ECOLE POLYTECHNIQUE
CENTRE DE MATHÉMATIQUES APPLIQUÉES

UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tel.: 01 69 33 46 00. Fax: 01 69 33 46 46
http : //www.cmap.polytechnique.fr/

On stability estimates in the Gel'fand-Calderon inverse problem

R.G. Novikov

R.I. 675 *Février 2010*

On stability estimates in the Gel'fand-Calderon inverse problem

R. G. Novikov

CNRS (UMR 7641), Centre de Mathématiques Appliquées, Ecole Polytechnique,
91128 Palaiseau, France
e-mail: novikov@cmap.polytechnique.fr

Abstract. We prove new global stability estimates for the Gel'fand-Calderon inverse problem in 3D.

1. Introduction

We consider the equation

$$-\Delta\psi + v(x)\psi = 0, \quad x \in D, \tag{1.1}$$

where

$$D \text{ is an open bounded domain in } \mathbb{R}^d, \ \ d \geq 2, \ \partial D \in C^2, \ v \in L^\infty(D). \tag{1.2}$$

Equation (1.1) arises, in particular, in quantum mechanics, acoustics, electrodynamics. Formally, (1.1) looks as the Schrödinger equation with potential v at zero energy.

We consider the map Φ such that

$$\frac{\partial\psi}{\partial\nu}\big|_{\partial D} = \Phi(\psi\big|_{\partial D}) \tag{1.3}$$

for all sufficiently regular solutions ψ of (1.1) in $\bar{D} = D \cup \partial D$, where ν is the outward normal to ∂D. Here we assume also that

$$0 \text{ is not a Dirichlet eigenvalue for the operator } \ -\Delta + v \ \text{ in } \ D. \tag{1.4}$$

The map Φ is called the Dirichlet-to-Neumann map for equation (1.1) and is considered as boundary measurements for (physical model described by) (1.1).

We consider the following inverse boundary value problem for equation (1.1):

Problem 1.1. Given Φ, find v.

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at zero energy (see [9], [16]). This problem can be also considered as a generalization of the Calderon problem of the electrical impedance tomography (see [5], [23], [16]).

Concerning results given in the literature on Problem 1.1 (in its Calderon or Gel'fand form) see [6], [11], [23], [10] (note added in proof), [16], [1], [14], [15], [3], [22], [13], [17], [19], [12], [4], [21], [20] and references therein.

In particular, in [21] it was shown that the Alessandrini stability estimates of [1] for Problem 1.1 in dimension $d \geq 3$ (see Theorem 2.1 of the next section) admit some principle improvement. These new stability estimates (see Theorem 2.2 of the next section) were found in [21] using methods developed in [17], [18], [19]. These methods include, in particular: (1) the $\bar{\partial}$-approach to inverse "scattering" at zero energy in dimension $d \geq 3$, going back to [2], [10], and (2) the reduction of Problem 1.1 to inverse "scattering" at zero energy, going back to [16].

However, a complete proof of the aforementioned new stability estimates for Problem 1.1 in dimension $d \geq 2$ was given in [21] in the Born approximation (that is in the linear approximation near zero potential) only. Besides, a scheme of proof of these estimates was also mentioned in [21] for potentials with sufficiently small norm in dimension $d = 3$. (In this scheme [21] refers, in particular, to results of [19].)

In the present work we give a complete proof of these new stability estimates (Theorem 2.2 of the next section) in the general (or by other words global) case in dimension $d = 3$. In this proof we use, in particular, results of the recent work [20].

2. Stability estimates

As in [21] we assume for simplicity that

$$\begin{aligned} &D \ \text{ is an open bounded domain in } \ \mathbb{R}^d, \ \partial D \in C^2, \\ &v \in W^{m,1}(\mathbb{R}^d) \ \text{ for some } m > d, \ \mathit{supp}\, v \subset D, \ d \geq 2, \end{aligned} \tag{2.1}$$

where

$$W^{m,1}(\mathbb{R}^d) = \{v : \ \partial^J v \in L^1(\mathbb{R}^d), \ |J| \leq m\}, \quad m \in \mathbb{N} \cup 0, \tag{2.2}$$

where

$$J \in (\mathbb{N} \cup 0)^d, \quad |J| = \sum_{i=1}^{d} J_i, \quad \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}.$$

Let

$$\|v\|_{m,1} = \max_{|J| \leq m} \|\partial^J v\|_{L^1(\mathbb{R}^d)}. \tag{2.3}$$

Let

$$\begin{aligned} &\|A\| \ \text{ denote the norm of an operator} \\ &A : \ L^\infty(\partial D) \to L^\infty(\partial D). \end{aligned} \tag{2.4}$$

We recall that if v_1, v_2 are potentials satisfying (1.2), (1.3), where D is fixed, then

$$\Phi_1 - \Phi_2 \ \text{ is a compact operator in } \ L^\infty(\partial D), \tag{2.5}$$

where Φ_1, Φ_2 are the DtN maps for v_1, v_2 respectively, see [16], [17]. Note also that (2.1) $\Rightarrow$ (1.2).

Theorem 2.1 (variation of the result of [1]). *Let conditions* (1.4), (2.1) *hold for potentials v_1 and v_2, where D is fixed, $d \geq 3$. Let $\|v_j\|_{m,1} \leq N$, $j = 1, 2$, for some $N > 0$. Let Φ_1, Φ_2 denote the DtN maps for v_1, v_2, respectively. Then*

$$\|v_1 - v_2\|_{L^\infty(D)} \leq c_1 (\ln(1 + \|\Phi_1 - \Phi_2\|^{-1}))^{-\alpha_1}, \tag{2.6}$$

where $c_1 = c_1(N, D, m)$, $\alpha_1 = (m - d)/m$, $\|\Phi_1 - \Phi_2\|$ is defined according to (2.4).

As it was mentioned in [21], Theorem 2.1 follows from formulas (3.9)-(3.11), (4.1) (of Sections 3 and 4).

A disadvantage of estimate (2.6) is that

$$\alpha_1 < 1 \quad \text{for any} \quad m > d \quad \text{even if} \quad m \quad \text{is very great.} \tag{2.7}$$

Theorem 2.2. *Let the assumptions of Theorem 2.1 hold. Then*

$$\|v_1 - v_2\|_{L^\infty(D)} \le c_2(\ln(1 + \|\Phi_1 - \Phi_2\|^{-1}))^{-\alpha_2}, \tag{2.8}$$

where $c_2 = c_2(N, D, m)$, $\alpha_2 = m - d$, $\|\Phi_1 - \Phi_2\|$ *is defined according to* (2.4).

A principal advantage of estimate (2.8) in comparison with (2.6) is that

$$\alpha_2 \to +\infty \quad \text{as} \quad m \to +\infty, \tag{2.9}$$

in contrast with (2.7).

In the Born approximation, that is in the linear approximation near zero potential, Theorem 2.2 was proved in [21].

For sufficiently small N in dimension $d = 3$, a scheme of proof of Theorem 2.2 was also mentioned in [21]. This scheme involves, in particular, results of [17], [19].

In the general (or by other words global) case Theorem 2.2 in dimension $d = 3$ is proved in Section 7. This proof involves, in particular, results of [17], [20].

3. Faddeev functions

We consider the Faddeev functions G, ψ and h (see [7], [8], [10], [16]):

$$\psi(x, k) = e^{ikx} + \int_{\mathbb{R}^d} G(x - y, k)v(y)\psi(y, k)dy, \tag{3.1}$$

$$G(x, k) = e^{ikx} g(x, k), \quad g(x, k) = -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 + 2k\xi}, \tag{3.2}$$

where $x \in \mathbb{R}^d$, $k \in \Sigma$,

$$\Sigma = \{k \in \mathbb{C}^d : \quad k^2 = k_1^2 + \ldots + k_d^2 = 0\}; \tag{3.3}$$

$$h(k, l) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ilx} v(x)\psi(x, k)dx, \tag{3.4}$$

where $(k, l) \in \Theta$,

$$\Theta = \{k \in \Sigma, \quad l \in \Sigma : \; Im\, k = Im\, l\}. \tag{3.5}$$

One can consider (3.1), (3.4) assuming that v is a sufficiently regular function on $\mathbb{R}^d$ with sufficient decay at infinity. For example, one can consider (3.1), (3.4) assuming that (1.2) holds.

We recall that:

$$\Delta G(x, k) = \delta(x), \quad x \in \mathbb{R}^d, \quad k \in \Sigma; \tag{3.6}$$

formula (3.1) at fixed k is considered as an equation for

$$\psi = e^{ikx}\mu(x, k), \tag{3.7}$$

where μ is sought in $L^\infty(\mathbb{R}^d)$; as a corollary of (3.1),(3.2), (3.6), ψ satisfies (1.1); h of (3.4) is a generalized "scattering" amplitude in the complex domain at zero energy.

Note that, actually, G, ψ, h of (3.1)-(3.5) are zero energy restrictions of functions introduced by Faddeev as extentions to the complex domain of some functions of the classical scattering theory for the Schrödinger equation at positive energies. In addition, G, ψ, h in their zero energy restriction were considered for the first time in [2]. The Faddeev functions G, ψ, h were, actually, rediscovered in [2].

We recall also that, under the assumptions of Theorem 2.1,

$$\mu(x,k) \to 1 \quad \text{as} \quad |Im\, k| \to \infty \quad \text{(uniformly in } \ x) \tag{3.8}$$

and, for any $\sigma > 1$,

$$|\mu(x,k)| < \sigma \quad \text{for} \quad |Im\, k| \geq r_1(N, D, m, \sigma), \tag{3.9}$$

where $x \in \mathbb{R}^d$, $k \in \Sigma$;

$$\hat{v}(p) = \lim_{\substack{(k,l)\in\Theta,\ k-l=p \\ |Im\, k|=|Im\, l|\to\infty}} h(k,l) \quad \text{for any} \quad p \in \mathbb{R}^d, \tag{3.10}$$

$$\begin{aligned} &|\hat{v}(p) - h(k,l)| \leq \frac{c_3(D,m)N^2}{\rho} \quad \text{for} \quad (k,l) \in \Theta, \ p = k - l, \\ &|Im\, k| = |Im\, l| = \rho \geq r_2(N, D, m), \end{aligned} \tag{3.11}$$

where

$$\hat{v}(p) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d. \tag{3.12}$$

Results of the type (3.8), (3.9) go back to [2]. Results of the type (3.10), (3.11) (with less precise right-hand side in (3.11)) go back to [10]. Estimates (3.8), (3.11) are related also with some important L_2-estimate going back to [23] on the Green function g of (3.1).

Note also that in some considerations it is convenient to consider h on Θ as H on Ω, where

$$\begin{aligned} h(k,l) &= H(k, k-l), \quad (k,l) \in \Theta, \\ H(k,p) &= h(k, k-p), \quad (k,p) \in \Omega, \end{aligned} \tag{3.13}$$

$$\Omega = \{k \in \mathbb{C}^d, \ p \in \mathbb{R}^d : \ k^2 = 0, \ p^2 = 2kp\}. \tag{3.14}$$

For more information on properties of the Faddeev functions G, ψ, h, see [10], [17], [20] and references therein.

In the next section we recall that Problem 1.1 (of Introduction) admits a reduction to the following inverse "scattering" problem:

Problem 3.1. Given h on Θ, find v on $\mathbb{R}^d$.

4. Reduction of [16], [17]

Let conditions (1.2), (1.4) hold for potentials v_1 and v_2, where D is fixed. Let Φ_i, ψ_i, h_i denote the DtN map Φ and the Faddeev functions ψ, h for $v = v_i$, $i = 1, 2$. Let also

$\Phi_i(x,y)$ denote the Schwartz kernel $\Phi(x,y)$ of the integral operator Φ for $v=v_i$, $i=1,2$. Then (see [17] for details):

$$h_2(k,l)-h_1(k,l)=\Big(\frac{1}{2\pi}\Big)^d\int\limits_{\partial D}\int\limits_{\partial D}\psi_1(x,-l)(\Phi_2-\Phi_1)(x,y)\psi_2(y,k)dydx, \tag{4.1}$$

where $(k,l)\in\Theta$;

$$\psi_2(x,k)=\psi_1(x,k)+\int_{\partial D}A(x,y,k)\psi_2(y,k)dy,\ x\in\partial D, \tag{4.2a}$$

$$A(x,y,k)=\int_{\partial D}R_1(x,z,k)(\Phi_2-\Phi_1)(z,y)dz,\ x,y\in\partial D, \tag{4.2b}$$

$$R_1(x,y,k)=G(x-y,k)+\int_{\mathbb{R}^d}G(x-z,k)v_1(z)R_1(z,y,k)dz,\ x,y\in\mathbb{R}^d, \tag{4.3}$$

where $k\in\Sigma$. Note that: (4.1) is an explicit formula, (4.2a) is considered as an equation for finding ψ_2 on ∂D from ψ_1 on ∂D and A on $\partial D\times\partial D$ for each fixed k, (4.2b) is an explicit formula, (4.3) is an equation for finding R_1 from G and v_1, where G is the function of (3.2).

Note that formulas and equations (4.1)-(4.3) for $v_1\equiv 0$ were given in [16] (see also [10] (Note added in proof), [14], [15]). In this case $h_1\equiv 0$, $\psi_1=e^{ikx}$, $R_1=G(x-y,k)$. Formulas and equations (4.1)-(4.3) for the general case were given in [17].

Formulas and equations (4.1)-(4.3) with fixed background potential v_1 reduce Problem 1.1 (of Introduction) to Problem 3.1 (of Section 3).

5. Some considerations related with Θ and Ω

5.1 Some subsets of Θ *and* Ω. Let

$$\begin{aligned}&\mathcal{B}_r=\{p\in\mathbb{R}^d:\ |p|<r\},\quad \partial\mathcal{B}_r=\{p\in\mathbb{R}^d:\ |p|=r\},\\ &\bar{\mathcal{B}}_r=\mathcal{B}_r\cup\partial\mathcal{B}_r,\quad \text{where}\ \ r>0.\end{aligned} \tag{5.1}$$

In addition to Θ of (3.5), we consider, in particular, the following its subsets:

$$\begin{aligned}&\Theta_\rho=\{(k,l)\in\Theta:\ |Im\,k|=|Im\,l|<\rho\},\\ &b\Theta_\rho=\{(k,l)\in\Theta:\ |Im\,k|=|Im\,l|=\rho\},\\ &\bar\Theta_\rho=\Theta_\rho\cup b\Theta_\rho,\\ &\Theta^\infty_{\rho,\tau}=\{(k,l)\in\Theta\backslash\bar\Theta_\rho:\ k-l\in\mathcal{B}_{2\rho\tau}\},\\ &b\Theta_{\rho,\tau}=\{(k,l)\in b\Theta_\rho:\ k-l\in\mathcal{B}_{2\rho\tau}\},\end{aligned} \tag{5.2}$$

where $\rho>0$, $0<\tau<1$, and $\mathcal{B}_r$ is defined in (5.1).

In addition to Ω of (3.14), we consider, in particular, the following its subsets:

$$\begin{aligned}&\Omega_\rho=\{(k,p)\in\Omega:\ |Im\,k|<\rho\},\\ &b\Omega_\rho=\{(k,p)\in\Omega:\ |Im\,k|=\rho\},\\ &\bar\Omega_\rho=\Omega_\rho\cup b\Omega_\rho,\\ &\Omega^\infty_{\rho,\tau}=\{(k,p)\in\Omega\backslash\bar\Omega_\rho:\ p\in\mathcal{B}_{2\rho\tau}\},\\ &b\Omega_{\rho,\tau}=\{(k,p)\in b\Omega_\rho:\ p\in\mathcal{B}_{2\rho\tau}\},\end{aligned} \tag{5.3}$$

where $\rho > 0$, $0 < \tau < 1$, and $\mathcal{B}_r$ is defined in (5.1).

Note that

$$\begin{aligned} &\Omega \approx \Theta,\ \Omega_\rho \approx \Theta_\rho,\ b\Omega_\rho \approx b\Theta_\rho, \\ &\Omega^\infty_{\rho,\tau} \approx \Theta^\infty_{\rho,\tau},\ b\Omega_{\rho,\tau} \approx b\Theta_{\rho,\tau}, \end{aligned} \tag{5.4}$$

or, more precisely,

$$\begin{aligned} &(k,p) \in \Omega \Rightarrow (k, k-p) \in \Theta,\quad (k,l) \in \Theta \Rightarrow (k, k-l) \in \Omega \\ &\text{and the same for}\ \ \Omega_\rho,\ b\Omega_\rho,\ \Omega^\infty_{\rho,\tau},\ b\Omega_{\rho,\tau} \\ &\text{and}\ \ \Theta_\rho,\ b\Theta_\rho,\ \Theta^\infty_{\rho,\tau},\ b\Theta_{\rho,\tau},\ \ \text{respectively, in place of}\ \ \Omega\ \ \text{and}\ \ \Theta. \end{aligned} \tag{5.5}$$

We consider also, in particular,

$$\begin{aligned} &\Omega_\nu = \{(k,p) \in \Omega :\ p \notin \mathcal{L}_\nu\}, \\ &\Omega^\infty_{\rho,\tau,\nu} = \Omega^\infty_{\rho,\tau} \cap \Omega_\nu,\ b\Omega_{\rho,\tau,\nu} = b\Omega_{\rho,\tau} \cap \Omega_\nu, \end{aligned} \tag{5.6}$$

where

$$\mathcal{L}_\nu = \{p \in \mathbb{R}^d :\ p = t\nu,\ t \in \mathbb{R}\}, \tag{5.7}$$

$\nu \in \mathbb{S}^{d-1}$, $\rho > 0$, $0 < \tau < 1$.

5.2. Coordinates on Ω for $d = 3$. In this subsection we assume that $d = 3$ in formulas (3.5), (3.14), (5.1)-(5.7).

For $p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu$ we consider $\theta(p)$ and $\omega(p)$ such that

$$\begin{aligned} &\theta(p), \omega(p)\ \ \text{smoothly depend on}\ \ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu, \\ &\text{take values in}\ \ \mathbb{S}^2,\ \ \text{and} \\ &\theta(p)p = 0,\ \omega(p)p = 0,\ \theta(p)\omega(p) = 0, \end{aligned} \tag{5.8}$$

where $\mathcal{L}_\nu$ is defined by (5.7) (for $d = 3$).

Assumptions (5.8) imply that

$$\omega(p) = \frac{p \times \theta(p)}{|p|}\ \ \text{for}\ \ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu \tag{5.9a}$$

or

$$\omega(p) = -\frac{p \times \theta(p)}{|p|}\ \ \text{for}\ \ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu, \tag{5.9b}$$

where $\times$ denotes vector product.

To satisfy (5.8), (5.9a) we can take

$$\theta(p) = \frac{\nu \times p}{|\nu \times p|},\ \omega(p) = \frac{p \times \theta(p)}{|p|},\ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu. \tag{5.10}$$

Let θ, ω satisfy (5.8). Then (according to [19]) the following formulas give a diffeomorphism between Ω_ν and $(\mathbb{C}\backslash 0) \times (\mathbb{R}^3 \backslash \mathcal{L}_\nu)$:

$$(k,p) \to (\lambda, p),\ \ \text{where}\ \ \lambda = \lambda(k,p) = \frac{2k(\theta(p) + i\omega(p))}{i|p|}, \tag{5.11a}$$

$$
\begin{aligned}
&(\lambda,p)\to(k,p), \quad \text{where} \;\; k=k(\lambda,p)=\kappa_1(\lambda,p)\theta(p)+\kappa_2(\lambda,p)\omega(p)+\frac{p}{2},\\
&\kappa_1(\lambda,p)=\frac{i|p|}{4}(\lambda+\frac{1}{\lambda}), \quad \kappa_2(\lambda,p)=\frac{|p|}{4}(\lambda-\frac{1}{\lambda}),
\end{aligned}
\tag{5.11b}
$$

where $(k,p)\in\Omega_\nu$, $(\lambda,p)\in(\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu)$. In addition, formulas (5.11a), (5.11b) for $\lambda(k)$ and $k(\lambda)$ at fixed $p\in\mathbb{R}^3\backslash\mathcal{L}_\nu$ give a diffeomorphism between $Z_p=\{k\in\mathbb{C}^3:\ (k,p)\in\Omega\}$ for fixed p and $\mathbb{C}\backslash 0$.

In addition, for k and λ of (5.11) we have that

$$
|Im\, k|=\frac{|p|}{4}\big(|\lambda|+\frac{1}{|\lambda|}\big), \quad |Re\, k|=\frac{|p|}{4}\big(|\lambda|+\frac{1}{|\lambda|}\big),
\tag{5.12}
$$

where $(k,p)\in\Omega_\nu$, $(\lambda,p)\in(\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu)$.

Let

$$
\Lambda_{\rho,\nu}=\{(\lambda,p):\ \lambda\in\mathcal{D}_{\rho/|p|},\ \ p\in\mathbb{R}^3\backslash\mathcal{L}_\nu\},
\tag{5.13}
$$

$$
\Lambda_{\rho,\tau,\nu}=\{(\lambda,p):\ \lambda\in\mathcal{D}_{\rho/|p|},\ \ p\in\mathbb{R}^3\backslash\mathcal{L}_\nu,\ \ |p|<2\tau\rho\},
$$

$$
b\Lambda_{\rho,\tau,\nu}=\{(\lambda,p):\ \lambda\in\mathcal{T}_{\rho/|p|},\ \ p\in\mathbb{R}^3\backslash\mathcal{L}_\nu,\ \ |p|<2\tau\rho\},
$$

where $\rho>0$, $0<\tau<1$, $\nu\in\mathbb{S}^2$,

$$
\mathcal{D}_r=\{\lambda\in\mathbb{C}\backslash 0:\ \frac{1}{4}(|\lambda|+|\lambda|^{-1})>r\},\ r>0,
\tag{5.14}
$$

$$
\mathcal{T}_r=\{\lambda\in\mathbb{C}:\ \frac{1}{4}(|\lambda|+|\lambda|^{-1})=r\},\ r\geq 1/2,
\tag{5.15}
$$

$\mathcal{L}_\nu$ is defined by (5.7) (for $d=3$).

Note that

$$
\begin{aligned}
&\Lambda_{\rho,\tau,\nu}=\Lambda^+_{\rho,\tau,\nu}\cup\Lambda^-_{\rho,\tau,\nu},\ \Lambda^+_{\rho,\tau,\nu}\cap\Lambda^-_{\rho,\tau,\nu}=\emptyset,\\
&b\Lambda_{\rho,\tau,\nu}=b\Lambda^+_{\rho,\tau,\nu}\cup b\Lambda^-_{\rho,\tau,\nu},
\end{aligned}
\tag{5.16}
$$

where

$$
\begin{aligned}
&\Lambda^\pm_{\rho,\tau,\nu}=\{(\lambda,p):\ \ \lambda\in\mathcal{D}^\pm_{\rho/|p|},\ \ p\in\mathcal{B}_{2\tau\rho}\backslash\mathcal{L}_\nu\},\\
&b\Lambda^\pm_{\rho,\tau,\nu}=\{(\lambda,p):\ \ \lambda\in\mathcal{T}^\pm_{\rho/|p|},\ \ p\in\mathcal{B}_{2\tau\rho}\backslash\mathcal{L}_\nu\},
\end{aligned}
\tag{5.17}
$$

$$
\begin{aligned}
&\mathcal{D}^\pm_r=\{\lambda\in\mathbb{C}\backslash 0:\ \ \frac{1}{4}(|\lambda|+|\lambda|^{-1})>r,\ \ |\lambda|^{\pm1}<1\},\\
&\mathcal{T}^\pm_r=\{\lambda\in\mathbb{C}:\ \ \frac{1}{4}(|\lambda|+|\lambda|^{-1})=r,\ \ |\lambda|^{\pm1}\leq 1\},\ r>1/2,
\end{aligned}
\tag{5.18}
$$

where $\rho>0$, $\tau\in]0,1[$, $\nu\in\mathbb{S}^2$.

Using (5.12) one can see that formulas (5.11) give also the following diffeomorphisms

$$
\begin{aligned}
&\Omega_\nu\backslash\bar\Omega_\rho\approx\Lambda_{\rho,\nu},\ \ \Omega^\infty_{\rho,\tau,\nu}\approx\Lambda_{\rho,\tau,\nu},\\
&b\Omega_{\rho,\tau,\nu}\approx b\Lambda_{\rho,\tau,\nu},\\
&Z^\infty_{p,\rho}=\{k\in\mathbb{C}^3:\ (k,p)\in\Omega_\nu\backslash\bar\Omega_\rho\}\approx\mathcal{D}_{\rho/|p|}\ \text{ for fixed }\ p,
\end{aligned}
\tag{5.19}
$$

where $\rho > 0$, $0 < \tau < 1$, $\nu \in \mathbb{S}^2$ (and where we use the definitions (5.3), (5.6), (5.13)).

In [19] λ, p of (5.11) were used as coordinates on Ω. In the present work we use them also as coordinates on $\Omega\backslash\Omega_\rho$ (or more precisely on $\Omega_\nu\backslash\Omega_\rho$).

6. An integral equation of [20] and some related formulas

In the main considerations of [20] it is assumed that $d = 3$ and the basic assumption on v consists in the following condition on its Fourier transform:

$$\hat{v} \in L^\infty_\mu(\mathbb{R}^3) \cap \mathcal{C}(\mathbb{R}^3) \ \text{ for some real } \ \mu \geq 2, \tag{6.1}$$

where $\hat{v}$ is defined by (3.12) (for $d = 3$),

$$\begin{aligned} &L^\infty_\mu(\mathbb{R}^d) = \{u \in L^\infty(\mathbb{R}^d) : \ \ \|u\|_\mu < +\infty\}, \\ &\|u\|_\mu = ess \sup_{p \in \mathbb{R}^d} (1 + |p|)^\mu |u(p)|, \ \ \mu > 0, \end{aligned} \tag{6.2}$$

and $\mathcal{C}$ denotes the space of continuous functions.

Note that

$$\begin{aligned} &v \in W^{m,1}(\mathbb{R}^d) \Longrightarrow \hat{v} \in L^\infty_\mu(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d), \\ &\|\hat{v}\|_\mu \leq c_4(m, d)\|v\|_{m,1} \ \text{ for } \ \mu = m, \end{aligned} \tag{6.3}$$

where $W^{m,1}$, L^∞_μ are the spaces of (2.2), (6.2).

Let

$$H(\lambda, p) = H(k(\lambda, p), p), \ \ (\lambda, p) \in (\mathbb{C}\backslash 0) \times (\mathbb{R}^3 \backslash \mathcal{L}_\mu), \tag{6.4}$$

where H is the function of (3.13), λ, p are the coordinates of Subsection 5.2 under assumption (5.9a).

Let

$$\begin{aligned} &L^\infty_\mu(\Lambda_{\rho,\tau,\nu}) = \{U \in L^\infty(\Lambda_{\rho,\tau,\nu}) : \ \ |||U|||_{\rho,\tau,\mu} < \infty\}, \\ &|||U|||_{\rho,\tau,\mu} = ess \sup_{(\lambda,p) \in \Lambda_{\rho,\tau,\nu}} (1 + |p|)^\mu |U(\lambda, p)|, \ \ \mu > 0, \end{aligned} \tag{6.5}$$

where $\Lambda_{\rho,\tau,\nu}$ is defined in (5.13), $\rho > 0$, $\tau \in]0, 1[$, $\nu \in \mathbb{S}^2$, $\mu > 0$.

Let v satisfy (6.1) and $\|\hat{v}\|_\mu \leq C$. Let

$$\eta(C, \rho, \mu) \stackrel{\text{def}}{=} a(\mu)C(\ln \rho)^2 \rho^{-1} < 1, \ \ln \rho \geq 2, \tag{6.6}$$

where $a(\mu)$ is the constant $c_2(\mu)$ of [20]. Let $H(\lambda, p)$ be defined by (6.4) and be considered as a function on $\Lambda_{\rho,\tau,\nu}$ of (5.13). Then (see Section 4 of [20]):

$$H = H^0 + M_{\rho,\tau}(H) + Q_{\rho,\tau}, \ \ \tau \in]0, 1[, \tag{6.7}$$

where

$$H^0(\lambda, p) = \frac{1}{2\pi i} \int\limits_{\mathcal{T}^+_{\rho/|p|}} H(\zeta, p) \frac{d\zeta}{\zeta - \lambda}, \ (\lambda, p) \in \Lambda^+_{\rho,\tau,\nu}, \tag{6.8a}$$

$$H^0(\lambda, p) = -\frac{1}{2\pi i} \int\limits_{\mathcal{T}^-_{\rho/|p|}} H(\zeta, p) \frac{\lambda d\zeta}{\zeta(\zeta - \lambda)}, \ (\lambda, p) \in \Lambda^-_{\rho,\tau,\nu}, \tag{6.8b}$$

where $\Lambda^{\pm}_{\rho,\tau,\nu}$, $\mathcal{T}^{\pm}_r$ are defined in (5.17), (5.18) (and where the integrals along $\mathcal{T}^{\pm}_r$ are taken in the counter-clock wise direction);

$$M_{\rho,\tau}(U)(\lambda,p) = M^{+}_{\rho,\tau}(U)(\lambda,p) = \\ -\frac{1}{\pi}\int\int_{\mathcal{D}^{+}_{\rho/|p|}} (U,U)_{\rho,\tau}(\zeta,p)\frac{dRe\,\zeta dIm\,\zeta}{\zeta-\lambda},\ (\lambda,p)\in\Lambda^{+}_{\rho,\tau,\nu}, \tag{6.9a}$$

$$M_{\rho,\tau}(U)(\lambda,p) = M^{-}_{\rho,\tau}(U)(\lambda,p) = \\ -\frac{1}{\pi}\int\int_{\mathcal{D}^{-}_{\rho/|p|}} (U,U)_{\rho,\tau}(\zeta,p)\frac{\lambda dRe\,\zeta dIm\,\zeta}{\zeta(\zeta-\lambda)},\ (\lambda,p)\in\Lambda^{-}_{\rho,\tau,\nu}, \tag{6.9b}$$

$$\begin{aligned}&(U_1,U_2)_{\rho,\tau}(\zeta,p) = \{\chi_{2\tau\rho}U_1',\chi_{2\tau\rho}U_2'\}(\zeta,p),\ (\zeta,p)\in\Lambda_{\rho,\tau,\nu},\\ &\chi_{2\tau\rho}U_j'(k,p) = U_j(\lambda(k,p),p),\ (k,p)\in\Omega^{\infty}_{\rho,\tau,\nu},\\ &\chi_{2\tau\rho}U_j'(k,p) = 0,\quad |p|\ge 2\tau\rho,\quad j=1,2,\end{aligned} \tag{6.10}$$

where U,U_1,U_2 are test functions on $\Lambda_{\rho,\tau,\nu}$, $\Omega^{\infty}_{\rho,\tau,\nu}$ is defined in (5.6), $\lambda(k,p)$ is defined in (5.11a), $\{\cdot,\cdot\}$ is defined by the formula

$$\begin{aligned}\{F_1,F_2\}(\lambda,p) = -\frac{\pi}{4}\int_{-\pi}^{\pi}\Big(\frac{|p|}{2}\frac{|\lambda|^2-1}{\bar\lambda|\lambda|}(\cos\varphi-1)-\frac{|p|}{\bar\lambda}\sin\varphi\Big)\times\\ F_1(k(\lambda,p),-\xi(\lambda,p,\varphi))F_2(k(\lambda,p)+\xi(\lambda,p,\varphi),p+\xi(\lambda,p,\varphi))d\varphi,\end{aligned} \tag{6.11}$$

for $(\lambda,p)\in\Lambda_{\rho,\nu}$, where F_1,F_2 are test functions on $\Omega\backslash\bar\Omega_\rho$, $k(\lambda,p)$ is defined in (5.11b), $\Lambda_{\rho,\nu}$ is defined in (5.13),

$$\xi(\lambda,p,\varphi) = Re\,k(\lambda,p)(\cos\varphi-1)+k^{\perp}(\lambda,p)\sin\varphi, \tag{6.12}$$

$$k^{\perp}(\lambda,p) = \frac{Im\,k(\lambda,p)\times Re\,k(\lambda,p)}{|Im\,k(\lambda,p)|}, \tag{6.13}$$

where $\times$ in (6.13) denotes vector product;

$$H, H^0, Q_{\rho,\tau}\in L^{\infty}_{\mu}(\Lambda_{\rho,\tau,\nu}), \tag{6.14}$$

$$|||H|||_{\rho,\tau,\mu_0} \le \frac{C}{1-\eta(C,\rho,\mu)}, \tag{6.15a}$$

$$|||H^0|||_{\rho,\tau,\mu_0} \le \frac{C}{1-\eta(C,\rho,\mu)}\Big(1+\frac{c_5(\mu_0)C}{1-\eta(C,\rho,\mu)}\Big), \tag{6.15b}$$

$$|||Q_{\rho,\tau}|||_{\rho,\tau,\mu_0} \le \frac{3c_5(\mu_0)C^2}{(1-\eta(C,\rho,\mu))^2(1+2\tau\rho)^{\mu-\mu_0}}, \tag{6.15c}$$

where $2\le\mu_0\le\mu$, c_5 is the constant b_4 of [20], $\eta(C,\rho,\mu)$ is defined by (6.6).

Following [20] we consider (6.7) as an approxiate integral equation for finding H on $\Lambda_{\rho,\tau,\nu}$ from H^0 on $\Lambda_{\rho,\tau,\nu}$ with unknown remainder $Q_{\rho,\tau}$.

Note also that if $\hat{v}$ satisfies (6.1), then (see [19], [20])

$$\begin{aligned} &H(\lambda,p)\to\hat{v}(p) \ \text{ as } \ \lambda\to 0,\\ &H(\lambda,p)\to\hat{v}(p) \ \text{ as } \ \lambda\to\infty, \end{aligned} \tag{6.16}$$

where $p\in\mathcal{B}_{2\tau\rho}\backslash\mathcal{L}_\nu$, H is defined by (6.4) and is considered as a function on $\Lambda_{\rho,\tau,\nu}$, $\rho>0$, $0<\tau<1$, $\nu\in\mathbb{S}^2$.

7. Proof of Theorem 2.2 for $d=3$

Lemma 7.1. *Let $\hat{v}_i$ satisfy (6.1) and $\|\hat{v}_i\|_\mu<C$, where $i=1,2$. Let*

$$0<\tau\le\tau_1(\mu,\mu_0,C,\delta),\ \ \rho\ge\rho_1(\mu,\mu_0,C,\delta), \tag{7.1}$$

where τ_1,ρ_1 are the constants of Section 4 of [20] and where $\delta=1/2$, $2\le\mu_0<\mu$. Then

$$|||H_2-H_1|||_{\rho,\tau,\mu_0}\le 2(|||H_2^0-H_1^0|||_{\rho,\tau,\mu_0}+|||Q^2_{\rho,\tau}-Q^1_{\rho,\tau}|||_{\rho,\tau,\mu_0}), \tag{7.2}$$

where H_i, H_i^0, $Q^i_{\rho,\tau}$ are the functions of (6.4), (6.7), (6.8), (6.14), (6.15) for $v=v_i$, $i=1,2$. In addition,

$$|||Q^2_{\rho,\tau}-Q^1_{\rho,\tau}|||_{\rho,\tau,\mu_0}\le\frac{24c_5(\mu_0)C^2}{(1+2\tau\rho)^{\mu-\mu_0}}. \tag{7.3}$$

In connection with (7.1) we remind that $\tau_1\in]0,1[$ is sufficiently small and ρ_1 is sufficiently great, see [20].

Lemma 7.1 follows from estimates mentioned as estimates (6.14), (6.15) of the present paper (see estimates (3.3), (4.20), (4.22) of [20]) and from Lemmas 4.4, 4.5 and estimates (4.36) of [20].

Lemma 7.2. *Let $\hat{v}_i$ satisfy (6.1) and $\|\hat{v}_i\|_\mu<C$, where $i=1,2$. Let*

$$\eta(C,\rho_0,\mu)\le 1/2,\quad \ln\rho_0\ge 2, \tag{7.4}$$

where η is defined in (6.6). Let

$$0<\tau_0<1,\quad 2\le\mu_0<\mu,\quad \rho=2\rho_0,\quad \tau=\tau_0/2.$$

Then

$$|||H_2^0-H_1^0|||_{\rho,\tau,\mu_0}\le(c_6+4c_7(\mu_0,\tau_0,\rho_0)C)|||\chi_{\rho_0,\tau_0,\rho,\tau}(H_2-H_1)|||_{\rho_0,\tau_0,\mu_0}, \tag{7.5}$$

where H_i, H_i^0 are the functions of (6.4), (6.8) for $v=v_i$, $i=1,2$, $\chi_{\rho_0,\tau_0,\rho,\tau}$ is the characteristic function of $\Lambda_{\rho_0,\tau_0,\nu}\backslash\Lambda_{\rho,\tau,\nu}$, c_6 is defined by (8.9), c_7 is the constant c_8 of [20] (that is $c_7(\mu,\tau,\rho)=3b_1(\mu)\tau^2+4b_2(\mu)\rho^{-1}+4b_3(\mu)\tau$, where b_1, b_2, b_3 are the constants of [20]).

Note that

$$\Lambda_{\rho,\tau,\nu} \subset \Lambda_{\rho_0,\tau_0,\nu} \ \ \textit{under the assumptions that} \ \ \rho = 2\rho_0,\ \tau = \tau_0/2,\ 0 < \tau_0 < 1,\ \rho > 0. \tag{7.6}$$

Lemma 7.2 is proved in Section 8.

Lemma 7.3. *Let the assumptions of Theorem* 2.1 *hold (for* $d = 3$*). Let*

$$\rho_0 \geq r_1(N, D, m, \sigma) \ \ \textit{for some} \ \ \sigma > 1, \tag{7.7}$$

where r_1 *is the number of* (3.9)*. Let* $0 < \tau_0 < 1$, $0 < \mu_0$, $\rho = 2\rho_0$, $\tau = \tau_0/2$*. Then*

$$|||\chi_{\rho_0,\tau_0,\rho,\tau}(H_2 - H_1)|||_{\rho_0,\tau_0,\mu_0} \leq c_8\sigma^2 e^{2\rho L}\|\Phi_2 - \Phi_1\|(1+\rho)^{\mu_0}, \tag{7.8}$$

where

$$c_8 = (2\pi)^{-d}\int\limits_{\partial D} dx, \quad L = \max_{x\in\partial D}|x|, \tag{7.9}$$

$\|\Phi_2 - \Phi_1\|$ *is defined according to* (2.4), $\chi_{\rho_0,\tau_0,\rho,\tau}$, H_1, H_2 *are the same that in* (7.5).
Lemma 7.3 is proved in Section 8.

Lemma 7.4. *Let the assumptions of Theorem* 2.1 *hold (for* $d = 3$*). Let*

$$0 < \tau \leq \tau_2(m, \mu_0, N), \quad \rho \geq \rho_2(m, \mu_0, N, D, \sigma), \tag{7.10}$$

where $2 \leq \mu_0 < m$, $\sigma > 1$, *and* τ_2, ρ_2 *are constants such that* (7.10) *implies that*

$$\tau \leq \tau_1(m, \mu_0, c_4(m,3)N, 1/2), \quad \rho \geq \rho_1(m, \mu_0, c_4(m,3)N, 1/2), \tag{7.11a}$$
$$\tau < 1/2,\ \eta(c_4(m,3)N, \rho/2, m) \leq 1/2,\ \ln(\rho/2) \geq 2, \tag{7.11b}$$
$$\rho/2 \geq r_1(N, D, m, \sigma), \tag{7.11c}$$

where τ_1, ρ_1, η, r_1 *are the same that in* (7.1), (7.4), (7.7), c_4 *is the constant of* (6.3)*. Then*

$$|||H_2 - H_1|||_{\rho,\tau,\mu_0} \leq c_9(N, D, m, \mu_0, \sigma, \tau)e^{2L\rho}\rho^{\mu_0}\|\Phi_2 - \Phi_1\| + c_{10}(N, m, \mu_0, \tau)\rho^{-(m-\mu_0)}, \tag{7.12}$$

where c_9, c_{10} *are some constants which can be given explicitly.*

Lemma 7.4 follows from formula (6.3) and Lemmas 7.1, 7.2, 7.3.

The final part of the proof of Theorem 2.2 for $d = 3$ consists of the following. Under the assumptions of Lemma 7.4 for $\mu_0 = 2$, we have that

$$\begin{aligned} &\|v_1 - v_2\|_{L^\infty(D)} \leq |||H_2 - H_1|||_{\rho,\tau,2}\int\limits_{|p|\leq 2\rho\tau}\frac{dp}{(1+|p|)^2} + \\ &2c_4(m,3)N\int\limits_{|p|\geq 2\rho\tau}\frac{dp}{(1+|p|)^m} \leq \\ &8\pi\rho\tau|||H_2 - H_1|||_{\rho,\tau,2} + \frac{8\pi c_4(m,3)N}{(m-3)(2\tau)^{m-3}}\frac{1}{\rho^{m-3}} \leq \\ &c_{11}(N, D, m, \sigma, \tau)e^{2L\rho}\rho^3\|\Phi_2 - \Phi_1\| + c_{12}(N, m, \tau)\rho^{-(m-3)}, \end{aligned} \tag{7.13}$$

where c_{11}, c_{12} are related in a simple way with c_9, c_{10} for $\mu_0 = 2$. To obtain (7.13) we used also (6.3), (6.5), (6.16) and the inverse Fourier transform formula

$$v(x) = \int\limits_{\mathbb{R}^3} e^{-ipx}\hat{v}(p)dp, \quad x \in \mathbb{R}^3. \tag{7.14}$$

Let now

$$\alpha \in]0,1[, \ \beta = \frac{1-\alpha}{2L}, \ \delta = \|\Phi_1 - \Phi_2\|, \ \rho = \beta\ln(1+\delta^{-1}), \tag{7.15}$$

where δ is so small that $\rho \geq \rho_2(m, 2, N, D, \sigma)$, where ρ_2 is the constant of (7.10). Then, due to (7.13),

$$\begin{aligned}
&\|v_1 - v_2\|_{L^\infty(D)} \leq c_{11}(N, D, m, \sigma, \tau)(1+\delta^{-1})^{2L\beta}(\beta\ln(1+\delta^{-1}))^3\delta + \\
&c_{12}(N, D, m, \tau)(\beta\ln(1+\delta^{-1}))^{-(m-3)} = \\
&c_{11}(N, D, m, \sigma, \tau)\beta^3(1+\delta)^{1-\alpha}\delta^\alpha(\ln(1+\delta^{-1}))^3 + \\
&c_{12}(N, D, m, \tau)\beta^{-(m-3)}(\ln(1+\delta^{-1}))^{-(m-3)},
\end{aligned} \tag{7.16}$$

where σ, τ are the same that in (7.10) for $\mu_0 = 2$ and where α, β and δ are the same that in (7.15).

Using (7.16) we obtain that

$$\|v_1 - v_2\|_{L^\infty(D)} \leq c_{13}(N, D, m)(\ln(1 + \|\Phi_1 - \Phi_2\|^{-1}))^{-(m-3)} \tag{7.17}$$

for $\delta = \|\Phi_1 - \Phi_2\| \leq \delta_0(N, D, m)$, where δ_0 is sufficiently small positive constant. Estimate (7.17) in the general case (with modified c_{13}) follows from (7.17) for $\delta = \|\Phi_1 - \Phi_2\| \leq \delta_0(N, D, m)$ and the property that $\|v_j\|_{L^\infty(D)} \leq c_{14}(m)N$ (for $d = 3$).

Thus, Theorem 2.2 for $d = 3$ is proved.

8. Proofs of Lemmas 7.2 and 7.3

Proof of Lemma 7.2. Using the maximum principle for holomorphic functions it is sufficient to prove that

$$\begin{aligned}
&\sup_{(\lambda,p)\in b\Lambda^{\pm}_{\rho,\tau,\nu}} (1+|p|)^{\mu_0}|H_2^0(\lambda(1\mp 0),p) - H_1^0(\lambda(1\mp 0),p)| \leq \\
&(c_6 + 4c_7(\mu_0, \tau_0, \rho_0)C)|||\chi_{\rho_0,\tau_0,\rho,\tau}(H_2 - H_1)|||_{\rho_0,\tau_0,\mu_0},
\end{aligned} \tag{8.1}$$

where $b\Lambda^+_{\rho,\tau,\nu}$, $b\Lambda^-_{\rho,\tau,\nu}$ are defined in (5.17) (and where $H_i^0(\lambda(1-0),p)$, $i = 1,2$, are considered for $(\lambda,p) \in b\Lambda^+_{\rho,\tau,\nu}$, $H_i^0(\lambda(1+0),p)$, $i = 1,2$, are considered for $(\lambda,p) \in b\Lambda^-_{\rho,\tau,\nu}$).

Using (6.8) and the Sohotsky-Plemelj formula, we have that

$$H^0(\lambda(1-0),p) = \frac{1}{2\pi i}\int\limits_{\mathcal{T}^+_{\rho/|p|}} H(\zeta,p)\frac{d\zeta}{\zeta - \lambda(1+0)} + H(\lambda,p), \ (\lambda,p) \in b\Lambda^+_{\rho,\tau,\nu}, \tag{8.2a}$$

$$H^0(\lambda(1+0),p) = -\frac{1}{2\pi i}\int\limits_{\mathcal{T}^-_{\rho/|p|}} H(\zeta,p)\frac{\lambda d\zeta}{\zeta(\zeta - \lambda(1-0))} + H(\lambda,p), \ (\lambda,p) \in b\Lambda^-_{\rho,\tau,\nu} \tag{8.2b}$$

In addition, using the Cauchy-Green formula we have that

$$H(\lambda,p) = -\frac{1}{2\pi i}\int\limits_{\mathcal{T}^+_{\rho/|p|}} H(\zeta,p)\frac{d\zeta}{\zeta-\lambda(1+0)} + \frac{1}{2\pi i}\int\limits_{\mathcal{T}^+_{\rho/|p|}} H(\zeta,p)\frac{d\zeta}{\zeta-\lambda} - \tag{8.3a}$$

$$\frac{1}{\pi}\int\int\limits_{\mathcal{D}^+_{\rho_0/|p|}\setminus\mathcal{D}^+_{\rho/|p|}} \frac{\partial H(\zeta,p)}{\partial\bar\zeta}\frac{dRe\zeta dIm\zeta}{\zeta-\lambda},\ (\lambda,p)\in b\Lambda^+_{\rho,\tau,\nu},$$

$$H(\lambda,p) = \frac{1}{2\pi i}\int\limits_{\mathcal{T}^-_{\rho/|p|}} H(\zeta,p)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda(1-0))} - \frac{1}{2\pi i}\int\limits_{\mathcal{T}^-_{\rho/|p|}} H(\zeta,p)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda)} - \tag{8.3b}$$

$$\frac{1}{\pi}\int\int\limits_{\mathcal{D}^-_{\rho_0/|p|}\setminus\mathcal{D}^-_{\rho/|p|}} \frac{\partial H(\zeta,p)}{\partial\bar\zeta}\frac{\lambda dRe\zeta dIm\zeta}{\zeta(\zeta-\lambda)},\ (\lambda,p)\in b\Lambda^-_{\rho,\tau,\nu}$$

(where the integrals along $\mathcal{T}^\pm_r$ are taken in the counter- clockwise direction). In addition (see formulas (3.22), (3.23), (4.8), (4.14) of [20]),

$$\frac{\partial H(\zeta,p)}{\partial\bar\zeta} = (H,H)_{\rho_0,\tau_0}(\zeta,p),\ (\zeta,p)\in\Lambda_{\rho_0,\tau_0,\nu}, \tag{8.4}$$

where $(\cdot,\cdot)_{\rho,\tau}$ is defined by (6.10).

Using (8.2), (8.3) we obtain that

$$H^0(\lambda(1-0),p) = \frac{1}{2\pi i}\int\limits_{\mathcal{T}^+_{\rho_0/|p|}} H(\zeta,p)\frac{d\zeta}{\zeta-\lambda} - \tag{8.5a}$$

$$\frac{1}{\pi}\int\int\limits_{\mathcal{D}^+_{\rho_0/|p|}\setminus\mathcal{D}^+_{\rho/|p|}} \frac{\partial H(\zeta,p)}{\partial\bar\zeta}\frac{dRe\zeta dIm\zeta}{\zeta-\lambda},\ (\lambda,p)\in b\Lambda^+_{\rho,\tau,\nu},$$

$$H^0(\lambda(1+0),p) = -\frac{1}{2\pi i}\int\limits_{\mathcal{T}^-_{\rho_0/|p|}} H(\zeta,p)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda)} - \tag{8.5b}$$

$$\frac{1}{\pi}\int\int\limits_{\mathcal{D}^-_{\rho_0/|p|}\setminus\mathcal{D}^-_{\rho/|p|}} \frac{\partial H(\zeta,p)}{\partial\bar\zeta}\frac{\lambda dRe\zeta dIm\zeta}{\zeta(\zeta-\lambda)},\ (\lambda,p)\in b\Lambda^-_{\rho,\tau,\nu}.$$

Using (8.5), (8.4) for $H^0 = H^0_i$, $H = H_i$, $i = 1,2$, we obtain that:

$$(H^0_2 - H^0_1)(\lambda(1-0),p) = A^+(\lambda,p) + B^+(\lambda,p), \tag{8.6a}$$

$$A^+(\lambda,p) = \frac{1}{2\pi i}\int\limits_{\mathcal{T}^+_{\rho_0/|p|}} (H_2-H_1)(\zeta,p)\frac{d\zeta}{\zeta-\lambda},$$

$$B^+(\lambda,p) = -\frac{1}{\pi}\int\int\limits_{\mathcal{D}^+_{\rho_0/|p|}\setminus\mathcal{D}^+_{\rho/|p|}} \big((H_2-H_1,H_2)_{\rho_0,\tau_0} + (H_1,H_2-H_1)_{\rho_0,\tau_0}\big)(\zeta,p)\frac{dRe\zeta dIm\zeta}{\zeta-\lambda},$$

where $(\lambda,p)\in b\Lambda^+_{\rho,\tau,\nu}$;

$$(H_2^0-H_1^0)(\lambda(1+0),p)=A^-(\lambda,p)+B^-(\lambda,p), \tag{8.6b}$$
$$A^-(\lambda,p)=-\frac{1}{2\pi i}\int\limits_{\mathcal{T}^-_{\rho_0/|p|}}(H_2-H_1)(\zeta,p)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda)},$$
$$B^-(\lambda,p)=-\frac{1}{\pi}\int\int\limits_{\mathcal{D}^-_{\rho_0/|p|}\setminus\mathcal{D}^-_{\rho/|p|}}\big((H_2-H_1,H_2)_{\rho_0,\tau_0}+(H_1,H_2-H_1)_{\rho_0,\tau_0}\big)(\zeta,p)\frac{\lambda dRe\zeta dIm\zeta}{\zeta(\zeta-\lambda)},$$

where $(\lambda,p)\in b\Lambda^-_{\rho,\tau,\nu}$.

Estimates (8.1) follow from formulas (8.6) and from the estimates

$$|A^\pm(\lambda,p)|\le c_6(1+|p|)^{-\mu_0}\Delta,\ \ (\lambda,p)\in b\Lambda^\pm_{\rho,\tau,\nu}, \tag{8.7}$$
$$|B^\pm(\lambda,p)|\le 4c_7(\mu_0,\tau_0,\rho_0)C\Delta,\ \ (\lambda,p)\in b\Lambda^\pm_{\rho,\tau,\nu}, \tag{8.8}$$

where

$$c_6=\sup_{r\in]1/2,+\infty[}\frac{q(r)}{q(r)-q(2r)},\qquad q(r)=2r\big(1-\big(1-\frac{1}{4r^2}\big)^{1/2}\big), \tag{8.9}$$

$$\Delta=|||\chi_{\rho_0,\tau_0,\rho,\tau}(H_2-H_1)|||_{\rho_0,\tau_0,\mu_0}. \tag{8.10}$$

Note that

$$0<c_6\le(2\sqrt{3}-3)^{-1}, \tag{8.11}$$

where c_6 is defined by (8.9). Estimate (8.11) follows from the formulas

$$c_6=\frac{1}{1-2\sigma},\ \ \sigma=\sup_{\tau\in]0,1[}\frac{1-(1-(1/4)\tau)^{1/2}}{1-(1-\tau)^{1/2}}, \tag{8.12}$$

$$(1-\tau)^{1/2}\le 1-(1/2)\tau,\ \ 1-(1/4)\tau\ge a(1-(1/4)\tau)+1-a, \tag{8.13}$$
$$a=2(2-\sqrt{3}),\ \tau\in]0,1[.$$

Estimates (8.7) follow from formula (7.6), the properties that

$$H_i\in\mathcal{C}(\Lambda_{\rho_0,\tau_0,\nu}\cup b\Lambda_{\rho_0,\tau_0,\nu}),\qquad |H_i(\lambda,p)|\le(1+|p|)^{-\mu}|||H_i|||_{\rho_0,\tau_0,\mu},\ \ (\lambda,p)\in\Lambda_{\rho_0,\tau_0,\nu}\cup b\Lambda_{\rho_0,\tau_0,\nu},\ i=1,2, \tag{8.14}$$

(see formulas (3.2), (3.3) of [20] for H and formulas (5.3), (5.6), (5.13), (5.19), (6.4), (6.5) of the present paper for $\Omega^\infty_{\rho,\tau,\nu}$, $\Lambda_{\rho,\tau,\nu}$, H and $L^\infty_\mu(\Lambda_{\rho,\tau,\nu})$) and from the formulas

$$\frac{1}{2\pi}\int\limits_{\mathcal{T}^-_r}\frac{|\lambda||d\zeta|}{|\zeta||\zeta-\lambda|}=\frac{1}{2\pi}\int\limits_{\mathcal{T}^+_r}\frac{|d\zeta'|}{|\zeta'-\lambda'|}\le\frac{q(r)}{q(r)-q(2r)}, \tag{8.15}$$

$\lambda \in \mathcal{T}_{2r}^{-}$, $\lambda' \in \mathcal{T}_{2r}^{+}$, $r > 1/2$. In turn, formulas (8.15) follow from the property that $z^{-1} \in \mathcal{T}_r^{+}$ if $z \in \mathcal{T}_r^{-}$ and from the formula that $q(r)$ is the radius of $\mathcal{T}_r^{+}$, where $r > 1/2$.

Estimates (8.8) follow from the proof of estimates (7.8), (7.9) of [20] and from the formulas (6.15a) (for $\rho = \rho_0$, $\tau = \tau_0$), (7.4), (7.6), (8.14) of the present paper.

Lemma 7.2 is proved.

Proof of Lemma 7.3. Using formulas (6.4), (6.5), the formulas

$$\Lambda_{\rho,\tau,\nu} \approx \Omega^{\infty}_{\rho,\tau,\nu} = \Omega^{\infty}_{\rho,\tau} \cap \Omega_{\nu}, \quad \Omega^{\infty}_{\rho,\tau} \approx \Theta^{\infty}_{\rho,\tau}$$

(see (5.19), (5.6), (5.4)), and formulas (5.2), (3.13), we have that

$$\Delta \le \sup_{\substack{(k,l)\in\bar\Theta_\rho\backslash\Theta_{\rho_0},\\ |k-l|\le 2\tau\rho}} |h_2(k,l) - h_1(k,l)|, \tag{8.16}$$

where Δ is defined by (8.10), h_1, h_2 are the functions of Section 4.

Estimate (7.8) follows from formulas (8.16), (4.1), (3.7), (3.9), the property that $|k-l| \le \rho$ in (8.16) (since $\tau < 1/2$ due to the assumptions of Lemma 7.3) and the property that $|e^{ikx}| \le e^{\rho L}$, $|e^{ilx}| \le e^{\rho L}$ for $k, l \in \bar\Theta_\rho$, $x \in \partial D$.

Lemma 7.3 is proved.

References

[1] G.Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal. **27** (1988), 153-172.

[2] R.Beals and R.R.Coifman, *Multidimensional inverse scattering and nonlinear partial differential equations*, Proc. Symp. Pure Math. **43** (1985), 45-70.

[3] R.M.Brown and G.Uhlmann, *Uniqueness in the inverse conductivity problem for non-smooth conductivities in two dimensions*, Comm. Partial Diff. Eq. **22** (1997), 1009-1027.

[4] A.L.Bukhgeim, *Recovering a potential from Cauchy data in the two-dimensional case*, J.Inverse Ill-Posed Probl. **16**(1) (2008), 19-33.

[5] A.-P.Calderón, *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980), pp.65-73, Soc. Brasil. Mat. Rio de Janeiro, 1980.

[6] L.D.Druskin, *The unique solution of the inverse problem in electrical surveying and electrical well logging for piecewise-constant conductivity*, Physics of the Solid Earth **18**(1) (1982), 51-53.

[7] L.D.Faddeev, *Growing solutions of the Schrödinger equation*, Dokl. Akad. Nauk SSSR **165** (1965), 514-517 (in Russian); English Transl.: Sov. Phys. Dokl. **10** (1966), 1033-1035.

[8] L.D.Faddeev, *Inverse problem of quantum scattering theory II*, Itogi Nauki i Tekhniki, Sovr. Prob. Math. **3** (1974), 93-180 (in Russian); English Transl.: J.Sov. Math. **5** (1976), 334-396.

[9] I.M.Gelfand, *Some problems of functional analysis and algebra*, Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, pp.253-276.

[10] G.M.Henkin and R.G.Novikov, *The $\bar{\partial}$- equation in the multidimensional inverse scattering problem*, Uspekhi Mat. Nauk **42(3)** (1987), 93-152 (in Russian); English Transl.: Russ. Math. Surv. **42(3)** (1987), 109-180.

[11] R.Kohn and M.Vogelius, *Determining conductivity by boundary measurements II, Interior results*, Comm. Pure Appl. Math. **38** (1985), 643-667.

[12] M.Lassas, J.L.Mueller and S.Siltanen, *Mapping properties of the nonlinear Fourier transform in dimension two*, Comm.Partial Differential Equations **32**(4-6) (2007), 591-610.

[13] N.Mandache, *Exponential instability in an inverse problem for the Schrödinger equation*, Inverse Problems **17** (2001), 1435-1444.

[14] A.I.Nachman, *Reconstructions from boundary measurements*, Ann. Math. **128** (1988), 531-576.

[15] A.I.Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem*, Ann, Math. **142** (1995), 71-96.

[16] R.G.Novikov, *Multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$*, Funkt. Anal. i Pril. **22(4)** (1988), 11-22 (in Russian); English Transl.: Funct. Anal. and Appl. **22** (1988), 263-272.

[17] R.G.Novikov, *Formulae and equations for finding scattering data from the Dirichlet-to-Neumann map with nonzero background potential*, Inverse Problems **21** (2005), 257-270.

[18] R.G.Novikov, *The $\bar{\partial}$- approach to approximate inverse scattering at fixed energy in three dimensions*, International Mathematics Research Papers, **2005:6**, (2005), 287-349.

[19] R.G.Novikov, *On non-overdetermined inverse scattering at zero energy in three dimensions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **5** (2006), 279-328

[20] R.G.Novikov, *An effectivization of the global reconstruction in the Gel'fand-Calderon inverse problem in three dimensions*, Contemporary Mathematics, **494** (2009), 161-184.

[21] R.G.Novikov and N.N.Novikova, *On stable determination of potential by boundary measurements*, ESAIM: Proceedings **26** (2009), 94-99.

[22] V.P.Palamodov, *Gabor analysis of the continuum model for impedance tomography*, Ark.Mat. **40** (2002), 169-187.

[23] J.Sylvester and G.Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. Math. **125** (1987), 153-169.