{N}$ of such processes is given in matrix form by

$$A = (A(i,j))_{i,j\in\mathbb{N}} = \begin{pmatrix} -(\lambda_0+\mu_0) & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \qquad (1)$$

and its *potential coefficients* is given by the row vector

$$\alpha = (\alpha(n))_{n\in\mathbb{N}}\,, \qquad \alpha(n) = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\,, \tag{2}$$

which solves the detailed balance equations $\alpha(x)A(x,y) = \alpha(y)A(y,x)$ for $x,y \in \mathbb{N}$, so that A is self-adjoint in $L^2(\alpha)$. If $\mu_0 = 0$ then α is an invariant measure, but not if $\mu_0 > 0$.

The notation of [12, 13] for α is π, but we reserve it for the invariant law when it exists, which implies $\mu_0 = 0$.

We are interested in the existence and uniqueness for the (possibly defective) process and for its backward and forward Kolmogorov equations. The forward equation $\dot\nu_t = \nu_t A$ is developed, with the convention $\lambda_{-1}\nu_t(-1) = 0$, into

$$\dot\nu_t(n) \;=\; \lambda_{n-1}\nu_t(n-1) - (\lambda_n + \mu_n)\nu_t(n) + \mu_{n+1}\nu_t(n+1)\,, \qquad n \geq 0\,. \tag{3}$$

We assume from now on that

$$\sum_{n=0}^{\infty}\left(\alpha(n) + \frac{1}{\lambda_n\alpha(n)}\right) = \infty\,, \qquad \|\alpha\|_{\mathrm{TV}} = \sum_{k\in\mathbb{N}}\alpha(k) < \infty\,. \tag{4}$$

The first condition is necessary and sufficient for these existence and uniqueness results, in particular for (3) in $\mathscr{M}$ and if ν_0 is in $\mathscr{P}(\mathbb{N})$ then ν_t is a sub-probability measure, see Karlin and McGregor [13, Introduction], [12, Theorems 14,15].

The second is the "ergodic" case: either $\mu_0 = 0$, the process is recurrent positive, and

$$\pi := \frac{\alpha}{\|\alpha\|_{\mathrm{TV}}} \in \mathscr{M}^2(\alpha) \tag{5}$$

is its unique invariant law [13, Theorem 2], or else $\mu_0 > 0$, the process is absorbed ergodically at 0, and there is no invariant law [13, Sect. 5].

The condition [5, (1.24) p. 9] (in which $\mu_0 = 0$ and the notations are different) is obviously equivalent to (4).

In Callaert and Keilson [3, p. 209] the conditions in (4) are respectively called $\bar{A}$ and $\bar{B}$, which together imply $\bar{C}$ and $\bar{D}$ (denoted by C and D in [6]), and then the process has a natural boundary at infinity (intuitively, there is no loss or gain of probability at infinity).

3.2 Karlin and McGregor's spectral decomposition

The representation of Karlin and McGregor [12, 13] yields in particular the results in the previous subsection. Moreover, van Doorn and Zeifman [8] pointed out that this representation is valid for a birth and death process with killing in *every* state (not only 0) by relating it to a conservative one (without any killing).

The equation $AQ(x) = -xQ(x)$ for an eigenvector $Q(x) = (Q_n(x))_{n\geq 0}$ of eigenvalue $-x$ (where $x \in \mathbb{R}_+$) is developed into

$$\lambda_0 Q_1(x) = (\lambda_0+\mu_0-x)Q_0(x)\,, \quad \lambda_n Q_{n+1}(x) = (\lambda_n+\mu_n-x)Q_n(x) - \mu_n Q_{n-1}(x)\,,\ n\geq 1\,.$$

With the natural choice $Q_0 = 1$ and convention $Q_{-1} = 0$ we obtain inductively Q_n as the polynomial of degree n satisfying

$$-xQ_n(x) = \mu_n Q_{n-1}(x) - (\lambda_n+\mu_n)Q_n(x) + \lambda_n Q_{n+1}(x)\,, \qquad n\geq 0\,. \tag{6}$$

These recursions correspond to [12, (2.1)] and [7, (2.15)]. A crucial fact is that a sequence of polynomials satisfying such a recursion is orthogonal with respect to a probability measure ψ on $\mathbb{R}_+$, and precisely

$$\int_0^\infty Q_i(x)^2\,\psi(dx) = \alpha(i)^{-1}\,, \qquad \int_0^\infty Q_i(x)Q_j(x)\,\psi(dx) = 0\,, \qquad i\neq j\in\mathbb{N}\,,$$

or in matrix notation, with Q considered as a row vector,

$$\int_0^\infty Q(x)Q(x)^*\,\psi(dx) = \mathrm{diag}\left(\alpha^{-1}\right).$$

Let $P_t = (P_t(i,j))_{i,j\in\mathbb{N}}$ denote the sub-stochastic transition matrix for A, in semigroup notation $P_t = \mathrm{e}^{At}$. The fundamental solution for the forward Kolmogorov equation (3) is given by $P_t^* = \mathrm{e}^{A^*t}$, or by $P_t = \mathrm{e}^{At}$ with left-multiplication by row vectors. Karlin and McGregor's representation formula [12, (1.7)], [13, (0.12)], [7, (1.2),(2.18)], is

$$P_t(i,j) = \alpha(j)\int_0^\infty \mathrm{e}^{-xt}Q_i(x)Q_j(x)\,\psi(dx)\,, \qquad i,j\in\mathbb{N}\,, \tag{7}$$

or in matrix notation

$$P_t = \mathrm{e}^{At} = \int_0^\infty \mathrm{e}^{-xt}Q(x)Q(x)^*\,\psi(dx)\,\mathrm{diag}(\alpha)\,.$$

The probability measure ψ is called the spectral measure, and its support S, called the spectrum, is intimately related to the set of zeros of the orthogonal polynomials. Since $P_t = \mathrm{e}^{At}$ is self-adjoint in $L^2(\alpha)$, the spectral representation yields

$$\|\nu P_t\|^2_{\mathscr{M}^2(\alpha)} := \|\nu\mathrm{e}^{At}\|^2_{\mathscr{M}^2(\alpha)} = \left(\nu\mathrm{e}^{2At},\nu\right)_{\mathscr{M}^2(\alpha)} = \nu\int_S \mathrm{e}^{-2xt}Q(x)Q(x)^*\,\psi(dx)\,\mathrm{diag}(\alpha)\nu^*\,. \tag{8}$$

Karlin and McGregor's spectral decomposition is not a resolution of the identity, see Rudin [20, pp. 301–311], but the spectrum is obviously the same.

3.3 Long time behavior and exponential stability

We obtain from (7) and dominated convergence that

$$\lim_{t\to\infty} P_t(i,j) = \psi(0)\alpha(j)\,, \qquad i,j\in\mathbb{N}\,,$$

so that (8) yields for

$$\gamma = \min(S-\{0\})\,, \qquad \mathscr{H} = \perp_{\mathscr{M}^2(\alpha)} \psi(0)\alpha\,,$$

that

$$\|\nu P_t\|^2_{\mathscr{M}^2(\alpha)} := \|\nu \mathrm{e}^{At}\|^2_{\mathscr{M}^2(\alpha)} \le \mathrm{e}^{-2\gamma t}\|\nu\|^2_{\mathscr{M}^2(\alpha)}\,, \qquad \nu\in\mathscr{H}\,. \tag{9}$$

The alternative surrounding (5) then yields the following alternative:

- if $\mu_0 = 0$ then $\psi(0) > 0$ and hence

  $$\mathscr{H} = \left\{\nu\in\mathscr{M}^2(\alpha) : \sum_{n\in\mathbb{N}} \nu(n) = 0\right\} = \mathrm{Span}\big(\mathscr{P}(\mathbb{N}) - \mathscr{P}(\mathbb{N})\big)\cap\mathscr{M}^2(\alpha)$$

  so that instantaneous laws $p_t = p_0P_t = p_0\mathrm{e}^{At}$ and $q_t = q_0P_t = q_0\mathrm{e}^{At}$, $t\ge 0$, satisfy

  $$\|p_t - q_t\|_{\mathscr{M}^2(\alpha)} \le \mathrm{e}^{-\gamma t}\|p_0-q_0\|_{\mathscr{M}^2(\alpha)}\,, \qquad p_0,q_0\in\mathscr{P}(\mathbb{N})\cap\mathscr{M}^2(\alpha)\,,$$

  which is true in particular for $q_t = q_0 = \pi$ given in (5), the invariant law,

- if $\mu_0 > 0$ then $\psi(0) = 0$ and

  $$\mathscr{H} = \mathscr{M}^2(\alpha)$$

  so that

  $$\|p_t\|_{\mathscr{M}^2(\alpha)} \le \mathrm{e}^{-\gamma t}\|p_0\|_{\mathscr{M}^2(\alpha)}\,, \qquad p_0\in\mathscr{P}(\mathbb{N})\cap\mathscr{M}^2(\alpha)\,.$$

More generally, the forward Kolmogorov equation $\dot\nu_t = \nu_t A$ starting at $\nu_0 = \nu$, made explicit in (3), has solution $\nu_t = \nu\mathrm{e}^{At}$ for $t\ge 0$, so that (9) implies that if $\gamma > 0$ then this equation is globally exponentially stable at 0 in $\mathscr{H}$ and globally exponentially stable at

$$\big(\|\nu_0^+\|_{\mathrm{TV}} - \|\nu_0^-\|_{\mathrm{TV}}\big)\,\mathbb{1}_{\mu_0=0}\pi \in \mathscr{M}^2(\alpha)\,.$$

3.4 Spectral gap criteria and estimates

Many tractable spectral gap criteria and upper and lower bounds exist, see Callaert [3, 4] and van Doorn [6, 7, 8] who use Karlin and McGregor's spectral decomposition, and the impressive wealth of information and bibliography in Chen [5, Index p. 226, birth-death process].

In particular, a beautiful result of Liggett [16, Cor. 3.8], see also [5, Theorem 5.5 p. 93], sheds titillating light on the assumption of quasi-geometrical decrease.

Theorem 3 (T. Liggett). *Let an irreducible birth and death process satisfy*

$$\mu_0 = 0\,, \qquad 0 < \inf_{n\geq 0} \lambda_n < \sup_{n\geq 0} \lambda_n < \infty\,,$$

and have an invariant law π*. Then there exists a spectral gap if and only if* π *is quasi-geometrically decreasing: with the notations of Theorems 1 and 2,*

$$\gamma > 0 \Leftrightarrow C(\pi) < \infty \Leftrightarrow C(\alpha) < \infty\,.$$

Under these assumptions, Lemma 2 implies that $C(\alpha) < \infty \Rightarrow \inf_{n\geq 1} \mu_n > 0$.

Also, following Van Doorn [6, Sect. 2.2], [7, Sect. 2.3], Q_n has n increasing zeros $(x_{n,i})_{1\leq i\leq n}$ such that

$$0 < \cdots < x_{n+1,i} < x_{n,i} < x_{n+1,i+1} < \cdots$$

and hence $\xi_i = \lim_{n\to\infty} x_{n,i} \geq 0$ exists, $\xi_i \leq \xi_{i+1}$, and $\sigma = \lim_{i\to\infty} \xi_i$ exists in $[0,\infty]$. Moreover

$$\gamma > 0 \Leftrightarrow \sigma > 0$$

and σ is *not* affected by a *finite* number of changes in the birth and death rates, see [7, Theorem 5.1] and the explanation thereafter, whereas γ may vary *greatly*.

Many practical upper and lower bounds for σ exist, such as

$$\sigma \geq \liminf_{n\to\infty} \{\lambda_n + \mu_n - \sqrt{\lambda_{n-1}\mu_n} - \sqrt{\lambda_n \mu_{n+1}}\} \tag{10}$$

given in [7, Theorem 5.3 (i)] which implies for instance that $\gamma > 0$ if

$$\liminf_{n\to\infty} \mu_n > 0\,, \quad \liminf_{n\to\infty} \frac{\lambda_n}{\mu_n} = \rho > 0\,, \quad \limsup_{n\to\infty} \left\{ \sqrt{\frac{\lambda_{n-1}}{\lambda_n}} + \sqrt{\frac{\mu_{n+1}}{\mu_n}} \right\} < \frac{\rho+1}{\sqrt{\rho}}\,. \tag{11}$$

3.5 An application to operators which are not necessarily self-adjoint

The infinitesimal generator $B = (B(i,j))_{i,j\in\mathbb{N}}$ of a sub-Markovian process on $\mathbb{N}$ satisfies

$$B(i,j) \ge 0 \ (i \ne j)\,, \qquad b(i) := -\sum_{j\in\mathbb{N}} B(i,j) \ge 0\,,$$

and the killing rate $b(i)$ can be interpreted as the absorption rate into a cemetery state -1.

We assume that for a birth and death sub-Markovian generator A satisfying the assumptions in Subsection 3.1, we have

$$\begin{aligned} B(i,j) = 0 \ (j > i+1)\,, \qquad B(i,i+1) \le \lambda_i := A(i,i+1)\,, \\ b(i) + \sum_{j<i} B(i,j) := -B(i,i) - B(i,i+1) \ge \mu_i := A(i,i-1)\,. \end{aligned}$$

Such generators B are widely studied, and called single birth Q-matrices by Chen [5].

A simple coupling argument shows that if $p_0 \le^d q_0$ are in $\mathscr{P}(\mathbb{N})$ then $p_0 \mathrm{e}^{Bt} \le^d q_0 \mathrm{e}^{At}$ for all $t \ge 0$, and this extends by linearity and sign preservation to initial data in $\mathscr{M}(\mathbb{N})$, so that we may apply Theorem 2 for any α satisfying its assumptions.

We assume that $C(\alpha) < \infty$ so as to use Theorem 2, and that there is a spectral gap $\gamma > 0$, for which there are numerous tractable criteria and lower bounds, see Section 3.4. Then Subsection 3.3 shows that B is exponentially stable on $\mathscr{H}$: there is $K < \infty$ with an explicit upper bound in terms of the rates such that

$$\begin{aligned} \|\nu \mathrm{e}^{Bt}\|_{\mathscr{M}^2(\alpha)} \le \mathrm{e}^{-\gamma t} K \|\nu\|_{\mathscr{M}^2(\alpha)}\,, \qquad \nu \in \mathscr{H}\,, \\ \|p_0 \mathrm{e}^{Bt} - q_0 \mathrm{e}^{Bt}\|_{\mathscr{M}^2(\alpha)} \le \mathrm{e}^{-\gamma t} K \|p_0 - q_0\|_{\mathscr{M}^2(\alpha)}\,, \qquad p_0, q_0 \in \mathscr{P}(\mathbb{N}) \cap \mathscr{M}^2(\alpha)\,. \end{aligned}$$

The stochastic domination assumption implies using positive recurrence that if $\mu_0 = 0$ then B is Markovian and has a unique invariant law, which is in $\mathscr{M}^2(\alpha)$ (this also follows from a classical fixed-point argument), and we have exponential convergence of the instantaneous laws p_t to this invariant law in $\mathscr{M}^2(\alpha)$ for any initial law p_0 in this space.

This result implies strong ergodicity for the process in the sense of Chen [5].

Note that B may *only* be self-adjoint when it is itself the infinitesimal generator of a birth-death process, and even then we may thus obtain a result for a weaker scalar product than the one for which it is self-adjoint. Results for such *weaker* norms can actually be *stronger* in the sense that they are applicable to much more general initial values.

This concept was essential in Graham [10], where the global exponential stability result was used to prove tightness of the initial values of the fluctuations in equilibrium, interpreted as long-time limits, see the discussion therein. The situation was such that $\mu_0 > 0$.

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