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Abstract

This paper deals with the homogenization of a second order parabolic operator with a large nonlinear potential and periodically oscillating coefficients of both spatial and temporal variables. Under a centering condition for the nonlinear zero-order term, we obtain the effective problem and prove a convergence result. The main feature of the homogenized equation is the appearance of a non-linear convection term.

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1 Introduction

We consider the homogenization of a reaction-diffusion equation with a large non-linear reaction term in periodic porous media. We assume that the coefficients of the equations are periodically oscillating in both space and time with a parabolic (or diffusive) scaling; namely, the spatial period is ε and temporal period is ε^2, ε being a small positive parameter. The corresponding Cauchy problem reads

$$\begin{cases} \rho\Big(\dfrac{x}{\varepsilon}\Big)\partial_t u^\varepsilon = \operatorname{div}\Big(a\big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\big)\nabla u^\varepsilon\Big) + \dfrac{1}{\varepsilon} g\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2},u^\varepsilon(x,t)\Big) \quad \text{in } \mathbb{R}^n \times (0,T), \\ u^\varepsilon(x,0) = u_0(x) \in L^2(\mathbb{R}^n), \end{cases} \tag{1}$$

where u_0 is the initial data, and the coefficients $a(y,s)$, $\rho(y)$ and $g(y,s,u)$ are 1-periodic in y and s. The unknown u^ε is the concentration of some chemical species diffusing in a porous medium of porosity $\rho(y)$, with diffusivity $a(y,s)$ and reacting with the background medium (for example by absorption/desorption) through the nonlinear term $g(y,s,u)$. The fact that the coefficients vary periodically in time can be interpreted as a crude modelling of some exterior forcing (like another chemical reaction). In addition to usual assumptions on the coefficients (see the next section for details), we make the crucial assumption on centering of the nonlinear term, namely we assume that, for each $u \in \mathbb{R}$,

$$\int\limits_{[0,1]^{n+1}} g(y,s,u)\,ds\,dy = 0.$$

This condition can be interpreted as a local equilibrium of the reaction for any concentration level, which allows us to expect a non-trivial limit, as ε goes to zero. Our main result is the convergence of the sequence u^ε of solutions of problem (1) to the unique solution of the following homogenized problem (see Theorem 3 for the exact statement):

$$\begin{cases} \partial_t u = \operatorname{div}\big(\hat{a}\nabla u\big) + F(u)\cdot\nabla u + V(u) \quad \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x) \in L^2(\mathbb{R}^n), \end{cases} \tag{2}$$

where the effective velocity $F(u)$ and effective potential $V(u)$ are explicit non-linear functions defined in terms of correctors or solutions of the so-called cell problems. There are two interesting features of the above effective operator. The first one is the appearance of the first order term in the limit operator, even if the original equation is formally self-adjoint. This effect is due to the time oscillation of the coefficients in (1). Indeed, if the coefficients do not depend on the temporal variable s, then simple computations show that the velocity $F(u)$ vanishes, see Remark 4 below.

The second feature is that, in contrast with the original equation which has nonlinearity only in the zero order term, the limit operator also includes a nonlinear first order term, in other words the nonlinearity can jump to the next higher order term. This asymptotic phenomenon is well-known in physics and mechanics. For several models it has also been justified in the mathematical literature, see for instance [10].

The fact that convection can arise from the homogenization of a purely diffusion-reaction problem is already known and has several interesting applications. This effect was first discovered in nuclear reactor physics in [9] and later rigorously justified by homogenization arguments in [6], [7]. It is also an important phenomena in reactive transport through porous media where convection can be enhanced by chemical reactions [3], [4], [11], [12]. Finally, it is an explanation for the origin of bio-motors [14]. On the other hand, the transmission of the nonlinearity from the reactive term in the original equation (1) to the convective term in the effective equation (2) is another evidence of the strong coupling between convection and chemical reactions in reactive transport through porous media.

Large zero-order terms have already been homogenized in the linear case when they scale like $1/\varepsilon^2$ [5]. In such a case the factorization technique allows us to separate a periodically oscillating part $\psi(x/\varepsilon)$ of solutions so that the remaining part $u^\varepsilon/\psi(x/\varepsilon)$ has a regular behaviour, see [2], [4], [8]. However, this technique fails to work if the problem under consideration is nonlinear.

On the contrary, for linear equations with centered zero order term of order $1/\varepsilon$, solutions exhibit a regular asymptotic behaviour (see, for instance, [5], Chapter 1, Section 12). The goal of the present paper is to prove an homogenization result for equations with a nonlinear potential.

The paper is organized as follows. In section 2 we introduce the problem and specify the conditions on the coefficients of equation (1). Section 3 is aimed at obtaining uniform a priori estimates. In Sections 4 and 5 we characterize the two-scale limit of solutions, define the limit problem and prove the convergence result. Finally, in Section 6 the properties of the limit problem are studied. It is shown, in particular, that this problem has a unique solution.

2 Statement of the problem

Instead of (1) we consider a slightly more general initial boundary value problem

$$\begin{cases} \rho\Big(\dfrac{x}{\varepsilon}\Big)\partial_t u^\varepsilon = \operatorname{div}\Big(a\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\Big)\nabla u^\varepsilon\Big) + \dfrac{1}{\varepsilon} g\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}, u^\varepsilon(x,t)\Big) \quad \text{in } Q\times(0,T), \\ u^\varepsilon(x,t) = 0 \quad \text{on } \partial Q\times(0,T), \\ u^\varepsilon(x,0) = u_0(x) \in L^2(Q), \end{cases} \tag{3}$$

in a Lipschitz domain $Q \subset \mathbb{R}^n$, for some given final time $T > 0$. Notice that the domain Q might be either bounded or unbounded. In the case $Q = \mathbb{R}^n$ problem (3) turns into problem (1).

We assume that the coefficients of (3) satisfy the following properties:

A1. Uniform ellipticity. The matrix a_{ij} is real, not necessary symmetric, positive definite: there exists $\Lambda > 0$ such that

$$\|a_{ij}\|_{L^\infty(\mathbb{R}^{n+1})} \le \Lambda^{-1},\ 1 \le i, j \le n,$$

$$a_{ij}(y,s)\xi_i\xi_j \geq \Lambda|\xi|^2 \quad \text{for all } (s,y) \in \mathbb{R}^{n+1},\ \xi \in \mathbb{R}^n.$$

A2. Positivity. There exists $\Lambda > 0$ such that

$$\Lambda \leq \rho(y) \leq \Lambda^{-1} \quad \text{for all } y \in \mathbb{R}^n.$$

A3. Periodicity. The density $\rho(y)$ is $[0,1]^n$-periodic, and the entries of the matrix $a(s,y)$ are $[0,1]^{n+1}$-periodic. Without loss of generality we assume that

$$\int_{[0,1]^n} \rho(y)dy = 1.$$

Here and in the sequel Y stands for the periodicity cell $[0,1]^n$. We also denote $\mathcal{Y} = [0,1]^{n+1}$.

A4. Centering condition. We assume that, for any $u \in \mathbb{R}$,

$$< g > \stackrel{\text{def}}{=} \int\limits_{[0,1]^{n+1}} g(y,s,u)dsdy = 0.$$

A5. Lipschitz continuity. We assume that there exists a finite constant $0 < C < +\infty$ such that, for any $y, s \in [0,1]^{n+1}$ and $u \in \mathbb{R}$,

$$|\partial_u g(y,s,u)| \leq C\,,$$

$$|\partial_u g(y,s,u_1) - \partial_u g(y,s,u_2)| \leq C|u_1 - u_2|\big(1 + |u_1| + |u_2|\big)^{-1}.$$

The last bound implies that, for all $y, s \in [0,1]^{n+1}$, the function $\partial_u g(y,s,\cdot)$ belongs to $W^{1,\infty}(\mathbb{R})$, and, moreover,

$$\Big|\frac{\partial^2 g}{\partial u^2}(y,s,u)\Big| \leq \frac{C}{1+|u|}.$$

In particular, **A5** is fulfilled if $g(y,s,u)$ is two times differentiable in u, and $\Big|\dfrac{\partial^2 g}{\partial u^2}(y,s,u)\Big| \leq C(1+|u|)^{-1}$.

A6. Equilibrium condition. We assume that 0 is a possible solution of (1) or (3), i.e.,

$$g(y,s,0) = 0 \qquad \text{for all } y, s \in [0,1]^{n+1}.$$

In a standard way one can show that under the above assumptions for each $\varepsilon > 0$ problems (1) and (3) have a unique solution u^ε, moreover $u^\varepsilon \in L^2(0,T;H^1(Q)) \cap C(0,T;L^2(Q)$. However, due to the presence of the factor $1/\varepsilon$ in front of the zero order term in the studied equation, the standard energy estimates are not uniform in ε. In the next section we improve these estimates and show that uniform in ε a priori estimates hold.

3 A priori estimates and compactness

We begin this section by obtaining uniform a priori energy estimate for the solution u^ε. In the sequel, Q stands either for $\mathbb{R}^n$, as in the case of problem (1), or for a Lipschitz open subset of $\mathbb{R}^n$, in the case of the boundary-value problem (3).

Lemma 1. *Under assumptions* **A1.**–**A6.** *the following estimates hold true*

$$\|u^\varepsilon\|_{L^2(0,T;H^1(Q))} \le C, \qquad \|u^\varepsilon\|_{L^\infty(0,T;L^2(Q))} \le C$$

with a constant C which does not depend on ε.

Proof. The desired estimate relies on the following representation of the large nonlinear potential term

$$\begin{aligned} \frac{1}{\varepsilon} g\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u^\varepsilon\Big) \ &= \ \mathrm{div}\Big[G\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u^\varepsilon\Big)\Big] + \varepsilon\rho\Big(\frac{x}{\varepsilon}\Big)\partial_t\Big[\overline{G}\Big(\frac{t}{\varepsilon^2}, u^\varepsilon\Big)\Big] \\ &- \ \partial_u G\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u^\varepsilon\Big) \cdot \nabla u^\varepsilon - \varepsilon\rho\Big(\frac{x}{\varepsilon}\Big)\partial_u \overline{G}\Big(\frac{t}{\varepsilon^2}, u^\varepsilon\Big)\partial_t u^\varepsilon \end{aligned} \tag{4}$$

with

$$\overline{G}(s,u) = \int_0^s \bar{g}(\tau, u)\, d\tau, \qquad \bar{g}(s,u) = \int_Y g(y,s,u)dy, \tag{5}$$

and $G(y,s,u) = \nabla_y R(y,s,u)$ with R defined as the solution of

$$\begin{cases} \Delta_y R(y,s,u) = g(y,s,u) - \rho(y)\bar{g}(s,u) \text{ in } Y, \\ R(y,s,u) \text{ is } Y-\text{periodic in } y. \end{cases} \tag{6}$$

The right hand side of (6) has zero average in Y so R exists and is unique up to an additive function of variables s and u (which does not matter in the definition of G). This representation can be checked by straightforward computations. By construction and due to **A5** and **A6**, we have

$$\overline{G}(s,u)\le C|u|,\quad |\partial_u\overline{G}(s,u)|\le C,\quad |\partial_u\overline{G}(s,u_1)-\partial_u\overline{G}(s,u_2)|\le\frac{C|u_1-u_2|}{1+|u_1|+|u_2|}.$$

and

$$G(y,s,u)\le C|u|,\quad |\partial_u G(y,s,u)|\le C,$$
$$|\partial_u G(y,s,u_1)-\partial_u G(y,s,u_2)|\le\frac{C|u_1-u_2|}{1+|u_1|+|u_2|}.$$

In particular, $\frac{\partial^2\overline{G}}{\partial u^2}(s,u)$ belongs to $L^\infty((0,1)\times\mathbb{R})$, and $\left|\frac{\partial^2}{\partial u^2}\overline{G}(s,u)\right|\le C(1+|u|)^{-1}$.

Multiplying equation (3) by u^ε, integrating the resulting relation over the cylinder $Q\times(0,t)$ and making use of representation (4), we get after straightforward rearrangements

$$\frac{1}{2}\int\limits_Q\rho\Big(\frac{x}{\varepsilon}\Big)(u^\varepsilon(x,t))^2\,dx+\int\limits_0^t\int\limits_Q a\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}\Big)\nabla u^\varepsilon(x,s)\cdot\nabla u^\varepsilon(x,s)\,dxds$$

$$=\frac{1}{2}\int\limits_Q\rho\Big(\frac{x}{\varepsilon}\Big)(u_0)^2\,dx+\int\limits_0^t\int\limits_Q G\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2},u^\varepsilon\Big)\cdot\nabla u^\varepsilon(x,s)\,dxds$$

$$-\varepsilon\int\limits_Q\overline{G}\Big(\frac{t}{\varepsilon^2},u^\varepsilon(x,t)\Big)u^\varepsilon(x,t)\,dx+\varepsilon\int\limits_Q\rho\Big(\frac{x}{\varepsilon}\Big)\overline{G}\big(0,u_0(x)\big)u_0(x)\,dx \tag{7}$$

$$+\varepsilon\int\limits_0^t\int\limits_Q\rho\Big(\frac{x}{\varepsilon}\Big)\overline{G}\Big(\frac{s}{\varepsilon^2},u^\varepsilon(x,s)\Big)\partial_t u^\varepsilon(x,s)\,dxds+\int\limits_0^t\int\limits_Q\partial_u G\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2},u^\varepsilon\Big)\cdot\nabla u^\varepsilon u^\varepsilon\,dxds$$

$$+\varepsilon\int\limits_0^t\int\limits_Q\rho\Big(\frac{x}{\varepsilon}\Big)\partial_u\overline{G}\Big(\frac{s}{\varepsilon^2},u^\varepsilon(x,s)\Big)\big(\partial_t u^\varepsilon(x,s)\big)u^\varepsilon(x,s)\,dxds\equiv I_1+I_2+I_3+I_4+I_5+I_6+I_7.$$

The terms on the right-hand side can be estimated as follows

$$I_1\le\Lambda^{-1}\|u_0\|^2_{L^2(Q)},\qquad |I_2|\le C\int_0^t\int_Q|u^\varepsilon|\,|\nabla u^\varepsilon|\,dxds,$$

$$|I_3| + |I_4| \le C\varepsilon\big(\|u^\varepsilon(\cdot,t)\|^2_{L^2(Q)} + \|u_0\|^2_{L^2(Q)}\big),$$

$$|I_5| = \varepsilon\Big|\int_0^t\int_Q \rho\Big(\frac{x}{\varepsilon}\Big)\overline{G}\Big(\frac{s}{\varepsilon^2}, u^\varepsilon(x,s)\Big)\partial_t u^\varepsilon(x,s)\,dxds\Big|$$

$$= \varepsilon\Big|\int_0^t\int_Q \overline{G}\Big(\frac{s}{\varepsilon^2}, u^\varepsilon(x,s)\Big)\Big\{\mathrm{div}\big(a(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2})\nabla u^\varepsilon(x,s)\big) + \frac{1}{\varepsilon}g\big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2},u^\varepsilon(x,s)\big)\Big\}\,dxds\Big|$$

$$\le C\Big(\varepsilon\|\nabla u^\varepsilon\|^2_{L^2(Q\times(0,T))} + \|u^\varepsilon\|^2_{L^2(Q\times(0,T))}\Big)$$

(with no boundary terms because $\overline{G}(s,0)=0$),

$$|I_6| \le C\|u^\varepsilon\|_{L^2(Q\times(0,T))}\|\nabla u^\varepsilon\|_{L^2(Q\times(0,T))},$$

$$|I_7| = \varepsilon\Big|\int_0^t\int_Q \partial_u\overline{G}\Big(\frac{s}{\varepsilon^2}, u^\varepsilon(x,s)\Big)u^\varepsilon(x,s)\mathrm{div}\big(a(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2})\nabla u^\varepsilon(x,s)\big)\,dxds\Big| +$$

$$+\Big|\int_0^t\int_Q \partial_u\overline{G}\Big(\frac{s}{\varepsilon^2}, u^\varepsilon(x,s)\Big)g\big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2},u^\varepsilon(x,s)\big)u^\varepsilon(x,s)\,dxds\Big|$$

$$\le C\varepsilon\|\nabla u^\varepsilon\|^2_{L^2(Q\times(0,T))} + C\|u^\varepsilon\|^2_{L^2(Q\times(0,T))}.$$

Substituting these bounds in (7), we obtain

$$\int_Q \rho\Big(\frac{x}{\varepsilon}\Big)(u^\varepsilon(x,t))^2\,dx + \int_0^t\int_Q a\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}\Big)\nabla u^\varepsilon(x,s)\cdot\nabla u^\varepsilon(x,s)\,dxds$$

$$\le C\|u_0\|^2_{L^2(Q)} + C\varepsilon\int_0^t\|\nabla u^\varepsilon(\cdot,s)\|^2_{L^2(Q)}ds$$

$$+\int_0^t\|\nabla u^\varepsilon(\cdot,s)\|_{L^2(Q)}\|u^\varepsilon(\cdot,s)\|_{L^2(Q)}ds + \int_0^t\|u^\varepsilon(\cdot,s)\|^2_{L^2(Q)}ds.$$

This yields the desired bound by a standard application of Gronwall's lemma.

□

In the case of an unbounded domain Q we also need to show that the solution u^ε remains localized in space as $\varepsilon\to 0$.

Lemma 2. *In the case of Cauchy problem (1) or unbounded domain Q in (3), for any $\delta>0$ there exists $R=R(\delta)$ such that, uniformly in ε,*

$$\|u^\varepsilon\|_{L^2(\{x\in Q\,:\,|x|\ge R\})} \le \delta.$$

Proof. Let $\tilde{\varphi}_R$ be a continuous, piecewise linear function $\tilde{\varphi}_R : \mathbb{R}^+ \mapsto \mathbb{R}$ such that $\tilde{\varphi}_R(r) = 0$ for $r \le R$, $\tilde{\varphi}_R(r) = 1$ for $r \ge 2R$, and $\tilde{\varphi}'(r) = 1/R$ for $r \in (R, 2R)$. We denote $\varphi_R(x) = \tilde{\varphi}(|x|)$, multiply equation (1) by $\varphi_R(x)u^\varepsilon(x)$ and integrate the resulting relation over the set $\mathbb{R}^n \times (0,\tau)$. After simple rearrangements this yields

$$\frac{1}{2}\int\limits_Q \varphi_R(x)(u^\varepsilon(x,\tau))^2\rho^\varepsilon(x)dx + \int\limits_0^\tau\int\limits_Q \varphi_R(x)a^\varepsilon(x,t)\nabla u^\varepsilon(x,t)\cdot\nabla u^\varepsilon(x,t)\,dxdt$$

$$= \frac{1}{2}\int\limits_Q \varphi_R(x)(u_0(x))^2\rho^\varepsilon(x)dx - \int\limits_0^\tau\int\limits_Q u^\varepsilon(x,t)a^\varepsilon(x,t)\nabla u^\varepsilon(x,t)\cdot\nabla\varphi_R(x)\,dxdt$$

$$+\int\limits_0^\tau\int\limits_Q \{u^\varepsilon(x,t)G^\varepsilon(x,t)\cdot\nabla\varphi_R(x) + \varphi_R(x)G^\varepsilon(x,t)\cdot\nabla u^\varepsilon(x,t)\}\,dxdt$$

$$+\int\limits_0^\tau\int\limits_Q \varphi_R(x)u^\varepsilon(x,t)\partial_u G^\varepsilon(x,t)\cdot\nabla u^\varepsilon(x,t)dxdt \tag{8}$$

$$+\varepsilon\int\limits_0^\tau\int\limits_Q \rho^\varepsilon(x)\left\{\overline{G}^\varepsilon(x,t)\partial_t u^\varepsilon(x,t) + u^\varepsilon(x,t)\partial_u\overline{G}^\varepsilon(x,t)\partial_t u^\varepsilon(x,t)\right\}\varphi_R(x)dxdt$$

$$-\varepsilon\int\limits_Q \rho^\varepsilon(x)\overline{G}^\varepsilon(x,\tau)u^\varepsilon(x,\tau)\varphi_R(x)\,dx + \varepsilon\int\limits_Q \rho^\varepsilon(x)\overline{G}^\varepsilon(x,0)u_0(x)\varphi_R(x)\,dx;$$

here $a^\varepsilon(x,t)$ and $\rho^\varepsilon(x)$ stand for $a(x/\varepsilon, t/\varepsilon^2)$ and $\rho(x/\varepsilon)$, respectively, and

$$G^\varepsilon(x,t) = G\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},u^\varepsilon(x,t)\Big),\quad \overline{G}^\varepsilon(x,t) = \overline{G}\Big(\frac{t}{\varepsilon^2},u^\varepsilon(x,t)\Big),$$

$$\partial_u\overline{G}^\varepsilon(x,t) = \partial_u\overline{G}\Big(\frac{t}{\varepsilon^2},u^\varepsilon(x,t)\Big).$$

Considering the assumptions on g and the definition of φ_R, we get

$$\Big|\int\limits_0^\tau\int\limits_Q u^\varepsilon(x,t)a^\varepsilon(x)\nabla u^\varepsilon(x,t)\cdot\nabla\varphi_R(x)\,dxdt\Big| \le \frac{C}{R}\|u^\varepsilon\|^2_{L^2(0,T;H^1(Q))} \le \frac{C}{R},$$

$$\Big|\int\limits_0^\tau\int\limits_Q u^\varepsilon(x,t)G^\varepsilon(x,t)\cdot\nabla\varphi_R(x)\,dxdt\Big| \le \frac{C}{R}\|u^\varepsilon\|^2_{L^2(0,T;H^1(Q))} \le \frac{C}{R},$$

$$\Big|\int_0^\tau\int_Q \varphi_R(x)G^\varepsilon(x,t)\cdot\nabla u^\varepsilon(x,t)\,dxdt\Big| \le \mu\int_0^\tau\int_Q \varphi_R|\nabla u^\varepsilon|^2\,dxdt + \frac{C}{\mu}\int_0^\tau\int_Q \varphi_R|u^\varepsilon|^2\,dxdt,$$

$$\Big|\int_0^\tau\int_Q \varphi_R(x)u^\varepsilon(x,t)\partial_u G^\varepsilon(x,t)\cdot\nabla u^\varepsilon(x,t)\,dxdt\Big| \le \mu\int_0^\tau\int_Q \varphi_R|\nabla u^\varepsilon|^2\,dxdt + \frac{C}{\mu}\int_0^\tau\int_Q \varphi_R|u^\varepsilon|^2\,dxdt$$

$$\Big|\varepsilon\int_Q \rho^\varepsilon(x)\overline{G}^\varepsilon(x,\tau)u^\varepsilon(x,\tau)\varphi_R(x)\,dx\Big| \le C\varepsilon\int_Q (u^\varepsilon(x,\tau))^2\varphi_R(x)\,dx$$

with an arbitrary $\mu>0$; here we have also used the bounds $|\partial_u G(y,s,u)|\le C$ and $|G(y,s,u)|\le C|u|$. It remains to estimate the integral on the right-hand side of (8) which contains the time derivative of u^ε. To this end we use the original equation (1). This gives

$$\varepsilon\Big|\int_0^\tau\int_Q \rho^\varepsilon(x)\overline{G}^\varepsilon(x,t)\partial_t u^\varepsilon(x,t)\varphi_R(x)\,dxdt\Big| \le$$

$$\le \varepsilon\Big|\int_0^\tau\int_Q \overline{G}^\varepsilon(x,t)\varphi_R(x)\mathrm{div}\big(a^\varepsilon(x,t)\nabla u^\varepsilon(x,t)\big)\,dxdt\Big|$$

$$+\Big|\int_0^\tau\int_Q \overline{G}^\varepsilon(x,t)\varphi_R(x)g^\varepsilon(x,t)\,dxdt\Big| \le$$

$$\le C\int_0^\tau\int_Q \Big\{\varepsilon\varphi_R(x)|\nabla u^\varepsilon(x,t)|^2 + \frac{\varepsilon}{R}|u^\varepsilon(x,t)||\nabla u^\varepsilon(x,t)| + \varphi_R(x)|u^\varepsilon(x,t)|^2\Big\}\,dxdt$$

$$\le C\frac{\varepsilon}{R} + C\int_0^\tau\int_Q \big\{\varepsilon\varphi_R(x)|\nabla u^\varepsilon(x,t)|^2 + \varphi_R(x)|u^\varepsilon(x,t)|^2\big\}\,dxdt.$$

Similarly,

$$\varepsilon\Big|\int_0^\tau\int_Q \varphi_R(x)u^\varepsilon(x,t)\partial_u\overline{G}^\varepsilon(x,t)\partial_t u^\varepsilon(x,t)\,dxdt\Big|$$

$$\leq C\frac{\varepsilon}{R} + C\int_0^\tau\!\!\int_Q \left\{\varepsilon\varphi_R(x)|\nabla u^\varepsilon(x,t)|^2 + \varphi_R(x)|u^\varepsilon(x,t)|^2\right\} dxdt.$$

Combining the above estimates and choosing an appropriate value of μ, say $\mu = 1/4$, we conclude that for all sufficiently small $\varepsilon > 0$ the inequality holds

$$\int_Q \varphi_R(x)(u^\varepsilon(x,\tau))^2 dx + \int_0^\tau\int_Q \varphi_R(x)|\nabla u^\varepsilon(x,t)|^2\, dxdt \leq$$

$$\leq C\int_Q \varphi_R(x)(u_0(x))^2 dx + \frac{C}{R} + C\int_0^\tau\int_Q \varphi_R(x)|u^\varepsilon(x,t)|^2\, dxdt$$

From this estimate the desired statement follows by Gronwall's lemma. $\square$

The previous two statements are not sufficient for obtaining the compactness of $\{u^\varepsilon\}$ in $L^2(Q\times(0,T))$. We need in addition a uniform estimate for the modulus of continuity of $\{u^\varepsilon\}$.

Lemma 3. *For any $\varphi\in C_0^\infty(Q)$ and any $T>0$ there exist $c_1>0$ and $c_2>0$ such that for all $\gamma>0$ the inequality holds*

$$\sup_{\substack{0\leq t_1<t_2\leq T\\ t_2-t_1\leq\gamma}} \left|(\rho^\varepsilon u^\varepsilon(\cdot,t_2),\varphi)_{L^2(Q)} - (\rho^\varepsilon u^\varepsilon(\cdot,t_1),\varphi)_{L^2(Q)}\right| \leq c_1\sqrt{\gamma} + c_2\varepsilon.$$

Proof. We have

$$(\rho^\varepsilon u^\varepsilon(\cdot,t_2),\varphi)_{L^2(Q)} - (\rho^\varepsilon u^\varepsilon(\cdot,t_1),\varphi)_{L^2(Q)} = \int_{t_1}^{t_2}\!\!\int_Q \rho^\varepsilon\varphi\partial_t u^\varepsilon\, dxdt$$

$$= -\int_{t_1}^{t_2}\!\!\int_Q \{a^\varepsilon\nabla u^\varepsilon\cdot\nabla\varphi - G^\varepsilon\cdot\nabla\varphi - \varphi\partial_u G^\varepsilon\cdot\nabla u^\varepsilon\}\, dxdt$$

$$-\,\varepsilon\int_Q \rho^\varepsilon\varphi\big(\overline{G}^\varepsilon(x,t_2) - \overline{G}^\varepsilon(x,t_1)\big)\, dx + \varepsilon\int_{t_1}^{t_2}\!\!\int_Q \rho^\varepsilon\partial_u\overline{G}^\varepsilon\varphi\partial_t u^\varepsilon\, dxdt.$$

The last term on the right-hand side can be rearranged as follows

$$\varepsilon\int_{t_1}^{t_2}\!\!\int_Q \rho^\varepsilon\partial_u\overline{G}^\varepsilon\varphi\partial_t u^\varepsilon\, dxdt = -\varepsilon\int_{t_1}^{t_2}\!\!\int_Q \partial_u\overline{G}^\varepsilon a^\varepsilon\nabla u^\varepsilon\cdot\nabla\varphi\, dxdt$$

$$-\varepsilon \int_{t_1}^{t_2}\int_Q \varphi \frac{\partial^2}{\partial u^2}\overline{G}^\varepsilon a^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon \, dxdt - \int_{t_1}^{t_2}\int_Q \varphi g^\varepsilon \partial_u \overline{G}^\varepsilon \, dxdt.$$

Recalling the estimate of Lemma 1 and the properties of $\overline{G}$, one derives from the above relations that

$$|(\rho^\varepsilon u^\varepsilon(\cdot,t_2),\varphi)_{L^2(Q)} - (\rho^\varepsilon u^\varepsilon(\cdot,t_1),\varphi)_{L^2(Q)}| \le C(\varphi)\sqrt{t_2-t_1}\|u^\varepsilon\|_{L^2(0,T;H^1(Q))}$$

$$+\varepsilon C(\varphi)\|u^\varepsilon\|^2_{L^2(0,T;H^1(Q))} \le C(\varphi)(\sqrt{t_2-t_1}+\varepsilon),$$

and the required bound follows. □

We proceed with the compactness result.

Lemma 4. *The family $(u^\varepsilon)_{\varepsilon>0}$ is relatively compact in the space $L^2(Q\times(0,T))$.*

Proof. The fact that the estimates of Lemmata 1–3 imply the compactness of u^ε has been proved in [10]. For the sake of completeness we simply explain the main idea of the proof. Introducing a smooth orthonormal basis $\big(\mathbf{e_1}(x)\big)_{i\ge 1}$ of $L^2(Q)$, we use the representation

$$u^\varepsilon(x,t) = \sum_{j=1}^{\infty} \alpha_j(t)\mathbf{e}_j(x), \qquad \alpha_j(t) = (u^\varepsilon(\cdot,t),\mathbf{e}_j)_{L^2(Q)},$$

and denote

$$u^\varepsilon_N(x,t) = \sum_{j=1}^{N} \alpha_j(t)\mathbf{e}_j(x), \qquad U^\varepsilon_N(x,t) = \sum_{j=N+1}^{\infty} \alpha_j(t)\mathbf{e}_j(x).$$

From the estimates of Lemma 1 it follows that $\|U^\varepsilon_N\|_{L^2(Q\times(0,T))}$ goes to zero as $N\to\infty$ uniformly in ε. Then one can derive from Lemma 3 and, in the case of unbounded domain Q, Lemma 2, that u_N is compact in $L^2(Q\times(0,T))$ for any N. This implies the desired compactness. The reader can find a detailed proof in [10]. □

4 Two-scale convergence and correctors

In this section we study the two-scale limits of u^ε and its gradient, and introduce the correctors required for passing to the limit in the original problem.

As a first step we apply the two-scale compactness arguments (see [1, 13]). It follows from Lemmata 1, 4 that there exist a subsequence and limits $u(x,t) \in L^2((0,T); H^1(Q))$, $w \in L^2(Q \times (0,T); L^2_\#(0,1; H^1_\#(Y)))$ (the symbol # indicates periodicity of the corresponding functions) such that, along this subsequence,

$$u^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} u(x,t) \qquad \text{in } L^2(Q \times (0,T)) \text{ strongly}, \tag{9}$$

$$\nabla u^\varepsilon \underset{\varepsilon \to 0}{\rightharpoonup} \nabla_x u(x,t) + \nabla_y w(x,t,y,s) \quad \text{in the sense of two-scale convergence.} \tag{10}$$

Notice that w is defined up to an arbitrary additive function of x, t and s. In order to fix its choice, we assume that

$$\langle \rho w \rangle_y \overset{\text{def}}{=} \int\limits_Y \rho(y) w(x,t,y,s)\, dy = 0. \tag{11}$$

Denote $\tilde{g}(y,s,u) = g(y,s,u) - \rho(y)\langle g(\cdot,s,u)\rangle_y$, which satisfies $\langle \tilde{g}(\cdot,s,u)\rangle_y = 0$ since $\langle \rho \rangle_y = 1$. The rest of this section is devoted to the proof of the following characterization of w.

Lemma 5. *The function w is a solution of the boundary value problem*

$$\begin{cases} \rho(y)\partial_s w = \operatorname{div}_y\big(a(y,s)(\nabla_y w + \nabla_x u(x,t))\big) + \tilde{g}(y,s,u(x,t)) \quad \text{in } \mathcal{Y} \\ \qquad (y,s) \to w \text{ is } \mathcal{Y}\text{-periodic.} \end{cases} \tag{12}$$

Proof. We first check that (12) is well-posed. According to [8], equation (12) has a $\mathcal{Y}$-periodic solution which is unique up to an additive constant. Thus, under the normalization condition $\int_{\mathcal{Y}} \rho(y) w(y,s,u) dy ds = 0$, the solution of (12) is unique. Moreover, due to the definition of $\tilde{g}$, integrating (12) with respect to the spatial variable y, we find that condition (11) is also fulfilled by this solution of (12).

Let us now establish (12) by passing to the limit in the equation for u^ε with a test function of the form

$$\phi^\varepsilon = \varepsilon\varphi(x,t)\psi\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\Big)$$

with $\varphi \in C_0^\infty(Q\times(0,T))$ and $\psi \in C_\#^\infty([0,1]^{n+1})$ such that $\langle\rho\psi\rangle_y = 0$ for any s. By virtue of the last condition, there is a smooth periodic vector-function $\Psi = \Psi(y,s)$ such that $\mathrm{div}_y\Psi = \rho\psi$. Differentiating in s gives $\mathrm{div}_y\partial_s\Psi = \rho\partial_s\psi$. After straightforward rearrangements we obtain

$$-\varepsilon\int_0^T\!\!\int_Q \rho^\varepsilon u^\varepsilon\psi^\varepsilon\partial_t\varphi\,dxdt - \int_0^T\!\!\int_Q u^\varepsilon\varphi\,\mathrm{div}(\partial_s\Psi^\varepsilon)dxdt$$

$$= \varepsilon\int_0^T\!\!\int_Q \rho\big(\frac{x}{\varepsilon}\big)\big(\partial_t u^\varepsilon(x,t)\big)\varphi(x,t)\psi\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\big)dxdt$$

$$= -\varepsilon\int_0^T\!\!\int_Q \psi^\varepsilon a^\varepsilon\nabla u^\varepsilon\cdot\nabla\varphi\,dxdt - \int_0^T\!\!\int_Q \varphi a^\varepsilon\nabla u^\varepsilon\cdot\nabla_y\psi^\varepsilon\,dxdt + \int_0^T\!\!\int_Q \varphi\psi^\varepsilon g^\varepsilon\,dxdt;$$

here and later on we use the notation $\psi^\varepsilon = \psi\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\big)$, $\partial_s\Psi^\varepsilon = \partial_s\Psi(y,s)\Big|_{y=x/\varepsilon,\,s=t/\varepsilon^2}$, $\nabla_y\psi^\varepsilon = \nabla_y\psi(y,s)\Big|_{y=x/\varepsilon,\,s=t/\varepsilon^2}$ and $g^\varepsilon = g\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},u^\varepsilon(x,t)\big)$. Notice that by the Lebesgue theorem, the difference $(g^\varepsilon - g(x/\varepsilon,t/\varepsilon^2,u(x,t)))$ converges to zero in $L^2(Q\times(0,T))$. Integrating by parts, passing to the limit as $\varepsilon\to 0$, and considering the properties of g, we get

$$\lim_{\varepsilon\to 0}\int_0^T\!\!\int_Q \varphi\nabla u^\varepsilon\cdot\partial_s\Psi^\varepsilon\,dxdt \;+\; \lim_{\varepsilon\to 0}\int_0^T\!\!\int_Q u^\varepsilon\nabla\varphi\cdot\partial_s\Psi^\varepsilon\,dxdt$$

$$= -\int_0^T\!\!\int_Q\!\int_{\mathcal{Y}} a(y,s)\big(\nabla u + \nabla_y w(x,t,y,s)\big)\varphi(x,t)\nabla_y\psi(y,s)\,dydsdxdt$$

$$+\int_0^T\!\!\int_Q\!\int_{\mathcal{Y}} \varphi(x,t)\psi(y,s)g(y,s,u(x,t))\,dydsdxdt.$$

Making further transformations yields

$$\int_0^T \int_Q \int_{\mathcal{Y}} \varphi(x,t)\big(\nabla u + \nabla_y w(x,t,y,s)\big) \cdot \partial_s \Psi(y,s)\, dydsdxdt$$

$$+ \int_0^T \int_Q \int_{\mathcal{Y}} u(x,t)\nabla\varphi(x,t) \cdot \partial_s \Psi(y,s)\, dydsdxdt$$

$$= \int_0^T \int_Q \int_{\mathcal{Y}} a(y,s)\big(\nabla u + \nabla_y w(x,t,y,s)\big)\varphi(x,t)\nabla_y \psi(y,s)\, dydsdxdt$$

$$+ \int_0^T \int_Q \int_{\mathcal{Y}} \varphi(x,t)\psi(y,s)g(y,s,u(x,t))\, dydsdxdt.$$

The second integral on the left-hand side is equal to zero, and, since φ is an arbitrary test function, we end up with the following relation

$$\int_{\mathcal{Y}} \nabla_y w \cdot \partial_s \Psi\, dyds = -\int_{\mathcal{Y}} (\nabla_x u + \nabla_y w) \cdot a\nabla_y \psi\, dyds$$

$$+ \int_{\mathcal{Y}} \psi(y,s)g(y,s,u(x,t))\, dyds,$$

which holds for almost all $(x,t) \in Q \times (0,T)$. The integral on the left-hand side can be rewritten as follows

$$\int_{\mathcal{Y}} \nabla_y w \cdot \partial_s \Psi\, dyds = \int_{\mathcal{Y}} \mathrm{div}_y \Psi \partial_s w\, dyds = \int_{\mathcal{Y}} \rho\psi\partial_s w\, dyds.$$

Finally, we get

$$\int_{\mathcal{Y}} \psi(y,s)\big\{\rho(y)\partial_s w(x,t,y,s) - \mathrm{div}_y\big(a(y,s)(\nabla_y w(x,t,y,s) + \nabla_x u(x,t))\big)\big\}\, dyds$$

$$- \int_{\mathcal{Y}} \psi(y,s)g(y,s,u(x,t))dyds = 0$$

for almost all $(x,t) \in Q \times (0,T)$ and for any $\psi = \psi(y,s)$ such that $\langle \rho\psi \rangle_y = 0$ for all s. Considering the definition of $\tilde{g}$, we conclude that

$$\int\limits_{\mathcal{Y}} \psi(y,s)\big\{\rho(y)\partial_s w(x,t,y,s) - \mathrm{div}_y\big(a(y,s)(\nabla_y w(x,t,y,s) + \nabla_x u(x,t))\big)\big\} dyds \\ - \int\limits_{\mathcal{Y}} \psi(y,s)\tilde{g}(y,s,u(x,t))dyds = 0 \tag{13}$$

It is straightforward to check that the last identity also holds true for any periodic $\psi = \psi(s)$. Indeed, since $\int_Y \rho(y)w(x,t,y,s)dy = 0$, then

$$\int_Y \rho(y)w(x,t,y,s)\partial_s\psi(s)\, dy = 0,$$

and thus $\int_Y \rho(y)\partial_s w(x,t,y,s)\psi(s)\, dy = 0$. Finally, since any $\mathcal{Y}$-periodic function $\psi(y,s)$ can be represented as

$$\psi(y,s) = \psi_1(y,s) + \psi_2(s)$$

with

$$\psi_2(s) = \int\limits_Y \rho(y)\psi(y,s)dx, \qquad \psi_1(y,s) = \psi(y,s) - \psi_2(s),$$

then the relation (13) holds for any $\psi \in C_\#^\infty(\mathcal{Y})$, which implies that w is a solution of (12). $\square$

Remark 1. *By linearity of (12) it is straightforward to check that its solution satisfies*

$$w(y,s,x,t) = \chi(y,s) \cdot \nabla u(x,t) + w_1(y,s,u(x,t)) \tag{14}$$

with χ and w_1 being $\mathcal{Y}$-periodic solutions of the following problems

$$\rho\partial_s\chi - \mathrm{div}_y(a\nabla_y\chi) = \mathrm{div}_y a, \qquad \langle \rho\chi(\cdot,s)\rangle_y = 0 \ \ \textit{for all } s, \tag{15}$$

and

$$\rho\partial_s w_1(y,s,u) - \mathrm{div}_y(a\nabla_y w_1(y,s,u)) = \tilde{g}(y,s,u), \\ \langle \rho w_1(\cdot,s,u)\rangle_y = 0 \ \ \textit{for all } s, u \in \mathbb{R}. \tag{16}$$

5 Passage to the limit

The goal of this section is to derive the effective macroscopic model and to prove the convergence result, namely Theorem 3.

For this aim we multiply equation (1) (or (3)) by a smooth function $\varphi = \varphi(x,t)$ which is compactly supported in $\mathbb{R}^n \times [0,T)$ (respectively, in $Q \times [0,T)$), integrate the resulting relation over $\mathbb{R}^n \times [0,T)$ and pass to the two-scale limit as $\varepsilon \to 0$. After straightforward rearrangements this yields

$$-\int_0^T\int_{\mathbb{R}^n} u(x,t)\partial_t\varphi(x,t)\,dxdt - \int_{\mathbb{R}^n} u_0(x)\varphi(x,0)\,dx$$

$$= -\int_0^T\int_{\mathbb{R}^n}\int_{\mathcal{Y}} a(y,s)\big(\nabla_y\chi(y,s)+\mathbf{I}\big)\nabla u(x,t)\cdot\nabla\varphi(x,t)\,dxdtdyds$$

$$-\int_0^T\int_{\mathbb{R}^n}\int_{\mathcal{Y}} \big(a(y,s)\nabla_y w_1(y,s,u(x,t)) - G(y,s,u(x,t))\big)\cdot\nabla\varphi(x,t)\,dxdtdyds$$

$$+\int_0^T\int_{\mathbb{R}^n}\int_{\mathcal{Y}} \partial_u G(y,s,u(x,t))\cdot\big(\nabla_y\chi(y,s)+\mathbf{I}\big)\nabla u(x,t)+\nabla_y w_1(y,s,u(x,t))\big)\varphi(x,t)\,dxdtdyds$$

$$+\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\mathbb{R}^n}\rho\big(\frac{x}{\varepsilon}\big)\Big\{\partial_u\overline{G}\Big(\frac{t}{\varepsilon^2},u^\varepsilon(x,t)\Big)\partial_t u^\varepsilon(x,t)+\partial_t\Big[\overline{G}\Big(\frac{t}{\varepsilon^2},u^\varepsilon(x,t)\Big)\Big]\Big\}\varphi(x,t)\,dxdt;$$

here $\mathbf{I}$ stands for the unit $n\times n$ matrix, and, as above (see Section 3), $\partial_s\overline{G}(s,u) = \langle g(\cdot,s,u)\rangle_y$, $\langle\overline{G}(\cdot,u)\rangle_s = 0$, and $\mathrm{div}G(y,s,u) = g(y,s,u) - \rho(y)\langle g(\cdot,s,u)\rangle_y$, $\langle G(\cdot,s,u)\rangle_y = 0$. We transform further the last term on the right-hand side as follows

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\mathbb{R}^n}\rho\big(\frac{x}{\varepsilon}\big)\Big\{\partial_u\overline{G}\Big(\frac{t}{\varepsilon^2},u^\varepsilon(x,t)\Big)\partial_t u^\varepsilon(x,t)+\partial_t\Big[\overline{G}\Big(\frac{t}{\varepsilon^2},u^\varepsilon(x,t)\Big)\Big]\Big\}\varphi(x,t)\,dxdt;$$

$$=\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\mathbb{R}^n}\rho^\varepsilon(x)\partial_u\overline{G}\Big(\frac{t}{\varepsilon^2},u^\varepsilon(x,t)\Big)\partial_t u^\varepsilon(x,t)\varphi(x,t)\,dxdt =$$

$$-\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\mathbb{R}^n}\partial_u\overline{G}\Big(\frac{t}{\varepsilon^2},u^\varepsilon(x,t)\Big)a\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\big)\nabla u^\varepsilon(x,t)\cdot\nabla\varphi(x,t)\,dxdt$$

$$-\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\mathbb{R}^n}\varphi(x,t)\frac{\partial^2}{\partial u^2}\overline{G}\Big(\frac{t}{\varepsilon^2},u^\varepsilon(x,t)\Big)a\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\big)\nabla u^\varepsilon(x,t)\cdot\nabla u^\varepsilon(x,t)\,dxdt$$

$$-\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^n}\varphi(x,t)\partial_u\overline{G}\Big(\frac{t}{\varepsilon^2},u^\varepsilon(x,t)\Big)g\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},u^\varepsilon(x,t)\Big)\,dxdt$$

$$=\int_0^T\int_{\mathbb{R}^n}\int_{\mathcal{Y}}\varphi(x,t)\partial_u\overline{G}(s,u(x,t))g(y,s,u(x,t))\,dxdtdyds.$$

Combining the above relations we arrive at the following limit equation

$$-\int_0^T\int_{\mathbb{R}^n}u\partial_t\varphi\,dxdt-\int_{\mathbb{R}^n}u_0\varphi(\cdot,0)dx=-\int_0^T\int_{\mathbb{R}^n}\hat{a}\nabla u\cdot\nabla\varphi\,dxdt$$

$$-\int_0^T\int_{\mathbb{R}^n}\int_{\mathcal{Y}}a(y,s)\nabla_y w_1(y,s,u(x,t))\cdot\nabla\varphi(x,t)\,dxdtdyds$$

$$+\int_0^T\int_{\mathbb{R}^n}\int_{\mathcal{Y}}\partial_u\tilde{g}(y,s,u(x,t))\big(\chi(y,s)\cdot\nabla u(x,t)+w_1(x,y,u(x,t))\big)\varphi(x,t)\,dxdtdyds$$

$$+\int_0^T\int_{\mathbb{R}^n}\int_{\mathcal{Y}}\varphi(x,t)\partial_u\overline{G}(s,u(x,t))g(y,s,u(x,t))\,dxdtdyds$$

with $\hat{a}$ the usual homogenized tensor, defined by $\hat{a}=\langle a(\nabla_y\chi+\mathbf{I})\rangle_{\mathcal{Y}}$, and $\tilde{g}(y,s,u)=g(y,s,u)-\rho(y)\langle g(\cdot,s,u)\rangle_y$. If we denote

$$F_1(u)=\int_{\mathcal{Y}}a(y,s)\nabla_y w_1(y,s,u)dyds,\quad F_2(u)=\int_{\mathcal{Y}}\partial_u\tilde{g}(y,s,u)\chi(y,s)dyds,$$

$$F_3(u)=\int_{\mathcal{Y}}\partial_u\tilde{g}(y,s,u)w_1(y,s,u)dyds,\quad F_4(u)=\int_{\mathcal{Y}}\partial_u\overline{G}(s,u)g(y,s,u)dyds,$$

then the limit equation takes the form

$$\begin{cases}\partial_t u=\mathrm{div}\big(\hat{a}\nabla u\big)+\mathrm{div}F_1(u)-F_2(u)\cdot\nabla u-F_3(u)-F_4(u),\\ \qquad u(x,t)=0\text{ on }\partial Q\times(0,T),\\ \qquad u(x,0)=u_0(x)\text{ in }Q.\end{cases}\tag{17}$$

The Dirichlet boundary condition on ∂Q come from the fact that the sequence u^ε is weakly converging in the space $L^2((0,T); H_0^1(Q))$. Of course, there is no such boundary condition if the domain Q is the entire space $\mathbb{R}^n$.

We summarize the above statements in the following proposition.

Proposition 2. *Under hypotheses* **A1**–**A6** *the sequence u^ε of solutions of problem (1) or (3) is relatively compact in the space $L^2(Q \times (0,T))$. Any subsequential limit of u^ε is a solution of the limit problem (17).*

6 Properties of the effective equation

This section is aimed at proving the uniqueness of the solution of the homogenized problem (17). For this we shall prove that all nonlinearities in (17) are Lipschitz continuous. The desired uniqueness will then easily follow.

It is clear from their very definitions that the functions $F_1(u)$ is Lipschitz continuous while $F_2(u)$ is uniformly bounded and Lipschitz continuous:

$$|F_1(u_1) - F_1(u_2)| \le C|u_1 - u_2|, \quad |F_2(u_2)| \le C,$$

$$|F_1(u_1) - F_1(u_2)| \le C|u_1 - u_2| \quad \text{for all } u_1, u_2 \in \mathbb{R}.$$

The case of F_3 and F_4 is slightly more involved.

Lemma 6. *Under the assumptions* **A1**–**A6** *the functions F_3 and F_4 are Lipschitz continuous.*

Proof. As an immediate consequence of definition (5) of $\overline{G}$ we have

$$|\overline{G}(s,u)| \le C|u|, \quad |\partial_u \overline{G}(s,u)| \le C,$$

$$|\partial_u \overline{G}(s,u_1) - \partial_u \overline{G}(s,u_2)| \le C \frac{|u_1 - u_2|}{1 + \min(|u_1|, |u_2|)}.$$

If $\partial_u \overline{G}$ is differentiable in u, then the latter inequality implies that

$$|\partial_u \partial_u \overline{G}(s,u)| \le C(1 + |u|)^{-1}.$$

In this case,

$$\left|\partial_u\Big(g(y,s,u)\partial_u\overline{G}(s,u)\Big)\right| \le C,$$

and thus

$$|g(y,s,u_1)\partial_u\overline{G}(s,u_1) - g(y,s,u_2)\partial_u\overline{G}(s,u_2)| \le C|u_1-u_2|. \tag{18}$$

In the general case (18) can be obtained by means of smoothing $\partial_u\overline{G}$. Clearly, (18) implies the desired Lipschitz continuity of F_4. The Lipschitz continuity of F_3 can be justified in exactly the same way. □

We proceed with our main result.

Theorem 3. *Under assumptions* **A1**–**A6** *the sequence* $(u^\varepsilon)_{\varepsilon>0}$ *of solutions of problem (1) converges, as* $\varepsilon \to 0$, *in the space* $L^2(Q\times(0,T))$ *towards the unique solution of problem (17).*

Proof. The convergence has been proved in the preceding section. The uniqueness of the solution of (17) follows from Lemma 6 by means of Gronwall's theorem. By uniqueness of the limit the entire sequence converges without any extraction of a subsequence. □

Remark 4. *If the coefficients of equation (1) or (3) do not depend on the temporal variable but only on the spacial variables, then* $\partial_u F_1(u) - F_2(u) = 0$ *so that the effective equation does not contain first order terms. Indeed, for* $a = a(y)$ *and* $g = g(y,u)$ *the functions* χ *and* w_1*, defined in Remark 1, are solutions of the elliptic equations*

$$-\mathrm{div}_y(a(y)\nabla_y\chi(y)) = \mathrm{div}_y a(y)$$

and

$$-\mathrm{div}_y(a(y)\nabla_y w_1(y,u)) = g(y,u) \tag{19}$$

respectively. Thus, we have

$$\partial_u F_1(u) - F_2(u) = \int\limits_Y a(y)\nabla_y\partial_u w_1(y,u)dy - \int\limits_Y \partial_u g(y,u)\chi(y)dy.$$

Differentiating (19) in u we get

$$-\mathrm{div}_y(a(y)\nabla_y\partial_u w_1(y,u)) = \partial_u g(y,u).$$

Therefore,

$$\int\limits_Y \partial_u g(y,u)\chi(y)dy = -\int\limits_Y \mathrm{div}_y(a(y)\nabla_y\partial_u w_1(y,u))\chi(y)dy$$

$$= -\int\limits_Y \partial_u w_1(y,u)\mathrm{div}_y(a(y)\nabla_y\chi(y))dy = \int\limits_Y \partial_u w_1(y,u)\mathrm{div}_y(a(y))dy$$

$$= -\int\limits_Y a(y)\nabla_y\partial_u w_1(y,u)dy,$$

and the desired relation $\partial_u F_1(u) - F_2(u) = 0$ *follows.*

Remark 5. *The two-scale limit of the gradient* ∇u^ε *is given by (10) with* w *specified in (14)–(16).*

References

[1] G. Allaire, *Homogenization and two-scale convergence,* SIAM J. Math. Anal. 23, 6, pp.1482-1518 (1992).

[2] G. Allaire, Y. Capdeboscq, A. Piatnitski, V. Siess, M. Vanninathan, *Homogenization of periodic systems with large potentials,* Archive Rat. Mech. Anal. 174, pp.179-220 (2004).

[3] G. Allaire, A. Mikelić, A. Piatnitski, *Homogenization approach to the dispersion theory for reactive transport through porous media*, to appear in SIAM J. Math. Anal.

[4] G. Allaire, A.L. Raphael, Homogenization of a convection-diffusion model with reaction in a porous medium. C.R.Acad.Sci.Paris, t.344, Série I, pp.523-528 (2007).

[5] A. Bensoussan, J.-L. Lions, G. Papanicolaou, *Asymptotic analysis for periodic structures,* North-Holland, Amsterdam, 1978.

[6] Y. Capdeboscq, *Homogenization of a diffusion with drift*, C.R.Acad.Sci.Paris, t.327, Série I, pp.807-812 (1998).

[7] Y. Capdeboscq, *Homogenization of a neutronic critical diffusion problem with drift*, Proc. Roy. Soc. Edinburgh Sect. A 132 (2002), no. 3, pp.567-594.

[8] P. Donato, A. Piatnitski, *Averaging of nonstationary parabolic operators with large lower order terms*, GAKUTO International Series, Math. Sci. Appl., Vol. 24 (2005), Multi Scale Problems and Asymptotic Analysis, pp.153-165.

[9] E.W. Larsen and M. Williams, *Neutron Drift in Heterogeneous Media,* Nuclear Sci. Engrg. 65 (1978) pp.290-302.

[10] E. Marusic-Paloka, A. Piatnitski, Homogenization of a nonlinear convection-diffusion equation with rapidly oscillating coefficients and strong convection. *Journal of London Math. Soc.* **72** (2005), No.2, pp.391-409.

[11] R. Mauri, *Dispersion, convection and reaction in porous media*, Phys. Fluids A, Vol. 3 (1991), pp.743-755.

[12] A. Mikelić, V. Devigne, C.J. van Duijn, Rigorous upscaling of the reactive flow through a pore, under dominant Peclet and Damkohler numbers, *SIAM J. Math. Anal.,* Vol. 38, Issue 4 (2006), pp.1262-1287.

[13] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal. **20**(3), pp.608-623 (1989).

[14] B. Perthame, P. Souganidis, *Asymmetric potentials and motor effect: a homogenization approach,* Annales de l'Institut Henri Poincare (C) Non Linear Analysis Volume 26, Issue 6, pp.2055-2071 (2009).