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CENTRE DE MATHÉMATIQUES APPLIQUÉES

UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 46 00. Fax: 01 69 33 46 46
http://www.cmap.polytechnique.fr/

Uniform estimates for metastable transition times in a coupled bistable system

Florent Barret, Anton Bovier and Sylvie Méléard

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UNIFORM ESTIMATES FOR METASTABLE TRANSITION TIMES IN A COUPLED BISTABLE SYSTEM

FLORENT BARRET, ANTON BOVIER, AND SYLVIE MÉLÉARD

ABSTRACT. We consider a coupled bistable N-particle system on $\mathbb{R}^N$ driven by a Brownian noise, with a strong coupling corresponding to the synchronised regime. Our aim is to obtain sharp estimates on the metastable transition times betwen the two stable states, both for fixed N and in the limit when N tends to infinity. These estimates would be the main step for a rigorous understanding of the metastable behavior of infinite dimensional systems, as the stochastically perturbed Ginzburg-Landau equation. The quantities of interest are objects of potential theory, as capacities and equilibrium measure. We prove estimates with error bounds that are uniform in the dimension of the system.

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1. INTRODUCTION

Our aim in this paper is to analyze the behavior of metastable transition times for a gradient diffusion model independently of the dimension.

More precisely, we deal here with a model of a chain of coupled particles subjected to a double well potential driven by Brownian noise (see e.g. [2]). We consider the system of stochastic differential equations

$$\mathrm{d}X_\epsilon(t) = -\nabla F_{\gamma,N}(X_\epsilon(t))\mathrm{d}t + \sqrt{2\epsilon}\mathrm{d}B(t) \tag{1.1}$$

where $X_\varepsilon(t) \in \mathbb{R}^N$ and

$$F_{\gamma,N}(x) = \sum_{i\in\Lambda} \frac{1}{4}x_i^4 - \frac{1}{2}x_i^2 + \frac{\gamma}{4}\sum_{i\in\Lambda}(x_i - x_{i+1})^2, \tag{1.2}$$

with $\Lambda = \mathbb{Z}/N\mathbb{Z}$ and $\gamma > 0$ is a parameter. B is a N dimensional Brownian motion and $\varepsilon > 0$ is the intensity of the noise. Each equation of this particle system has a bistable one dimensional drift. The equations are coupled to their closest neighbor with intensity γ and perturbed by independent noises of uniform magnitude ε. While the system without noise, i.e. $\varepsilon = 0$, has several stable fixpoints, for $\varepsilon > 0$, transitions between these fixpoints will occur at a suitable timescale. Such a situation is called metastability.

For fixed N and small ε, this problem has been widely studied in the literature and we refer to the books by Freidlin and Wentzell [8] and Olivieri and Vares [12] for further discussions. In recent years, the potential theory approach, initiated by Bovier, Eckhoff, Gayrard and Klein [5] (see [4] for a review) has allowed to give very precise results on such transition times and notably led to a proof of the so-called Eyring-Kramers formula which gives sharper asymptotics for this transition times, for any fixed dimension. Nevertheless, the techniques used in [5] cannot be directly applied to get results that are uniform in the dimension of the system.

Our aim in this paper is to obtain such uniform estimates. These estimates would be the main step for a rigorous understanding of the metastable behavior of infinite dimensional systems, i.e. stochastic partial differential equations (SPDE) such as the stochastically perturbed Ginzburg-Landau equation. Indeed the deterministic part of the system (1.1) can be seen as the discretization of the drift part of the SPDE, as it has been noticed in [3]. For a heuristic discussion of the metastable behaviour of this SPDE, see e.g. [11] and [14].

In the present paper, we consider only the simplest case, the so-called synchronisation regime, where the coupling γ between the particles is so strong that there are only three relevant critical points of the potential $F_{\gamma,N}$ (1.2). A generalization to more complex situations is however possible and will be treated elsewhere. As in [5], we will express in this paper the quantities of interest (hitting probabilities and transition times) in terms of objects of potential theory as capacities and equilibrium measure.

The remainder of this paper is organised as follows. In Section 2 we recall briefly the main results from the potential theory approach, we recall the key properties of the potential $F_{\gamma,N}$, and we state the results on metastability that follow from the results of [5] for fixed N. In Section 3 we deal with the case when N tends to infinity and state our main result, Theorem 3.1. In Section 4 we prove the main theorem through sharp estimates on the relevant capacities.

In the remainder of the paper we adopt the following notations:

- for $t \in \mathbb{R}$, $\lfloor t \rfloor$ denotes the unique integer k such that $k \leq t < k+1$;
- $\tau_D \equiv \inf\{t > 0 : X_t \in D\}$ is the hitting time of the set D for the process (X_t);
- $B_r(x)$ is the ball of radius $r > 0$ and center $x \in \mathbb{R}^N$;
- $V(A)$ denotes the volume of a set $A \subset \mathbb{R}^N$;
- for $p \geq 1$, and $(x_k)_{k=1}^N$ a sequence, we denote the L^P-norm of x by

$$\|x\|_p = \left(\sum_{k=1}^{N} |x_k|^p \right)^{1/p}. \tag{1.3}$$

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2. PRELIMINARIES

2.1. **Key formulas from the potential theory approach.** We recall briefly the basic formulas from potential theory that we will need here. For A, D regular open subsets of $\mathbb{R}^N$, let $h_{A,D}(x)$ be the harmonic function (with respect to the generator, L, of the diffusion) with boundary conditions 1 in A and 0 in D. Then, for $x \in (A \cup D)^c$, one has $h_{A,D}(x) = \mathbb{P}_x[\tau_A < \tau_D]$. Let us firstly define, (cf. [7]), the equilibrium measure $e_{A,D}$ as the unique measure on ∂A such that

$$h_{A,D}(x) = \int_{\partial A} e^{-F_{\gamma,N}(y)/\varepsilon} G_{D^c}(x,y) e_{A,D}(dy), \tag{2.1}$$

where G_{D^c} is the Green function associated with the generator L on the domain D^c. The following formula for the hitting time of D has been proved in [5]:

$$\int_{\partial A} \mathbb{E}_z[\tau_D] e^{-F_{\gamma,N}(z)/\varepsilon} e_{A,D}(dz) = \int_{D^c} h_{A,D}(y) e^{-F_{\gamma,N}(y)/\varepsilon} dy. \tag{2.2}$$

The capacity, $\operatorname{cap}(A,D)$, is defined as

$$\operatorname{cap}(A,D) = \int_{\partial A} e^{-F_{\gamma,N}(z)/\varepsilon} e_{A,D}(dz). \tag{2.3}$$

Therefore,

$$\nu_{A,D}(dz) = \frac{e^{-F_{\gamma,N}(z)/\varepsilon} e_{A,D}(dz)}{\operatorname{cap}(A,D)} \tag{2.4}$$

is a probability measure on ∂A, that we may call the equilibrium probability. The equation (2.2) then reads

$$\int_{\partial A} \mathbb{E}_z[\tau_D] \nu_{A,D}(dz) = \mathbb{E}_{\nu_{A,D}}[\tau_D] = \frac{\int_{D^c} h_{A,D}(y) e^{-F_{\gamma,N}(y)/\varepsilon} dy}{\operatorname{cap}(A,D)}. \tag{2.5}$$

The strength of this formula comes from the fact that the capacity has an alternative representation through the Dirichlet variational principle as in [9],

$$\operatorname{cap}(A,D) = \inf_{h \in \mathcal{H}} \Phi(h), \tag{2.6}$$

where

$$\mathcal{H} = \left\{ h \in W^{1,2}(\mathbb{R}^N, e^{-F_{\gamma,N}(u)/\varepsilon} du) \,|\, \forall z\,, h(z) \in [0,1]\,, h_{|A} = 1\,, h_{|D} = 0 \right\}, \tag{2.7}$$

and the Dirichlet form Φ is given for $h \in \mathcal{H}$ as

$$\Phi(h) = \varepsilon \int_{(A \cup D)^c} e^{-F_{\gamma,N}(u)/\varepsilon} \|\nabla h(u)\|_2^2 du. \tag{2.8}$$

Remark. Formula (2.5) gives an average of the transition time expectation with respect to the equilibrium measure, that we will extensively use in what follows. A way to obtain the quantity $\mathbb{E}_z[\tau_D]$ consists in using Hölder and Harnack estimates (as developed in Corollary 2.3), but it is far from obvious whether this can be extended to give estimates that are uniform in N.

Formula (2.5) highlights the two terms for which we will prove uniform estimates: the capacity (Theorem 4.3) and the mass of $h_{A,D}$ (Theorem 4.9).

2.2. **Description of the Potential.** Let us describe in details the potential $F_{\gamma,N}$, its stationary points, and in particular the minima and the 1-saddle points, through which the transitions occur.

The coupling strength γ specifies the geometry of $F_{\gamma,N}$. For instance, if we set $\gamma = 0$, we get a set of N bistable independent particles, thus the stationary points are

$$x^* = (\xi_1, \ldots, \xi_N) \quad \forall i \in [\![1,N]\!],\, \xi_i \in \{-1,0,1\}. \tag{2.9}$$

To characterize their stability, we have to look to their Hessian matrix whose signs of the eigenvalues give us the index saddle of the point. It can be easily shown that, for $\gamma = 0$, the minima are those of the form (2.9) with no zero coordinates and the 1-saddle points have just one zero coordinate. As γ increases the structure of the potential evolves and the number of stationary points decreases from 3^N to 3. We notice that, for all γ, the points

$$I_{\pm} = \pm(1, 1, \cdots, 1) \quad O = (0, 0, \cdots, 0) \tag{2.10}$$

are stationary, furthermore $I_{\pm}$ are minima. If we calculate the Hessian at the point O, we have

$$\nabla^2 F_\gamma(O) = \begin{pmatrix} -1+\gamma & -\frac{\gamma}{2} & 0 & \cdots & 0 & -\frac{\gamma}{2} \\ -\frac{\gamma}{2} & -1+\gamma & -\frac{\gamma}{2} & & & 0 \\ 0 & -\frac{\gamma}{2} & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & \ddots & \ddots & -\frac{\gamma}{2} \\ -\frac{\gamma}{2} & 0 & \cdots & 0 & -\frac{\gamma}{2} & -1+\gamma \end{pmatrix} \tag{2.11}$$

whose eigenvalues are, for all $\gamma > 0$ and for $0 \le k \le N-1$,

$$\lambda_{k,N} = -\left(1 - 2\gamma \sin^2\left(\frac{k\pi}{N}\right)\right). \tag{2.12}$$

Let us define, for $k \ge 1$, $\gamma_k^N = \frac{1}{2\sin^2(k\pi/N)}$, then these eigenvalues take the form

$$\begin{cases} \lambda_{k,N} & = \lambda_{N-k,N} = -1 + \frac{\gamma}{\gamma_k^N},\ 1 \le k \le N-1 \\ \lambda_{0,N} & = \lambda_0 = -1. \end{cases} \tag{2.13}$$

Let us observe that $(\gamma_k^N)_{k=1}^{\lfloor N/2 \rfloor}$ is a decreasing sequence, therefore as γ increases the number of non-positive eigenvalues $(\lambda_{k,N})_{k=0}^{N-1}$ decreases. When $\gamma > \gamma_1^N$, the only negative eigenvalue is -1. Thus

$$\gamma_1^N = \frac{1}{2\sin^2(\pi/N)} \tag{2.14}$$

is the threshold of the synchronization regime.

Lemma 2.1 (Synchronization Regime). *If $\gamma > \gamma_1^N$, the only stationary points are $I_{\pm}$ and O. $I_{\pm}$ are minima, O is a 1-saddle.*

This lemma is proved in [2] by using a Lyapunov function. This configuration is called the synchronization regime because the coupling between the particles is so strong that they all pass simultaneously through their respective saddle points in a transition between the stable equilibria ($I_{\pm}$).

In this paper, we will focus on the synchronization regime.

2.3. **Results for fixed N.** Let $\rho > 0$, we set $B_{\pm} = B_\rho(I_{\pm})$. In this setting, Equation (2.5) gives with $A = B_-$ and $D = B_+$

$$\mathbb{E}_{\nu_{B_-,B_+}}[\tau_{B_+}] = \frac{\int_{B_+^c} h_{B_-,B_+}(y) e^{-F_{\gamma,N}(y)/\varepsilon} dy}{\operatorname{cap}(B_-, B_+)}. \tag{2.15}$$

First, we obtain a sharp estimate for this transition time for fixed N:

Theorem 2.2. *Let $N > 2$ be given, for $\gamma > \gamma_1^N = \frac{1}{2\sin^2(\pi/N)}$, let $\sqrt{N} > \rho \geq \epsilon > 0$, we have*

$$\mathbb{E}_{\nu_{B_-,B_+}}[\tau_{B_+}] = 2\pi c_N e^{\frac{N}{4\epsilon}}(1 + O(\sqrt{\varepsilon|\ln\varepsilon|})) \tag{2.16}$$

with

$$c_N = \left[1 - \frac{3}{2+2\gamma}\right]^{\frac{e(N)}{2}} \prod_{k=1}^{\lfloor\frac{N-1}{2}\rfloor} \left[1 - \frac{3}{2+\frac{\gamma}{\gamma_k^N}}\right] \tag{2.17}$$

where $e(N) = 1$ if N is even and 0 if N is odd.

Remark. As mentioned above, for any fixed dimension, we can replace the probability measure ν_{B_-,B_+} by the single point I_-, using Hölder and Harnack inequalities. We get the following corollary:

Corollary 2.3. *Under the assumptions of Theorem 2.2, there exists $\alpha > 0$ such that,*

$$\mathbb{E}_{I_-}[\tau_{B_+}] = 2\pi c_N e^{\frac{N}{4\epsilon}}(1 + O(\sqrt{\varepsilon|\ln\varepsilon|}, \varepsilon^\alpha)). \tag{2.18}$$

Proof of the theorem. We apply Theorem 3.2 in [5]. For $\gamma > \gamma_1^N = \frac{1}{2\sin^2(\pi/N)}$, let us recall that there are only three stationary points: two minima $I_\pm$ and one saddle point O. One easily checks that F_γ satisfies the following assumptions:

- F_γ is polynomial in the $(x_i)_{i\in\Lambda}$ and so clearly C^3 on $\mathbb{R}^N$.
- $F_\gamma(x) \geq \frac{1}{4}\sum_{i\in\Lambda} x_i^4$ so $F_\gamma \underset{x\to\infty}{\longrightarrow} +\infty$.
- $\|\nabla F_\gamma(x)\|_2 \sim \|x\|_3^3$ as $\|x\|_2 \to \infty$.
- As $\Delta F_\gamma(x) \sim 3\|x\|_2^2$ ($\|x\|_2 \to \infty$), then $\|\nabla F_\gamma(x)\| - 2\Delta F_\gamma(x) \sim \|x\|_3^3$.

The Hessian matrix at the minima $I_\pm$ has the form

$$\nabla^2 F_\gamma(I_\pm) = \begin{pmatrix} 2+\gamma & -\frac{\gamma}{2} & 0 & \cdots & 0 & -\frac{\gamma}{2} \\ -\frac{\gamma}{2} & 2+\gamma & -\frac{\gamma}{2} & & & 0 \\ 0 & -\frac{\gamma}{2} & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & \ddots & \ddots & -\frac{\gamma}{2} \\ \frac{\gamma}{2} & 0 & \cdots & 0 & -\frac{\gamma}{2} & 2+\gamma \end{pmatrix} \tag{2.19}$$

whose eigenvalues are

$$\begin{cases} \nu_{k,N} & = \nu_{N-k,N} = 2 + \frac{\gamma}{\gamma_k^N},\ 1 \leq k \leq N-1 \\ \nu_{0,N} & = \nu_0 = 2. \end{cases} \tag{2.20}$$

Observe that unlike the $(\lambda_{k,N})$, those eigenvalues are always positive. The stationary points $I_\pm$ are always minima. Then the so-called Eyring-Kramers formula applies and gives the following result (cf. [5]), for $\sqrt{N} > \rho > \epsilon > 0$,

$$\mathbb{E}_{\nu_{B_-,B_+}}[\tau_{B_+}] = \frac{2\pi e^{\frac{N}{4\epsilon}}}{\sqrt{\det(\nabla^2 F_{\gamma,N}(I_-))}\frac{1}{\sqrt{|\det(\nabla^2 F_{\gamma,N}(O))|}}}(1 + O(\sqrt{\epsilon}|\ln\epsilon|)). \tag{2.21}$$

Finally, (2.13) and (2.20) give:

$$\det(\nabla^2 F_{\gamma,N}(I_-)) = \prod_{k=0}^{N-1} \nu_{k,N} = 2\nu_{N/2,N}^{e(N)} \prod_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \nu_{k,N}^2 = 2^N (1+\gamma)^{e(N)} \prod_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \left(1 + \frac{\gamma}{2\gamma_k^N}\right)^2 \tag{2.22}$$

$$|\det(\nabla^2 F_{\gamma,N}(O))| = \prod_{k=0}^{N-1} \lambda_{k,N} = \lambda_{N/2,N}^{e(N)} \prod_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \lambda_{k,N}^2 = (2\gamma-1)^{e(N)} \prod_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \left(1 - \frac{\gamma}{\gamma_k^N}\right)^2. \tag{2.23}$$

Then,

$$c_N = \frac{\sqrt{\det(\nabla^2 F_{\gamma,N}(I_-))}}{\sqrt{|\det(\nabla^2 F_{\gamma,N}(O))|}} = \left[1 - \frac{3}{2+2\gamma}\right]^{\frac{e(N)}{2}} \prod_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \left[1 - \frac{3}{2+\frac{\gamma}{\gamma_k^N}}\right] \tag{2.24}$$

and Theorem 2.2 is proved. ☐

Proof of the corollary. We refer to Proposition 6.1 in [5]: $w_{B_+}(x) = \mathbb{E}_x[\tau_{B_+}]$ solves an inhomogeneous Dirichlet problem with the generator L of the diffusion:

$$\begin{cases} Lw_{B_+}(x) & = 1, \quad x \in B_+^c \\ w_{B_+}(x) & = 0, \quad x \in B_+. \end{cases} \tag{2.25}$$

Then, we can apply usual pointwise estimates from elliptic partial differential equations theory such as Harnack and Hölder estimates (e.g. Corollaries 9.24, 9.25 in [10]) to obtain: there is $\alpha > 0$ such that, for $x \in \partial B_-$

$$w_{B_+}(x) = w_{B_+}(I_-)\left(1 + O(\varepsilon^\alpha)\right). \tag{2.26}$$

We conclude by an integration with respect to the equilibrium measure. ☐

Let us point out that the use of these estimates is a major obstacle to obtain a mean transition time starting from a single stable point with uniform error terms. That is the reason why we have introduced the equilibrium probability. Although, there are several difficulties to be overcome if we want to pass to the limit $N \uparrow \infty$.

(i) We must show that the prefactor c_N has a limit as $N \uparrow \infty$.
(ii) The exponential term in the hitting time tends to infinity with N. This suggests that one needs to rescale the potential $F_{\gamma,N}$ by a factor $1/N$, or equivalently, to increase the noise strength by a factor N.
(iii) One will need uniform control of error estimates in N to be able to infer the metastable behavior of the infinite dimensional system. This will be the most subtle of the problems involved.

3. Large N limit

As mentioned above, in order to prove a limiting result as N tends to infinity, we need to rescale the potential to eliminate the N dependence in the exponential. Thus henceforth we replace $F_{\gamma,N}(x)$ by

$$G_{\gamma,N}(x) = N^{-1} F_{\gamma,N}(x). \tag{3.1}$$

This choice actually has a very nice side effect. Namely, as we always want to be in the regime where $\gamma \sim \gamma_1^N \sim N^2$, it is natural to parameterize the coupling constant with a fixed $\mu > 1$ as

$$\gamma^N = \mu\gamma_1^N = \frac{\mu}{2\sin^2(\frac{\pi}{N})} = \frac{\mu N^2}{2\pi^2}(1+o(1)) \quad (N \to +\infty). \tag{3.2}$$

Then, if we replace the lattice by a lattice of spacing $1/N$ i.e. $(x_i)_{i\in\Lambda}$ is the discretization of a real function x on $[0,1]$ ($x_i = x(i/N)$), the resulting potential converges formally to

$$F_{\gamma^N,N}(x) \underset{N\to\infty}{\longrightarrow} \int_0^1 \left(\frac{1}{4}[x(s)]^4 - \frac{1}{2}[x(s)]^2\right) ds + \frac{\mu}{4\pi^2}\int_0^1 \frac{[x'(s)]^2}{2} ds \tag{3.3}$$

with $x(0) = x(1)$.

In the Euclidean norm, we have $\|I_\pm\|_2 = \sqrt{N}$ which suggests to rescale the size of neighborhoods. We consider for $\rho > 0$, henceforward the neighborhoods $B_\pm^N = B_{\rho\sqrt{N}}(I_\pm)$. The volume $V(B_-^N) = V(B_+^N)$ goes to 0 if and only if $\rho < 1/2\pi e$, so given such a ρ, the balls $B_\pm^N$ are not as large as we could think. Let us also observe that,

$$\frac{1}{\sqrt{N}}\|x\|_2 \underset{N\to\infty}{\longrightarrow} \|x\|_{L^2[0,1]} = \int_0^1 |x(s)|^2 ds. \tag{3.4}$$

Therefore if $x \in B_+^N$ for all N, we get at the limit, $\|x-1\|_{L^2[0,1]} < \rho$.

The main result of this paper is the following uniform version of Theorem 2.2 with a rescaled potential $G_{\gamma,N}$.

Theorem 3.1. *Let $\mu \in]1,\infty[$, there exists a function $\Psi(\varepsilon, N)$ such that $|\Psi(\varepsilon,N)| \le C$ uniformly in N and for all $\varepsilon < \varepsilon_0$, such that*

$$\frac{1}{N}\mathbb{E}_{\nu_{B_-^N,B_+^N}}[\tau_{B_+^N}] = 2\pi c_N e^{1/4\varepsilon}(1+\varepsilon^{1/8}\Psi(\varepsilon,N)). \tag{3.5}$$

Moreover,

$$\lim_{N\uparrow\infty}\frac{1}{N}\mathbb{E}_{\nu_{B_-^N,B_+^N}}[\tau_{B_+^N}] = 2\pi V(\mu)e^{1/4\varepsilon}(1+O(\varepsilon^{1/8})) \tag{3.6}$$

where

$$V(\mu) = \prod_{k=1}^{+\infty}\left[\frac{\mu k^2 - 1}{\mu k^2 + 2}\right] < \infty. \tag{3.7}$$

The proof of this theorem will be decomposed in two parts:

- convergence of the sequence c_N (Proposition 3.2);
- uniform control of the denominator (Theorem 4.3) and the numerator (Theorem 4.9) of Formula (2.15).

Convergence of the prefactor c_N

Our first step will be to control the behavior of c_N as $N \uparrow \infty$. We prove the following:

Proposition 3.2. *The sequence c_N converges: for $\mu > 1$, we set $\gamma = \mu\gamma_1^N$, then*

$$\lim_{N\uparrow\infty} c_N = V(\mu), \tag{3.8}$$

with $V(\mu)$ defined in (3.7).

Remark. This proposition immediately leads to

Corollary 3.3. *For $\mu \in]1,\infty[$, we set $\gamma = \mu\gamma_1^N$, then*

$$\lim_{N\uparrow\infty}\lim_{\varepsilon\downarrow 0}\frac{e^{-\frac{1}{4\varepsilon}}}{N}\mathbb{E}_{\nu_{B_-^N,B_+^N}}[\tau_{B_+^N}] = 2\pi V(\mu). \tag{3.9}$$

Of course such a result is unsatisfactory, since it does not tell us anything about the large system with specified fixed noise strength. To be able to interchange the limits regarding ε and N, we need a uniform control on the error terms.

Proof of the proposition. The rescaling of the potential introduces a factor $\frac{1}{N}$ for the eigenvalues, so that (2.21) becomes

$$\begin{aligned}\mathbb{E}_{\nu_{B_-^N,B_+^N}}[\tau_{B_+^N}] &= \frac{2\pi e^{\frac{1}{4\epsilon}}}{N^{-N/2}\sqrt{\det(\nabla^2 F_{\gamma,N}(I_-))}\frac{N^{-1}}{N^{-N/2}\sqrt{|\det(\nabla^2 F_{\gamma,N}(O))|}}}(1+O(\sqrt{\epsilon}|\ln\epsilon|))\\ &= 2\pi N c_N e^{\frac{1}{4\epsilon}}(1+O(\sqrt{\epsilon}|\ln\epsilon|)).\end{aligned} \tag{3.10}$$

Then, with $u_k^N = \frac{3}{2+\mu\frac{\gamma_1^N}{\gamma_k^N}}$,

$$c_N = \left[1-\frac{3}{2+2\mu\gamma_1^N}\right]^{\frac{e(N)}{2}}\prod_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\left[1-u_k^N\right]. \tag{3.11}$$

To prove the convergence, let us consider the $(\gamma_k^N)_{k=1}^{N-1}$. For all $k\geq 1$, we have

$$\frac{\gamma_1^N}{\gamma_k^N} = \frac{\sin^2(\frac{k\pi}{N})}{\sin^2(\frac{\pi}{N})} = k^2 + (1-k^2)\frac{\pi^2}{3N^2} + o\left(\frac{1}{N^2}\right). \tag{3.12}$$

Hence, $u_k^N \underset{N\to+\infty}{\longrightarrow} v_k = \frac{3}{2+\mu k^2}$. Thus we want to show that

$$c_N \underset{N\to+\infty}{\longrightarrow} \prod_{k=1}^{+\infty}(1-v_k) = V(\mu). \tag{3.13}$$

Using the following inequalities: for $0\leq t\leq\frac{\pi}{2}$,

$$0 < t^2(1-\frac{t^2}{3}) \leq \sin^2(t) \leq t^2, \tag{3.14}$$

we get some estimates for $\frac{\gamma_1^N}{\gamma_k^N}$: set $a = \left(1-\frac{\pi^2}{12}\right)$, for $1\leq k\leq N/2$,

$$ak^2 = \left(1-\frac{\pi^2}{12}\right)k^2 \leq k^2\left(1-\frac{k^2\pi^2}{3N^2}\right) \leq \frac{\gamma_1^N}{\gamma_k^N} = \frac{\sin^2(\frac{k\pi}{N})}{\sin^2(\frac{\pi}{N})} \leq \frac{k^2}{1-\frac{\pi^2}{3N^2}}. \tag{3.15}$$

Then, for $N\geq 2$ and for all $1\leq k\leq N/2$,

$$-\frac{k^4\pi^2}{3N^2} \leq \frac{\gamma_1^N}{\gamma_k^N} - k^2 = \frac{\sin^2(\frac{k\pi}{N})}{\sin^2(\frac{\pi}{N})} \leq \frac{k^2\pi^2}{3N^2\left(1-\frac{\pi^2}{3N^2}\right)} \leq \frac{k^2\pi^2}{N^2}. \tag{3.16}$$

Let us introduce

$$V_m = \prod_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}(1-v_k), \quad U_{N,m} = \prod_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}\left(1-u_k^N\right). \tag{3.17}$$

Thus

$$\Big|\ln\frac{U_{N,N}}{V_N}\Big| = \Big|\ln\prod_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\frac{1-u_k^N}{1-v_k}\Big| \leq \sum_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\Big|\ln\frac{1-u_k^N}{1-v_k}\Big|. \tag{3.18}$$

Using (3.15) and (3.16), we obtain, for all $1 \leq k \leq N/2$,

$$\left|\frac{v_k - u_k^N}{1-v_k}\right| = \frac{3\mu\left|\frac{\gamma_1^N}{\gamma_k^N} - k^2\right|}{(-1+\mu k^2)\left(2+\mu\frac{\gamma_1^N}{\gamma_k^N}\right)} \leq \frac{\mu k^4\pi^2}{N^2\,(-1+\mu k^2)\,(2+\mu a k^2)} \leq \frac{C}{N^2} \tag{3.19}$$

with C a constant independent of k. Therefore, for $N > N_0$,

$$\Big|\ln\frac{1-u_k^N}{1-v_k}\Big| = \left|\ln\left(1+\frac{v_k-u_k^N}{1-v_k}\right)\right| \leq \frac{C'}{N^2}. \tag{3.20}$$

Hence

$$\Big|\ln\frac{U_{N,N}}{V_N}\Big| \leq \frac{C'}{N} \underset{N\to+\infty}{\longrightarrow} 0. \tag{3.21}$$

As $\sum|v_k| < +\infty$, we get $\lim_{N\to+\infty} V_N = V(\mu) > 0$, and thus (3.13) is proved. $\square$

4. Estimates on capacities

To prove Theorem 3.1, we prove uniform estimates of the denominator and numerator of (2.5), namely the capacity and the mass of the equilibrium potential.

4.1. **Uniform control in large dimensions for capacities.** A crucial step is the control of the capacity. This will be done with the help of the Dirichlet principle (2.6). We will obtain the asymptotics by using a Laplace-like method. The exponential factor in the integral (2.8) is largely predominant at the points where h is likely to vary the most, that is around the saddle point O. Therefore we need some good estimates of the potential near O.

4.1.1. *Local Taylor approximation.* This subsection is devoted to the quadratic approximations of the potential which are quite subtle. We will make a change of basis in the neighborhood of the saddle point O that will diagonalize the quadratic part.

Recall that the potential $G_{\gamma,N}$ is of the form

$$G_{\gamma,N}(x) = -\frac{1}{2N}(x,[1-\mathbb{D}]x) + \frac{1}{4N}\|x\|_4^4. \tag{4.1}$$

where the operator $\mathbb{D}$ is given by $\mathbb{D} = \gamma[\mathrm{Id} - 1/2(\Sigma+\Sigma^*)]$ and $(\Sigma x)_j = x_{j+1}$. The linear operator $(1-\mathbb{D}) = -\nabla^2 F_\gamma(O)$ has eigenvalues $-\lambda_k$ and eigenvectors v_k with components $v_k(j) = \omega^{jk}$, with $\omega = e^{i2\pi/N}$.

Let us change the basis by setting

$$\hat{x}_j = \sum_{k=0}^{N-1}\omega^{-jk}x_k. \tag{4.2}$$

Then the inverse transformation is given by

$$x_k = \frac{1}{N}\sum_{j=0}^{N-1} \omega^{jk}\hat{x}_j = x_k(\hat{x}). \tag{4.3}$$

Note that the map $x \to \hat{x}$ maps $\mathbb{R}^N$ to the set

$$\widehat{\mathbb{R}}^N = \left\{\hat{x} \in \mathbb{C}^N : \hat{x}_k = \overline{\hat{x}_{N-k}}\right\} \tag{4.4}$$

endowed with the classical inner product on $\mathbb{C}^N$.
Remark that, expressed in $\hat{x}$, the potential (1.2) takes the form

$$G_{\gamma,N}(x(\hat{x})) = \frac{1}{2N^2}\sum_{k=0}^{N-1} \lambda_{k,N}|\hat{x}_k|^2 + \frac{1}{4N}\|x(\hat{x})\|_4^4. \tag{4.5}$$

Our main concern will be the control of the non-quadratic term in the new coordinates. To that end, we introduce the following norms on the Fourier space:

$$\|\hat{x}\|_{p,\mathcal{F}} = \left(\frac{1}{N}\sum_{i=0}^{N-1} |\hat{x}|^p\right)^{1/p} = \frac{1}{N^{1/p}}\|\hat{x}\|_p. \tag{4.6}$$

The factor $1/N$ is the suitable choice to make the map $x \to \hat{x}$ a bounded map between L^p spaces, as implied by the Hausdorff-Young inequalities (see [13], Vol. 1, Theorem IX.8). More precisely, the following lemma holds.

Lemma 4.1. *With the norms defined above, we have*

(i) *the Parseval identity,*

$$\|x\|_2 = \|\hat{x}\|_{2,\mathcal{F}}, \tag{4.7}$$

and

(ii) *the Hausdorff-Young inequalities: for $1 \le q \le 2$ and $p^{-1} + q^{-1} = 1$, there exists a finite, N-independent constant C_q such that*

$$\|x\|_p \le C_q\|\hat{x}\|_{q,\mathcal{F}}. \tag{4.8}$$

In particular

$$\|x\|_4 \le C_{4/3}\|\hat{x}\|_{4/3,\mathcal{F}}. \tag{4.9}$$

Let us introduce the change of variable, defined by the complex vector z, as

$$z = \frac{\hat{x}}{N}. \tag{4.10}$$

Let us remark that $z_0 = \frac{1}{N}\sum_{k=1}^{N-1} x_k \in \mathbb{R}$. In the variable z, the potential takes the form

$$\widetilde{G}_{\gamma,N}(z) = G_{\gamma,N}\left(x(Nz)\right) = \frac{1}{2}\sum_{k=0}^{N-1} \lambda_{k,N}|z_k|^2 + \frac{1}{4N}\|x(Nz)\|_4^4. \tag{4.11}$$

Moreover, by (4.7) and (4.10)

$$\|x(Nz)\|_2^2 = \|Nz\|_{2,\mathcal{F}}^2 = \frac{1}{N}\|Nz\|_2^2. \tag{4.12}$$

In the new coordinates the minima are now given by

$$I_\pm = \pm(1, 0, \dots, 0). \tag{4.13}$$

In addition, $z(B_-^N) = z(B_{\rho\sqrt{N}}(I_-)) = B_\rho(I_-)$ where the last ball is in the new coordinates.

Lemma 4.1 will allow us to prove the following important estimates. For $\delta > 0$, we set

$$C_\delta = \left\{ z \in \widehat{\mathbb{R}}^N : |z_k| \le \delta \frac{r_{k,N}}{\sqrt{|\lambda_{k,N}|}},\ 0 \le k \le N-1 \right\} \tag{4.14}$$

where $\lambda_{k,N}$ are the eigenvalues of the Hessian at O as given in (2.13). Using (3.15), we have, for $3 \le k \le N/2$,

$$\lambda_{k,N} \ge k^2 \left(1 - \frac{\pi^2}{12}\right) \mu - 1. \tag{4.15}$$

Thus $(\lambda_{k,N})$ verifies $\lambda_{k,N} \ge ak^2$, for $1 \le k \le N/2$, with some a, independent of N.

The sequence $(r_{k,N})$ is constructed as follows. Choose an increasing sequence, $(\rho_k)_{k\ge1}$, and set

$$\begin{cases} r_{0,N} &= 1 \\ r_{k,N} &= r_{N-k,N} = \rho_k, \quad 1 \le k \le \left\lfloor \frac{N}{2} \right\rfloor. \end{cases} \tag{4.16}$$

Let, for $p \ge 1$,

$$K_p = \left(\sum_{k\ge1} \frac{\rho_k^p}{k^p} \right)^{1/p}. \tag{4.17}$$

Note that if K_{p_0} is finite then, for all $p_1 > p_0$, K_{p_1} is finite. With this notation we have the following key estimate.

Lemma 4.2. *For $p \ge 2$, there exists a constant B_p, such that, for $z \in C_\delta$,*

$$\|x(Nz)\|_p^p \le \delta^p N B_p \tag{4.18}$$

if K_q is finite, with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The Hausdorff-Young inequality (Lemma 4.1) gives us:

$$\|x(Nz)\|_p \le C_q \|Nz\|_{q,\mathcal{F}}. \tag{4.19}$$

Since $z \in C_\delta$, we get

$$\|Nz\|_{q,\mathcal{F}}^q \le \delta^q N^{q-1} \sum_{k=0}^{N-1} \frac{r_{k,N}^q}{\lambda_k^{q/2}}. \tag{4.20}$$

Then

$$\sum_{k=0}^{N-1} \frac{r_{k,N}^q}{\lambda_k^{q/2}} = \frac{1}{\lambda_0^{q/2}} + 2 \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{r_{k,N}^q}{\lambda_k^{q/2}} \le \frac{1}{\lambda_0^{q/2}} + \frac{2}{a^{q/2}} \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{\rho_k^q}{k^q} \le \frac{1}{\lambda_0^{q/2}} + \frac{2}{a^{q/2}} K_q^q = D_q^q \tag{4.21}$$

which is finite if K_q is finite. Therefore,

$$\|x(Nz)\|_p^p \le \delta^p N^{(q-1)\frac{p}{q}} C_q^p D_q^p, \tag{4.22}$$

which gives us the result since $(q-1)\frac{p}{q} = 1$. □

We have all what we need to estimate the capacity.

4.1.2. *Capacity Estimates.* Let us now prove our main theorem.

Theorem 4.3. *There exists a function Ψ_1 bounded uniformly in ε and in N such that, for all $\varepsilon < \varepsilon_0$ and for all N,*

$$\frac{\operatorname{cap}\left(B_+^N, B_-^N)\right)}{N^{N/2-1}} = \varepsilon\sqrt{2\pi\varepsilon}^{N-2} \frac{1}{\sqrt{|\det(\nabla F_{\gamma,N}(0))|}} \left(1 + \varepsilon^{1/8}\Psi_1(\varepsilon, N)\right). \tag{4.23}$$

The proof will be decomposed into two lemmata, one for the upper bound and the other for the lower bound. The proofs are quite different but follow the same idea. We have to estimate some integrals. We isolate a neighborhood around the point O of interest. We get an approximation of the potential on this neighborhood, we bound the remainder and we estimate the integral on the suitable neighborhood.

In what follows, constants independent of N are denoted A_i.

Upper bound

The first lemma we prove is the upper bound for Theorem 4.3.

Lemma 4.4. *There exists a constant A_0 such that for all ε and for all N,*

$$\frac{\operatorname{cap}\left(B_+^N, B_-^N\right)}{N^{N/2-1}} \leq \varepsilon\sqrt{2\pi\varepsilon}^{N-2} \frac{1}{\sqrt{|\det(\nabla F_{\gamma,N}(0))|}} \left(1 + A_0\sqrt{\varepsilon}|\ln\varepsilon|\right). \tag{4.24}$$

Proof. This lemma is proved in [5] in the finite dimension setting. We use the same strategy, but here we take care to control the integrals appearing uniformly in the dimension.

We will denote the quadratic approximation of $\widetilde{G}_{\gamma,N}$ by F_0, i.e.

$$F_0(z) = \sum_{k=0}^{N-1} \frac{\lambda_{k,N}|z_k|^2}{2} = -\frac{z_0^2}{2} + \sum_{k=1}^{N-1} \frac{\lambda_{k,N}|z_k|^2}{2}. \tag{4.25}$$

On C_δ, we can control the non quadratic part through Lemma 4.2.

Lemma 4.5. *There exists a constant A_1 and δ_0, such that for all N, $\delta < \delta_0$ and all $z \in C_\delta$,*

$$\left|\widetilde{G}_{\gamma,N}(z) - F_0(z)\right| \leq A_1\delta^4. \tag{4.26}$$

Proof. Using (4.11), we see that

$$\widetilde{G}_{\gamma,N}(z) - F_0(z) = \frac{1}{4N}\|x(Nz)\|_4^4. \tag{4.27}$$

We choose a sequence $(\rho_k)_{k\geq 1}$ such that $K_{4/3}$ is finite.

Thus, it follows from Lemma 4.2, with $A_1 = B_4$, that

$$\left|\widetilde{G}_{\gamma,N}(z) - \frac{1}{2}\sum_{k=0}^{N-1} \lambda_{k,N}|z_k|^2\right| \leq A_1\delta^4, \tag{4.28}$$

as desired. □

We obtain the upper bound of Lemma 4.4 by choosing a test function h^+. We change coordinates from x to z as explained in (4.10). A simple calculation shows that

$$\|\nabla h(x)\|_2^2 = N^{-1}\|\nabla \tilde{h}(z)\|_2^2, \tag{4.29}$$

where $\tilde{h}(z) = h(x(z))$ under our coordinate change.

For δ sufficiently small, we can ensure that, for $z \notin C_\delta$ with $|z_0| \le \delta$,

$$G_{\gamma,N}(z) \ge F_0(z) = -\frac{z_0^2}{2} + \frac{1}{2}\sum_{k=1}^{N-1}\lambda_{k,N}|z_k|^2 \ge -\frac{\delta^2}{2} + 2\delta^2 \ge \delta^2. \tag{4.30}$$

Therefore, the strip $\{x |\, x = x(Nz),\, |z_0| < \delta\}$ separates $\mathbb{R}^N$ into two disjoint sets, one containing I_- and the other one containing I_+. Thus, setting

$$S_\delta = \widetilde{G}_{\gamma,N}^{-1}([\delta^2, +\infty]), \tag{4.31}$$

we have

$$\{x |\, x = x(Nz),\, |z_0| < \delta\} \setminus C_\delta \subset S_\delta. \tag{4.32}$$

Hence the complement of $S_\delta \cup C_\delta$ is made of two connected components Γ_+, Γ_- which contain I_+ and I_-, respectively. Then, we define

$$\tilde{h}^+(z) = \begin{cases} 1 & \text{for } z \in \Gamma_- \\ 0 & \text{for } z \in \Gamma_+ \\ f(z_0) & \text{for } z \in C_\delta \\ \text{arbitrary} & \text{on } S_\delta \setminus C_\delta \text{ but } \|\nabla h^+\|_2 \le \frac{c}{\delta}. \end{cases} \tag{4.33}$$

where f satisfies $f(\delta) = 0$ and $f(-\delta) = 1$ and will be specified later.

Then, taking care of the change of coordinates, the Dirichlet form (2.8) evaluated on h^+ provides the upper bound

$$\begin{aligned} \Phi(h^+) &= N^{N/2-1}\varepsilon \int_{z((B_-^N \cup B_+^N)^c)} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon}\|\nabla \tilde{h}^+(z)\|_2^2 dz \qquad (4.34) \\ &\le N^{N/2-1}\left[\varepsilon \int_{C_\delta} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon}\left(f'(z_0)\right)^2 dz + \varepsilon\delta^{-2}c^2 \int_{S_\delta \setminus C_\delta} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} dz\right]. \end{aligned}$$

The first term will be the predominant term, let us focus on it. We replace $\widetilde{G}_{\gamma,N}$ by F_0, using the bound (4.26), we get for well chosen δ,

$$\begin{aligned} \int_{C_\delta} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon}\left(f'(z_0)\right)^2 dz &\le \left(1 + 2A_1\frac{\delta^4}{\varepsilon}\right)\int_{C_\delta} e^{-F_0(z)/\varepsilon}\left(f'(z_0)\right)^2 dz \qquad (4.35) \\ &= \int_{D_\delta} e^{-\frac{1}{2\varepsilon}\sum_{k=1}^{N-1}\lambda_{k,N}|z_k|^2} dz_1 \ldots dz_{N-1} \int_{-\delta}^{\delta}\left(f'(z_0)\right)^2 e^{z_0^2/2\varepsilon} dz_0. \end{aligned}$$

Here we have used that we can write C_δ in the form $[-\delta, \delta] \times D_\delta$. As we want to calculate an infimum, we choose a function f which minimizes the integral $\int_{-\delta}^{\delta}\left(f'(z_0)\right)^2 e^{z_0^2/2\varepsilon} dz_0$. A simple computation leads to the choice

$$f(z_0) = \frac{\int_{z_0}^{\delta} e^{-t^2/2\varepsilon} dt}{\int_{-\delta}^{\delta} e^{-t^2/2\varepsilon} dt}. \tag{4.36}$$

Therefore

$$\int_{C_\delta} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} (f'(z_0))^2 \, dz \leq \frac{\int_{C_\delta} e^{-\frac{1}{2\varepsilon}\sum_{k=0}^{N-1}|\lambda_{k,N}||z_k|^2} dz}{\left(\int_{-\delta}^{\delta} e^{-\frac{1}{2\varepsilon}z_0^2} dz_0\right)^2} \left(1 + 2A_1 \frac{\delta^4}{\varepsilon}\right). \tag{4.37}$$

Choosing $\delta = K\sqrt{\varepsilon|\ln\varepsilon|}$, a simple calculation shows that there exists A_2 such that

$$\frac{\int_{C_\delta} e^{-\frac{1}{2\varepsilon}\sum_{k=0}^{N-1}|\lambda_{k,N}||z_k|^2} dz}{\left(\int_{-\delta}^{\delta} e^{\frac{1}{2}z_0^2/\varepsilon} dz_0\right)^2} \leq \sqrt{2\pi\varepsilon}^{N-2} \frac{1}{\sqrt{|\det(\nabla F_{\gamma,N}(0))|}} (1 + A_2\varepsilon). \tag{4.38}$$

The second term in (4.34) is bounded above by the following lemma.

Lemma 4.6. *For $\delta > K\sqrt{\varepsilon|\ln(\varepsilon)|}$ and $\rho_k = k^\alpha$, with $0 < \alpha < 1/4$, there exists $A_3 < \infty$, such that for all N*

$$\int_{S_\delta \setminus C_\delta} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} dz \leq A_3 e^{-\delta^2/\varepsilon} \tag{4.39}$$

where $S_\delta = \left\{z : \widetilde{G}_{\gamma,N}(z) \geq \delta^2\right\}$.

Proof. By using Cauchy-Schwartz inequality, $\|x\|_2^4 \leq N\|x\|_4^4$ for $x \in \mathbb{R}^N$, then we have (cf. (4.12))

$$\|x(Nz)\|_4^4 \geq \frac{1}{N}\|x(Nz)\|_2^4 = \frac{1}{N}\|Nz\|_{2,\mathcal{F}}^4 = \frac{1}{N^3}\|Nz\|_2^4 = N\|z\|_2^4. \tag{4.40}$$

We get (cf. (4.11))

$$\widetilde{G}_{\gamma,N}(z) - \frac{z_0^2}{2} - \frac{1}{2}\sum_{k=1}^{N-1}\lambda_{k,N}|z_k|^2 = -z_0^2 + \frac{1}{4N}\|x(Nz)\|_4^4 \geq -\|z\|_2^2 + \frac{\|z\|_2^4}{4}. \tag{4.41}$$

Therefore, for $\|z\|_2 > 2$, the right hand side of (4.41) is non-negative, and

$$\widetilde{G}_{\gamma,N}(z) \geq \frac{z_0^2}{2} + \frac{1}{2}\sum_{k=1}^{N-1}\lambda_{k,N}|z_k|^2 = \frac{1}{2}\sum_{k=0}^{N-1}|\lambda_{k,N}||z_k|^2. \tag{4.42}$$

Thus

$$\begin{aligned} \int_{S_\delta \setminus C_\delta} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} dz &\leq \int_{S_\delta \setminus C_\delta, \|z\|_2 > 2} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} dz + \int_{S_\delta \setminus C_\delta, \|z\|_2 \leq 2} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} dz \\ &\leq \int_{C_\delta^c} e^{-\frac{1}{2\varepsilon}\sum_{k=0}^{N-1}|\lambda_{k,N}||z_k|^2} dz + e^{-\delta^2/\varepsilon} V(B_2(O)). \end{aligned} \tag{4.43}$$

The first term of (4.43) satisfies

$$\begin{aligned} \int_{C_\delta^c} e^{-\frac{1}{2\varepsilon}\sum_{k=0}^{N-1}|\lambda_{k,N}||z_k|^2} dz &\leq \int_{\mathbb{R}} e^{-z_0^2/2\varepsilon} dz_0 \sum_{k=1}^{N-1} e^{-\delta^2 r_{k,N}^2/2\varepsilon} \int_{\mathbb{R}^{N-2}} e^{-\frac{1}{2\varepsilon}\sum_{i=1,i\neq k}^{N-1}\lambda_{i,N}|z_i|^2} dz \\ &\leq \sqrt{\frac{(2\pi\varepsilon)^{N-1}}{\prod_{i=1}^{N-1}\lambda_{i,N}}} \sum_{k=1}^{N-1}\sqrt{\lambda_{k,N}} e^{-\delta^2 r_{k,N}^2/2\varepsilon}. \end{aligned} \tag{4.44}$$

Using the standard inequality $\frac{2}{\pi}t \leq \sin t \leq t$, we see that, for $2 \leq k \leq N/2$,

$$\frac{1}{10}k^2\mu \leq \frac{4}{\pi^2}k^2\mu - 1 \leq \lambda_{k,N} = \mu\frac{\gamma_1^N}{\gamma_k^N} - 1 \leq \frac{\pi^2}{4}\mu k^2. \tag{4.45}$$

Hence,

$$\left(\prod_{i=1}^{N-1}\lambda_{i,N}\right)^{1/2} \geq \prod_{i=1}^{\lfloor\frac{N}{2}\rfloor-1}\lambda_{i,N} \geq (\mu-1)\left(\frac{\mu}{10^2}\right)^{\lfloor\frac{N}{2}\rfloor-2}\left[\left(\left\lfloor\frac{N}{2}\right\rfloor-2\right)!\right]^2. \tag{4.46}$$

Moreover,

$$\begin{aligned}\sum_{k=1}^{N-1}\sqrt{\lambda_{k,N}}e^{-\delta^2 r_{k.N}^2/2\varepsilon} &= \sum_{k=1}^{\lfloor\frac{N}{2}\rfloor}\sqrt{\lambda_{k,N}}e^{-\delta^2 r_{k,N}^2/2\varepsilon} + \sum_{k=\lfloor\frac{N}{2}\rfloor+1}^{N-1}\sqrt{\lambda_{N-k,N}}e^{-\delta^2 r_{N-k,N}^2/2\varepsilon} \\ &\leq \frac{\sqrt{\mu}\pi}{2}\sum_{k=1}^{N/2}ke^{-\delta^2\rho_k^2/2\varepsilon}. \end{aligned} \tag{4.47}$$

We choose $\rho_k = k^\alpha$ with $0 < \alpha < 1/4$ to ensure that $K_{4/3}$ is finite. Then, setting $a = e^{-\delta^2/(2\varepsilon)}$, we get

$$\sum_{k=1}^{N/2}ka^{\rho_k^2} \leq a + \frac{N^2}{4}\sum_{k=2}^{N/2}\frac{a^{\rho_k^2}}{k} \leq a + \frac{N^2}{4}\int_1^{N/2}\frac{e^{-x^{2\alpha}\ln(\frac{1}{a})}}{x}dx \leq a + \frac{N^2}{8\alpha}\int_{\ln(\frac{1}{a})}^{(\frac{N}{2})^{2\alpha}\ln(\frac{1}{a})}\frac{e^{-t}}{t}dt. \tag{4.48}$$

Here we used that $f(x) = a^{x^{2\alpha}}/x$ is decreasing for $x \geq 1$, and the last inequality follows from the change of variables $t = x^{2\alpha}\ln(1/a)$. For $1 \leq r < s$, we have

$$\int_r^s \frac{e^{-t}}{t}dt \leq \int_r^s e^{-t}dt \leq e^{-r}. \tag{4.49}$$

Then, as $\delta > K\sqrt{\varepsilon|\ln(\varepsilon)|}$, a goes to 0 with ε, hence $\ln(1/a) \geq 1$. Then (4.49) with (4.48) gives

$$\sum_{k=1}^{N/2}ka^{\rho_k^2} \leq a + \frac{N^2}{8\alpha}a = a\left[1 + \frac{N^2}{8\alpha}\right]. \tag{4.50}$$

Putting all the parts together, we get that

$$\int_{S_\delta - C_\delta} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon}dz \leq C_N a \tag{4.51}$$

with

$$\begin{aligned} C_N &= V(B_2) + \frac{\sqrt{\mu}\pi}{4(\mu-1)}\left[1+\frac{N^2}{8\alpha}\right]\frac{\sqrt{(2\pi\varepsilon)^{N-1}}}{(\frac{\mu}{10^2})^{\lfloor\frac{N}{2}\rfloor-2}}\frac{1}{\left[(\lfloor\frac{N}{2}\rfloor-2)!\right]^2} \\ &= \frac{(4\pi)^{N/2}}{\Gamma(N/2+1)} + \frac{\sqrt{\mu}\pi}{4(\mu-1)}\left(\frac{\mu}{10^2}\right)^{i_N}\left[1+\frac{N^2}{8\alpha}\right]\left(\frac{2\pi^2\varepsilon 10^2}{\mu}\right)^{\frac{N-1}{2}}\frac{1}{\left[(\lfloor\frac{N}{2}\rfloor-2)!\right]^2} \end{aligned} \tag{4.52}$$

and $i_N = \lfloor\frac{N}{2}\rfloor - 2 - \frac{N-1}{2}$. Therefore C_N is bounded by some constant A_3, and Lemma 4.6 is proven. □

Finally, using (4.35), (4.38) and (4.39), we obtain the upper bound

$$\frac{\Phi(h^+)}{N^{N/2-1}} \le \varepsilon\sqrt{2\pi\varepsilon}^{N-2}\frac{1}{\sqrt{|\det(\nabla F_{\gamma,N}(0))|}}(1+A_2\varepsilon)\left(1+2A_1\frac{\delta^4}{\varepsilon}\right)+A_3e^{-\delta^2/\varepsilon} \quad (4.53)$$

with the choice $\rho_k = k^\alpha$, $0<\alpha<1/4$ and $\delta = K\sqrt{\varepsilon|\ln\varepsilon|}$. Thus Lemma 4.4 is proven. □

Lower Bound The idea here (as already used in [6]), is to get a lower bound by restricting the state space to a narrow corridor from I_- to I_+ that contains the relevant paths and along which the potential is well controlled.
We will prove the following lemma.

Lemma 4.7. *There exists a constant $A_4<\infty$ such that for all ε and for all N,*

$$\frac{\operatorname{cap}\left(B_+^N,B_-^N\right)}{N^{N/2-1}} \ge \varepsilon\sqrt{2\pi\varepsilon}^{N-2}\frac{1}{\sqrt{|\det(\nabla F_{\gamma,N}(0))|}}\left(1-A_4\varepsilon^{1/8}\right). \quad (4.54)$$

Proof. Given a sequence $(\rho_k)_{k\ge1}$, $r_{k,N}$ is defined as is (4.16),

$$\widehat{C}_\delta = \left\{z_0\in]-1+\rho,1-\rho[,\ |z_k|\le\delta\ r_{k,N}/\sqrt{\lambda_{k,N}}\right\}. \quad (4.55)$$

The restriction $|z_0|<1-\rho$ is made to ensure that $\widehat{C}_\delta$ is disjoint from $B_\pm$ since in the new coordinates (4.10) $I_\pm=\pm(1,0,\dots,0)$.

Clearly, if h^* is the minimizer of the Dirichlet form, then

$$\operatorname{cap}\left(B_-^N,B_+^N\right)=\Phi(h^*)\ge\Phi_{\widehat{C}_\delta}(h^*), \quad (4.56)$$

where $\Phi_{\widehat{C}_\delta}$ is the Dirichlet form for the process on $\widehat{C}_\delta$,

$$\Phi_{\widehat{C}_\delta}(h)=\varepsilon\int_{\widehat{C}_\delta}e^{-G_{\gamma,N}(x)/\varepsilon}\|\nabla h(x)\|_2^2dx = N^{N/2-1}\varepsilon\int_{z(\widehat{C}_\delta)}e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon}\|\nabla\tilde{h}(z)\|_2^2dx. \quad (4.57)$$

Then, since

$$\|\nabla\tilde{h}(z)\|_2^2=\sum_{k=0}^{N-1}\left|\frac{\partial\tilde{h}^*}{\partial z_k}\right|^2\ge\left|\frac{\partial\tilde{h}^*}{\partial z_0}\right|^2, \quad (4.58)$$

we keep only the derivative with respect to z_0

$$\frac{\Phi(h^*)}{N^{N/2-1}}\ge\varepsilon\int_{z(\widehat{C}_\delta)}e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon}\left|\frac{\partial\tilde{h}^*}{\partial z_0}(z)\right|^2dz=\widetilde{\Phi}_{\widehat{C}_\delta}(\tilde{h}^*)\ge\min_{h\in\mathcal{H}}\widetilde{\Phi}_{\widehat{C}_\delta}(\tilde{h}). \quad (4.59)$$

Thus we minimize along the first (real) coordinate z_0, the other ones, $z_\perp=(z_i)_{1\le i\le N-1}$, are considered as parameters. The corresponding minimizer is readily found explicitly as

$$\tilde{h}^-(z_0,z_\perp)=\frac{\int_{z_0}^{1-\rho}e^{\widetilde{G}_{\gamma,N}(z_0,z_\perp)/\varepsilon}dz_0}{\int_{-1+\rho}^{1-\rho}e^{\widetilde{G}_{\gamma,N}(z_0,z_\perp)/\varepsilon}dz_0} \quad (4.60)$$

and hence the capacity is bounded from below by

$$\frac{\operatorname{cap}\left(B_-^N,B_+^N\right)}{N^{N/2-1}}\ge\widetilde{\Phi}_{\widehat{C}_\delta}(\tilde{h}^-)=\varepsilon\int_{\widehat{C}_\delta^\perp}\left(\int_{-1+\rho}^{1-\rho}e^{\widetilde{G}_{\gamma,N}(z_0,z_\perp)/\varepsilon}dz_0\right)^{-1}dz_\perp. \quad (4.61)$$

To go further, we have to evaluate the r.h.s. integral above. To this aim, we show in the next lemma an approximation of the potential on $\widehat{C}_\delta$. Since z_0 is no longer small, we only expand in the coordinates $z_\perp$.

Lemma 4.8. *There exists a constant, A_5, such that, for all N and $\delta < \delta_0$, on $\widehat{C}_\delta$,*

$$\left| \widetilde{G}_{\gamma,N}(z) - \Big(-\frac{1}{2}z_0^2 + \frac{1}{4}z_0^4 + \frac{1}{2}\sum_{k=1}^{N-1} \lambda_{k,N}|z_k|^2 + \frac{3}{2}z_0^2 \sum_{k=1}^{N-1} |z_k|^2 \Big) \right| \leq A_5 \delta^3 \tag{4.62}$$

provided that we choose (ρ_k) such that $K_{4/3}$ is finite.

Proof. We study the non-quadratic part of the potential on $\widehat{C}_\delta$, using (4.11) and (4.3)

$$\frac{1}{N}\|x(Nz)\|_4^4 = \frac{1}{N}\sum_{i=0}^{N-1} |x_i(Nz)|^4 = \frac{1}{N}\sum_{i=0}^{N-1} \left| z_0 + \sum_{k=1}^{N-1} \omega^{ik} z_k \right|^4 = \frac{z_0^4}{N} \sum_{i=0}^{N-1} |1+u_i|^4 \tag{4.63}$$

where $u_i = \frac{1}{z_0}\sum_{k=1}^{N-1} \omega^{ik} z_k$. Remark that $\sum_{i=0}^{N-1} u_i = 0$ and $u = \frac{1}{z_0} x\left(N(0, z_\perp)\right)$.

Then, using

$$|1+u|^4 = 1 + 2(u+\bar{u}) + 2u\bar{u} + (u+\bar{u})^2 + 2(u+\bar{u})u\bar{u} + (u\bar{u})^2, \tag{4.64}$$

we get that

$$\left| \frac{1}{N}\|x(Nz)\|_4^4 - z_0^4 \Big(1 + \frac{1}{N}\sum_i 2u_i\bar{u}_i + (u_i+\bar{u}_i)^2 \Big) \right| \leq \frac{z_0^4}{N}\left(4\|u\|_3^3 + \|u\|_4^4 \right). \tag{4.65}$$

A simple computation shows that

$$\frac{1}{N}\sum_i 2u_i\bar{u}_i + (u_i+\bar{u}_i)^2 = \frac{6}{z_0^2}\sum_{k\neq 0} |z_k|^2. \tag{4.66}$$

Thus as $|z_0| \leq 1$, we see that

$$\left| \frac{1}{N}\|x(Nz)\|_4^4 - z_0^4 - 6z_0^2 \sum_{k\neq 0} |z_k|^2 \right| \leq \frac{1}{N}\big(4\|x(N(0,z_\perp))\|_3^3 + \|x(N(0,z_\perp))\|_4^4 \big). \tag{4.67}$$

Since $K_{4/3}$ is finite, $K_{3/2}$ also, then Lemma 4.2 for $p = 3$ and 4 shows that:

$$\begin{aligned} \|x(N(0,z_\perp))\|_3^3 &\leq B_3 N \delta^3 \\ \|x(N(0,z_\perp))\|_4^4 &\leq B_4 N \delta^4. \end{aligned} \tag{4.68}$$

Therefore, Lemma 4.8 is proved, with $A_5 = B_3 + B_4\delta_0$. □

We use Lemma 4.8 to obtain the upper bound

$$\int_{-1+\rho}^{1-\rho} e^{\widetilde{G}_{\gamma,N}(z_0,z_\perp)/\varepsilon} dz_0 \leq \exp\left(\tfrac{1}{2\varepsilon}\textstyle\sum_{k\neq 0} \lambda_{k,N}|z_k|^2 + \frac{A_5\delta^3}{\varepsilon} \right) g(z_\perp)\sqrt{2\pi\varepsilon}, \tag{4.69}$$

where

$$g(z_\perp)\sqrt{2\pi\varepsilon} = \int_{-1+\rho}^{1-\rho} \exp\left(-\frac{1}{2\varepsilon}z_0^2 + \frac{1}{4\varepsilon}z_0^4 + \frac{3}{2\varepsilon}z_0^2 \sum_{k\neq 0} |z_k|^2 \right) dz_0. \tag{4.70}$$

We first deal with $g(z_\perp)$. We fix $z_\perp$, and by the change of variable $t = z_0/\sqrt{\varepsilon}$, we get

$$g(z_\perp) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{1-\rho}{\sqrt{\varepsilon}}}^{\frac{1-\rho}{\sqrt{\varepsilon}}} e^{-\frac{1}{2}t^2(1-3\sum_{k\neq 0}|z_k|^2)+\frac{\varepsilon}{4}t^4} dt. \tag{4.71}$$

Then, we set $\sigma(z_\perp) = 1 - 3\sum_{k\neq 0} |z_k|^2$, and split the integral into

$$\begin{aligned}
g(z_\perp) &= \frac{2}{\sqrt{2\pi}} \left[\int_0^{(1-\rho)\varepsilon^{-1/4}} e^{-\frac{\sigma(z_\perp)}{2}t^2+\frac{\varepsilon}{4}t^4} dt + \int_{(1-\rho)\varepsilon^{-1/4}}^{(1-\rho)\varepsilon^{-1/2}} e^{-\frac{\sigma(z_\perp)}{2}t^2+\frac{\varepsilon}{4}t^4} dt \right] \\
&\leq \frac{2}{\sqrt{2\pi}} \left[\int_0^{(1-\rho)\varepsilon^{-1/4}} e^{-\frac{\sigma(z_\perp)}{2}t^2+\frac{(1-\rho)^2\sqrt{\varepsilon}}{4}t^2} dt + \int_{(1-\rho)\varepsilon^{-1/4}}^{(1-\rho)\varepsilon^{-1/2}} e^{-\frac{\sigma(z_\perp)}{2}t^2+\frac{(1-\rho)^2}{4}t^2} dt \right] \\
&\leq \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{t^2}{2}(\sigma(z_\perp)-\frac{(1-\rho)^2\sqrt{\varepsilon}}{2})} dt + \frac{2}{\sqrt{2\pi}} \int_{(1-\rho)\varepsilon^{-1/4}}^{+\infty} e^{-\frac{t^2}{2}(\sigma(z_\perp)-\frac{(1-\rho)^2}{2})} dt \\
&\leq \left(\sigma(z_\perp) - \frac{(1-\rho)^2\sqrt{\varepsilon}}{2} \right)^{-1/2} + \frac{2(1-\rho)\varepsilon^{1/4}}{\sqrt{2\pi}(\sigma(z_\perp)-1/2)} e^{-\frac{\sigma(z_\perp)-(1-\rho)^2/2}{2\sqrt{\varepsilon}}}. \qquad (4.72)
\end{aligned}$$

Since $z_\perp \in \widehat{C}_\delta$, by the same procedure as in (4.21), there exists A_6 s.t.

$$\sum_{k\neq 0} |z_k|^2 \leq \delta^2 \sum_{k\neq 0} \frac{r_{k,N}^2}{\lambda_{k,N}} \leq A_6 \delta^2 \sum_{k\neq 0} \frac{\rho_k^2}{k^2} = A_6 K_2^2 \delta^2. \tag{4.73}$$

Then, because $K_{4/3}$ is finite, K_2 also. Hence $\sigma(z_\perp) = 1 + O(\delta^2)$, therefore from (4.72), there exists A_7 s.t.

$$g(z_\perp) \leq (1 + A_7(\delta^2 + \sqrt{\varepsilon})) \tag{4.74}$$

uniformly on $(0, z_\perp) \in \widehat{C}_\delta^\perp$.

Then changing variables $t_k = |z_k|\sqrt{\lambda_{k,N}/\varepsilon}$, the right hand side of (4.61) can now be written as

$$\begin{aligned}
N^{1-N/2}\mathrm{cap}(B_-^N, B_+^N) &\geq \frac{\sqrt{2\pi\varepsilon}}{2\pi} \int_{\widehat{C}_\delta^\perp} e^{-\frac{1}{2\varepsilon}\sum_{k\neq 0}\lambda_{k,N}|z_k|^2} dz_\perp \frac{e^{-\frac{A_5\delta^3}{\varepsilon}}}{1 + A_7(\delta^2+\sqrt{\varepsilon})} \\
&\geq \frac{\sqrt{2\pi\varepsilon}^N}{2\pi} \frac{1}{\sqrt{|\det(\nabla^2 F_{\gamma,N}(O))|}} \prod_{k\neq 0} \frac{2}{\sqrt{2\pi}} \left[\int_0^{\eta_k} e^{-\frac{t_k^2}{2}} dt_k \right] \\
&\quad \times \frac{e^{-\frac{A_5\delta^3}{\varepsilon}}}{1 + A_7(\delta^2+\sqrt{\varepsilon})} \qquad (4.75)
\end{aligned}$$

with $\eta_k = \delta r_{k,N}/\sqrt{\varepsilon}$.

To bound the product of integrals in (4.75), we use that

$$\frac{2}{\sqrt{2\pi}} \int_0^{\eta_k} e^{-\frac{t^2}{2}} dt = 1 - \frac{2}{\sqrt{2\pi}} \int_{\eta_k}^{+\infty} e^{-\frac{t^2}{2}} dt \geq 1 - \frac{2}{\sqrt{2\pi}\eta_k} e^{-\frac{\eta_k^2}{2}}. \tag{4.76}$$

To conclude, we specify the sequence (ρ_k). As in the case of the upper bound, we choose $\rho_k = k^\alpha$ with $0 < \alpha < 1/4$, such that $K_{4/3}$ is finite. Thus, taking

$a = \exp(-\delta^2/2\varepsilon)$

$$\sum_{k=1}^{N-1} \frac{1}{r_{k,N}} e^{-\frac{r_k^2\delta^2}{2\varepsilon}} \leq 2\sum_{k=1}^{N/2} \frac{1}{\rho_k} e^{-\frac{\rho_k^2\delta^2}{2\varepsilon}} \leq 2\sum_{k=1}^{+\infty} \frac{1}{k^\alpha} a^{k^{2\alpha}} = S_a < S_{a_0} < +\infty \tag{4.77}$$

uniformly for $a < a_0 < 1$.

Then, if we choose $\delta^2 = K\varepsilon^{3/4}$, we have $a = e^{-K/2\varepsilon^{1/4}} \to 0$ as ε goes to 0. Therefore

$$\prod_{k=1}^{N-1} \frac{2}{\sqrt{2\pi}} \int_0^{\eta_k} e^{-\frac{t^2}{2}} dt \geq \prod_{k=1}^{N-1} [1 - \frac{2}{\sqrt{2\pi}\eta_k} e^{-\frac{\eta_k^2}{2}}] = 1 - \frac{2S_a}{\sqrt{2\pi}} \varepsilon^{1/8} + o(\varepsilon^{1/8}) \tag{4.78}$$

uniformly in N. At last, we have an error term from the approximation (4.62). Setting $A_8 = K^{3/2}A_5$,

$$e^{-A_5\delta^3/\varepsilon} = e^{-A_8\varepsilon^{1/8}} = 1 - A_8\varepsilon^{1/8} + o(\varepsilon^{1/8}). \tag{4.79}$$

Thus, (4.75) becomes

$$\frac{\mathrm{cap}(B_-^N, B_+^N)}{N^{N/2-1}} \geq \frac{\sqrt{2\pi\varepsilon}^N}{2\pi} \frac{1}{\sqrt{|\det(\nabla^2 F_{\gamma,N}(O))|}} (1 - A_4\varepsilon^{1/8}). \tag{4.80}$$

□

4.2. **Uniform estimate of the mass of the equilibrium potential.** We will prove the following theorem.

Theorem 4.9. *There exists a function Ψ_2 such that, for all $\varepsilon < \varepsilon_0$ and all N,*

$$\frac{1}{N^{N/2}} \int_{B_+^{N^c}} h^*_{B_-^N, B_+^N}(x) e^{-G_{\gamma,N}(x)/\varepsilon} dx = \sqrt{2\pi\varepsilon}^N \frac{e^{\frac{1}{4\varepsilon}}}{\sqrt{\det(\nabla F_{\gamma,N}(I_-))}} (1 + \varepsilon^{1/8}\Psi_2(\varepsilon, N)) \tag{4.81}$$

with Ψ_2 bounded uniformly in ε and in N.

Proof. The idea of the proof is to use the Laplace method. The predominant contribution to the integral comes from the point where the argument of the exponent realises its minimum. Since $\widetilde{G}_{\gamma,N}$ reaches its minima at $I_\pm$, with the value $-\frac{1}{4}$, the mass is concentrated around I_- (I_+ is not in the domain). We introduce

$$\widehat{G}_{\gamma,N} = \widetilde{G}_{\gamma,N} + \frac{1}{4}. \tag{4.82}$$

Changing variables as before, we get that

$$\int_{B_+^{N^c}} h^*_{B_-^N, B_+^N}(x) e^{-G_{\gamma,N}(x)/\varepsilon} dx = N^{N/2} e^{\frac{1}{4\varepsilon}} \int_{z(B_+^{N^c})} \widetilde{h}^*_{B_-^N, B_+^N}(z) e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz. \tag{4.83}$$

We will split the integral in two parts: one over a suitable neighborhood of I_- and the remainder. The idea is to use the level set $\widehat{S}_\delta = \widehat{G}_{\gamma,N}^{-1}([\delta^2, \infty[)$ to bound the second part. Let us recall that the diameter of $B_\pm^N$ depends on N by $B_\pm^N = B_{\sqrt{N}\rho}(I_\pm)$ with $\rho > 0$ and that $z(B_-^N) = B_\rho(I_-)$ where the last ball is in the new coordinates. We define a neighborhood of I_-, $C_\delta(I_-)$, as in (4.14),

$$C_\delta(I_-) = \left\{ z \in \widehat{\mathbb{R}}^N : |z_0 - 1| \leq \frac{\delta}{\sqrt{\nu_0}},\ |z_k| \leq \delta\, \frac{r_{k,N}}{\sqrt{\nu_{k,N}}}\ 1 \leq k \leq N-1 \right\}. \tag{4.84}$$

Here $(\nu_{k,N})$ are the eigenvalues of the Hessian at I_-. These eigenvalues have the following property that allows us to use Lemma 4.2 with $(\nu_{k,N})$ instead of $(\lambda_{k,N})$: for $1 \leq k \leq N/2$

$$\nu_{k,N} \geq \frac{4\mu}{\pi^2}k^2 + 2 \geq \frac{4}{\pi^2}k^2. \tag{4.85}$$

Therefore, provided K_2 is finite, for $z + I_- \in C_{2\delta}(I_-)$ there exists A_9 s.t.

$$\|z\|_2^2 \leq \delta^2 \sum_{k=0}^{N-1} \frac{r_{k,N}^2}{\nu_k} \leq \delta^2 A_9 K_2^2. \tag{4.86}$$

This means that, for δ small enough $C_{2\delta}(I_-) \subset z(B_-)$.

We prove a suitable approximation of the potential on $C_{2\delta}(I_-)$.

Lemma 4.10. *For all N,*

$$\widehat{G}_{\gamma,N}(z) - \frac{1}{2}\sum_{k=0}^{N-1} \nu_k |z_k|^2 = R(z) \tag{4.87}$$

and there exists a constant A_{10} and δ_0 such that, for $\delta < \delta_0$, on $C_{3\delta}(I_-)$

$$|R(z)| \leq A_{10}\delta^3 \tag{4.88}$$

provided that we choose (ρ_k) such that $K_{4/3}$ is finite.

Proof. In the neighborhood of I_-, since $\nabla^2 F_{\gamma,N} = 2\mathrm{Id} + \mathbb{D}$ (cf. (4.1)) and have eigenvalues $(\nu_{k,N})_k$ (2.20) associated with the eigenvectors $(v_k)_k$, we use the same change of coordinate as around O. In this setting, the potential takes the form

$$\widehat{G}_{\gamma,N}(z + I_-) = G_{\gamma,N}\left(x(Nz) + I_-\right) = \frac{1}{2}\sum_{k=0}^{N-1} \nu_{k,N}|z_k|^2 + R(z) \tag{4.89}$$

with $R(z) = -\frac{1}{N}\sum_{i=0}^{N-1} x_i(Nz)^3 + \frac{1}{4N}\|x(Nz)\|_4^4$. Therefore,

$$|R(z)| \leq \frac{1}{N}\|x(Nz)\|_3^3 + \frac{1}{4N}\|x(Nz)\|_4^4 \tag{4.90}$$

then, provided $K_{3/2}$ and $K_{4/3}$ are finite, Lemma 4.2 shows that, for $z + I_- \in C_{3\delta}(I_-)$,

$$|R(z)| \leq A_{10}\delta^3 \tag{4.91}$$

with $A_{10} = B_3 + B_4\delta_0$. □

Lemma 4.10 allows us to show that, for δ small enough, if $z \in C_{3\delta}(I_-) \setminus C_{2\delta}(I_-)$, then

$$\widehat{G}_{\gamma,N}(z + I_-) = \frac{1}{2}\sum_{k=0}^{N-1} \nu_{k,N}|z_k|^2 + R(z) \geq 2\delta^2 - A_{12}\delta^3 \geq \delta^2. \tag{4.92}$$

Thus, $C_{2\delta}^c \subset \widehat{S}_\delta$.

We split the integral (4.83) into:

$$\begin{aligned}\int_{z(B_+^{N\,c})} h^*_{B_-^N,B_+^N}(z) e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz \;&=\; \int_{C_{2\delta}(I_-)} h^*_{B_-^N,B_+^N}(z) e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz \\ &\quad + \int_{C_{2\delta}(I_-)^c} h^*_{B_-^N,B_+^N}(z) e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz \end{aligned}\tag{4.93}$$

The first integral is the predominant one, the second is a remainder and will be treated as in the proof of Lemma 4.6. First, since $0 \leq h^* \leq 1$,

$$\int_{C_{2\delta}(I_-)^c} h^*_{B^N_-,B^N_+}(z) e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz \leq \int_{C_{2\delta}(I_-)^c} e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz. \tag{4.94}$$

Then, since $C_{2\delta}(I_-)^c \subset \widehat{S}_\delta$, we get

$$\begin{aligned} \int_{C_{2\delta}(I_-)^c} h^*_{B^N_-,B^N_+}(z) e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz &\leq \int_{\widehat{S}_\delta} e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz \qquad (4.95)\\ &\leq \int_{\widehat{S}_\delta \cap B_2(O)} e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz + \int_{\widehat{S}_\delta \cap B_2(O)^c} e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz \\ &\leq V(B_2(O)) e^{-\delta^2/\varepsilon} + \int_{B_2(O)^c} e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz. \end{aligned}$$

With the notation of Lemma 4.6, for S_δ and C_δ, we will show that $B_2(O)^c \subset S_\delta \setminus C_\delta$ for δ small enough. This follows from (4.42) and the estimates (4.45). For $\|z\|_2 > 2$, we have, since $\mu > 1$

$$\widetilde{G}_{\gamma,N}(z) \geq \frac{1}{2} \sum_{k=0}^{N-1} |\lambda_{k,N}| |z_k|^2 \geq \frac{z_0^2}{2} + \frac{1}{2} \sum_{k=1}^{N-1} \frac{2}{\pi^2} \mu k^2 |z_k|^2 \geq \frac{1}{\pi^2} \|z\|_2^2 \geq \frac{4}{\pi^2}. \tag{4.96}$$

Thus, if $\delta < 2/\pi$, we get $\widetilde{G}_{\gamma,N}(z) > \delta^2$. For δ sufficiently small, by the same arguments as in (4.86), C_δ is included in $B_2(O)$. Then, Lemma 4.6 gives us

$$\int_{B_2(O)^c} e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz \leq \int_{S_\delta \setminus C_\delta} e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz \leq e^{-\frac{1}{4\varepsilon}} A_3 e^{-\delta^2/\varepsilon}. \tag{4.97}$$

The second integral is therefore bounded uniformly in N: there exists A_{11} s.t.

$$\int_{C_{2\delta}(I_-)^c} h^*_{B^N_-,B^N_+}(z) e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz \leq A_{11} e^{-\delta^2/\varepsilon}. \tag{4.98}$$

Let us now focus on the first integral in (4.94). By definition of h^*, $h^*_{B^N_-,B^N_+} = 1$ on B^N_-, and since $C_{2\delta}(I_-) \subset z(B^N_-)$,

$$\int_{C_{2\delta}(I_-)} \tilde{h}^*_{B_-,B_+}(z) e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz = \int_{C_{2\delta}(I_-)} e^{-\widehat{G}_{\gamma,N}(z)/\varepsilon} dz = \mathcal{I}. \tag{4.99}$$

Due to Lemma 4.10

$$e^{-\frac{A_{10}\delta^3}{\varepsilon}} \int_{C_{2\delta}(I_-)} e^{-\frac{1}{2\varepsilon}\sum_{k=0}^{N-1} \nu_{k,N}|z_k|^2} dz \leq \mathcal{I} \leq e^{\frac{A_{10}\delta^3}{\varepsilon}} \int_{C_{2\delta}(I_-)} e^{-\frac{1}{2\varepsilon}\sum_{k=0}^{N-1} \nu_{k,N}|z_k|^2} dz. \tag{4.100}$$

Finally, by the change of variable $t_k = |z_k|\sqrt{\lambda_{k,N}/\varepsilon}$ and with $\eta_k = \delta r_{k,N}/\sqrt{\varepsilon}$,

$$\int_{C_{2\delta}(I_-)} e^{-\frac{1}{2\varepsilon}\sum_{k=0}^{N-1} \nu_{k,N}|z_k|^2} dz = \frac{\sqrt{2\pi\varepsilon}^N}{\sqrt{\det(\nabla^2 F_{\gamma,N}(I_-))}} \prod_{k=0}^{N-1} \frac{2}{\sqrt{2\pi}} \left[\int_0^{\eta_k} e^{-\frac{t^2}{2}} dt\right]. \tag{4.101}$$

We conclude by the same arguments as in (4.78): choosing $\delta = K\varepsilon^{3/8}$,

$$\prod_{k=1}^{N-1} \frac{2}{\sqrt{2\pi}} \int_0^{\eta_k} e^{-\frac{t^2}{2}} dt = 1 - A_{12}\varepsilon^{1/8} + \varepsilon^{1/8}\phi_1(\varepsilon, N) \tag{4.102}$$

with $\phi_1(\varepsilon, N)$ goes to 0 with ε and uniformly bounded in N. At last, (4.93) becomes

$$\begin{aligned} &\frac{1}{N^{N/2}} \int_{B_+^{N^c}} h^*_{B_-^N, B_+^N}(x) e^{-\widehat{G}_{\gamma,N}(x)/\varepsilon} dx \\ &= \frac{\sqrt{2\pi\varepsilon}^N}{\sqrt{\det(\nabla^2 F_{\gamma,N}(I_-))}} \left(1 - A_{12}\varepsilon^{\frac{1}{8}} + \varepsilon^{\frac{1}{8}}\phi_1(\varepsilon, N)\right)\left(1 + A_{10}\varepsilon^{\frac{1}{8}} + \varepsilon^{\frac{1}{8}}\phi_2(\varepsilon, N)\right) \\ &\quad + A_{11} e^{-K/\varepsilon^{1/2}} \end{aligned} \tag{4.103}$$

where $\phi_2(\varepsilon, N)$ goes to 0 with ε and is uniformly bounded in N. ϕ_2 represents the remainder due to the approximation realized at (4.100). This concludes the proof of Theorem 4.9. □

4.3. **Proof of Theorem 3.1.** .

Proof. The proof of Theorem 3.1 is now an obvious consequence of (2.15) together with Theorems 4.3 and 4.9. □

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CMAP UMR 7641, École Polytechnique CNRS, Route de Saclay, 91128 Palaiseau Cedex France; email: barret@cmap.polytechnique.fr

Institut für Angewandte Mathematik, Rheinische Friedrich-Wilhelms-Universität, Endenicher Allee 60, 53115 Bonn, Germany; email: bovier@uni-bonn.de

CMAP UMR 7641, École Polytechnique CNRS, Route de Saclay, 91128 Palaiseau Cedex France; email: sylvie.meleard@polytechnique.edu