Information Asymmetry in Pricing of Credit Derivatives

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Abstract

We study the pricing of credit derivatives with asymmetric information. The managers have complete information on the value process of the firm and on the default threshold, while the investors on the market have only partial observations, especially about the default threshold. Different information structures are distinguished using the framework of enlargement of filtrations. We specify risk neutral probabilities and we evaluate default sensitive contingent claims in these cases.

Keywords : asymmetric information, enlargement of filtrations, default threshold, risk neutral probability measures, pricing of credit derivatives.

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1 Introduction

The modelling of a default event is an important subject from both economic and financial point of view. There exist a large literature on this issue and mainly two modelling approaches: the structural one and the reduced-form one. In the structural approach, where the original idea goes back to the pioneer paper of Merton [22], the default is triggered when a fundamental process X of the firm passes below a threshold level L. The fundamental process may represent the asset value or the total cash flow of the firm where the debt value of the firm can also be taken into consideration. This provides a convincing economic interpretation for this approach. The default threshold L is in general supposed to be constant or deterministic. Its level is chosen by the managers of the firm according to some criterions — maximizing the equity value for example as in [21].

For an agent on the financial market, the vision on the default is quite different: on one hand, he possesses merely a limited information of the basic data (the process X for example) of the firm; on the other hand, to deal with financial products written on the firm, he needs to update his estimations of the default probability in a dynamic manner. This leads to the reduced-form approach for default modeling where the default arrives in a more "surprising" way and the model parameters can be daily calibrated by using the market data such as the CDS spreads.

The default time constructed in the classical structural approach is a stopping time with respect to the filtration $\mathbb{F}$ generated by the fundamental process. The intensity of such predictable stopping times does not exist. In the credit risk literature, it is also interpreted by the fact that the default intensity (or the credit spread) tends to zero when the time to maturity decreases to zero (we shall make precise the meanings of these two intensities later on). The links between the structural and the intensity approaches have been investigated in the literature. If the default threshold L is a random variable instead of constant or deterministic, then the default time admits the intensity. One important example is the well known Cox process model introduced in [20] where L is supposed to be an exponentially distributed random variable independent with $\mathbb{F}$ (see also [9]). Another class of models is the incomplete information models (e.g. [8, 6, 18, 5, 4]) where the agent only has a partial observation of the fundamental process X and thus his available information is represented by some subfiltration of $\mathbb{F}$. The intensity can then be deduced for the subfiltration.

In this paper, we are interested in the impact of information accessibility of an agent on the pricing of credit derivatives. In particular, we aim to study the information concerning the default threshold L in addition to the partial observation of the process X. This case has been studied in [11] where investors anticipate the distribution of L (following for example the Beta distribution) whose parameters are calibrated through market data. Our approach is different and is related to the insider's information problems. Indeed, when the managers make decisions on whether the firm will default or not, he has supplementary information on the default threshold L compared to an ordinary investor on the market.

Facing the financial crisis, this study is also motivated by some recent "technical default events", where the bankruptcy occurs although the firm is still capable to repay its debts.

We present our model in the standard setting. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space which represents the financial market. We consider a firm and model its default time as the first time that a continuous time process $(X_t)_{t\geq 0}$ reaches some default barrier L, i.e.,

$$\tau = \inf\{t : X_t \leq L\} \quad \text{where } X_0 > L \tag{1.1}$$

with the convention that $\inf \emptyset = +\infty$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by the process X, i.e., $\mathcal{F}_t = \sigma(X_s, s \leq t) \vee \mathcal{N}$ satisfying the usual conditions where $\mathcal{N}$ denotes the $\mathbb{P}$ null sets. Such construction of a default time adapts to both the structural approach and the reduced form approach of the default modelling, according to the specification of the process X and the threshold L.

In the structural approach models, L is a constant or a deterministic function $L(t)$, then τ defined in (1.1) is an $\mathbb{F}$-stopping time as in the classical first passage models. In the reduced-form approach, the default barrier L is unknown and is described as a random variable in $\mathcal{A}$. We introduce the decreasing process X^* defined as

$$X_t^* = \inf\{X_s,\, s \leq t\}.$$

Then (1.1) can be rewritten as

$$\tau = \inf\{t : X_t^* = L\}. \tag{1.2}$$

This formulation gives a general reduced-form model of default (see [9]). In particular, when the barrier L is supposed to be independent of $\mathcal{F}_\infty$, then

$$\mathbb{P}(\tau > t|\mathcal{F}_\infty) = \mathbb{P}(X_t^* > L|\mathcal{F}_\infty) = F_L(X_t^*),$$

where F_L denotes the distribution function of L. Note that the (H)-hypothesis is satisfied in this case, that is, $\mathbb{P}(\tau > t|\mathcal{F}_\infty) = \mathbb{P}(\tau > t|\mathcal{F}_t)$. We may also recover the Cox-process model using a similar construction.

In most papers concerning the information-based credit models, the process X is partially observed, making an impact on the conditional default probabilities and on the credit spreads. In this paper, we let L to be a random variable and take into consideration the information on L. Such information modelling is closely related to the enlargement of filtrations theory. Generally speaking, the information of a manager is represented by the initial enlargement of the filtration $(\mathcal{F}_t)_{t\geq 0}$ and the information of an investor is modelled by the progressive enlargement of $(\mathcal{F}_t)_{t\geq 0}$ or of some of its subfiltration. We shall also consider the case of an insider who may have some extra knowledge on L compared to an investor and whose knowledge is however perturbed compared to the manager.

The rest of this paper is organized as follows. In Section 2, we introduce the pricing problem and the different information structures for various agents on the market, notably

the information on the default barrier L. We shall distinguish the role of the manager, the investor and the insider, who have different level of information on L. In successively Sections 3, 4 and 5, we make precise the mathematical hypothesis for these cases, using the languages of enlargement of filtrations. We also discuss the risk-neutral probabilities in each case for further pricing purposes. In order to distinguish the impact of the different filtrations from the impact of the different pricing probabilities, we first give the price of a contingent claim under the historical probability measure $\mathbb{P}$ for each information in Section 3, 4, 5, the calculus under the corresponding pricing (or "risk-neutral") probability being done in the last section. Finally, we end the last section with numerical illustrations.

2 Pricing framework and information structures

On the financial market, the available information for each agent is various. There exists in general information asymmetry between different market investors, and moreover between the managers of a firm and the investors. In particular, the managers may have information on whether the firm will default or not, or when the default may happen. The pricing of credit-sensitive derivative depends strongly on the information flow of the agent. We begin by introducing the general pricing principle and then we precise different information.

2.1 General pricing principle

We fix in the sequel a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ of $\mathcal{A}$, representing the default-free information. Let τ be a strictly positive and finite random time on $(\Omega, \mathcal{A}, \mathbb{P})$, modelling the default time. The information flow of the agent is described by a filtration $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ such that τ is an $\mathbb{H}$-stopping time, that is, all agents observe at time t whether the default has occurred or not. Without loss of generality, we assume that all the filtrations we consider satisfy the usual conditions of completeness and right-continuity.

We describe a general credit-sensitive derivative claim of maturity T as in [2], by a triplet (C, G, Z) where C is an $\mathcal{F}_T$-measurable random variable representing the payment at the maturity T if no default occurs before the maturity, G is an $\mathbb{F}$-adapted, continuous process of finite variation with $G_0 = 0$ and represents the dividend payment, Z is an $\mathbb{F}$-predictable process and represents the recovery payment at the default time τ.

The triplet for a CDS, viewed by a protection buyer, satisfy $C = 0$, $G_t = -\kappa t$ and $Z = 1 - \alpha$ where κ is the spread of CDS and α is the recovery rate of the underlying name. The triplet for a defaultable zero-coupon satisfy $C = 1$, $G = 0$ and $Z = 1 - \alpha$.

The value process of the claim at time $t < \tau \wedge T$ is given by

$$V_t = R_t E_{\mathbb{Q}}\Big[CR_T^{-1}1_{\{\tau>T\}} + \int_t^T 1_{\{\tau>u\}}R_u^{-1}dG_u + Z_\tau 1_{\{\tau\leq T\}}R_\tau^{-1}\,\Big|\,\mathcal{H}_t\Big] \tag{2.1}$$

where $\mathbb{Q}$ denotes the pricing probability measure which we shall precise later, and R is the discount factor process. We note that both the filtration and the pricing probability depend on the information level of the agent.

In the credit risk analysis, one often tries to establish a relationship between the market filtration and the default-free one. The main advantage is that the default-free filtration is often supposed to have nice regularity conditions, while the global market filtration which contains the default information is often difficult to work with directly. Indeed, due to the default information, the processes adapted to the global filtration have in general a jump at the default time (except in the structural approach) and this makes it difficult to propose explicit models in this filtration. In our model with insider's information, we need to make precise the filtration $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ in (2.1) for different types of agents. Our objective, similar as mentioned above, is to establish a pricing formula with respect to the default-free filtration in each case.

2.2 Information structures

We now describe the different information flows and the corresponding filtration $\mathbb{H}$ for different agents on the market. Recall that the default time is modelled by

$$\tau = \inf\{t : X_t^* = L\},$$

where L is a random variable and X^* is the infimum process of an $\mathbb{F}$-adapted process X. We assume that L is chosen by the managers of the firm who hence have the total knowledge on L. The information of X_t^* is contained in the σ-algebra $\mathcal{F}_t$. However, the process X^* can not give us full information on $\mathcal{F}_t$.

- Manager's information.

The manager has complete information on X and on L. The filtration of the manager's information, denoted by $\mathbb{G}^M = (\mathcal{G}_t^M)_{t\geq 0}$, is then

$$\mathcal{G}_t^M := \mathcal{F}_t \vee \sigma(L).$$

Note that $\mathbb{G}^M$ is in fact the initial enlargement of the filtration $\mathbb{F}$ with respect to L and we call it the full information on L. It is obvious that τ is a $\mathbb{G}^M$-stopping time. We shall precise some technical hypothesis in the next section.

- Investor's information.

In the credit risk literature, the accessible information on the market is often modelled by the progressive enlargement $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ of $\mathbb{F}$. More precisely, let $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$ be the minimal filtration which makes τ a $\mathbb{D}$-stopping time, i.e. $\mathcal{D}_t = \mathcal{D}_{t+}^0$ with $\mathcal{D}_t^0 = \sigma(\tau \wedge t)$, then

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t.$$

In our model (1.2), this is interpreted as $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{L \leq X_t^*\})$ and we call this information the progressive (enlargement) information on L. Together with the information

flow of the filtration $(\mathcal{F}_t)_{t\geq 0}$, an investor who observes the filtration $(\mathcal{G}_t)_{t\geq 0}$ knows at time t whether or not the default has occurred up to t and the default time τ once it occurs. We see that the manager's information $\mathcal{G}_t^M$ is larger than $\mathcal{G}_t$.

• Investor's incomplete information.
In many incomplete information credit risk models, the process X driving the default risk is not totally observable for the investors. In this paper, we will only consider the example of a delayed information on X : the information of such an investor is described by a progressive enlargement $\mathbb{G}^D = (\mathcal{G}_t^D)_{t\geq 0}$ of a delayed filtration of $\mathbb{F}$, where

$$\mathcal{G}_t^D := \mathcal{F}_{t-\delta(t)} \vee \mathcal{D}_t,$$

and $\delta(t)$ being a function valued in $[0,t]$ such that $t-\delta(t)$ is increasing. The above formulation covers the constant delay time model where $\delta(t) = \delta$ (see [6], [13]) and the discrete observation model where $\delta(t) = t - t_i^{(m)}$ and $t_i^{(m)} \leq t < t_{i+1}^{(m)}$, $0 = t_0^{(m)} < t_1^{(m)} < \cdots < t_m^{(m)} = T$ being the discrete dates on which the $(\mathcal{F}_t)_{t\geq 0}$ information may be renewed (for example, the release dates of the accounting reports of the firm, see [8], [18]).

• Insider's information.
Finally, we shall consider the insiders who have as supplementary information a partial observation on L compared to the investor's information $\mathcal{G}_t$. Namely, the agent has the knowledge on a noisy default threshold: $(L_t)_{t\geq 0}$, $L_s = f(L, \epsilon_s)$ with ϵ being an independent noise perturbing the information on L. The corresponding information flow is then modelled by $\mathbb{G}^I = (\mathcal{G}_t^I)_{t\geq 0}$ where

$$\mathcal{G}_t^I := \mathcal{F}_t \vee \sigma(L_s,\, s \leq t) \vee \mathcal{D}_t.$$

Notice that $\mathcal{G}_t^I = \mathcal{G}_t \vee \sigma(L_s,\, s \leq t)$. We call this information the "noisy full information" on L. It is a successive enlargement of $\mathcal{F}_t$, firstly by the noised information of the default threshold and then by the default occurrence information.

For the different types of information described above, we observe that the following relations hold:

$$\mathbb{G}^M \supset \mathbb{G}^I \supset \mathbb{G} \supset \mathbb{G}^D.$$

They correspond to the pricing filtration $\mathbb{H}$ in (2.1) for different agents on the market. We shall concentrate on the pricing problem with the above filtrations and we begin by making precise the mathematical hypothesis on these types of information on L, with which we introduce the risk-neutral probabilities $\mathbb{Q}$ in each case.

3 Full information

In this section, we work with the manager information flow $\mathbb{G}^M = \mathbb{F} \vee \sigma(L)$, which is an initial enlargement of the filtration $\mathbb{F}$. Recall that the default barrier is fixed at date 0 by the manager as the realization of a random variable L. We assume in addition that the filtration $\mathbb{F}$ is generated by a Brownian motion B.

3.1 Initial enlargement of filtration

In the theory of initial enlargement of filtration, it is standard to work under the following density hypothesis due to Jacod [16, 17].

Assumption 3.1 *We assume that L is an $\mathcal{A}$-measurable random variable with values in $\mathbb{R}$, which satisfies the assumption :*

$$\mathbb{P}(L \in \cdot | \mathcal{F}_t)(\omega) \sim \mathbb{P}(L \in \cdot), \quad \forall t \geq 0, \mathbb{P} - a.s..$$

Remark: Jacod has shown that, if Assumption 3.1 is fulfilled, then any $\mathbb{F}$-local martingale is a $\mathbb{G}^M$-semimartingale.

We denote by $P_t^L(\omega, dx)$ a regular version of the conditional law of L given $\mathcal{F}_t$ and by P^L the law of L (under the probability $\mathbb{P}$). According to [17], there exists a measurable version of the conditional density

$$p_t(x)(\omega) = \frac{dP_t^L}{dP^L}(\omega, x) \tag{3.1}$$

which is an $(\mathbb{F}, \mathbb{P})$-martingale and hence can be written as

$$p_t(x) = p_0(x) + \int_0^t \beta_s(x) dB_s, \quad \forall x \in \mathbb{R}$$

for some $\mathbb{F}$-predictable process $(\beta_t(x))_{t \geq 0}$. Moreover, the fact that P_t^L is equivalent to P^L implies that $\mathbb{P}$-almost surely $p_t(L) > 0$. Let us introduce the $\mathbb{F}$-predictable process ρ^M where $\rho_t^M(x) = \beta_t(x)/p_t(x)$, the density process $p_t(L)$ satisfies the following stochastic differential equation

$$dp_t(L) = p_t(L) \rho_t^M(L) dB_t.$$

Note that $(\widetilde{B}_t^M := B_t - \int_0^t \rho_s^M(L) ds, t \geq 0)$ is a $(\mathbb{G}^M, \mathbb{P})$-Brownian motion.

It is proved in [12] that Assumption 3.1 is satisfied if and only if there exists a probability measure equivalent to $\mathbb{P}$ and under which $\mathcal{F}_\infty := \cup_{t \geq 0} \mathcal{F}_t$ and $\sigma(L)$ are independent. The probability $\mathbb{P}^L$ defined by the density process

$$E_{\mathbb{P}^L}\Big[\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^L} \Big| \mathcal{G}_t^M \Big] = p_t(L)$$

is the only one that is identical to $\mathbb{P}$ on $\mathcal{F}_\infty$.

We introduce the process Y^M by

$$Y^M = \mathcal{E}\Big(- \int_0^\cdot \rho_s^M(L) d\widetilde{B}_s^M \Big), \tag{3.2}$$

where $\mathcal{E}$ denotes the Doléans-Dade exponential. We assume in addition that Y^M is a $(\mathbb{G}^M, \mathbb{P})$ martingale. A straightforward computation yields $d((Y_t^M)^{-1}) = (Y_t^M)^{-1} \rho_t^M(L) dB_t$.

Thus, $Y_t^M = \frac{1}{p_t(L)}$, that is, Y_t^M is the Radon-Nikodym density of the change of probability $\mathbb{P}^L$ with respect to $\mathbb{P}$ on $\mathcal{G}_t^M$. The process Y^M is important in the study of risk-neutral probabilities on $\mathbb{G}^M$. Indeed, let ϕ be the price process of a default-free financial instrument. It is an $\mathbb{F}$-adapted process which is an $\mathbb{F}$-local martingale under certain $\mathbb{F}$ risk-neutral probability $\mathbb{Q}$ (which is equivalent to $\mathbb{P}$). In general ϕ is not an $(\mathbb{G}^M, \mathbb{Q})$-local martingale. However, if we define a new probability measure $\mathbb{Q}^M$ by

$$\mathrm{d}\mathbb{Q}^M = Y_t^M \mathrm{d}\mathbb{Q} \quad \text{on } \mathcal{G}_t^M,$$

then any $(\mathbb{F}, \mathbb{Q})$-local martingale is an $(\mathbb{G}^M, \mathbb{Q}^M)$-local martingale. In particular, B is a $(\mathbb{G}^M, \mathbb{Q}^M)$-Brownian motion. Moreover, one has the following martingale representation property by [1]: if A is a $(\mathbb{G}^M, \mathbb{Q}^M)$-local martingale, then there exists $\psi \in L^1_{loc}(B, \mathbb{G}^M, \mathbb{Q}^M)$ such that

$$A_t = A_0 + \int_0^t \psi_s dB_s.$$

This shows that the market is complete for the manager.

3.2 Pricing with full information

We consider now the pricing problem with the manager's information flow $\mathbb{H} = \mathbb{G}^M$ and we assume Assumption 3.1. In order to distinguish the impact of different filtrations and the impact of different pricing measures, we first assume that the pricing probability is $\mathbb{P}$ for all agents. The result under $\mathbb{Q}^M$, the risk-neutral probability for the manager, is computed in Section 6 by a change of probability measure.

Our objective is to establish the pricing formula for the manager with respect to the default-free filtration $\mathbb{F}$. We begin by giving the following useful result.

Proposition 3.1 *For any $\theta \geq t$ and any positive $\mathcal{F}_\theta \otimes \mathcal{B}(\mathbb{R})$-measurable function $\phi_\theta(\cdot)$, one has*

$$E_{\mathbb{P}}[\phi_\theta(L)1_{\{\tau>\theta\}} \,|\, \mathcal{G}_t^M] = \frac{1}{p_t(L)} E_{\mathbb{P}}[\phi_\theta(x)p_\theta(x)1_{\{X_\theta^*>x\}} \,|\, \mathcal{F}_t]_{x=L} \tag{3.3}$$

where $p_t(x)$ is defined in (3.1).

Proof: Let $\mathbb{P}^L$ be the equivalent probability measure of $\mathbb{P}$ of density $p_t(L)^{-1}$ on $\mathcal{G}_t^M$. By using the facts that $\mathcal{F}_\theta$ and $\sigma(L)$ are independent under $\mathbb{P}^L$ and that $\mathbb{P}^L$ is identical to $\mathbb{P}$ on $\mathcal{F}_\infty$, we have

$$\begin{aligned} E_{\mathbb{P}}[\phi_\theta(L)1_{\{\tau>\theta\}} \,|\, \mathcal{G}_t^M] &= E_{\mathbb{P}}[\phi_\theta(L)1_{\{X_\theta^*>L\}} \,|\, \mathcal{F}_t \vee \sigma(L)] \\ &= p_t(L)^{-1} E_{\mathbb{P}^L}[\phi_\theta(L)p_\theta(L)1_{\{X_\theta^*>L\}} | \mathcal{F}_t \vee \sigma(L)] \\ &= p_t(L)^{-1} E_{\mathbb{P}^L}[\phi_\theta(x)p_\theta(x)1_{\{X_\theta^*>x\}} | \mathcal{F}_t]_{x=L} \\ &= p_t(L)^{-1} E_{\mathbb{P}}[\phi_\theta(x)p_\theta(x)1_{\{X_\theta^*>x\}} | \mathcal{F}_t]_{x=L}. \end{aligned} \tag{3.4}$$

□

Remark: If $\mathcal{F}_\theta$ and $\sigma(L)$ are independent under $\mathbb{P}$, then $p_t(x) \equiv 1$, we obtain the simpler formula

$$E_{\mathbb{P}}[\phi_\theta(L)1_{\{\tau>\theta\}} \,|\, \mathcal{G}_t^M] = E_{\mathbb{P}}[\phi_\theta(x)1_{\{X_\theta^*>x\}}|\mathcal{F}_t]_{x=L}.$$

Proposition 3.2 *We keep the notation of Section 2 and define* $F_t^M(x) := p_t(x)1_{\{X_t^*>x\}}$. *The value process of the contingent claim* (C, G, Z) *given the full information* $(\mathcal{G}_t^M)_{t\geq 0}$ *is*

$$V_t^M = 1_{\{\tau>t\}} \frac{\widetilde{V}_t^M(L)}{p_t(L)} \tag{3.5}$$

where

$$\widetilde{V}_t^M(L) = R_t E_{\mathbb{P}}\left[CR_T^{-1}F_T^M(x) + \int_t^T F_s^M(x)R_s^{-1}dG_s - \int_t^T Z_s R_s^{-1} dF_s^M(x) \,\middle|\, \mathcal{F}_t \right]_{x=L}. \tag{3.6}$$

Proof: Using Proposition 3.1, the first part of (2.1) is given by

$$R_t E_{\mathbb{P}}\Big[C1_{\{\tau>T\}}R_T^{-1}|\mathcal{G}_t^M\Big] = \frac{R_t}{p_t(L)} E_{\mathbb{P}}\big[CR_T^{-1}p_T(x)1_{\{X_T^*>x\}}|\mathcal{F}_t\big]_{x=L}.$$

Let's see the third term

$$R_t E_{\mathbb{P}}\Big[Z_\tau R_\tau^{-1}1_{\{t<\tau\leq T\}}|\mathcal{G}_t^M\Big].$$

We begin by assuming that Z is a stepwise $\mathbb{F}$-predictable process as in [2], that is $Z_u = \sum_{i=0}^n Z_i 1_{t_i<u\leq t_{i+1}}$ for $t < u \leq T$ where $t_0 = t < \cdots < t_{n+1} = T$ and Z_i is $\mathcal{F}_{t_i}$-measurable for $i = 0, \cdots, n$. We have

$$\begin{aligned}
& E_{\mathbb{P}}\Big[Z_\tau 1_{\{t<\tau\leq T\}}|\mathcal{G}_t^M\Big] \\
= & \sum_{i=0}^n \left(\frac{1}{p_t(L)} E_{\mathbb{P}^L}\Big[Z_i p_{t_i}(L)1_{\{t_i<\tau\}}|\mathcal{G}_t^M\Big] - \frac{1}{p_t(L)} E_{\mathbb{P}^L}\Big[Z_i p_{t_{i+1}}(L)1_{\{t_{i+1}<\tau\}}|\mathcal{G}_t^M\Big] \right) \\
= & \sum_{i=0}^n \frac{1}{p_t(L)} \left(E_{\mathbb{P}}\Big[Z_i p_{t_i}(x)1_{\{X_{t_i}^*>x\}}|\mathcal{F}_t\Big] - E_{\mathbb{P}}\Big[Z_i p_{t_{i+1}}(x)1_{\{X_{t_{i+1}}^*>x\}}|\mathcal{F}_t\Big] \right)_{x=L} \\
= & \frac{1}{p_t(L)} E_{\mathbb{P}}\left[\sum_{i=0}^n Z_i \left(p_{t_i}(x)1_{\{X_{t_i}^*>x\}} - p_{t_{i+1}}(x)1_{\{X_{t_{i+1}}^*>x\}} \right) |\mathcal{F}_t \right]_{x=L}.
\end{aligned}$$

We define $F_t^M(x) = p_t(x)1_{\{X_t^*>x\}}$. For x fixed, $1_{\{X_t^*>x\}}$ is decreasing and right continuous, and according to [17], $(p_s(x))_{s\geq 0}$ is an $(\mathbb{F}, \mathbb{P})$-martingale. Thus $(F_t^M(x))_{t\geq 0}$ is a nonnegative $(\mathbb{F}, \mathbb{P})$-supermartingale, and we may deal with its right-continuous modification with finite left-hand limits. Therefore

$$\begin{aligned}
E_{\mathbb{P}}\Big[Z_\tau 1_{\{t<\tau\leq T\}}|\mathcal{G}_t^M\Big] &= -\frac{1}{p_t(L)} E_{\mathbb{P}}\left[\sum_{i=0}^n Z_i (F_{t_{i+1}}^M(x) - F_{t_i}^M(x))|\mathcal{F}_t \right]_{x=L} \\
&= -\frac{1}{p_t(L)} E_{\mathbb{P}}\left[\int_t^T Z_u dF_u^M(x)|\mathcal{F}_t \right]_{x=L}.
\end{aligned}$$

Finally, we get the third term of (2.1) by approximating $(Z_u R_u^{-1})_u$ by a suitable sequence of stepwise $\mathbb{F}$-predictable processes :

$$R_t E_{\mathbb{P}}\Big[Z_\tau R_\tau^{-1} 1_{\{t<\tau\leq T\}} | \mathcal{G}_t^M\Big] = -\frac{R_t}{p_t(L)} E_{\mathbb{P}}\Big[\int_t^T Z_u R_u^{-1} dF_u^M(x) | \mathcal{F}_t\Big]_{x=L}.$$

The second term of (2.1) can be decomposed in two parts as follows

$$\begin{aligned} & R_t E_{\mathbb{P}}\Big[\int_t^T 1_{\{\tau>u\}} R_u^{-1} dG_u |\, \mathcal{G}_t^M\Big] \\ = \;& R_t E_{\mathbb{P}}\Big[1_{\{\tau>T\}} \int_t^T R_u^{-1} dG_u + 1_{\{t<\tau\leq T\}} \int_t^\tau R_u^{-1} dG_u |\, \mathcal{G}_t^M\Big] \\ = \;& \frac{R_t}{p_t(L)} E_{\mathbb{P}}\Big[p_T(x) 1_{\{X_T^*>x\}} \int_t^T R_u^{-1} dG_u - \int_t^T \int_t^u R_s^{-1} dG_s dF_u^M(x) | \mathcal{F}_t\Big]_{x=L}. \end{aligned}$$

Putting the three terms all together leads to

$$V_t^M = \frac{R_t}{p_t(L)} E_{\mathbb{P}}\left[F_T^M(x)\left(C R_T^{-1} + \int_t^T R_u^{-1} dG_u\right) - \int_t^T \left(Z_s R_s^{-1} + \int_t^s R_u^{-1} dG_u\right) dF_s^M(x) \,\middle|\, \mathcal{F}_t\right]_{x=L}.$$

The equality (3.5) then follows by an integration by part. □

4 Progressive information

4.1 Pricing with progressive enlargement of filtration

The progressive information on L corresponds to the standard information modelling in the credit risk literature where an investor observes the default event when it occurs. Recall that

$$\mathbb{G} = (\mathcal{G}_t)_{t\geq 0} \;\text{ with }\; \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t,$$

where $\mathcal{D}_t = \mathcal{D}_{t+}^0$, $\mathcal{D}_t^0 = \sigma(\tau \wedge t)$. The pricing formula (2.1) when $\mathcal{H}_t$ is $\mathcal{G}_t$ is well known. We recall it briefly below and we refer to [2, 3] for a proof.

Recall that the $\mathbb{G}$-compensator of τ (under the probability $\mathbb{P}$) is the $\mathbb{G}$-predictable increasing process $\Lambda^{\mathbb{G}}$ such that the process $(1_{\{\tau\leq t\}} - \Lambda_t^{\mathbb{G}}, t \geq 0)$ is a $(\mathbb{G}, \mathbb{P})$-martingale. The process $\Lambda^{\mathbb{G}}$ coincides on the set $\{t \leq \tau\}$ with an $\mathbb{F}$-predictable process $\Lambda^{\mathbb{F}}$, called the $\mathbb{F}$-compensator of τ. We define $S_t := \mathbb{P}(\tau > t \,|\, \mathcal{F}_t) = \mathbb{P}(X_t^* > L \,|\, \mathcal{F}_t)$, which is the Azéma supermartingale of τ. The following result is classical (see [19, 2, 10]).

Proposition 4.1 *For any $\theta \geq t$ and any $\mathcal{F}_\theta$-measurable random variable ϕ_θ, one has*

$$E_{\mathbb{P}}[\phi_\theta 1_{\{\tau>\theta\}} \,|\, \mathcal{G}_t] = 1_{\{\tau>t\}} \frac{E_{\mathbb{P}}[\phi_\theta S_\theta \,|\, \mathcal{F}_t]}{S_t}. \tag{4.1}$$

where $S_t := \mathbb{P}(\tau > t \,|\, \mathcal{F}_t)$. *The value process for an investors given the progressive information flow* $\mathbb{G}$ *is*

$$V_t = 1_{\{\tau>t\}} \frac{R_t}{S_t} E_{\mathbb{P}}\left[R_T^{-1} S_T C + \int_t^T R_u^{-1} S_u dG_u - \int_t^T R_u^{-1} Z_u dS_u \,\middle|\, \mathcal{F}_t \right]. \tag{4.2}$$

Remark 4.1 It is interesting to note the similitude between the case of manager (Proposition 3.2) and the case of investor (Proposition 4.1). Comparing the pricing formulas (3.5),(3.6) and (4.2), we observe that F^M plays a similar role in the full information case as S does in the progressive information case.

The pricing formula for delayed information flow is similar since $\mathbb{G}^D$ is the progressive enlargement of $\mathbb{F}^D$ with respect to τ and $\mathbb{F}^D$ is a sub-filtration of $\mathbb{F}$. The only difference is that S_t and R_t are not $\mathcal{F}_t^D$-measurable.

Proposition 4.2 *For any* $\theta \geq t$ *and any* $\mathcal{F}_\theta$*-measurable random variable* ϕ_θ*, one has*

$$E_{\mathbb{P}}[\phi_\theta 1_{\{\tau>\theta\}} \,|\, \mathcal{G}_t^D] = 1_{\{\tau>t\}} \frac{E_{\mathbb{P}}[\phi_\theta S_\theta | \mathcal{F}_t^D]}{E_{\mathbb{P}}[S_t | \mathcal{F}_t^D]} \tag{4.3}$$

The value process for a delay-informed investors is

$$V_t^D = \frac{1_{\{\tau>t\}}}{E[S_t|\mathcal{F}_t^D]} E_{\mathbb{P}}\left[\frac{R_t}{R_T} S_T C + \int_t^T \frac{R_t}{R_u} S_u dG_u - \int_t^T \frac{R_t}{R_u} Z_u dS_u \,\middle|\, \mathcal{F}_t^D \right]. \tag{4.4}$$

4.2 Intensity hypothesis

In the reduced-form approach of credit risk modelling, the standard hypothesis is the existence of the intensity of default time τ. We say that τ has an $\mathbb{F}$-*intensity* if its $\mathbb{F}$-compensator $\Lambda^{\mathbb{F}}$ is absolutely continuous with respect to the Lebesgue measure, that is, there exists an $\mathbb{F}$-adapted process $\lambda^{\mathbb{F}}$ (called the $\mathbb{F}$-*intensity* of τ under $\mathbb{P}$) such that $(1_{\{\tau\leq t\}} - \int_0^{t\wedge\tau} \lambda_s^{\mathbb{F}} ds, t \geq 0)$ is a $(\mathbb{G}, \mathbb{P})$-martingale. The intensity hypothesis implies that τ avoids the $\mathbb{F}$-predictable stopping times and that τ is $\mathbb{G}$ totally inaccessible.

Under the intensity hypothesis, the Doob-Meyer decomposition of the supermartingale S has the explicit form: the process $(S_t + \int_0^t S_u \lambda_u^{\mathbb{F}} du, t \geq 0)$ is an $\mathbb{F}$-martingale. The pricing formulae (4.2) and (4.4) can be written as

$$V_t = \frac{1_{\{\tau>t\}} R_t}{S_t} E_{\mathbb{P}}\left[R_T^{-1} S_T C + \int_t^T R_u^{-1} S_u dG_u + \int_t^T R_u^{-1} Z_u S_u \lambda_u^{\mathbb{F}} du \,\middle|\, \mathcal{F}_t \right], \tag{4.5}$$

$$V_t^D = \frac{1_{\{\tau>t\}}}{E[S_t|\mathcal{F}_t^D]} E_{\mathbb{P}}\left[\frac{R_t}{R_T} S_T C + \int_t^T \frac{R_t}{R_u} S_u dG_u + \int_t^T \frac{R_t}{R_u} Z_u S_u \lambda_u^{\mathbb{F}} du \,\middle|\, \mathcal{F}_t^D \right]. \tag{4.6}$$

Note that the intensity does not always exist. For example, in the structural model where L is deterministic, τ is a $\mathbb{F}$ predictable stopping time. Hence its intensity does

not exist. It is in general a difficult problem to determine the existence of the intensity process (see [13], [14] for a detailed discussion).

In contrast to the notion of intensity as above, the *default intensity* in the credit analysis is often referred as the instantaneous probability of default at time t conditioned on some filtration $(\mathcal{H}_t)_{t\geq 0}$:

$$\lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{P}(t < \tau \leq t + \Delta t | \mathcal{H}_t) \quad a.s.$$

Under Aven's conditions (see [13], [14]), the two intensities coincide. But this is not true in general. For example, in the classical structural model, the default intensity equals to zero. However, the intensity process does not exist in this case. The default intensity when $\mathcal{H}_t = \mathcal{F}_t^D$ has been studied in many papers such as [8, 6, 18, 13], the default intensity is strictly positive in the delayed information case. We note that in the full information case where $\mathcal{H}_t = \mathcal{G}_t^M$, we encounter the same situation as in the structural model: the default intensity equals to zero since L is $\mathcal{G}_t^M$-measurable.

5 Noisy full information

In this section, we consider the insider's information flow. Recall that the insider has a perturbed information on the barrier L which changes through time. We assume that the perturbation is given by an independent noise, and is getting clearer as time evolves. To be more precise, the noised barrier is modeled by a process $(L_t = f(L, \epsilon_t))_{t\geq 0}$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a given Borel measurable function, and ϵ is a process independent of $\mathcal{F}_\infty$. The information flow $\mathbb{G}^I = (\mathcal{G}_t^I)_{t\geq 0}$ of the insider is then given by

$$\mathcal{G}_t^I := \mathcal{F}_t \vee \sigma(L_s,\, s \leq t) \vee \mathcal{D}_t.$$

5.1 Perturbed initial enlargement of filtration

We firstly make precise the mathematical assumptions in this case. We introduce an auxiliary filtration $\mathbb{F}^I = (\mathcal{F}_t^I)_{t\geq 0}$ defined as

$$\mathcal{F}_t^I := \mathcal{F}_t \vee \sigma(L_s,\, s \leq t).$$

Note that $\mathbb{G}^I$ is a progressive enlargement of $\mathbb{F}^I$ by the information on the default. The filtration $\mathbb{F}^I$ has been studied in [7] under Assumption 3.1. It has nice properties similarly to the filtration $\mathbb{G}^M$. With the notation of Section 3.1, assume that $\rho_t^I := E_{\mathbb{P}}[\rho_t^M(L)|\mathcal{F}_t^I]$ satisfies $\int_0^\infty |\rho_t^I| dt < +\infty$ $\mathbb{P}$-a.s. Then the process $\widetilde{B}^I$ defined as $\widetilde{B}_t^I := B_t - \int_0^t \rho_s^I ds$ is an $(\mathbb{F}^I, \mathbb{P})$-Brownian motion. Moreover, the Doléans-Dade integral

$$Y_\cdot^I = \mathcal{E}(-\int_0^\cdot \rho_s^I d\widetilde{B}_s^I)$$

is a positive $(\mathbb{F}^I, \mathbb{P})$-local martingale. We assume that Y^I is an $(\mathbb{F}^I, \mathbb{P})$-martingale and define the probability measure $\mathbb{Q}^I$ by

$$\mathrm{d}\mathbb{Q}^I = Y_t^I \mathrm{d}\mathbb{Q} \quad \text{on } \mathcal{F}_t^I$$

where $\mathbb{Q}$ is an equivalent probability of $\mathbb{P}$. Then any $(\mathbb{F}, \mathbb{Q})$-local martingale is an $(\mathbb{F}^I, \mathbb{Q}^I)$-local martingale. In particular, B is an $(\mathbb{F}^I, \mathbb{P}^I)$-Brownian motion.

5.2 Pricing with noisy information

We now consider the pricing problem for the insider information flow $\mathbb{G}^I$. We shall focus on the particular but useful case:

$$L_t = L + \epsilon_t,$$

where ϵ is a continuous process independent of $\mathcal{F}_\infty \vee \sigma(L)$ and is of backwardly independent increments whose marginal has a density with respect to the Lebesgue measure (example in [7] and [15]). We say that a process ϵ has *backwardly independent increments* if for all $0 \leq s \leq t \leq \theta$, the random variable $\epsilon_s - \epsilon_t$ is independent to ϵ_θ. For example, if one takes $\epsilon_t = W_{g(T-t)}$ with W an Brownian motion, and $g : [0, T] \to [0, +\infty)$ a strictly increasing bounded function with $g(0) = 0$, then ϵ is a process on $[0, T]$ which has backwardly independent increments. Another example with infinite horizon is $\epsilon_t = W_{g(\frac{1}{t+1})}$, where $g : [0, 1] \to [0, +\infty)$ a strictly increasing bounded function with $g(0) = 0$.

To compute the pricing formula (2.1) for the insider where $\mathcal{H}_t = \mathcal{G}_t^I$, our strategy is to combine the results in the two previous sections using the auxiliary filtration $\mathbb{F}^I$. More precisely, we present firstly in Proposition 5.1 a result for the filtration $\mathbb{F}^I$ which is similar to the one in Proposition 3.1 for the filtration $\mathbb{G}^M$. We then use it to obtain the pricing formula in Theorem 5.1. In fact, since $\mathbb{G}^I$ is the progressive enlargement of $\mathbb{F}^I$, applying (4.2) leads to the value process for insiders:

$$V_t^I = \frac{\mathbf{1}_{\{\tau > t\}} R_t}{S_t^I} E_{\mathbb{P}}\left[R_T^{-1} S_T^I C + \int_t^T R_u^{-1} S_u^I dG_u - \int_t^T R_u^{-1} Z_u dS_u^I \,\middle|\, \mathcal{F}_t^I \right] \tag{5.1}$$

where $S_t^I := E_{\mathbb{P}}[\mathbf{1}_{\{\tau > t\}} | \mathcal{F}_t^I]$. In the rest of the section, we aim to give a reformulation of (5.1) as a conditional expectation with respect to the default-free filtration $\mathbb{F}$. It is interesting to remark that although the formula (5.2) in Proposition 5.1 seems to be complicated, the final result (5.6) is given in a simple and coherent form similarly as for the full and progressive information.

We assume Assumption 3.1 in the sequel, that is, the conditional probability law of L given $\mathcal{F}_t$ has a density $p_t(\cdot)$ with respect to the unconditioned probability law of L.

Proposition 5.1 *We assume Assumption 3.1. Let ϵ be a continuous process, independent of $\mathcal{F}_\infty \vee \sigma(L)$, and with backwardly independent increments such that the probability law of ϵ_t has a density $q_t(\cdot)$ with respect to the Lebesgue measure. For any $t \geq 0$, let $L_t = L + \epsilon_t$*

and $\mathcal{F}_t^I = \mathcal{F}_t \vee \sigma(L_s, s \le t)$. *Then, for any* $\theta \ge t$ *and any positive* $\mathcal{F}_\theta \otimes \mathcal{B}(\mathbb{R})$*-measurable function* $\phi_\theta(\cdot)$, *one has*

$$E_{\mathbb{P}}[\phi_\theta(L_\theta)1_{\{\tau>\theta\}}|\mathcal{F}_t^I] = \frac{\iint E_{\mathbb{P}}[\phi_\theta(u+y)p_\theta(l)1_{\{X_\theta^*>l\}}|\mathcal{F}_t]_{u=L_t} q_t(L_t-l)\mu_{t,\theta}(dy)P^L(dl)}{\int_{\mathbb{R}} p_t(l)q_t(L_t-l)P^L(dl)} \tag{5.2}$$

where P^L *is the probability law of* L, $\mu_{t,\theta}$ *is the probability law of* $\epsilon_\theta - \epsilon_t$. *For any* $\mathcal{F}_\theta$*-measurable* ϕ_θ, *one has*

$$E_{\mathbb{P}}[\phi_\theta 1_{\{\tau>\theta\}}|\mathcal{F}_t^I] = \frac{\int E_{\mathbb{P}}[\phi_\theta p_\theta(l)1_{\{X_\theta^*>l\}}|\mathcal{F}_t]q_t(L_t-l)P^L(dl)}{\int_{\mathbb{R}} p_t(l)q_t(L_t-l)P^L(dl)}.$$

PROOF: Since ϵ has backwardly independent increment and is independent of $\mathcal{F}_\theta \vee \sigma(L)$, one has

$$\begin{aligned} E_{\mathbb{P}}[\phi_\theta(L_\theta)1_{\{\tau>\theta\}}|\mathcal{F}_t^I] &= E_{\mathbb{P}}[\phi_\theta(L+\epsilon_\theta)1_{\{X_\theta^*>L\}}|\mathcal{F}\vee\sigma(L_t)\vee\sigma(\epsilon_s-\epsilon_t,\, s\le t)] \\ &= E_{\mathbb{P}}[\phi_\theta(L+\epsilon_\theta)1_{\{X_\theta^*>L\}}|\mathcal{F}\vee\sigma(L_t)]. \end{aligned} \tag{5.3}$$

By the independence of $\mathcal{F}_\theta \vee \sigma(L)$ and ϵ, we obtain

$$\begin{aligned} & E_{\mathbb{P}}\left[\phi_\theta(L_\theta)1_{\{\tau>\theta\}}|\mathcal{F}_t\vee\sigma(L_t)\vee\sigma(L)\right] \\ &= E_{\mathbb{P}}\left[\phi_\theta(L_\theta)1_{\{X_\theta^*>L\}}|\mathcal{F}_t\vee\sigma(\epsilon_t)\vee\sigma(L)\right] \\ &= \int_{\mathbb{R}} E_{\mathbb{P}}[\phi_\theta(L_t+y)1_{\{X_\theta^*>L\}}|\mathcal{F}_t\vee\sigma(\epsilon_t)\vee\sigma(L)]\mu_{t,\theta}(dy) \\ &= \int_{\mathbb{R}} E_{\mathbb{P}}\left[\phi_\theta(L+z+y)1_{\{X_\theta^*>L\}}|\mathcal{F}_t\vee\sigma(L)\right]_{z=\epsilon_t}\mu_{t,\theta}(dy) \\ &= p_t(L)^{-1}\int_{\mathbb{R}} E_{\mathbb{P}}[\phi_\theta(x+y+z)p_\theta(x)1_{\{X_\theta^*>x\}}\,|\,\mathcal{F}_t]_{\substack{x=L\\z=\epsilon_t}}\mu_{t,\theta}(dy), \end{aligned}$$

where the last equality comes from Proposition 3.1. In the rest of the proof, we denote by

$$H_t(L,L_t) := p_t(L)^{-1}\int_{\mathbb{R}} E_{\mathbb{P}}[\phi_\theta(u+y)p_\theta(x)1_{\{X_\theta^*>x\}}\,|\,\mathcal{F}_t]_{\substack{x=L\\u=L_t}}\mu_{t,\theta}(dy).$$

By definition and similar argument as for (5.3), one has

$$\begin{aligned} E_{\mathbb{P}}[\phi_\theta(L_\theta)1_{\{\tau>\theta\}}|\mathcal{F}_t^I] &= E_{\mathbb{P}}[H_t(L,L_t)|\mathcal{F}_t\vee\sigma(L_t)\vee\sigma((\epsilon_t-\epsilon_s),s\le t)] \\ &= E\left[H_t(L,L_t)|\mathcal{F}_t\vee\sigma(L_t)\right]. \end{aligned}$$

Let $P_t^L(dl)$ be the regular conditional probability of L given $\mathcal{F}_t$. Then for $U \in \mathcal{B}(\mathbb{R}^2)$,

$$\mathbb{P}\left((L,L_t)\in U|\mathcal{F}_t\right) = \int_{\mathbb{R}^2} 1_U(l,x)q_t(x-l)P_t^L(dl)dx$$

Therefore

$$E\left[H_t(L,L_t)|\mathcal{F}_t^I\right] = \frac{\int_{\mathbb{R}} H_t(l,L_t)q_t(L_t-l)P_t^L(dl)}{\int_{\mathbb{R}} q_t(L_t-l)P_t^L(dl)}. \tag{5.4}$$

By the equality $P_t^L(dl) = p_t(l)P^L(dl)$, we obtain the desired result. The second equality is obtained in a similar way. □

As a consequence of Proposition 5.1, the conditional expectation $E_{\mathbb{P}}[1_{\{\tau>t\}}|\mathcal{F}_t^I]$ can be written as $S_t^I(L_t)$, where $S_t^I(\cdot)$ is the $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable function defined as

$$S_t^I(x) = \frac{\int_{\mathbb{R}} 1_{\{X_t^* > l\}} p_t(l) q_t(x-l) P^L(dl)}{\int_{\mathbb{R}} p_t(l) q_t(x-l) P^L(dl)}. \tag{5.5}$$

In the following result, we compute (5.1) as $\mathbb{F}$-conditional expectations.

Theorem 5.1 *We keep the notations and assumptions of Proposition 5.1 and recall that $\mathcal{G}_t^I = \mathcal{F}_t^I \vee \mathcal{D}_t$. Then the value process for the noisy full information flow $\mathbb{G}^I$ is given by*

$$V_t^I = \frac{1_{\{\tau>t\}}}{\int_{\mathbb{R}} F_t^M(l) q_t(L_t - l) P^L(dl)} \int \widetilde{V}_t^M(l) q_t(L_t - l) P^L(dl) \tag{5.6}$$

where $\widetilde{V}^M$ and F^M are defined in Proposition 3.2.

PROOF: To obtain results with respect to $\mathcal{F}_t$, we shall calculate respectively the three terms of (5.1) using Proposition 5.1. Let $N_t(x) := \int_{\mathbb{R}} 1_{\{X_t^* > l\}} p_t(l) q_t(x-l) P^L(dl) = \int_{\mathbb{R}} F_t^M(l) q_t(x-l) P^L(dl)$. Firstly,

$$E_{\mathbb{P}}[\frac{C}{R_T} 1_{\{\tau>T\}} | \mathcal{G}_t^I] = 1_{\{\tau>t\}} \frac{E_{\mathbb{P}}[\frac{C}{R_T} 1_{\{\tau>T\}} | \mathcal{F}_t^I]}{E_{\mathbb{P}}[1_{\{\tau>t\}} | \mathcal{F}_t^I]} = \frac{1_{\{\tau>t\}}}{N_t(L_t)} \int E_{\mathbb{P}}\big[\frac{C}{R_T} F_T^M(l) | \mathcal{F}_t\big] q_t(L_t - l) P^L(dl)$$

where the second equality comes from Proposition 5.1. Secondly, using the same argument,

$$\begin{aligned} E_{\mathbb{P}}[\int_t^T 1_{\{\tau>\theta\}} \frac{dG_\theta}{R_\theta} | \mathcal{G}_t^I] &= \int_t^T E_{\mathbb{P}}[1_{\{\tau>\theta\}} \frac{dG_\theta}{R_\theta} | \mathcal{G}_t^I] \\ &= \frac{1_{\{\tau>t\}}}{N_t(L_t)} \int \int_t^T E_{\mathbb{P}}[F_\theta^M(l) \frac{dG_\theta}{R_\theta} | \mathcal{F}_t] q_t(L_t - l) P^L(dl) \end{aligned}$$

Thirdly, similar as in the proof of Proposition 3.2, we assume $Z_u = \sum_{i=0}^n Z_i 1_{t_i < u \le t_{i+1}}$ for $t < u \le T$ where $t_0 = t < \cdots < t_{n+1} = T$ and Z_i is $\mathcal{F}_{t_i}$-measurable for $i = 0, \cdots, n$. We have

$$\begin{aligned} & E_{\mathbb{P}}\Big[Z_\tau 1_{\{t<\tau\le T\}} | \mathcal{G}_t^I\Big] \\ = \; & \frac{1_{\{\tau>t\}}}{S_t^I} \sum_{i=0}^n E_{\mathbb{P}}\Big[Z_i 1_{\{t_i<\tau\}} - Z_i 1_{\{t_{i+1}<\tau\}} | \mathcal{F}_t^I\Big] \\ = \; & \frac{1_{\{\tau>t\}}}{N_t(L_t)} \int \sum_{i=0}^n \Big(E_{\mathbb{P}}\Big[Z_i p_{t_i}(x) 1_{\{X_{t_i}^* > x\}} | \mathcal{F}_t\Big] - E_{\mathbb{P}}\Big[Z_i p_{t_{i+1}}(x) 1_{\{X_{t_{i+1}}^* > x\}} | \mathcal{F}_t\Big] \Big) q_t(L_t - l) P^L(dl) \\ = \; & \frac{1_{\{\tau>t\}}}{N_t(L_t)} \sum_{i=0}^n \int E_{\mathbb{P}} \left[\sum_{i=0}^n Z_i \left(F_{t_i}^M(l) - F_{t_{i+1}}^M(l) \right) | \mathcal{F}_t \right] q_t(L_t - l) P^L(dl) \\ = \; & -\frac{1_{\{\tau>t\}}}{N_t(L_t)} \int E_{\mathbb{P}} \left[\int_t^T Z_u dF_u^M(l) | \mathcal{F}_t \right] q_t(L_t - l) P^L(dl) \end{aligned}$$

We get the third term by approximating $(Z_u R_u^{-1})_u$ by a suitable sequence of stepwise $\mathbb{F}$-predictable processes :

$$E_{\mathbb{P}}\Big[Z_\tau R_\tau^{-1} 1_{\{t<\tau\leq T\}}|\mathcal{G}_t^I\Big] = -\frac{1_{\{\tau>t\}}}{N_t(L_t)}\int E_{\mathbb{P}}\Big[\int_t^T \frac{Z_u}{R_u} dF_u^M(l)|\mathcal{F}_t\Big] q_t(L_t-l)P^L(dl).$$

We combine the three terms to complete the proof. □

6 Risk-neutral pricing and numerical illustrations

6.1 Pricing under different probabilities

To evaluate a credit derivative, both the pricing filtration and the choice of risk-neutral probability measures depend on the information level of the market agent. In the previous sections, we have computed the pricing formula (2.1) for different information filtration under the same historical probability measure. In the following, our objective is to take into account the pricing probabilities for each type of information.

We have made precise different pricing probabilities. First of all, we assume that a pricing probability $\mathbb{Q}$ is given with respect to the filtration $\mathbb{F}$ of the fundamental process X. Usually, we choose $\mathbb{Q}$ such that X is an $(\mathbb{F},\mathbb{Q})$ local martingale. Since we shall focus on the change of probability measures due to the different sources of informations and on its impact on the pricing of credit derivatives, we may assume, without loss of generality, the historical probability $\mathbb{P}$ to be the benchmark pricing probability $\mathbb{Q}$ on $\mathbb{F}$. For the same reason, we will consider the same pricing probability for the filtration $\mathbb{F}$ and its progressive enlargement $\mathbb{G}$.[1] Given the pricing probability $\mathbb{Q}$ on $\mathbb{F}$ (and thus on $\mathbb{G}$), the pricing probability for the manager is $\mathbb{Q}^M$ where $\frac{d\mathbb{Q}^M}{d\mathbb{Q}} = Y^M(L)$ with $Y^M(L) = \mathcal{E}(-\int_0^\cdot \rho_s^M(L)(dB_s - \rho_s^M(L)ds))$ (see Subsection 3.1) and for the noisy full information is $\mathbb{Q}^I$ where $\frac{d\mathbb{Q}^I}{d\mathbb{Q}} = Y^I$ with $Y^I = \mathcal{E}(-\int_0^\cdot \rho_s^I(dB_s - \rho_s^I ds))$ (see Subsection 5.1). We also take $\mathbb{Q}$ as the pricing probability for the delayed information because the delayed information case is more complicated : indeed, the notion of a $\mathbb{F}^D$ Brownian motion is a widely open question that we do not want to investigate here and we assume that the pricing probability for the delayed case is the same as for the progressive information.

The following proposition gives the price of a credit derivative for the full and the noisy information if we take into account not only the enlargement of filtration but also the change of pricing probability due to this insiders' information. Since we take $\mathbb{P}$ as the pricing measure, note that for the investors with progressive or delayed information, there is no change of pricing probability, so the results of Propositions 4.1 and 4.2 still hold.

[1] In general, a $(\mathbb{F},\mathbb{Q})$ local martingale is not necessarily a $(\mathbb{G},\mathbb{Q})$ local martingale except under (H) hypothesis. However, since all the filtrations we consider contains the progressive enlargement, we prefer to concentrate on the change of probabilities due to different sources of information and we keep the same pricing probability for $\mathbb{F}$ and $\mathbb{G}$.

Proposition 6.1 *We assume Assumption 3.1.*
1) Define $F_t^{\mathbb{Q}^M}(l) = 1_{\{X_t^* > l\}}$. *Then the value process of a credit sensitive claim* (C, G, Z) *for the manager's full information under the risk neutral probability measure* $\mathbb{Q}^M$ *is given by*

$$V_t^{\mathbb{Q}^M} = R_t E_{\mathbb{P}}\big[CR_T^{-1}F_T^{\mathbb{Q}^M}(x) + \int_t^T F_s^{\mathbb{Q}^M}(x)R_s^{-1}dG_s - \int_t^T Z_s R_s^{-1} dF_s^{\mathbb{Q}^M}(x) \mid \mathcal{F}_t\big]_{x=L}.$$

2) Let ϵ *be a continuous process with backwardly independent increments such that the probability law of* ϵ_t *has a density* $q_t(\cdot)$ *w.r.t. the Lebesgue measure. Then the value process for the insider's noisy full information under* $\mathbb{Q}^I$ *is given by*

$$V_t^{\mathbb{Q}^I} = \frac{1_{\{\tau > t\}}}{\int_{\mathbb{R}} F_t^M(l) q_t(L_t - l) P^L(dl)} \int \widetilde{V}_t^{\mathbb{Q}^I}(l) q_t(L_t - l) P^L(dl) \tag{6.1}$$

where

$$\widetilde{V}_t^{\mathbb{Q}^I}(l) = R_t E_{\mathbb{P}}\big[CR_T^{-1}F_{t,T}^I(u,l) + \int_t^T F_{t,\theta}^I(u,l)R_\theta^{-1}dG_\theta - \int_t^T R_\theta^{-1}Z_\theta dF_{t,\theta}^I(u,l)|\mathcal{F}_t\big]_{u=L_t},$$

$$F_{t,\theta}^I(u,l) = \mathcal{E}\Big(\int_t^\theta \int \rho_\theta^I(u+y)\mu_{t,\theta}(dy)dB_u\Big)^{-1} F_\theta^M(l).$$

To prove the second assertion of the above proposition, we need the following lemma which is an extension of Proposition 5.1. We give the proof of Proposition 6.1 afterwards.

Lemma 6.1 *We keep the notations and assumptions of Proposition 5.1. Then, for any* $\theta \geq t$ *and any* $\mathcal{F}_\theta$*-measurable* ϕ_θ*, one has*

$$E_{\mathbb{P}}[Y_\theta^I \phi_\theta 1_{\{\tau > \theta\}} | \mathcal{F}_t^I] = Y_t^I \frac{\int E_{\mathbb{P}}[\phi_\theta F_{t,\theta}^I(u,l) \mid \mathcal{F}_t]_{u=L_t} q_t(L_t - l) P^L(dl)}{\int_{\mathbb{R}} p_t(l) q_t(L_t - l) P^L(dl)},$$

where P^L *is the probability law of* L, $\mu_{t,\theta}$ *is the probability law of* $\epsilon_\theta - \epsilon_t$ *and* $F_{t,\theta}^I(u,l)$ *is defined in Proposition 6.1.*

PROOF: First, let us recall, that $Y_T^I = \mathcal{E}(\int_0^T \rho_u^I dB_u)^{-1}$ and $\rho_t^I = E(\rho_t^M(L)|\mathcal{F}_t^I) = \frac{\int \rho_t^M(l) q_t(L_t - l) P_t^L(dl)}{\int q_t(L_t - l) P_t^L(dl)} = \rho_t^I(L_t)$. $(Y_t^I)_{t \geq 0}$ is an $(\mathbb{F}^I, \mathbb{P})$ martingale. Since ϵ has backwardly independent increment and is independent of $\mathcal{F}_\theta \vee \sigma(L)$, one has

$$\begin{aligned} E_{\mathbb{P}}[\phi_\theta Y_\theta^I 1_{\{\tau > \theta\}} | \mathcal{F}_t^I] &= Y_t^I E_{\mathbb{P}}[\phi_\theta \mathcal{E}(\int_t^\theta \rho_u^I(L_u) dB_u)^{-1} 1_{\{X_\theta^* > L\}} | \mathcal{F} \vee \sigma(L_t) \vee \sigma(\epsilon_s - \epsilon_t,\, s \leq t)] \\ &= Y_t^I E_{\mathbb{P}}[\phi_\theta \mathcal{E}(\int_t^\theta \rho_u^I(L + \epsilon_u) dB_u)^{-1} 1_{\{X_\theta^* > L\}} | \mathcal{F} \vee \sigma(L_t)]. \end{aligned}$$

By the independence of $\mathcal{F}_\theta \vee \sigma(L)$ and ϵ, we obtain

$$
\begin{aligned}
& E_{\mathbb{P}}\left[\phi_\theta \mathcal{E}(\int_t^\theta \rho_u^I(L+\epsilon_u)dB_u)^{-1}1_{\{\tau>\theta\}}|\mathcal{F}_t \vee \sigma(L_t) \vee \sigma(L)\right] \\
&= E_{\mathbb{P}}\left[\phi_\theta \mathcal{E}(\int_t^\theta \rho_u^I(L+\epsilon_u)dB_u)^{-1}1_{\{X_\theta^*>L\}}|\mathcal{F}_t \vee \sigma(\epsilon_t) \vee \sigma(L)\right] \\
&= \int_{\mathbb{R}} E_{\mathbb{P}}[\phi_\theta \mathcal{E}(\int_t^\theta \int \rho_u^I(L_t+y)\mu_{t,\theta}(dy)dB_u)^{-1}1_{\{X_\theta^*>L\}}|\mathcal{F}_t \vee \sigma(\epsilon_t) \vee \sigma(L)] \\
&= \int_{\mathbb{R}} E_{\mathbb{P}}\left[\phi_\theta \mathcal{E}(\int_t^\theta \int \rho_u^I(L+z+y)\mu_{t,\theta}(dy)dB_u)^{-1}1_{\{X_\theta^*>L\}}|\mathcal{F}_t \vee \sigma(L)\right]_{z=\epsilon_t} \\
&= p_t(L)^{-1}\int_{\mathbb{R}} E_{\mathbb{P}}[\phi_\theta p_\theta(x)\mathcal{E}(\int_t^\theta \int \rho_u^I(L+z+y)\mu_{t,\theta}(dy)dB_u)^{-1}1_{\{X_\theta^*>x\}} \mid \mathcal{F}_t]_{\substack{x=L \\ z=\epsilon_t}},
\end{aligned}
$$

where the last equality comes from Proposition 3.1. The rest of the proof is similar to the one of Proposition 5.1 , with

$$
H_t(L,L_t) := p_t(L)^{-1}\int_{\mathbb{R}} E_{\mathbb{P}}[\phi_\theta p_\theta(x)\mathcal{E}(\int_t^\theta \int \rho_u^I(u+y)\mu_{t,\theta}(dy)dB_u)^{-1}1_{\{X_\theta^*>x\}} \mid \mathcal{F}_t]_{\substack{x=L \\ u=L_t}}.
$$

□

PROOF: 1) For the full manager, the proof is similar as the one of Proposition 3.2 : by noting that $\mathbb{Q}$ is chosen to be $\mathbb{P}$, the probability measure $\mathbb{Q}^M$ coincides with $\mathbb{P}^L$ defined in Section 3.1. Thus, the end of the proof of Proposition 3.2 still holds, using $F_t^{\mathbb{Q}^M} = 1_{\{X_t^*>l\}}$ instead of F_t^M.
2) For the noisy information, $\frac{d\mathbb{Q}^I}{d\mathbb{P}} = Y_T^I$ with $Y_T^I = \mathcal{E}(\int_0^T \rho_u^I dB_u)^{-1}$ and $\rho_t^I = \rho_t^I(L_t)$. Let $N_t(x) = \int_{\mathbb{R}} 1_{\{X_t^*>l\}}p_t(l)q_t(x-l)P^L(dl) = \int_{\mathbb{R}} F_t^M(l)q_t(L_t-l)P^L(dl)$ and $F_{t,\theta}^I(u,l) = \mathcal{E}(\int_t^\theta \int \rho_\theta^I(u+y)\mu_{t,\theta}(dy)dB_u)^{-1}p_\theta(l)1_{\{X_\theta^*>l\}}$. For T, l and u fixed, $(F_{t,T}^I(u,l))_{0\le t\le T}$ is a nonnegative $(\mathbb{F},\mathbb{P})$-supermartingale, and we may deal with its right-continuous modification with finite left-hand limits. Firstly,

$$
E_{\mathbb{Q}^I}[\frac{C}{R_T}1_{\{\tau>T\}}|\mathcal{G}_t^I] = 1_{\{\tau>t\}}\frac{E_{\mathbb{Q}^I}[\frac{C}{R_T}1_{\{\tau>T\}}|\mathcal{F}_t^I]}{\mathbb{Q}^I[\tau>t|\mathcal{F}_t^I]} = 1_{\{\tau>t\}}\frac{E_{\mathbb{P}}[\frac{C}{R_T}Y_T^I 1_{\{\tau>T\}}|\mathcal{F}_t^I]}{E_{\mathbb{P}}[1_{\{\tau>t\}}Y_t^I|\mathcal{F}_t^I]}
$$

because on the event $\{\tau>t\}$, $\frac{d\mathbb{Q}^I}{d\mathbb{P}}|_{\mathcal{G}_t^I} = \frac{d\mathbb{Q}^I}{d\mathbb{P}}|_{\mathcal{F}_t^I} = Y_t^I$. Thus

$$
E_{\mathbb{Q}^I}[\frac{C}{R_T}1_{\{\tau>T\}}|\mathcal{G}_t^I] = \frac{1_{\{\tau>t\}}}{N_t(L_t)}\int E_{\mathbb{P}}\big[\frac{C}{R_T}F_{t,T}^I(u,l)|\mathcal{F}_t\big]_{u=L_t}q_t(L_t-l)P^L(dl)
$$

where the second equality comes from Lemma 6.1. Secondly, using the same argument,

$$\begin{aligned} E_{\mathbb{Q}^I}[\int_t^T 1_{\{\tau>\theta\}}\frac{dG_\theta}{R_\theta}|\mathcal{G}_t^I] &= 1_{\{\tau>t\}}\int_t^T \frac{E_{\mathbb{Q}^I}[1_{\{\tau>\theta\}}\frac{dG_\theta}{R_\theta}|\mathcal{F}_t^I]}{\mathbb{Q}^I[\tau>t|\mathcal{F}_t^I]} \\ &= 1_{\{\tau>t\}}\int_t^T \frac{E_{\mathbb{P}}[Y_\theta^I 1_{\{\tau>\theta\}}\frac{dG_\theta}{R_\theta}|\mathcal{F}_t^I]}{E_{\mathbb{P}}[Y_t^I 1_{\{\tau>\theta\}}|\mathcal{F}_t^I]} \\ &= \frac{1_{\{\tau>t\}}}{N_t(L_t)}\int E_{\mathbb{P}}\big[\int_t^T F_{t,\theta}^I(u,l)\frac{dG_\theta}{R_\theta}|\mathcal{F}_t\big]_{u=L_t} q_t(L_t-l)P^L(dl) \end{aligned}$$

Thirdly, we assume $Z_u = \sum_{i=0}^n Z_i 1_{t_i<u\le t_{i+1}}$ for $t<u\le T$ where $t_0 = t < \cdots < t_{n+1} = T$ and Z_i is $\mathcal{F}_{t_i}$-measurable for $i = 0, \cdots, n$. We have

$$\begin{aligned} & E_{\mathbb{Q}^I}\Big[Z_\tau 1_{\{t<\tau\le T\}}|\mathcal{G}_t^I\Big] \\ & 1_{\{\tau>t\}}\frac{E_{\mathbb{Q}^I}\Big[Z_\tau 1_{\{t<\tau\le T\}}|\mathcal{F}_t^I\Big]}{\mathbb{Q}^I[\tau>t|\mathcal{F}_t^I]} \\ = & \frac{1_{\{\tau>t\}}}{N_t(L_t)}\int\sum_{i=0}^n \Big(E_{\mathbb{P}}\Big[Z_i Y_{t_i}^I 1_{\{X_{t_i}^*>x\}}|\mathcal{F}_t\Big] - E_{\mathbb{P}}\Big[Z_i Y_{t_{i+1}}^I 1_{\{X_{t_{i+1}}^*>x\}}|\mathcal{F}_t\Big]\Big)_{u=L_t} q_t(L_t-l)P^L(dl) \\ = & \frac{1_{\{\tau>t\}}}{N_t(L_t)}\sum_{i=0}^n\int E_{\mathbb{P}}\left[\sum_{i=0}^n Z_i\left(F_{t_i,T}^I(u,l) - F_{t_{i+1},T}^I(u,l)\right)|\mathcal{F}_t\right]_{u=L_t} q_t(L_t-l)P^L(dl) \\ = & -\frac{1_{\{\tau>t\}}}{N_t(L_t)}\int E_{\mathbb{P}}\left[\int_t^T Z_s dF_{s,T}^I(u,l)|\mathcal{F}_t\right]_{u=L_t} q_t(L_t-l)P^L(dl) \end{aligned}$$

We conclude in the same way as in Proposition 3.2.

6.2 Numerical examples

We present numerical examples to illustrate the pricing formulas obtained previously. We shall consider the following binomial model for the default barrier L.

Example 6.2 (Binomial Model)
Let L be a random variable taking two values $l_i, l_s \in \mathbb{R}$, $l_i \le l_s$ such that

$$\mathbb{P}(L = l_i) = \alpha, \quad \mathbb{P}(L = l_s) = 1-\alpha \quad (0<\alpha<1).$$

Note that L is independent of $(\mathcal{F}_t)_{t\ge0}$.

We suppose that the asset values process X satisfies the Black Scholes model :

$$\frac{dX_t}{X_t} = \mu dt + \sigma dB_t, \quad t \ge 0.$$

It is classical in this model to calculate conditional probabilities ([2, Cor3.1.2]). In fact, for $t \geq 0$ and $h, l > 0$,

$$E_{\mathbb{P}}(1_{\{X_t^* > l\}} - 1_{\{X_{t+h}^* > l\}} | \mathcal{F}_t) \quad = \quad 1_{\{X_t^* > l\}} \left(\Phi\big(\frac{-Y_t^l - \nu h}{\sigma\sqrt{h}}\big) + e^{2\nu\sigma^{-2}Y_t} \Phi\big(\frac{-Y_t^l + \nu h}{\sigma\sqrt{h}}\big) \right)$$

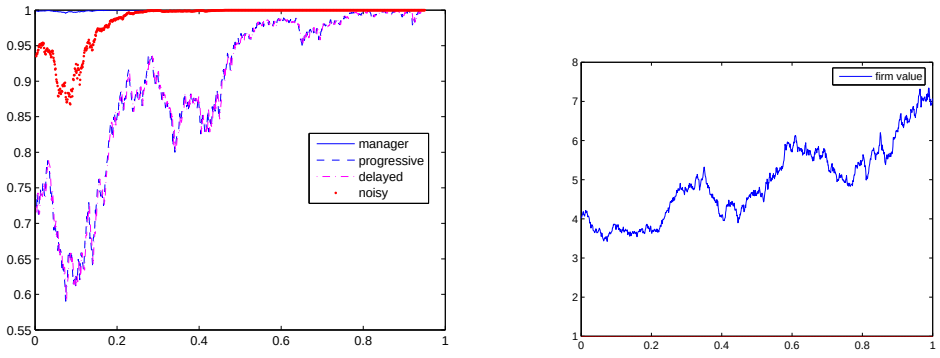
where Φ is the standard Gaussian cumulative distribution function and

$$Y_t^l = \nu t + \sigma B_t + \ln \frac{X_0}{l}, \quad \text{with} \qquad \nu = \mu - \frac{1}{2}\sigma^2.$$

This formula will allow us to obtain explicit pricing results in the binomial default barrier model.

We give numerical comparisons of the value process of a defaultable bond for different information, in Example 6.2 with the numerical values: $l_i = 1, l_s = 3, \alpha = \frac{1}{2}$. We have fixed a very small constant delayed time, which makes the pricing results for the delayed information very close to the ones for the progressive information. We present in each figure two graphs, one being the dynamic price of a defaultable bond with zero recovery rate in the scenario of the firm value presented in the second graph.

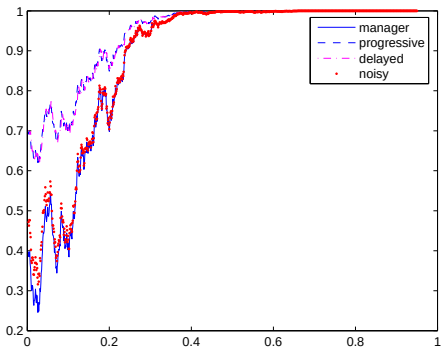
In the scenario of Figure 1, the manager has fixed the lower value for the default threshold. So he estimates smaller default probability and thus higher price for the defaultable bond, compared to the ones estimated by other agents on the market. We observe in addition that insider with noisy information has a better estimation of the price compared to the investors with progressive or delayed information.



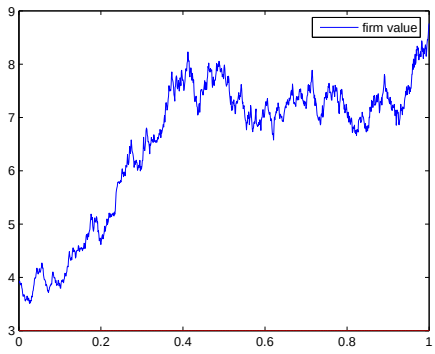
dynamic price of the defaultable bond firm value

Figure 1: $L = l_i$

We observe similar phenomena in Figure 2: the manager has fixed the upper value for the default threshold and thus estimates higher probability of default and smaller price of the defaultable bond. Note that in the particular case where L is constant ($l_i = l_s$), the price of the defaultable bond are the same, whatever the information we consider.



dynamic price of the defaultable bond



firm value

Figure 2: $L = l_s$

7 Conclusion

We have modelled the different levels of default information by several types of enlargement of filtrations, leading also to different pricing probability measures. We have taken into account these two aspects in the pricing of credit derivatives and obtained in all the cases coherent formulas given with respect to the "default-free" reference filtration. We have compared finally the pricing results by numerical illustrations.

References

[1] Amendinger, J., 2000. Martingale Representation Theorems for Initially Enlarged Filtrations, Stochastic Processes and their Applications 89, 101-116.

[2] Bielecki, T.R., Rutkowski, M., 2002. Credit Risk: Modeling, Valuation and Hedging, Springer-Verlag.

[3] Bielecki, T.R., Jeanblanc, M., Rutkowski, M., 2008. Pricing and Trading the Credit Default Swaps in a Hazard Process Model, Annals of Applied Probability, 18(6), 2495-2529.

[4] Çetin, U., Jarrow, R., Protter, P., Yıldırım, Y., 2004. Modeling credit risk with partial information, Annals of Applied Probability, 14(3), 1167-1178.

[5] Coculescu, D., Geman, H., Jeanblanc, M., 2006. Valuation of Default sensitive claims under imperfect Information, Finance and Stochastics.

[6] Collin-Dufresne, P., Goldstein, R., Helwege, J., 2003 Is credit event risk priced? Modelling contagion via the updating of beliefs, Working Paper, Carnegie Mellon University.

[7] Corcuera, J.M., Imkeller, P., Kohatsu-Higa, A., Nualart, D., 2004. Additional utility of insiders with imperfect dynamical information, Finance and Stochastics, 8, 437-450.

[8] Duffie, D., Lando, D. 2001. Term Structures of Credit Spreads with Incomplete Accounting Information, Econometrica, 69, 633-664.

[9] El Karoui, N., 1999. Modélisation de l'information, Lecture Notes, Ecole d'été CEA-EDF-INRIA.

[10] Elliott, R., Jeanblanc, M. and Yor, M., 2000. On models of default risk, Mathematical Finance 10, 179-195.

[11] Giesecke K, Goldberg L. R., 2008. The Market Price of credit risk : the impact of Asymmetric Information.

[12] Grorud, A., Pontier, M., 1998. Insider Trading in a Continuous Time Market Model, International Journal of Theorical and Applied Finance 1, 331-347.

[13] Guo, X., Jarrow, R., Zeng, Y., 2008. Credit Risk with Incomplete Information, to appear in Mathematics of Operations Research.

[14] Guo, X., Zeng, Y., 2008. Intensity process and compensator: a new filtration expansion approach and the Jeulin-Yor theorem, Annal of Applied Probability, 18, 120-142.

[15] Hillairet, C., 2005. Comparison of insiders' optimal strategies depending on the type of side-information. Stochastic Processes and Their Applications 115, 1603-1627.

[16] Jacod, J., 1979. Calcul stochastique et problèmes de martingales, Lecture Notes 714, Springer-Verlag, New York.

[17] Jacod, J., 1985. Grossissement Initial, Hypothèse (H') et Théorème de Girsanov, Lecture Notes 1118, Springer-Verlag, 15-35.

[18] Jeanblanc, M., Valchev, S., 2005. Partial information and hazard process, International Journal of Theoretical and Applied Finance, 8(6), 807–838.

[19] Jeulin, J., Yor, M., 1978. Grossissement d'une filtration et semi-martingales: formules explicites, Séminaire de Probabilités (Strasbourg), 12, 78-97, Springer-Verlag.

[20] Lando, D., 1998. On Cox processes and credit risky securities, Review of Derivatives Research, 2, 99-120.

[21] Leland, H., 1994. Corporate debt value, bond convenants and optimal capital structure, Journal of Finance 49(4), 1213-1252.

[22] Merton, R., 1974. On the pricing of corporate debt: the risk structure of interest rates, Journal of Finance 29, 449-470.