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Abstract

In this paper, we consider the control system Σ defined by the rolling of a strictly convex surface S on a plane without slipping or spinning. It is well known that Σ is completely controllable. The purpose of this paper is to present the numerical implementation of a constructive planning algorithm for Σ, which is based on a continuation method. The performances of that algorithm, both in robustness and convergence speed, are illustrated through several examples.

1 INTRODUCTION

In recent years, non-holonomic systems have attracted much attention due to the theoretical questions raised for their motion planning and to their importance in numerous applications (cf. [12, 13] and references therein). In particular, the planning of robotic manipulators for achieving high operational capability with low constructive complexity is a major issue for the control community in the last decade. Non-holonomy is exploited for the design of such manipulators but to ensure both hardware reduction and controllability performances yield serious difficulties, requiring more elaborate analysis and efficient algorithm. The rolling-body problem illustrates well all the aforementioned aspects.

Recall that the rolling-body problem (without slipping or spinning) is a control system Σ modelling the rolling of a connected surface S_1 of the Euclidean space $\mathbb{R}^3$ on another one S_2 so that the relative speed of the contact point is zero and the relative angular velocity has zero component along the common normal direction at the contact point. Five parameters are needed to describe the state of Σ: two for parameterizing the contact point as element of S_1, two others for parameterizing the contact point as element of S_2 and finally one more parameter for the relative orientation of S_1 with respect to S_2. Once an absolutely continuous curve c_1 is chosen on S_1, it is possible to define the rolling of S_2 on S_1 along c_1 without slipping or spinning. Therefore, the controls correspond to the choice of c_1 and can be represented by

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$\dot{c}_1$. It follows that, in local coordinates, Σ can be written as a drifless control system of the type $\dot{x} = u_1 F_1(x) + u_2 F_2(x)$, where $(u_1, u_2) \in I\!R^2$ is the control and F_1, F_2 vector fields (cf. [13, 12] and references therein).

Several works (cf. [12] and references therein) studied the controllability issue. Agrachev and Sachkov ([1]) proved that Σ is completely controllable if and only if S_1 and S_2 are not isometric. Regarding the motion planning problem (MPP for short) associated to the rolling-body problem, most of the attention focused on the rolling of a convex surface S on a flat one, due to the fact that the latter models dexterous robotic manipulation of a convex object by means of a robotic hand with as few as three motors and flat finger, see [12, 13] and references therein. Moreover, in [12], several prototype dexterous grippers are exhibited. Recall that the MPP is the problem of finding a procedure that, for every pair (p, q) of the state space of a control system $\mathcal{S}$, *effectively* produces a control $u_{p,q}$ giving rise to an admissible trajectory steering p to q. Until now, the most significant result is the ingenious algorithm proposed by Li and Canny ([11]), only treating the case where S is a ball. However, it is not possible to generalize that algorithm to more general convex surfaces S.

In [4], two different approaches to address the motion planning for convex surfaces M rolling on a plane were proposed. The first one is based on the Liouvillian character of Σ. One can show that, if S admits a symmetry of revolution, the MPP can be reduced to a purely inverse algebraic problem. However, such an approach presents a serious numerical drawback: the resulting inverse problem requires that implicit functions must be determined through transcendental equations involving local charts for S.

The second approach proposed in [4] to tackle the MPP for convex surfaces M rolling on a plane is based on the well-known continuation method (also called homotopy method or continuous Newton's algorithm-[2]-) which goes back to Poincaré. The MPP is therefore addressed as a pure inverse problem. Let us briefly recall how the continuation method (CM for short) works. It is used for solving nonlinear equations of the form $F(x) = y$, where x is the unknown and $F : X \to Y$ is surjective. Consider some $x_0 \in X$ and $y_0 = F(x_0)$. Pick a differentiable path $\pi : [0, 1] \to X$ joining y_0 to the given y. Then, as explained next, the CM is an iterative procedure which lifts π to a path Π so that $F \circ \Pi = \pi$ and the "iteration" occurs by the use of an ordinary differential equation in X. It starts by differentiating $F(\Pi(s)) = \pi(s)$ to get $DF(\Pi(s))\dot{\Pi}(s) = \dot{\pi}(s)$. The latter is satisfied by setting $\dot{\Pi}(s) := P(\Pi(s))\dot{\pi}(s)$ where $P(x)$ is a right inverse of $DF(x)$. Therefore, solving $F(x) = y$ amounts to first show that $P(\Pi(s))$ exists (for instance if $DF(\Pi(s))$ is surjective) and second to prove that the ODE in X, $\dot{\Pi}(s) = P(\Pi(s))\dot{\pi}(s)$, which is a "highly" non-linear equation (also called the Wazewski equation -[19]-), admits a global solution. In the context of the MPP, the CM was introduced in [9] and [15, 16], and further developed in [6, 7, 8, 17, 18]. The map F is now an end-point map from the space of admissible inputs to the state space. Its singularities are exactly the abnormal extremals of the sub-Riemannian metric induced by the dynamics of the system, which are usually a major obstacle for the CM to apply efficiently to the MPP (cf. [5]). In the case of Σ, non trivial abnormal extremals and their trajectories were determined in [4] and they exactly correspond to the horizontal geodesics of Σ. Despite that obstacle, assuming that the surface S is strictly convex and possesses a stable periodic geodesic, it was shown in [4] that the CM provides complete answers to the MPP. More precisely, it was shown that there exist enough paths π in the state space of Σ that can be lifted to paths Π in the control space by showing global existence of solutions to the Wazewski equation.

In this paper, we explain how we can implement numerically the continuation method presented above for resolving a very large class of rolling-body problems.

The paper is organized as follows: in Section 2, we present the kinematic equations of motion of a convex body S_1 *rolling without slipping or spinning* on top of another one S_2. We describe in Section 3 how we apply the continuation method to the motion planning problem and give sufficient conditions guaranteeing the existence of $P(\Pi)$ and the existence of a global solution of the Path Lifting Equation in the case of the rolling-body problem. Section 4 serves to detail some key points for numerical resolution of Path Lifting Equation. In Section 5, several numerical simulations are presented.

2 Description of the rolling-body problem

In this section, we briefly recall how to derive the equations of motion for the rolling-body problem with no slipping or spinning of a connected surface S_1 of the Euclidean space $\mathbb{R}^3$ on top of another one S_2. This section does not bring new results but we provide it for sake of completeness and also to exhibit the numerical challenges raised by trying to implement ordinary differential equations on a manifold. These results were already obtained in [1, 12, 13].

We start by the intrinsic formulation of the problem, we first assume that S_1 and S_2 are two-dimensional, connected, oriented, smooth, complete Riemannian manifolds.

2.1 Differential geometric notions and definitions

If P is a matrix, we use P^T and $tr(P)$ to denote respectively the transpose of P, and its the trace.

Let $(S, \langle\cdot,\cdot\rangle)$ be a two-dimensional, connected, oriented smooth complete Riemannian manifold for the Riemannian metric $\langle\cdot,\cdot\rangle$. We use TS to denote the tangent bundle over S and US the unit tangent bundle, i.e. the subset of TS of points (x, v) such that $x \in S$ and $v \in T_xS$, $\langle v, v\rangle = 1$.

Let $\{U_\alpha, \alpha\}_{\alpha\in\mathcal{A}}$ be an atlas on S. For $\alpha, \beta \in \mathcal{A}$ such that $U_\alpha \cap U_\beta$ is not empty, we denote by $J_{\beta\alpha}$ the jacobian matrix of $\varphi^\beta \circ (\varphi^\alpha)^{-1}$ the coordinate transformation on $\varphi^\alpha(U_\alpha \cap U_\beta)$. For $\alpha \in \mathcal{A}$, the Riemannian metric is represented by the symmetric definite positive matrix $\mathcal{I}^\alpha$ and set $M^\alpha := \sqrt{\mathcal{I}^\alpha}$.

For $x \in S$, a frame f at x is an ordered basis for T_xS and, for $\alpha, \beta \in \mathcal{A}$, we have $f^\beta = J_{\beta\alpha} f^\alpha$. The frame f is orthonormal if, in addition $M^\alpha f^\alpha$ is an orthogonal matrix. An Orthonormal Moving Frame (briefly OMF) defined on an open subset U of S is a smooth map assigning to each $x \in U$ a positively oriented orthonormal frame $f(x)$ of T_xS.

Let ∇ be the Riemannain connection on S (cf. [14]). For a given OMF f defined on $U \subset S$, the Christoffel symbols associated to $f = (f_1, f_2)$ are defined by

$$\nabla_{f_i} f_j = \sum_k \Gamma^k_{ij} f_k,$$

where $1 \le i, j, k \le 2$. The connection form ω is the mapping defined on U such that, for every $x \in U$, ω_x is the linear application from T_xS to the set of 2×2 skew-symmetric matrices given as follows. For $i, j, k = 1, 2$, the (i, j)–th coefficient of $\omega_x(f_k)$ is equal to Γ^k_{ij}.

Let $c : J \to S$ be an absolutely continuous curve in S with J compact interval of $\mathbb{R}$. Set $X(t) := \dot{c}(t)$ in J which defines a vector field along c. Let $Y : J \to TS$ be an absolutely continuous assignment such that, for every $t \in J$, $Y(t) \in T_{c(t)}S$. We say that Y is parallel along c if $\nabla_X Y = 0$ for almost all $t \in J$. In the domain of an OMF f, that equation can be written as follows

$$\dot{Y}^k = -\sum_{1\leq i,j\leq 2} \Gamma^k_{ij} X^i Y^j,$$

or equivalently,

$$\dot{Y} = -\omega(X)Y.$$

Recall that a curve c is a geodesic if the velocity $\dot{c}(t)$ is parallel along c, that is

$$\nabla_{\dot{c}}\dot{c} = 0. \tag{1}$$

2.2 Rolling body problem

2.2.1 Definition of the state space

Consider now the rolling-body problem with no slipping or spinning of S_1 on top of S_2. We adopt here the viewpoint presented in [1].

At the contact points of the bodies $x_1 \in S_1$ and $x_2 \in S_2$, their tangent spaces are identified by an orientation-preserving isometry

$$q : T_{x_1}S_1 \longrightarrow T_{x_2}S_2,$$

Such an isometry q is a state of the system, and the state space is

$$\mathcal{Q}(S_1, S_2) = \{q : T_{x_1}S_1 \longrightarrow T_{x_2}S_2 \, | x_1 \in S_1, \; x_2 \in S_2, q \text{ an isometry}\}.$$

As the set of all orientation-preserving isometries in $I\!R^2$ is $SO(2)$ which can be identified with the unit circle S^1 in $I\!R^2$, $\mathcal{Q}(S_1, S_2)$ is a 5-dimensional connected manifold. A point $q \in \mathcal{Q}(S_1, S_2)$ is locally parametrized by $(x_1, \; x_2, \; R)$ with $x_1 \in S_1$, $x_2 \in S_2$ and $R \in SO(2)$.

2.2.2 Rolling dynamics

We next describe the motion of one body rolling on top of another one so that the contact point of the first follows a prescribed absolutely continuous curve on the second body.

Let f_1 and f_2 be two OMFs defined on the chart domains of α_1, α_2. Let $b_i(t) = f_i(c_i(t))R_i(t)$ parallel along $c_i^{\alpha_i}$, $i = 1, 2$, and $R := R_2(t)R_1(t)^{-1} \in SO(2)$ which, by definition, measures the relative position of f_2 with respect to f_1 along $(c_1^{\alpha_1}, c_2^{\alpha_2})$. The variation of R_i along $c_i^{\alpha_i}$, for $i = 1, 2$, is given by $\dot{R}_i = -\omega_i(\dot{c}_i^{\alpha_i})R_i$.

Given an a.c. curve $c_1 : [0, T] \to S_1$, the rolling of S_2 on S_1 without slipping or spinning along c_1 is characterized by a curve $\Gamma = (c_1, c_2, R) : [0, T] \to \mathcal{Q}(S_1, S_2)$ defined the two following conditions.

Up to initial conditions, the no slipping condition amounts to

$$M^{\alpha_2}\dot{c}_2^{\alpha_2}(t) = RM^{\alpha_1}\dot{c}_1^{\alpha_1}(t), \tag{2}$$

and the no spinning one to

$$\dot{R}R^{-1} = R\omega_1(\dot{c}_1^{\alpha_1})R^{-1} - \omega_2(\dot{c}_2^{\alpha_2}). \tag{3}$$

Since $SO(2)$ is commutative, equation (3) reduces to

$$\dot{R}R^{-1} = \omega_1(\dot{c}_1^{\alpha_1}) - \omega_2(\dot{c}_2^{\alpha_2}). \tag{4}$$

If we fix a point $x = (x_1, x_2, R_0) \in \mathcal{Q}(S_1, S_2)$, a curve c_1 on S_1 starting at x_1 defines entirely the curve Γ by equations (2) and (4). Therefore, we can give the following definition:

Definition 1 *The surface S_2 rolls on the surface S_1 without slipping nor spinning if, for every $x = (x_1, x_2, R_0) \in \mathcal{Q}(S_1, S_2)$ and a.c. curve $c_1 : [0, T] \to S_1$ starting at x_1, there exists an a.c. curve $\Gamma : [0, T] \to \mathcal{Q}(S_1, S_2)$ with $\Gamma(t) = (c_1(t), c_2(t), R(t))$, $\Gamma(0) = x$ and for every $t \in [0, T]$, such that, on appropriate charts, equations (2) and (4) are satisfied. We call the curve $\Gamma(t)$ an admissible trajectory.*

If we consider f_1 and f_2 two OMFs and if the state x is represented (in coordinates) by the triple $x = (c_1, c_2, R)$, then for almost all t such that $x(t)$ remains in the domain of an appropriate chart, there exists a measurable function $u(\cdot)$ (called control) with values in $I\!R^2$ such that

$$\begin{aligned} \dot{c_1}(t) &= u_1(t) f_1^1(c_1(t)) + u_2(t) f_2^1(c_1(t)), \quad (5) \\ \dot{c_2}(t) &= u_1(t)(f^2(c_2(t))R(t))_1 + u_2(t)(f^2(c_2(t))R(t))_2, \quad (6) \\ \dot{R}(t)R^{-1}(t) &= \sum_{i=1}^{2} u_i(t)[\omega_1(f_i^1(c_1(t))) - \omega_2(f^2(c_2(t))R(t))_i]. \quad (7) \end{aligned}$$

Let us consider the vector fields F_1 and F_2 defined by

$$F_i = (f_i^1,\ (f^2 R)_i,\ [\omega_1(f_i^1) - \omega_2(f^2 R)_i])^T, \quad i = 1, 2.$$

Then, Eqs. (5), (6) and (7) have the following compact form in local coordinates,

$$\dot{x} = u_1 F_1(x) + u_2 F_2(x). \quad (8)$$

We recognize the classical form of a driftless control-affine system.

Remark 1 *In general, it is not possible to get a global basis for the distribution Δ and thus to define globally the dynamics of the control system using vector fields. One notable exception occurs when one of the manifolds is a plane, cf. [4]. Therefore, addressing the motion planning efficiently (i.e. as far as producing a numerical scheme) becomes a delicate issue since most of the standard techniques are based on global vector field expressions of the dynamics of a control system.*

The following proposition describes a fundamental property of the rolling problem. For more detail, see [4] for instance.

Proposition 1 *Let $u \in H$ be an admissible control that gives rise to the admissible trajectory*

$$\Gamma = (c_1, c_2, R) : \ [0, 1] \to M.$$

Then the following statements are equivalent:

(a) the curve $c_1 : [0, 1] \to S_1$ is a geodesic;

(b) the curve $c_2 : [0, 1] \to S_2$ is a geodesic;

(c) the curve $\Gamma : [0, 1] \to M$ is a horizontal geodesic.

2.2.3 Rolling body problem in $\mathbb{R}^3$

From now on, we will assume that the manifolds S_1 and S_2 are oriented surfaces of $\mathbb{R}^3$ with metrics induced by the Euclidean metric of $\mathbb{R}^3$.

We first note that there are two possible ways to define the rolling problem, depending on the respective (global) choice of normal vectors for S_1 and S_2. Indeed, the orientation of the tangents planes of an oriented surface S is determined by the choice of a Gauss map i.e. a continuous normal vector $n : S \to S^2$. There are two such normal vectors, n and $-n$. If S is (strictly) convex, these two normal vectors are called inward and outward.

Recall that the rolling-body problem assumes that the tangent spaces at the contact points are identified. In $\mathbb{R}^3$, this is equivalent to identify the normal vectors. Let n_i be the normal vector of S_i, then at contact points, we can either assign n_1 to n_2 or $-n_2$, i.e. we have $n_1 = \varepsilon n_2$ with $\varepsilon = \pm 1$. The physical meaning of this parameter ε is the following: if $\varepsilon = 1$, the two surfaces roll so that one is "inside" the other one, in other words, they are on the same side of their common tangent space at the contact point; if $\varepsilon = -1$, the two surfaces roll so that one is "outside" the other one, in other words, they are on opposite sides with respect to their common tangent space at the contact point. It is clear that the second situation is more physically feasible in general since, it holds true globally as soon as the two surfaces are convex. We will only deal with this second situation.

For computational ease, we will rewrite Eq. (8) in geodesic coordinates. Recall that the geodesic coordinates on a Riemannian manifold S are charts (v, w) defined such that the matrix $\mathcal{I}^\alpha$ is diagonal and equal to $\mathrm{diag}(1, B^2(v, w))$. The function B is defined in an open neighborhood of $(0, 0)$ (the domain of the chart) and satisfies $B(0, w) = 1$, $B_v(0, w) = 0$ and $B_{vv} + K\ B = 0$, where K denotes the Gaussian curvature of S at (v, w) and B_v (B_{vv}, respectively) is the (double, respectively) partial derivative of B with respect to v.

Using the fact that $\mathcal{Q}(S_1, S_2)$ is a circle bundle when S_1 and S_2 are two-dimensional manifolds, and taking geodesic coordinates B_1, B_2 for S_1 and S_2 at contact points x_1 and x_2 respectively, consider coordinates $x = (v_1, w_1, v_2, w_2, \psi)$ in some neighborhood of $(0, \psi_0)$ in $\mathbb{R}^4 \times S^1$. Then the control system (8) can be written locally as

$$\dot{x} = u_1 F_1(x) + u_2 F_2(x), \tag{9}$$

with

$$F_1(x) = (1,\ 0,\ \cos\psi,\ -\frac{\sin\psi}{B_2},\ -\frac{B_{2v_2}}{B_2}\sin\psi)^T, \tag{10}$$

$$F_2(x) = (0,\ \frac{1}{B_1},\ -\sin\psi,\ -\frac{\cos\psi}{B_2}, -\frac{B_{1v_1}}{B_1} - \frac{B_{2v_2}}{B_2}\cos\psi)^T, \tag{11}$$

see [4] for instance.

3 Continuation method applied to the rolling-body problem

3.1 General description of the continuation method

We start with a general description of the CM, see [6] for more details and complete justifications.

The state space $\mathcal{Q}(S_1, S_2)$ is simply denoted by M. The admissible inputs u are elements of $H = L^2([0,1], I\!R^2)$. We use $\|u(t)\|$ and $\|u\|_H$ respectively to denote $(\sum_{i=1}^2 u_i^2(t))^{1/2}$ and $(\int_0^1 \|u(t)\|^2 dt)^{1/2}$. If $u, v \in H$, then $(u,v)_H = \int_0^1 u^T(t)v(t)dt$.

From the brief description of the CM given in the introduction, the map F is equal to the end-point $\phi_p : H \to M$ associated to some fixed $p \in M$. (For more details and complete justifications regarding the CM cf. [5].) For $u \in H$ and $p \in M$, let $\gamma_{p,u}$ be the trajectory of Σ starting at p for $t = 0$ and corresponding to u. Then, for $v \in H$, $\phi_p(v)$ is given by

$$\phi_p(v) := \gamma_{p,v}(1).$$

Recall that $\phi_p(v)$ is defined for every $v \in H$. The MPP can be reformulated as follows: for every $p, q \in M$, exhibit a control $u_{p,q} \in H$ such that

$$\phi_p(u_{p,q}) = q. \tag{12}$$

In other words, for fixed p, we must find a map $i_p : M \to H$ such that $\phi_p \circ i_p =$ identity, *i.e.*, we are looking for a right inverse of ϕ_p. It can be shown that such a right inverse exists in a neighborhood of any point $u \in H$ such that $D\phi_p(u)$ is surjective. Therefore, it is reasonable to expect difficulties with the singular points of ϕ_p, *i.e.*, the controls $v \in H$ where rank $D\phi_p(v) < 5$. Let S_p and $\phi_p(S_p)$ be the set of singular points of ϕ_p and the set of singular values respectively.

The application of the CM to the MPP is thus decomposed in two steps. In the first one, we have to characterize (when possible) S_p and $\phi_p(S_p)$. The second step consists of lifting paths $\pi : [0,1] \to M$ avoiding $\phi_p(S_p)$ to paths $\Pi : [0,1] \to H$ such that, for every $s \in [0,1]$,

$$\phi_p(\Pi(s)) = \pi(s). \tag{13}$$

Differentiating Eq. (13) yields to

$$D\phi_p(\Pi(s)) \cdot \frac{d\Pi}{ds}(s) = \frac{d\pi}{ds}(s). \tag{14}$$

If $D\phi_p(\Pi(s))$ has full rank, then Eq. (14) can be solved for $\Pi(s)$ by taking Π such that

$$\frac{d\Pi}{ds}(s) = P\big(\Pi(s)\big) \cdot \frac{d\pi}{ds}(s), \tag{15}$$

where $P(v)$ is a right inverse of $D\phi_p(v)$ for $v \in H/S_p$. (For instance, we can choose $P(v)$ to be the Moore-Penrose pseudo-inverse of $D\phi_p(v)$.)

We are then led to study the Wazewski equation (15) called the Path Lifting Equation (PLE) as an ODE in H. To successfully apply the CM to the MPP, we have to resolve two issues:

(a) Non degeneracy: the path π has to be chosen so that, for every $s \in [0,1]$, $\pi(s) \notin \phi_p(S_p)$ and then $D\phi_p\big(\Pi(s)\big)$ has always full rank;

(b) Non explosion: to solve Eq. (12), the PLE defined in Eq. (15) must have a global solution on $[0,1]$.

We first note that local existence and uniqueness of the solution of the PLE hold as soon as ϕ_p is of class C^2.

A sufficient condition resolving (a) and (b) is given by

Condition 1 *We say that a closed subset $\mathcal{K}$ of M verifies Condition 1 if*

(i) *$\mathcal{K}$ is disjoint from $\overline{\phi_p(S_p)}$, where $\overline{\phi_p(S_p)}$ is the closure of $\phi_p(S_p)$;*

(ii) *there exists $c_{\mathcal{K}} > 0$ such that, for every $u \in H$ with $\phi_p(u) \in \mathcal{K}$, we have*

$$\|P(u)\| \leq c_{\mathcal{K}} \|u\|, \tag{16}$$

where

$$\|P(u)\| = (\inf_{\|z\|=1} z^T D\phi_p(u) D\phi_p(u)^T z)^{-1/2},$$

with $z \in T^*_{\phi_p(u)} M$.

Once the existence of a closed set $\mathcal{K}$ verifying Condition 1 is guaranteed, a simple application of Gronwall Lemma yields that, for every path $\pi : [0,1] \to \mathcal{K}$ of class C^1 and every control $\bar{u} \in H$ such that $\phi_p(\bar{u}) = \pi(0)$, the solution of the PLE defined in Eq. (15) with initial condition $\bar{u}$ exists globally on the interval $[0,1]$.

3.2 Existence of a “large" compact $\mathcal{K}$ verifying *Condition 1*

We now consider the MPP for the rolling-body problem of a strictly convex surface S_1 on a plane. If S_1 verifies a simple geometric property (see Condition 2 below), then it is shown in [4] that there exists a compact $\mathcal{K}$ of M verifying *Condition 1*, large enough to completely resolve the MPP.

We first describe, in the following proposition (for a proof, see [4] for instance), the structure of S_p and $\phi_p(S_p)$ for the rolling problem.

Proposition 2 *Let $p \in M$. Then, $S_p = \{(v\cos\theta, v\sin\theta) | v \in L^2([0,1], I\!R), \theta \in S^1\}$ and $\phi_p(S_p)$ is equal to the union of the end-points of all horizontal geodesics starting at p, i.e. all trajectories starting at p and corresponding to one control $u \in S_p$.*

The existence of a compact $\mathcal{K}$ verifying Condition 1 requires a small singular set $\phi_p(S_p)$. This condition is guaranteed by the existence of a periodic geodesic on S_1, stable by the geodesic flow of S_1. More precisely, let d_1 be the distance function associated to the Riemannian metric of S_1 induced by the usual metric of $I\!R^3$.

Condition 2 *We say that a surface S_1 verifies Condition 2 if the following holds true.*

There exists a geodesic curve $\gamma : I\!R^+ \to T_1S_1$, $L > 0$ and $\rho_0 > 0$ such that

(s) *$\gamma(t+L) = \gamma(t)$ for all $t \geq 0$ (cf. [10]);*

(p) *$\forall \rho < \rho_0$, $\exists \eta(\rho) > 0$, $\forall y_0 \in N_\rho(G)$, $\forall t \geq 0$, we have*

$$\phi(y_0, t) \in N_\eta(G),$$

and

$$lim_{\rho \to 0} \eta(\rho) = 0,$$

where $G := \gamma([0,L])$, $N_\rho(G)$ is the open set of points $y \in T_1S_1$ such $d_1(y,G) < \rho$ and $\phi(y,t)$ is the geodesic flow of T_1S_1.

It is shown in [10] that Condition 2 holds true for any convex compact surface having a symmetry of revolution and it is generic within the convex compact surfaces verifying $K_{min}/K_{max} > \frac{1}{4}$, where K_{min} and K_{max} denote the minimum and the maximum respectively of the Gaussian curvature over the surface.

Assume now that S_1 verifies Condition 2 and let G be the support of the periodic geodesic. Then, $\rho \in (0, \rho_0)$, define $\mathcal{K}_{\bar{\rho}}$ as the set complement in $T_1 S_1$ of $N_\rho(G) \times L$, where L is a fixed line in $I\!R^2$. The next proposition, proved in [4], tackles issue (b).

Proposition 3 *There exists a line $L \in I\!R^2$ and $\bar{\rho} > 0$ such that the corresponding $\mathcal{K}_{\bar{\rho}}$ verifying Condition* 1*.*

Then, we have the following proposition guaranteeing that the continuation method can be successfully applied for solving the rolling-body motion planning problem.

Proposition 4 *With above notations, for every path $\pi : [0, 1] \to \mathcal{K}_{\bar{\rho}}$ of class C^1 and every control $\bar{u} \in H$ such that $\pi(0) = \phi_p(\bar{u})$, the solution of path lifting equation* (15) *with initial condition equal to $\bar{u}$ exists globally over* $[0, 1]$*.*

We now describe how Proposition 4 can be applied to the rolling-body motion planning problem. Assume that one wants to roll the body from an initial position $p \in M$ to a final one $q \in M$.

Let us first assume that both p and q belong to $\mathcal{K}_{\bar{\rho}}$. We note that since γ is periodic, $N_\rho(G)$ is diffeomorphic to the product of a small two-dimensional ball and a closed path on S_1, $\mathcal{K}_{\bar{\rho}}$ is closed and arc-connected. We begin by taking an arbitrary control $\bar{u}$ which does not belong to S_p, then we choose a C^1-curve $\pi : [0, 1] \to \mathcal{K}_{\bar{\rho}}$ such that $\pi(0) := \phi_p(\bar{u})$ and $\pi(1) := q$. Proposition 4 guarantees that by integrating Eq. (15) over $[0, 1]$ with initial condition equal to $\bar{u}$, we obtain a curve $\Pi : [0, 1] \to H$ such that $\phi_p(\Pi(s)) = \pi(s)$ for $s \in [0, 1]$. In particular, we have $\phi_p(\Pi(1)) = \pi(1) = q$, which means that the control $u := \Pi(1)$ solves the motion planning problem. If for instance p does not belong to $\mathcal{K}_{\bar{\rho}}$, it suffices first to roll the body along one geodesic which brings it to a point $\tilde{p}$ belonging to $\mathcal{K}_{\bar{\rho}}$, then we consider $\tilde{p}$ as the new initial condition, and Continuation method applies. We recall that geodesic curves are admissible trajectories for rolling body problem by Proposition 1.

Therefore, Continuation method can be successfully applied for solving rolling-body motion planning problem. In the following section, we explain how this method is numerically implemented.

4 Numerical implementation

In this section, we detail how we implement numerically the continuation method in order to solve the MPP for rolling-bodies in the case where S_1 is a strictly convex surface of $I\!R^3$. For simplicity, we assume that S_1 is defined as one bounded connected component of the zero-level set of a smooth real-valued function $f : I\!R^3 \to I\!R$, and $S_2 = \mathbb{R}^2$.

The normal vector field to S_1 is denoted by $n : S_1 \to S^2$ and is given by

$$\frac{(\nabla f)^T}{\|\nabla f\|},$$

where $\nabla f = (f_1, f_2, f_3)$ denotes the gradient vector of f. The Gaussian curvature of S_1 is denoted by K and $K_{min} = \min_{S_1} K > 0$. In addition, set $K_{max} = \max_{S_1} K$.

With the above hypotheses, Eq. (9) simply becomes

$$\begin{array}{rcl} \dot{v}_1 &=& \cos\psi u_1 - \sin\psi u_2, \\ \dot{w}_1 &=& -\frac{1}{B}\sin(\psi)u_1 - \frac{1}{B}\cos(\psi)u_2, \\ \dot{v}_2 &=& u_1, \\ \dot{w}_2 &=& u_2, \\ \dot{\psi} &=& -\frac{B_{v_1}}{B}\sin(\psi)u_1 - \frac{B_{v_1}}{B}\cos(\psi)u_2, \end{array} \tag{17}$$

where B is used to define geodesic coordinates on S_1.

We first note that controls in this case are just plane curves $c_2 : [0,1] \to I\!R^2$ such that $\dot{c}_2 = (u_1, u_2)$ for all $t \in [0,1]$. We divide the interval $[0,1]$ into N parts, and we approximate the control space H by the $2N$-dimensional subspace $\hat{H}$ of piecewise linear functions. Then, c_2 can be approximated by $\hat{c}_2$, the linear interpolation of $(c_2^1, \ldots, c_2^N)$ where $c_2^i = c_2(\frac{i}{N-1}) = (x_i, y_i)^T$ and on each segment $[t_i, t_{i+1}] = [\frac{i}{N-1}, \frac{i+1}{N-1}]$, the approximate control $(\hat{u}_1^i, \hat{u}_2^i)^T$ is proportional to the vector $(x_{i+1} - x_i, y_{i+1} - y_i)^T$.

The Path Lifting Equation (15) tells us how we have to modify this piecewise constant control $(\hat{u}_1, \hat{u}_2)$ in order to obtain an appropriate control steering our system from an initial state to a preassigned final state. Under some general geometric assumptions for S_1, theoretical results presented in Section 3 guarantee that, whatever the starting control we choose, Eq. (15) is complete and provides the correct control law at the end of the integration. We use the classical Euler scheme to integrate Eq. (15). Note that Theorem 1 in [6] ensures that, once there exists a global solution to Eq. (15), then for any “reasonable" Galerkin approximation of the control space and “reasonable” numerical scheme for the derivatives, there exists a global solution for the corresponding numerical approximation of Eq. (15).

In the following two paragraphs, we give details about the two key points for the numerical implementation which are the evaluation of a right inverse of $D\phi_p(u)$ and the integration of Eq. (17).

4.1 Computing $D\phi_p(u)$

We first need to define a field of covectors along $\gamma_{p,u}$. For $z \in T^*_{\phi_p(u)}M$, let $\lambda_{z,u} : [0,1] \to T^*M$ be the field of covectors along $\gamma_{p,u}$ satisfying (in coordinates) the adjoint equation along $\gamma_{p,u}$ with terminal condition z, *i.e.*, $\lambda_{z,u}$ is a.c., $\lambda_{z,u}(1) = z$ and for a.e. $t \in [0,1]$,

$$\dot{\lambda}_{z,u}(t) = -\lambda_{z,u}(t) \cdot \left(\sum_{i=1}^{2} u_i(t) DF_i(\lambda_{z,u}(t)) \right). \tag{18}$$

If X is a smoothvector field over M, the switching function $\varphi_{X,z,u}(t)$ associated to X is the evaluation of $\lambda \cdot X(x)$, the Hamiltonian function of X along $(\gamma_{p,u}, \lambda_{z,u})$, *i.e.*, for $t \in [0,1]$,

$$\varphi_{X,z,u}(t) := \lambda_{z,u}(t) \cdot X(\gamma_{p,u}(t)),$$

(see for instance [6] for more details). Then $D\phi_p(u)$ can be computed as follows: for $z \in T^*_{\phi_p(u)}M$ and $u, v \in H$,

$$z \cdot D\phi_p(u)(v) = (v, \varphi_{z,u})_H, \tag{19}$$

where the switching function vector $\varphi_{z,u}(t)$ is the solution of the following Cauchy problem, defined (in coordinates) below, by (see [4])

$$\begin{array}{rcl} \dot{\varphi}_1 & = & -u_2 K \varphi_3, \\ \dot{\varphi}_2 & = & u_1 K \varphi_3, \\ \dot{\varphi}_3 & = & -u_2 \varphi_4 + u_1 \varphi_5, \\ \dot{\varphi}_4 & = & -u_2 K \varphi_3, \\ \dot{\varphi}_5 & = & u_1 K \varphi_3. \end{array} \tag{20}$$

with terminal condition $\varphi_{z,u}(1) = z$.

In practice, since the discrete $D\phi_p(u)$ is a 5×5 matrix and its image is given by Eq. (19), it suffices to take five independents vectors in $\mathbb{R}^5$ as final conditions z, for instance the five elements in the canonical basis of $\mathbb{R}^5$ and integrate Eq. (20) in reverse time.

In our simulations, a fourth-order Runge-Kutta numerical scheme is used for integration, the scalar product $(\cdot,\cdot)_H$ in control space H is evaluated by Gaussian quadrature and the Gaussian curvature K is computed by using the following proposition, cf. [3].

Proposition 5 *Let S be (a bounded connected component of) the zero-set of $f : \mathbb{R}^3 \to \mathbb{R}$, and define a, b, c by*

$$\det \begin{pmatrix} \nabla^2 f - \lambda I & \nabla f \\ (\nabla f)^T & 0 \end{pmatrix} = a + b\lambda + c\lambda^2, \tag{21}$$

where $\nabla^2 f$ is the matrix of the second derivatives of f and I_3 the identity 3×3 *matrix.*

With this notation, one has

$$K = \frac{a/c}{\|\nabla f\|^2}. \tag{22}$$

Explicit computations show that $c = -\|\nabla f\|^2$ and

$$a = \det \begin{pmatrix} \nabla^2 f & \nabla f \\ (\nabla f)^T & 0 \end{pmatrix}.$$

Hence, we have

$$K = -\frac{\det \begin{pmatrix} \nabla^2 f & \nabla f \\ (\nabla f)^T & 0 \end{pmatrix}}{\|\nabla f\|^4}. \tag{23}$$

The gradient vector ∇f is then evaluated by a classical right-shifting finite difference scheme, and $\nabla^2 f$ by a centered one. For example, if $X = (x, y, z)$, then $f_x(X)$ is given by

$$\frac{f(x+\varepsilon, y, z) - f(x, y, z)}{\varepsilon}, \tag{24}$$

and $f_{xx}(X)$ by

$$\frac{f(x+\varepsilon, y, z) - 2f(x, y, z) + f(x-\varepsilon, y, z)}{\varepsilon^2}, \tag{25}$$

with $\varepsilon > 0$ small enough.

4.2 Lifting the plane curve $\hat{c}_2$ on S_1

Note that the curvature K appearing in Eq. (20) is taken at the final contact point on the surface S_1 after it has rolled along the piecewise constant curve $\hat{c}_2$. Thus, in order to locate the final point, we need to "lift" the plane curve $\hat{c}_2$ on S_1, and the lifting dynamics are given by Eq. (17). However, since the geodesic coordinates involved in Eq. (17) are not given explicitly in practice, our numerical *lifting* method is based on Proposition 1.

On each interval $[t_i, t_{i+1}]$, the approximate control curve $\hat{c}_2$ is a straight line (i.e. a geodesic in $\mathbb{R}^2$), and then, by Proposition 1, the lifting curve $\hat{c}_1$ on S_1 is also a geodesic on each inerval $[t_i, t_{i+1}]$ for all $i = 0, \ldots, N-1$. Then, from the initial contact point X_0 on S_1, we can integrate successively the geodesic equation on each $[t_i, t_{i+1}]$ with initial conditions equal to $\hat{c}_1(t_i)$ and $(\hat{u}_1^i, \hat{u}_2^i)$, for $i = 0, \ldots, N-1$.

Let us write explicitly the geodesic equation to be integrated (see for instance [3] for more details). Recall that a curve $c : [0,1] \to S_1$ is a geodesic curve if it verifies Eq. (1). In the case where S_1 is an immersed surface in $\mathbb{R}^3$, Eq. (1) is equivalent to

$$\ddot{c}(t) \perp T_{c(t)} S_1, \tag{26}$$

for almost all t in $[0,1]$.

When S_1 is defined as (a bounded connected component of) the zero-level set of a real-valued function $f : \mathbb{R}^3 \to \mathbb{R}$, we have $\nabla f(x) \perp T_x S_1$ at every $x \in S_1$. Thus, Eq. (26) becomes

$$\ddot{c} = \langle \ddot{c}, \frac{\nabla f(c)}{\|\nabla f(c)\|} \rangle \frac{\nabla f(c)}{\|\nabla f(c)\|}, \tag{27}$$

where $\langle , \rangle$ is the scalar product in $\mathbb{R}^3$.

Furthermore, since c is a curve traced on S_1, we also have

$$\langle \dot{c}(t), \nabla f(c(t)) \rangle = 0, \tag{28}$$

for almost all t in $[0,1]$. Then, by deriving Eq. (28) with respect to t, we get

$$\langle \ddot{c}, \nabla f(c) \rangle + \langle \dot{c}, \nabla^2 f(c) \dot{c} \rangle = 0 \tag{29}$$

Finally, summing up Eq. (27) and Eq. (29) together, we get

$$\ddot{c} = -\frac{\dot{c}^T \nabla^2 f(c) \dot{c}}{\|\nabla f(c)\|^2} \nabla f(c). \tag{30}$$

We use again a fourth-order Runge-Kutta scheme for numerical integration of Eq. (30).

An additional difficulty is that the numerical integration is not performed in an Euclidean space, but on a manifold S_1. Assume that we are at point $x \in S_1$ at time t. Then at time $t + \delta t$, we move to $X_{\text{new}} = X + (\delta t) d$ with $d \in T_x S_1$, but X_{new} does not belong to S_1 if d is nonzero. Therefore, at each integration step, we have to "project" X_{new} on S_1.

More precisely, assume that the point $(0,0,0)$ is inside the convex body S_1. Since S_1 is defined as (a bounded connected component of) the zero-level set of a smooth function f, we assume that $|f(X_{\text{new}})| \leq \varepsilon$ for some $\varepsilon << 1$, i.e X_{new} is close to S_1. Then, there exists a unique real number μ close to 1 such that $f(\mu X_{\text{new}}) = 0$, as a simple consequence of the convexity of S_1. The "projection" issue to be addressed is clearly a local one and therefore, Newton's method is efficient for finding μ. The derivative with respect to μ is also needed, it is evaluated by a finite difference scheme similar to Eq. (24).

5 Simulations

We have applied the numerical continuation method presented above for motion planning problem of several bodies rolling on the Euclidean plane. We first present the rolling of a flattened ball and an egg. We then give simulation in a case where the rolling body does not have a symmetry of revolution. Still, the CM works quite efficiently.

The first figure of each subsection shows the initial and final positions of the contact point on S_1 as well as the initial and final orientations of the body. An arbitrary non singular control is also given. Two phases of motion planning, adjusting the contact point and re-orienting the body, are shown in the second and the third figures of each simulation. We note that control curves are modified in a smooth way. In the fourth figure, a “right" control curve has been found and the body is rolling along this curve. Finally, the last figure shows that the body arrives at the end of the curve with desired contact point and orientation.

We note that all these simulations were performed with Matlab. The computation time is on average 30 seconds (2.2 GHz Intel Core 2 Duo, 1.6 G memory) with $N = 70$ for the discretization of control space H, which already gives satisfying control laws.

5.1 Flattened ball rolling on the plane

This flattened ball is defined by the zero-level set of the function

$$f(x, y, z) = x^2 + y^2 + 5z^2 - 1. \tag{31}$$

The gradient $\nabla f(x, y, z)$ is equal to $(2x, 2y, 10z)^T$ and it is never equal to zero on the zero-level set of Eq. (31). Then Eq. (23) and Eq. (30) are always well defined.

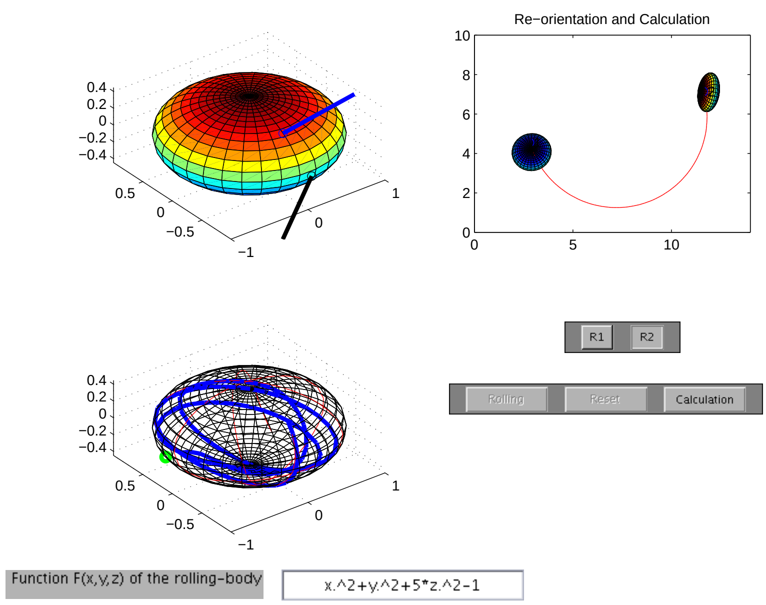

Figure 1: Initial and final positions of contact point and orientations of the flattened ball.

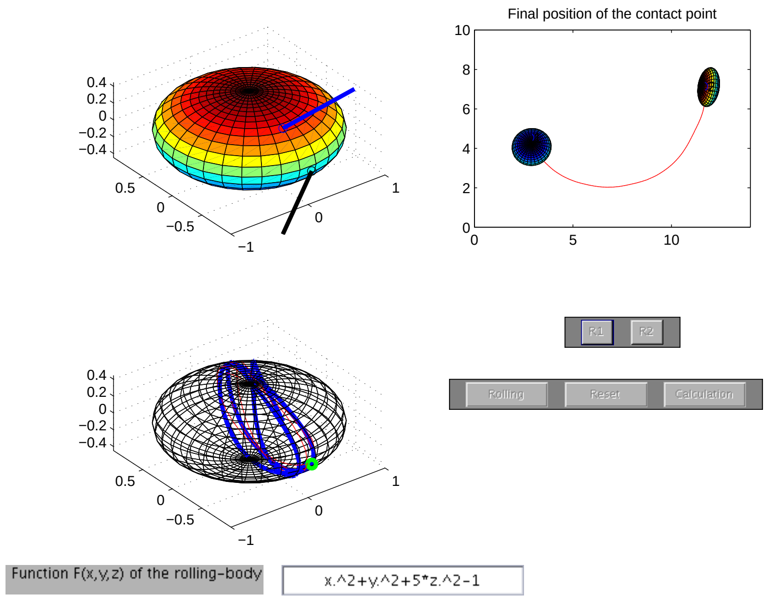

Figure 2: Computation for adjusting the final position of contact point by continuation method.

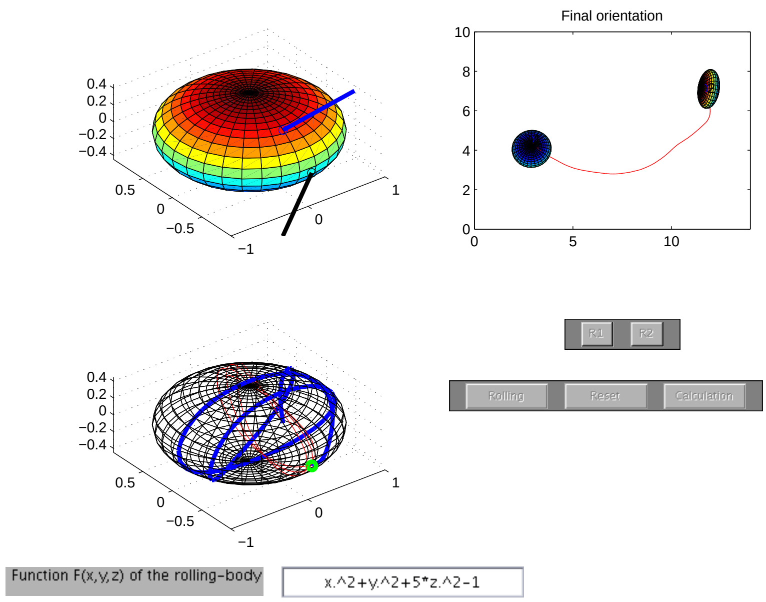

Figure 3: Computation for adjusting the final orientation of the flattened ball by continuation method.

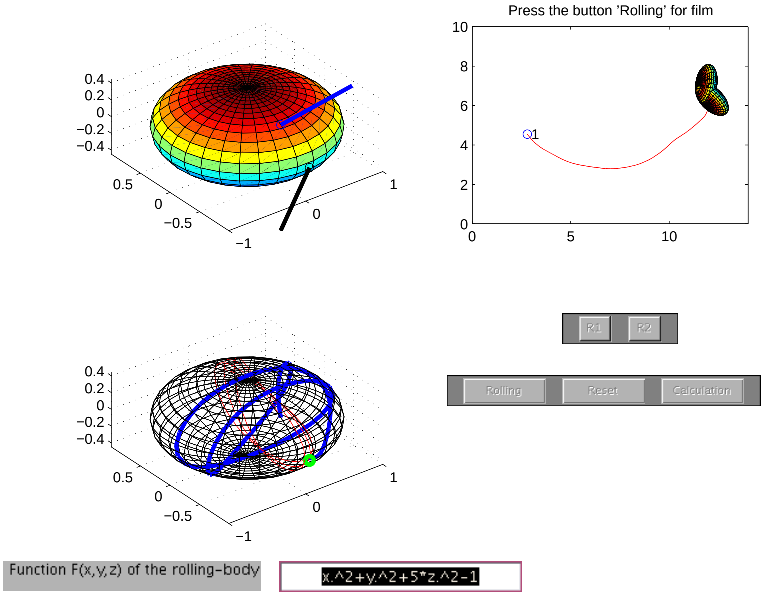

Figure 4: Flattened ball rolling along the curve before reaching the final position.

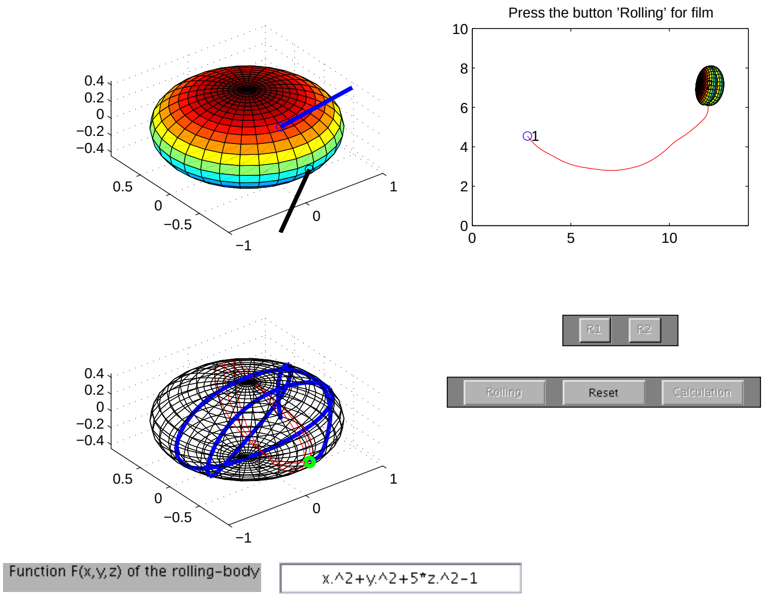

Figure 5: End of rolling.

5.2 Egg rolling on the plane

This "egg" is defined by one bounded connected component of the zero-level set of the function

$$f(x,y,z) = \frac{x^2+y^2}{1-0.4z} + \frac{z^2}{4} - 1. \tag{32}$$

We note that $\nabla f(x,y,z) = (\frac{2x}{1-0.4z}, \frac{2y}{1-0.4z}, \frac{0.4(x^2+y^2)}{(1-0.4z)^2} + \frac{z}{2})^T$, it is never equal to zero on the zero-level set of Eq. (32), then Eq. (23) and Eq. (30) are always well defined.

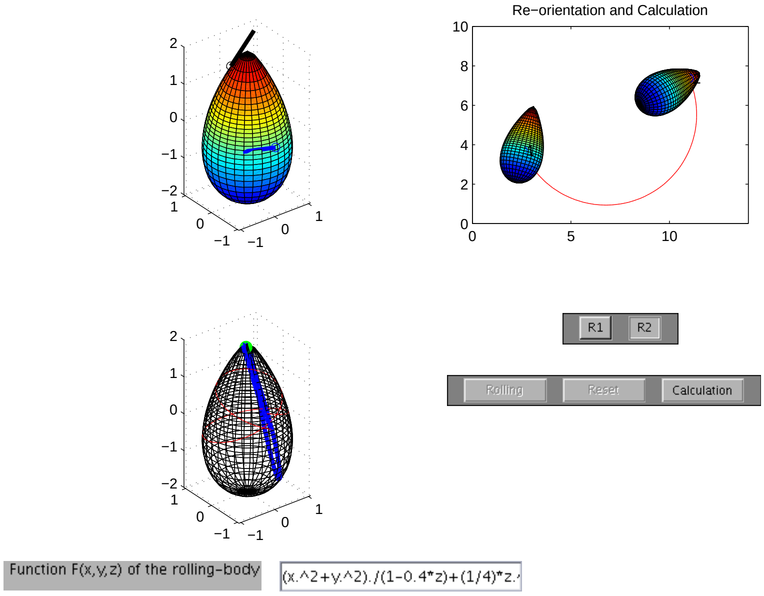

Figure 6: Initial and final positions of contact point and orientations of the egg.

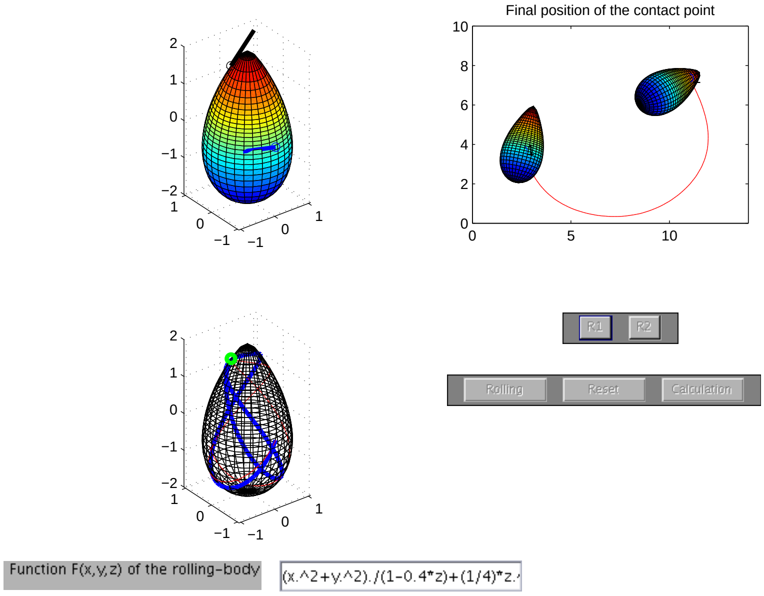

Figure 7: Computation for adjusting the final position of contact point by continuation method.

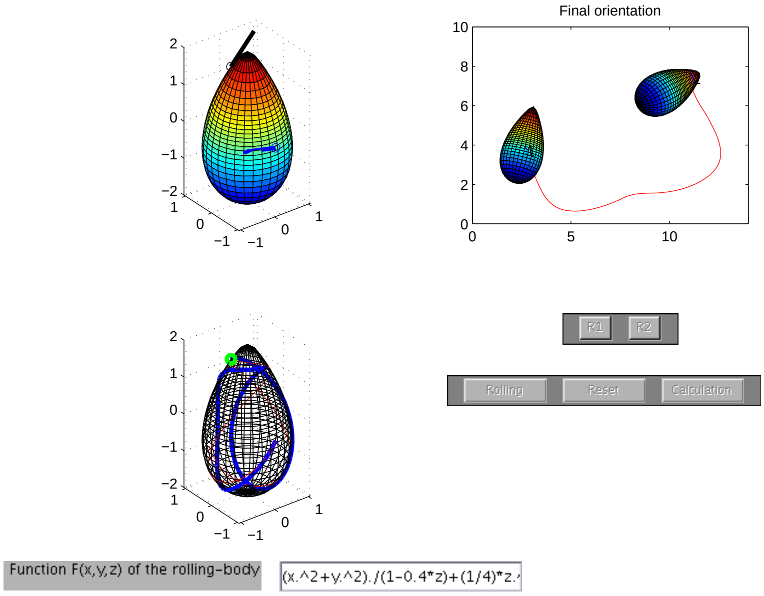

Figure 8: Computation for adjusting the final orientation of the egg by continuation method.

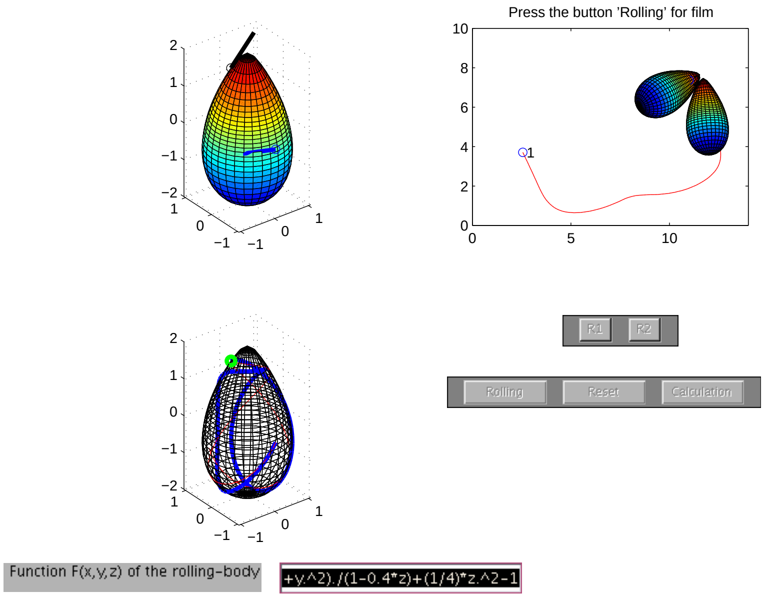

Figure 9: Egg rolling along the “right" curve before reaching the final position.

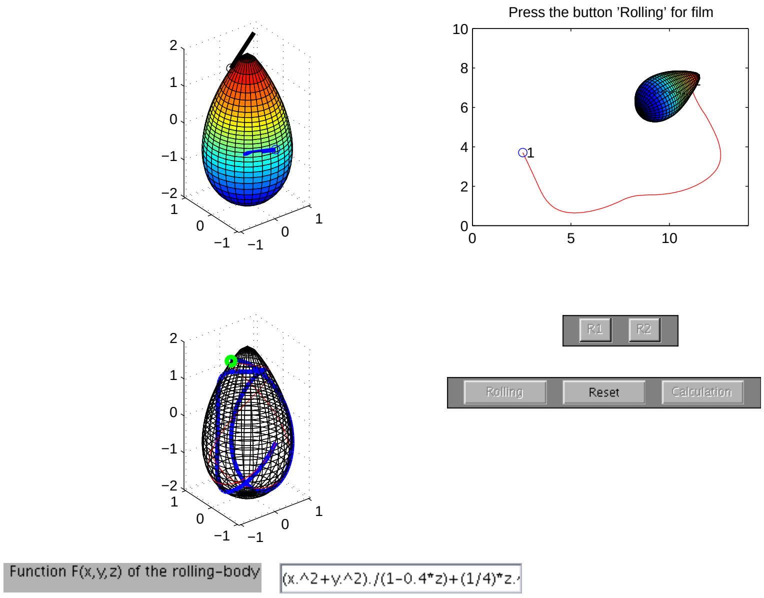

Figure 10: End of rolling.

5.3 More general case

In Section 3, the global convergence of continuation method has been proven for rolling of convex body with symmetric axis. However, we show in following simulations that continuation method still works numerically in more general cases, even though a theoretical convergence result is difficult to be obtained. This illustrates the robustness of the method.

For example, we take the convex body without symmetric axis, defined by one bounded connected component of the zero-level set of the function

$$f(x, y, z) = \frac{x^2}{1 - 0.5y} + \frac{2y^2}{1 - 0.1z} + \frac{0.5z^2}{1 - 0.3x - 0.1y} - 1. \tag{33}$$

We note that

$$\begin{aligned} & \nabla f(x, y, z) \\ = \; & \left(\frac{2x}{1 - 0.5y} + \frac{0.3}{(1 - 0.3x - 0.1y)^2}, \frac{4y}{1 - 0.1z} + \frac{0.1}{(1 - 0.3x - 0.1y)^2}, \frac{0.2y^2}{(1 - 0.1z)^2} + \frac{z}{1 - 0.3x - 0.1y}\right)^T, \end{aligned}$$

it is never equal to zero on the zero-level set of Eq. (33), then Eq. (23) and Eq. (30) are always well defined.

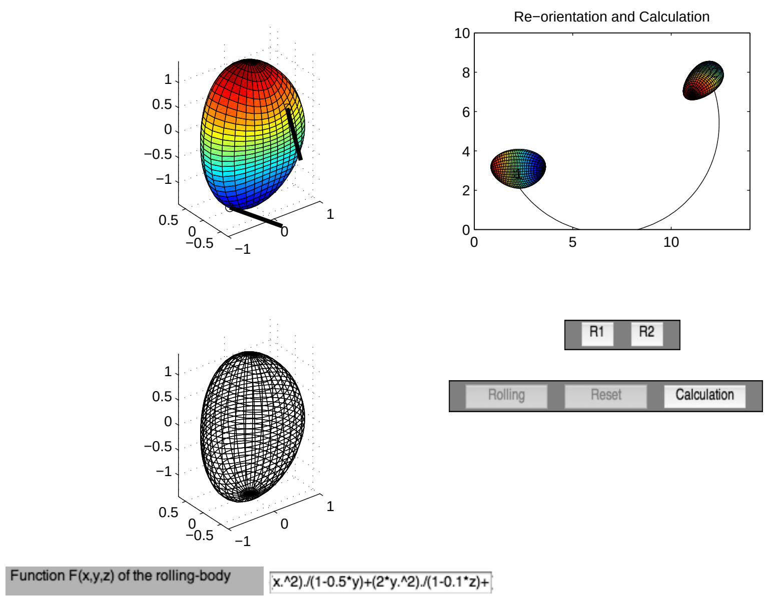

Figure 11: Initial and final positions of contact point and orientations of the convex body.

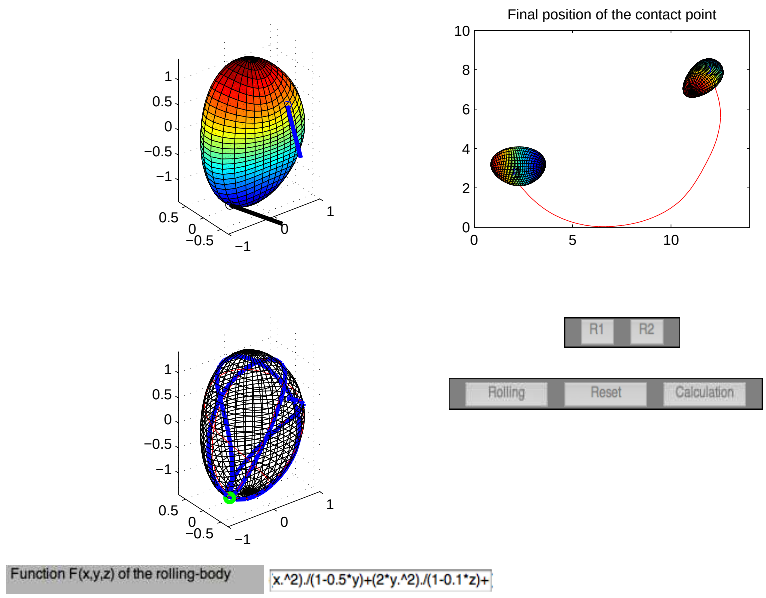

Figure 12: Computation for adjusting the final position of contact point by continuation method.

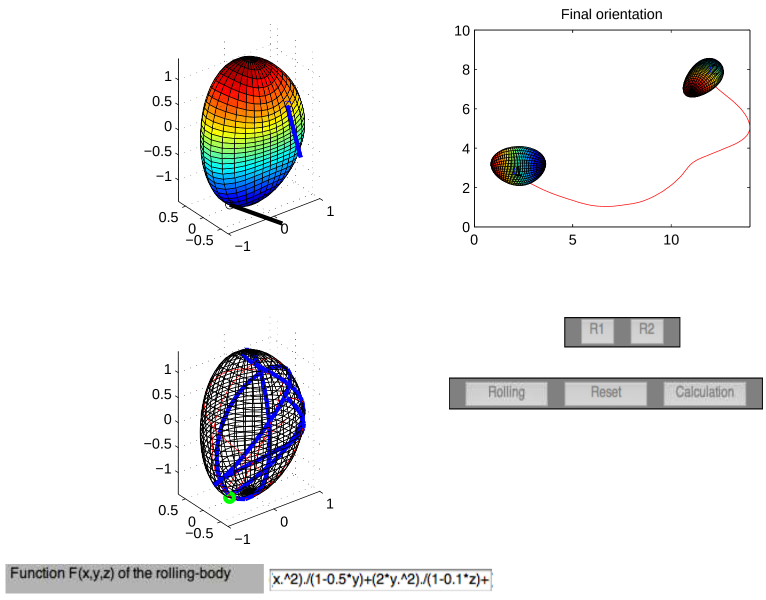

Figure 13: Computation for adjusting the final orientation of the convex body by continuation method.

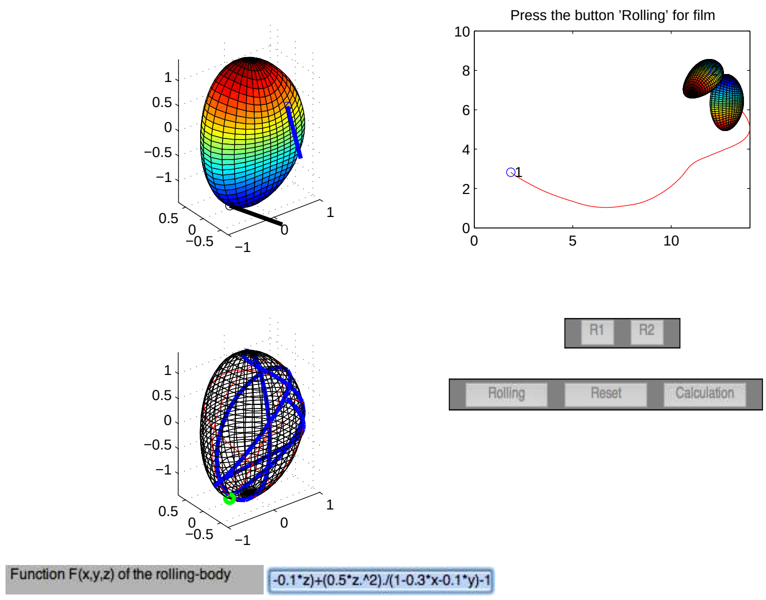

Figure 14: Convex body rolling along the “right" curve before reaching the final position.

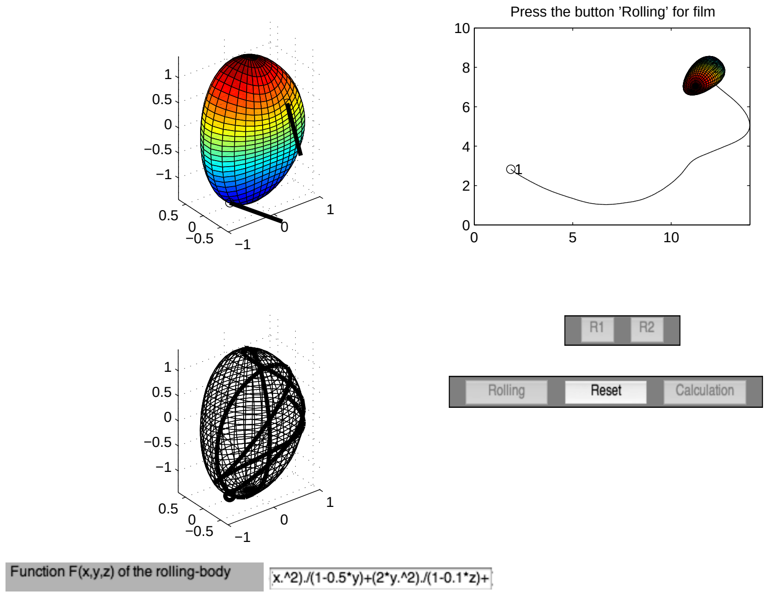

Figure 15: End of rolling.

6 Conclusion

In this paper, we have applied the continuation method to the motion planning problem of rolling bodies. Numerical simulations have been performed in the case where convex smooth body is rolling on the plane. We have shown through several examples the robustness and the convergence speed of this method. A possible direction of future work is the numerical implementation of this method for the motion planning problem of one convex smooth body rolling on the top of another one. It can be shown that the invertibility of $D\phi_p$ involved in the Path Lifting Equation (15) requires $K_2 - K_1 \neq 0$ at the contact point. However, this condition may not be globally verified for two general smooth convex bodies.

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