ECOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES APPLIQUÉES

UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 46 00. Fax: 01 69 33 46 46

http://www.cmap.polytechnique.fr/

Interacting particle systems and Yaglom limit approximation of diffusions with unbounded drift

Denis Villemonais

R.I. 684 *May 2010*

Interacting particle systems and Yaglom limit approximation of diffusions with unbounded drift

Denis Villemonais*

May 7, 2010

Abstract

We study the existence and the exponential ergodicity of a general interacting particle system, whose components are driven by independent diffusion processes with values in a bounded open subset of $\mathbb{R}^d$, $d \geq 1$. The interaction occurs when a particle hits the boundary: it jumps to a position chosen with respect to a probability measure depending on the position of the whole system.

Then we study the behavior of such a system when the number of particles goes to infinity. This leads us to an approximation method for the Yaglom limit of multi-dimensional diffusion processes with unbounded drift defined on an unbounded open set. While most of known results on such limits are obtained by spectral theory arguments and are concerned with existence and uniqueness problems, our approximation method allows us to get quantitative information on quasi-stationary distributions, which find applications to many disciplines. We end the paper with numerical illustrations of our approximation method for stochastic processes related to biological populations models.

Key words : diffusion process, interacting particle system, empirical process, quasi-stationary distribution, Yaglom limit.
MSC 2000 subject : Primary 82C22, 65C50, 60K35; secondary 60J60

1 Introduction

Let $D \subset \mathbb{R}^d$ be a bounded open set whose boundary is of class C^2. The first part of this paper is devoted to the study of interacting particle systems $(X^1,...,X^N)$, whose components X^i evolve in D as diffusion processes and jump when they hit the boundary ∂D. More precisely, let $N \geq 2$ be the number of particles in our system. Let us consider N independent d-dimensional Brownian motions $B^1,...,B^N$ and a jump measure $\mathcal{J}^{(N)}$: $\partial(D^N) \mapsto \mathcal{M}_1(D^N)$, where $\mathcal{M}_1(D^N)$ denotes the set of probability measures on D^N. We build the interacting particle system $(X^1,...,X^N)$ with values in D^N as follows. At the beginning, the particles X^i evolve as independent diffusion processes with values in D defined by

$$dX_t^{(i)} = dB_t^i + q_i^{(N)}(X_t^{(i)})dt,\ X_0^{(i)} \in D, \tag{1}$$

*villemonais@cmap.polytechnique.fr,
CMAP, École Polytechnique, route de Saclay, 91128 Palaiseau, France

where $q_i^{(N)}$ is continuous and bounded on D. When a particle hits the boundary, say at time τ_1, it jumps to a position chosen with respect to $\mathcal{J}^{(N)}(X^1_{\tau_1\text{-}},...,X^N_{\tau_n\text{-}})$. Then the particles evolve independently with respect to (1) until one of them hits the boundary and so on. In the whole study, we require the jumping particle to be, in some sense, attracted away from the boundary by the other ones during the jump (see Hypothesis 1 on $\mathcal{J}^{(N)}$ in Section 2.2). We emphasize the fact that the jumping position is allowed to be located strictly closer to the boundary than all other particles and that the diffusion processes which drive the particles between the jumps can depend on the particles. As a consequence, this construction is a generalization of the Fleming-Viot type model introduced in [4] for Brownian particles and in [18] for diffusion particles.

The first step of the study consists in proving that the interacting particle system is well defined for all $t \geq 0$, which means that there is no accumulation of jumps in finite time almost surely. In a second step, we prove that the system is exponentially ergodic. The whole study is made possible by a coupling between $(X^1,...,X^N)$ and a system of N independent 1-dimensional reflected diffusion processes, that we build in Section 2.3.

Assume now that, for all $N \geq 2$, we're given a jump measure $\mathcal{J}^{(N)}$ and a family of drifts $(q_i^{(N)})_{1\leq i\leq N}$ which is uniformly bounded by a constant $Q > 0$ that doesn't depend on N. Assume that the conditions for existence and ergodicity of the associated interacting process are fulfilled for all $N \geq 2$, and let M^N be its stationary distribution. We denote by $\mathcal{X}^N$ the associated empirical stationary distribution, which is defined by $\mathcal{X}^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, where $(x_1,...,x_N) \in D^N$ is distributed following M^N. We prove in Section 2.4 that the family of random measures $\mathcal{X}^N$ is uniformly tight.

In Section 3, we consider the following particular case: $q_i^{(N)} = q$ doesn't depend on i,N and

$$\mathcal{J}^{(N)}(x_1,...,x_N) = \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j},\ x_i \in \partial D. \tag{2}$$

It means that at each jump time, the jumping particle is sent to the position of a particle chosen uniformly between the $N-1$ remaining ones. In that case, we're able to identify the limit of the family of empirical stationary distributions $(\mathcal{X}^N)_{N\geq 2}$. This leads us to an approximation method of the limiting conditional distributions of diffusion processes absorbed at the boundary of an open set of $\mathbb{R}^d$ studied by Cattiaux and Méléard in [6] and defined as follows. Let $D_0 \subset \mathbb{R}^d$ be an open set and $\mathbb{P}^0$ be the law of the diffusion process defined by the SDE

$$dX^0_t = dB_t + \nabla V(X^0_t)dt,\ X^0 \in D_0 \tag{3}$$

and absorbed at the boundary ∂D_0. Here B is a d-dimensional Brownian motion and $V \in C^2(D,\mathbb{R})$. We denote by τ_∂ the absorption time of the diffusion process (3). As proved in [6], the limiting conditional distribution

$$\nu_0 = \lim_{t\to\infty}\mathbb{P}^0_x\left(X^0_t \in .|t < \tau_\partial\right) \tag{4}$$

exists and doesn't depend on $x \in D$, under suitable conditions which allow the drift ∇V and the set D to be unbounded (see Hypothesis 2 in Section 3). This probability is called the Yaglom limit associated with $\mathbb{P}^0$ and it is a quasi-stationary distribution for the diffusion process (3), which means that $\mathbb{P}^0_{\nu_0}(X_t \in dx|t < \tau_\partial) = \nu_0$ for all $t \geq 0$. We refer to [5, 21, 23] and references therein for existence or uniqueness results on such invariant conditional distributions in other settings (and to [27] for an extensive bibliography on quasi-stationary distributions).

Such distributions are an important tool in the theory of Markov processes with absorbing states, which are commonly used in stochastic models of biological populations, epidemics, chemical reactions and market dynamics (see the bibliography [27, Applications]). Indeed, while the long time behavior of a recurrent Markov process is well described by its stationary distribution, the stationary distribution of an absorbed Markov process is concentrated on the absorbing states, which is of poor interest. In contrast, the limiting distribution of the process conditioned to not being absorbed when it is observed can explain some complex behavior, as the mortality plateau at advanced ages (see [1] and [29]), which leads to new applications of Markov processes with absorbing states in biology (see [22]). As stressed in [26], such distributions are in most cases not explicitly computable. In our case, the existence of the Yaglom limit is proved in [6] by spectral theory arguments, which doesn't allow us to get its explicit value. The main motivation of Section 3 is to prove an approximation method of ν_0, even when the drift ∇V and the domain D_0 are unbounded and the boundary ∂D_0 isn't of class C^2.

The approximation method is based on a sequence of interacting particle systems with jumps from the boundary defined with the jump measures (2), for all $N \geq 2$. In the case of a Brownian motion killed at the boundary of a bounded open set (*i.e.* $q = 0$), Burdzy *et al.* conjectured in [3] that the unique limit measure of the sequence $(\mathcal{X}^N)_{N\in\mathbb{N}}$ is the Yaglom limit ν_0. This has been confirmed in the Brownian motion case (see [4], [17] and [24]) and proved in [15] for some Markov processes defined on discrete spaces. New difficulties arise from our case with unbounded drift and unbounded domain D_0. For instance, the interacting particle process introduced above isn't necessarily well defined, since it doesn't fulfill the conditions of Section 2. To avoid this difficulty, we introduce a cut-off of D_0 near its boundary. More precisely, let $(D_\epsilon)_{\epsilon>0}$ be a decreasing family of regular bounded subsets of D_0, such that ∇V is bounded on each D_ϵ and such that $D_0 = \bigcup_{\epsilon>0} D_\epsilon$. We define an interacting particle process $(X^{\epsilon,1},...,X^{\epsilon,N})$ on each subset D_ϵ^N, by setting $q_i^{(N)} = \nabla V$ and $D = D_\epsilon$ in (1). For all $\epsilon > 0$ and $N \geq 2$, $(X^{\epsilon,1},...,X^{\epsilon,N})$ is well defined and exponentially ergodic. Denoting by $\mathcal{X}^{N,\epsilon}$ its empirical stationary distribution, we prove that

$$\lim_{\epsilon\to 0}\lim_{N\to\infty} \mathcal{X}^{N,\epsilon} = \nu_0.$$

We conclude in Section 3.3 by some numerical illustrations of our method applied to the 1-dimensional Wright-Fisher diffusion conditionned to be killed at 0, to the Logistic Feller diffusion and to the 2-dimensional stochastic Lotka-Volterra diffusion.

2 A general interacting particle process with jumps from the boundary

2.1 Construction of the interacting process

Let D be a bounded open subset of $\mathbb{R}^d$, $d \geq 1$, with a boundary of class C^2. Let $N \geq 2$ be fixed. In what follows, we build a system of particles $(X^1,...,X^N)$ with values in D^N, which is càdlàg and whose components jump from the boundary ∂D. Between the jumps, each particle evolves independently of the other ones and follows the law $\mathbb{P}^i$ of the diffusion process defined on D by

$$dX_t^{(i)} = dB_t^i - q_i^{(N)}(X_t^{(i)})dt,\ X_0^{(i)} = x^i \in D \tag{5}$$

and absorbed at the boundary ∂D. Here B^1,...,B^N are N independent d-dimensional Brownian motions and $q_i^{(N)} = (q_{i,1}^{(N)},...,q_{i,d}^{(N)}) \in C(D,\mathbb{R}^d)$ is bounded. The infinitesimal generator associated with the diffusion process (5) will be denoted by $\mathcal{L}^i$, with

$$\mathcal{L}^i = \frac{1}{2}\sum_{j=1}^{d}\frac{\partial^2}{\partial x_j^2} - q_{i,j}^{(N)}\frac{\partial}{\partial x_j} \tag{6}$$

on its domain $\mathcal{D}_{\mathcal{L}^i}$.

For each $i \in \{1,...,N\}$, we set

$$\mathcal{D}_i = \{(x_1,...,x_N) \in \partial(D^N), \text{ such that } x_i \in \partial D, \text{ and}, \forall j \neq i,\ x_j \in D\}.$$

Let $\mathcal{J}^{(N)} : \bigcup_{i=0}^N \mathcal{D}_i \to \mathcal{M}_1(D)$ be the jump measure, which associates a probability measure $\mathcal{J}^{(N)}(x_1,...,x_N)$ on D to each point $(x_1,...,x_N) \in \bigcup_{i=1}^N \mathcal{D}_i$. Let $(X_0^1,...,X_0^N) \in D^N$ be the starting point of the interacting particle process $(X^1,...,X^N)$, which is built as follows:

- Each particle evolves following the SDE (5) independently of the other ones, until one particle, say X^{i_1}, hits the boundary at a time which is denoted by τ_1. In the one hand, we have $\tau_1 > 0$ almost surely, because each particle starts in D. In the other hand, the particle which hits the boundary at time τ_1 is unique, because the particles evolves as independent Itô's diffusion processes in D. It follows that $(X^1_{\tau_1\text{-}},...,X^N_{\tau_1\text{-}})$ belongs to $\mathcal{D}_{i_1}$.

- The position of X^{i_1} at time τ_1 is then chosen with respect to the probability measure $\mathcal{J}^{(N)}(X^1_{\tau_1\text{-}},...,X^N_{\tau_1\text{-}})$.

- At time τ_1 and after proceeding to the jump, all the particles are in D. Then the particles evolve with respect to (5) and independently of each other, until one of them, say X^{i_2}, hits the boundary, at a time which is denoted by τ_2. For the same reason as above, we have $\tau_1 < \tau_2$ and $(X^1_{\tau_2\text{-}},...,X^N_{\tau_2\text{-}}) \in \mathcal{D}_{i_2}$.

- The position of X^{i_2} at time τ_2 is then chosen with respect to the probability measure $\mathcal{J}^{(N)}(X^1_{\tau_2\text{-}},...,X^N_{\tau_2\text{-}})$.

- Then the particles evolve with law $\mathbb{P}^i$ and independently of each other, and so on.

The law of the interacting particle process with initial distribution $m \in \mathcal{M}_1(D^N)$ will be denoted by P_m^N, or by P_x^N if $m = \delta_x$, with $x \in D^N$. The associated expectation will be denoted by E_m^N, or by E_x if $m = \delta_x$.

The sequence of successive jumping particles is denoted by $(i_n)_{n\geq 1}$, and

$$0 < \tau_1 < \tau_2 < ...$$

denotes the strictly increasing sequence of jumping times. We set $\tau_\infty = \lim_{n\to\infty}\tau_n$. The process described above isn't necessarily well defined for all $t \in [0,+\infty[$, and we need more assumptions on the jump measure $\mathcal{J}^{(N)}$ to conclude that $\tau_\infty = \infty$ almost surely.

In the sequel, we denote by $\phi_D(x)$ the Euclidean distance from x to the boundary ∂D, which means that, for all $x \in D$,

$$\phi_D(x) = \inf_{y\in\partial D} \|y - x\|_2.$$

Hypothesis 1. *There exists $p_0^{(N)} > 0$ such that, $\forall i \in \{1,...,N\}$,*

$$\inf_{(x_1,...,x_N)\in\mathcal{D}_i} \mathcal{J}^{(N)}(x_1,...,x_N)(\{y \in D,\ \phi_D(y) \geq \min_{j\neq i}\phi_D(x_j)\}) \geq p_0^{(N)}$$

Informally, we assume that at each jump time τ_n, the probability that the jump position $X^{i_n}_{\tau_n}$ is chosen further from the boundary than at least one another particle is bounded below by a positive constant $p_0^{(N)}$. This assumption ensures that, at each jump, the jumping particle is attracted away from the boundary by the other ones.

Remark 1. Hypothesis 1 is very general and allows a lot of choices for $\mathcal{J}^{(N)}(x_1,...,x_N)$. For instance, for any choice of $\sigma^{(N)} : \bigcup_{i=0}^N \mathcal{D}_i \to \mathcal{M}_1(D)$, the jump measure

$$\mathcal{J}^{(N)}(x_1,...,x_N) = \frac{N-1}{N}\sigma^{(N)}(x_1,...,x_N) + \frac{1}{N(N-1)}\sum_{j=1,...,N,\ j\neq i}\delta_{x_j},\ \forall(x_1,...,x_N)\in\mathcal{D}_i,$$

fulfills the assumption with $p_0^{(N)} = 1/N$.

Hypothesis 1 also includes the case studied by Grigorescu and Kang in [18], where

$$\mathcal{J}^{(N)}(x_1,...,x_N) = \sum_{j\neq i} p_{ij}(x_i)\delta_{x_j},\ \forall(x_1,...,x_N)\in\mathcal{D}_i.$$

with $\sum_{j\neq i} p_{ij}(x_i) = 1$ and $\inf_{i\in\{1,...,N\},j\neq i,x_i\in\partial D} p_{ij}(x_i) > 0$, so that the particle on the boundary jumps to one of the other ones, with positive weights. In that case, Hypothesis 1 is fulfilled with $p_0^{(N)} = 1$. In Section 3, we will focus on the particular case

$$\mathcal{J}^{(N)}(x_1,...,x_N) = \frac{1}{N-1}\sum_{j=1,...,N,\ j\neq i}\delta_{x_j},\ \forall(x_1,...,x_N)\in\mathcal{D}_i.$$

That will lead us to an approximation method of the Yaglom limit (4).

Theorem 2.1. *Assume that Hypothesis 1 is fulfilled. Then*

1. *The process $(X^1,...,X^N)$ is well defined, which means that $\tau_\infty = +\infty$ almost surely.*
2. *Moreover, the process $(X^1,...,X^N)$ is exponentially ergodic, which means that there exists a probability measure M^N on D^N such that,*

$$||P^N_x((X^1_t,...,X^N_t)\in .) - M^N||_{TV} \leq C^{(N)}(x)\left(\rho^{(N)}\right)^t,\ \forall x \in]0,1[^N,\ \forall t\in\mathbb{R}_+,$$

where $C^{(N)}(x)$ is finite, $\rho^{(N)} < 1$ and $||.||_{TV}$ is the total variation norm. In particular, M^N is a stationary measure for the process $(X^1,...,X^N)$.

The main tool of the proof is a coupling between $(X^1,...,X^N)$ and a system of N independent one-dimensional diffusion processes $(Y^1,...,Y^N)$. The system is built in order to satisfy

$$0 \leq Y_t^i \leq \phi_D(X_t^i) \wedge \alpha \text{ a.s.}$$

for all $t \geq 0$ and each $i \in \{1,...,N\}$. We build this coupling in Subsection 2.2 and we conclude the proof of Theorem 2.1 in Subsection 2.3 .

In Subsection 2.4, we assume that, for all $N \geq 2$, we're given $\mathcal{J}^{(N)}$ which fulfills Hypothesis 1 and a family of drifts $(q_i^{(N)})_{1\leq i\leq N}$ which is uniformly bounded above by a constant that doesn't depend on N. We prove that the family of empirical distributions $(\mathcal{X}^N)_{N\geq 2}$ is uniformly tight.

2.2 Coupling's construction

Proposition 2.2. *There exists a N-dimensional Brownian motion $(W^1,...,W^N)$ and positive constants $\alpha, Q > 0$ such that, for each $i \in \{1,...,N\}$, the reflected diffusion process with values in $[0,\alpha]$ defined by*

$$Y_t^i = Y_0^i + W_t^i - Qt + L_t^{i,0} - L_t^{i,\alpha},\ Y_0^i = \min(\alpha,\phi_D(X_0^i)) \tag{7}$$

satisfies

$$0 \leq Y_t^i \leq \phi_D(X_t^i) \wedge \alpha \ a.s. \tag{8}$$

for all $t \in [0,\tau_\infty[$ (see Figure 1). In (7), $L^{i,0}$ (resp. $L^{i,\alpha}$) denotes the local time due to the reflecting property of the boundary $\{0\}$ (resp. $\{\alpha\}$), (cf. [8]).

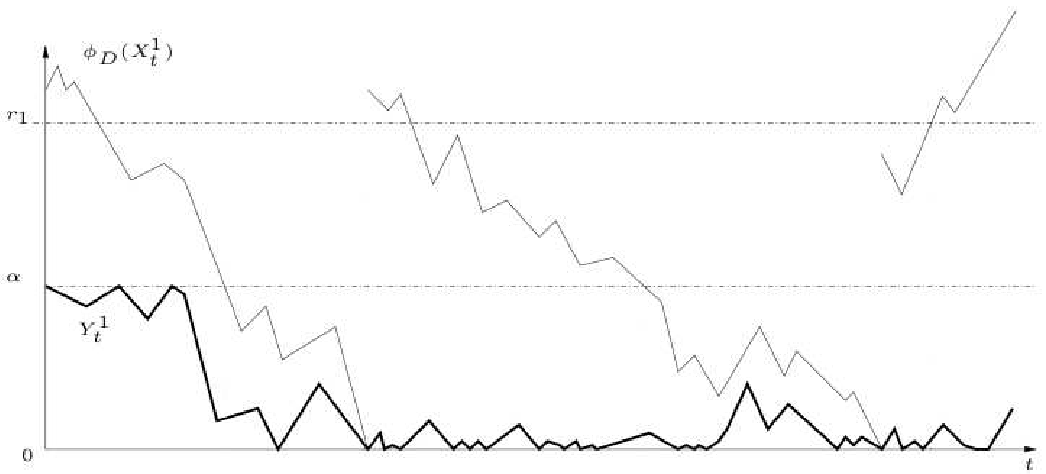

Figure 1: The particle X^1 and its coupled reflected diffusion process Y^1

Proof of Proposition 2.2 : The boundary of D is assumed to be of class C^2, which implies by [11, Theorem 4.3] that there exists a positive constant $r_1 > 0$, such that

$$\phi_D \text{ is of class } C^2 \text{ on } D_{r_1}^c, \tag{9}$$

where $D_r = \{x \in D, \phi_D(x) \geq r\}$. Moreover, we have

$$\|\nabla \phi_D(x)\|_2 = 1, \forall x \in D^c_{r_1}. \tag{10}$$

We set $\alpha = r_1/2$.

Fix $i \in \{1,...,N\}$. We define a sequence of stopping times $(t^i_n)_n$ such that $X^i_t \in D^c_{r_1}$ for all $t \in [t^i_{2n},t^i_{2n+1}[$ and $X^i_t \in D_\alpha$ for all $t \in [t^i_{2n+1},t^i_{2n+2}[$. More precisely, we set (see Figure 2)

$$t^i_0 = \inf\{t \in [0,+\infty],\, X^i_t \in D^c_\alpha\},$$
$$t^i_1 = \inf\{t \in [t_0,+\infty],\, X^i_t \in D_{r_1}\},$$

and, for $n \geq 1$,

$$t^i_{2n} = \inf\{t \in [t^i_{2n-1},+\infty],\, X^i_t \in D^c_\alpha\},$$
$$t^i_{2n+1} = \inf\{t \in [t^i_{2n},+\infty],\, X^i_t \in D_{r_1}\}.$$

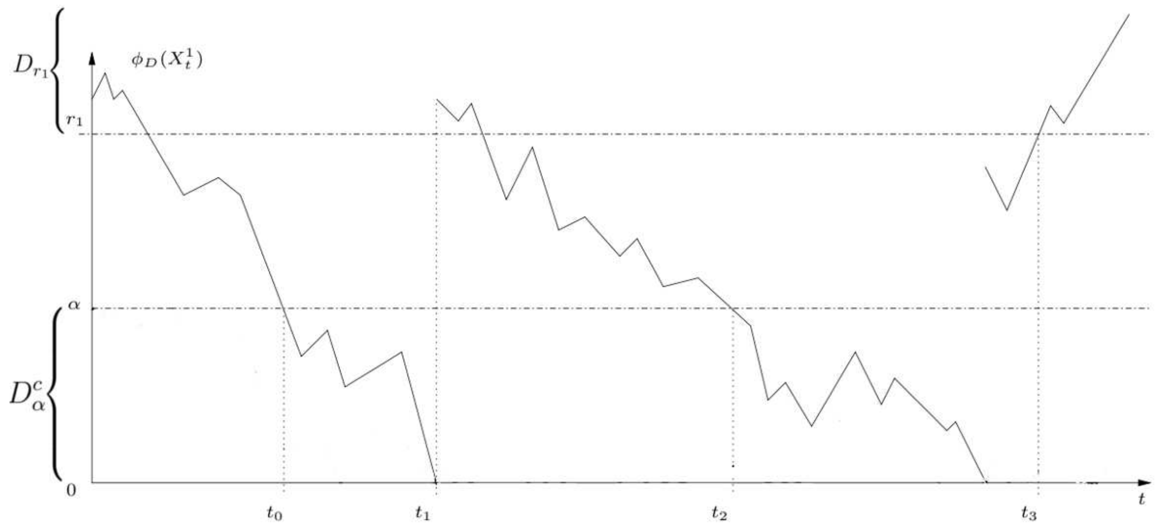

Figure 2: Definition of the sequence of stopping times $(t_n)_{n\geq 0}$

Let γ^i be a 1-dimensional Brownian motion independent of the process $(X^1,...,X^N)$. We set

$$W^i_t = \gamma^i_t, \text{ for } t \in [0,t_0[, \tag{11}$$

and, for all $n \geq 0$,

$$W^i_t = W^i_{t_{2n}} + \int_{t_{2n}}^t \nabla \phi_D(X^i_{s\text{-}}) \cdot dB^i_s \text{ for } t \in [t_{2n},t_{2n+1}],$$
$$W^i_t = W^i_{t_{2n+1}} + (\gamma^i_t - \gamma^i_{t_{2n+1}}) \text{ for } t \in [t_{2n+1},t_{2n+2}],$$

where $\int_{t_{2n}}^t \nabla \phi_D(X^i_{s\text{-}}) \cdot dB^i_s$ has the law of a Brownian motion between times t_{2n} and t_{2n+1}, thanks to (10). It can be obviously proved that $(W^1,...,W^N)$ is a N-dimensional Brownian motion.

Fix $i \in \{1,...,N\}$. Let us now prove that the process Y^i defined by (7) with a constant Q which satisfies

$$\max_{i=1,\dots,N} \|\mathcal{L}^i \phi_D\|_\infty < Q \tag{12}$$

fulfills the inequality (8). We define the time $\zeta = \inf\{0 \leq t < \tau_\infty, Y_t^i > \phi_D(X_t^i)\}$ and we work conditionally to $\zeta < \infty$. By right continuity of the two processes, $Y_\zeta^i \geq \phi_D(X_\zeta^i)$ a.s. If $\zeta = 0$, then $Y_\zeta^i \leq \phi_D(X_\zeta^i)$ by definition of Y_0^i. If $\zeta > 0$, then, by left continuity of Y^i and by the definition of ζ, $Y_\zeta^i \leq \phi_D(X_{\zeta\text{-}}^i)$. But $\phi_D(X_{\zeta\text{-}}^i) \leq \phi_D(X_\zeta^i)$, then $Y_\zeta^i \leq \phi_D(X_\zeta^i)$ almost surely. Finally, we get

$$Y_\zeta^i = \phi_D(X_\zeta^i) \ a.s. \tag{13}$$

We deduce from it that $\phi_D(X_\zeta^i) \leq \alpha$, then there exists $n \geq 0$, such that $\zeta \in [t_{2n}, t_{2n+1}[$. In particular, we can apply Itô's formula to $(\phi_D(X_t^i))_{t\in[\zeta,t_{2n+1}[}$, thanks to the regularity of ϕ_D on $D_{r_1}^c$ (9), and we get

$$\phi_D(X_t^i) = \phi_D(X_{t_{2n}}^i) + \int_{t_{2n}}^t \nabla\phi_D(X_s^i)\cdot dB_s^i + \int_{t_{2n}}^t \mathcal{L}^i\phi_D(X_s^i)ds$$

for all stopping time $t \in [\zeta, t_{2n+1} \wedge \tau^{(\zeta)}[$, where $\tau^{(\zeta)}$ denotes the first jumping time of i after ζ, which means $\tau^{(\zeta)} = \min_{n\geq 1}\{\tau_n, \tau_n > \zeta\}$. We deduce from (13) that $Y_\zeta^i > 0$ almost surely. Let $h > 0$ be a positive random variable, such that $\zeta + h < t_{2n+1} \wedge \tau^{(\zeta)}$ and $Y_t^i > 0$ for all $t \in [\zeta, \zeta + h[$ almost surely. Then, for all $t \in [\zeta, \zeta + h]$, we have, by the Itô's formula,

$$\phi_D(X_t^i) - Y_t^i = \int_\zeta^t (Q - \mathcal{L}^i\phi_D(X_s^i))ds - L_t^{i,0} + L_\zeta^{i,0} + L_t^{i,\alpha} - L_\zeta^{i,\alpha},$$

where $Q - \mathcal{L}^i\phi_D(X_s^i) \geq 0$, $(L_s^{i,\alpha})_{s\geq 0}$ is increasing and $L_t^{i,0} = L_\zeta^0$, since Y^i doesn't hit 0 between times ζ and t (see [8]). Then $\phi_D(X^i) - Y^i$ stays non-negative between times ζ and $\zeta + h$, what contradicts the definition of ζ. Finally, $\zeta = \infty$ almost surely, which means that the coupling inequality (8) remains true for all $t \in [0, \tau_\infty[$. ☐

2.3 Proof of Theorem 2.1

Proof that $(X^1,...,X^N)$ *is well defined.* Let $N \geq 2$ be the size of the interacting particle system and fix arbitrarily its starting point $x \in D^N$. We define the event $C = \{\tau_\infty < +\infty\}$. Conditionally to C, the total number of jumps is equal to $+\infty$ before the finite time τ_∞. There is a finite number of particles, then at least one particle makes an infinite number of jumps before τ_∞. We denote it by i_0 (which is a random index).

For each jumping time τ_n, we denote by $\sigma_n^{i_0}$ the next jumping time of i_0, with $\tau_n < \sigma_n^{i_0} < \tau_\infty$. Conditionally to C, we get $\sigma_n^{i_0} - \tau_n \to 0$ when $n \to \infty$. The process X^{i_0} being a continuous diffusion process with bounded drift between τ_n and $\sigma_n^{i_0}$-, we get

$$\phi_D(X_{\tau_n}^{i_0}) - \phi_D(X_{\sigma_n^{i_0}\text{-}}^{i_0}) \xrightarrow[n\to\infty]{} 0,\ a.s.$$

But $\phi_D(X_{\sigma_n^{i_0}\text{-}}^{i_0}) = 0$ by definition, then

$$\lim_{n\to\infty} \phi_D(X_{\tau_n}^{i_0}) = 0,\ a.s. \tag{14}$$

Let us denote by $(\tau_n^{i_0})_n$ the sequence of jumping times of the particle i_0. We denote by A_n the event

$$A_n = \left\{ \exists i \neq i_0 \,|\, \phi_D(X^i_{\tau_n^{i_0}}) \leq \phi_D(X^{i_0}_{\tau_n^{i_0}}) \right\}.$$

We have, for all $1 \leq k \leq l$,

$$\begin{aligned} P\left(\bigcap_{n=k}^{l+1} A_n^c \right) &= E\left(E\left(\prod_{n=k}^{l+1} \mathbf{1}_{A_n^c} \,|\, (X_t^1,\ldots X_t^N)_{0\leq t<\tau_{l+1}^{i_0}} \right) \right) \\ &= E\left(\prod_{n=k}^{l} \mathbf{1}_{A_n^c} E\left(\mathbf{1}_{A_{l+1}^c} \,|\, (X_t^1,\ldots X_t^N)_{0\leq t<\tau_{l+1}^{i_0}} \right) \right), \end{aligned}$$

where, by definition of the jump mechanism of the interacting particle system,

$$\begin{aligned} E\left(\mathbf{1}_{A_{l+1}^c} \,|\, (X_t^1,\ldots X_t^N)_{0\leq t<\tau_{l+1}^{i_0}} \right) &= \mathcal{J}^{(N)}(X^1_{\tau_{l+1}^{i_0}},\ldots,X^N_{\tau_{l+1}^{i_0}}) \left(A_{l+1}^c \right) \\ &\leq 1 - p_0^{(N)}, \end{aligned}$$

by Hypothesis 1. By induction on l, we get

$$P\left(\bigcap_{n=k}^{l} A_n^c \right) \leq (1-p_0^{(N)})^{l-k},\ \forall 1 \leq k \leq l.$$

Since $p_0^{(N)} > 0$, it yields that

$$P\left(\bigcup_{k\geq 1} \bigcap_{n=k}^{\infty} A_n^c \right) = 0.$$

It means that, for infinitely many jumps τ_n almost surely, one can find a particle j such that $\phi_D(X^j_{\tau_n}) \leq \phi_D(X^{i_0}_{\tau_n})$. Because there is only a finite number of other particles, one can find a particle, say j_0 (which is a random variable), such that

$$\phi_D(X^{j_0}_{\tau_n}) \leq \phi_D(X^{i_0}_{\tau_n}), \text{ for infinitely many } n \geq 1.$$

In particular, we get from (14) that

$$\lim_{n\to\infty} \left(\phi_D(X^{i_0}_{\tau_n}), \phi_D(X^{j_0}_{\tau_n}) \right) = (0,0)\ a.s.$$

Using the coupling inequality of Proposition 2.2, we deduce that

$$C \subset \left\{ \lim_{t\to\tau_\infty} (Y_t^{i_0}, Y_t^{j_0}) = (0,0) \right\}.$$

Then, conditionally to C, Y^{i_0} and Y^{j_0} are independent reflected diffusion processes with bounded drift, which hit 0 at the same time. This occurs for two independent reflected Brownian motions with probability 0, and then for Y^{i_0} and Y^{j_0} too, by the Girsanov's Theorem. That implies $P_x(C) = 0$. Finally, we have $\tau_\infty = +\infty$ almost surely. $\square$

Proof of the exponential ergodicity. It is sufficient to prove that there exists $n \geq 1$, $\epsilon > 0$ and a non-trivial probability ϑ on D^N such that

$$P_x((X_n^1,...,X_n^N) \in A) \geq \epsilon\vartheta(A),\ \forall x \in \overline{D_\alpha}^N,\ A \in \mathcal{B}(D^N) \tag{15}$$

and that

$$\sup_{x\in\overline{D_\alpha}^N} E_x(\kappa^{\tau'}) < \infty, \tag{16}$$

where κ is a positive constant and τ' is the return time to $\overline{D_\alpha}^N$ of the Markov chain $(X_n^1,...,X_n^N)_{n\in\mathbb{N}}$. Indeed, Down and Meyn proved in [12, Theorem 2.1 p.1673] that if the Markov chain $(X_n^1,...,X_n^N)_{n\in\mathbb{N}}$ is aperiodic (which is obvious in our case) and fulfills (15) and (16), then it is geometrically ergodic. But, thanks to [12, Theorem 5.3 p.1681], the geometric ergodicity of this Markov chain is a sufficient condition for $(X^1,...,X^N)$ to be exponentially ergodic.

Let us first prove that (15) is fulfilled. Let $\mathcal{F}$ be the event "the process $(X^1,...,X^N)$ has no jump between times 0 and 1". Conditionally to $\mathcal{F}$, the process X^i doesn't depend on the other particles before time 1. Then the law of X^i conditionally to $\mathcal{F}$ is the same as the law of the diffusion process $X^{(i)}$ defined by (5) at time 1 and conditioned to not be killed. It yields that

$$P_{(x_1,...,x_N)}(X_1^i \in dx|\mathcal{F}) = \mathbb{P}^i_{x_i}(X_1^{(i)} \in dx|1 < \tau_\partial) \geq \mathbb{P}^i_{x_i}(X_1 \in dx). \tag{17}$$

The law of $X_1^{(i)}$ has a density $p_1^i(x_i,y)$ with respect to the Lebesgue's measure and $p_1^i(x_i,y)$ depends continuously on x_i and y. It only vanishes when $y = 0$ or 1. Then

$$\epsilon_i = \inf_{(x_i,y)\in\overline{D_\alpha}\times\overline{D_\alpha}} p_1^i(x_i,y) > 0,$$

since $\overline{D_\alpha} \times \overline{D_\alpha}$ is compact. We get from (17) that

$$P_{(x_1,...,x_N)}(X_1^i \in dx|\mathcal{F}) \geq \epsilon_i \mathbf{1}_{\overline{D_\alpha}}(x)dx,\ \forall (x_1,...,x_N) \in \overline{D_\alpha}^N.$$

Conditionally to $\mathcal{F}$, the particles are independent from each other, so that

$$\begin{aligned}P_{(x_1,...,x_N)}((X_1^1,...,X_1^N) \in dy_1...dy_N|\mathcal{F}) &= \prod_{i=1}^N P_{(x_1,...,x_N)}(X_1^i \in dy_i|\mathcal{F})\\ &\geq \left(\prod_{i=1}^N \epsilon_i\right) \mathbf{1}_{\overline{D_\alpha}^N}(y_1,...,y_N)dy_1...dy_N.\end{aligned}$$

Define $p = \inf_{x\in\overline{D_\alpha}^N} P_x(\mathcal{F})$. Thanks to the coupling with $(Y^1,...,Y^N)$, we have $p > 0$. It yields that (15) is satisfied with $\vartheta(dx) = p\left(\prod_{i=1}^N \epsilon_i\right)\mathbf{1}_{\overline{D_\alpha}^N}(x)dx$.

Now we prove that $\exists\kappa > 0$ such that (16) holds. Let $p' > 0$ be the probability for $(Y^1,...,Y^N)$ to enter $\overline{D_\alpha}^N$ at time $n+1$, starting from 0 at time n. For all $x \in D^N$ and all $n \geq 1$, the probability for $(X^1,...,X^N)$ to be in $\overline{D_\alpha}^N$ at time $n+1$ starting from x at time n is bounded below by the probability p'. Hence, at each time $n \geq 1$, $(X^1,...,X^N)$ returns to $\overline{D_\alpha}^N$ at time $n+1$ with a probability greater than $p' > 0$. It implies that the return time to $\overline{D_\alpha}^N$ for $(X_n^1,...X_n^N)_{n\in\mathbb{N}}$ is bounded above by a time of geometrical law, independent of the starting point $x \in D^N$. Then condition (16) is fulfilled. □

2.4 Uniform tightness of the empirical stationary distributions

Assume that a jump measure $\mathcal{J}^{(N)}$ is given for each $N \geq 2$ and that Hypothesis 1 is fulfilled for all $N \geq 2$. Assume that we're given a family of drifts $(q_i^{(N)})_{i=1,\dots,N}$ for each $N \geq 2$ which is uniformly bounded, which means that there exists a constant $Q' > 0$ such that

$$\|q_i^{(N)}\|_\infty < Q',\ \forall\, i,N. \tag{18}$$

We denote by $M^N \in \mathcal{M}_1(D^N)$ the stationary distribution of the N-particles process described above. The empirical stationary distribution $\mathcal{X}^N$ denotes the random probability on D defined by

$$\mathcal{X}^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$$

where $(x^1,\dots,x^N)$ is a random vector in D^N distributed following M^N.

Theorem 2.3. *The family of empirical stationary distributions $\left(\mathcal{X}^N\right)_{N\geq 2}$ is uniformly tight.*

Proof. Let us consider the process $(X^1,\dots,X^N)$ starting with a distribution m^N and its coupled process $(Y^1,\dots,Y^N)$. For all $t \in [0,\tau_\infty[$, we denote by $\mu^N(t,dx)$ (resp. $\mu'^N(t,dx)$) the empirical measure of the process $(X^1,\dots,X^N)$ (resp. $(Y^1,\dots,Y^N)$) at time t:

$$\mu^N(t,dx) = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}(dx) \text{ and } \mu'^N(t,dx) = \frac{1}{N}\sum_{i=1}^{N} \delta_{Y_t^i}(dx).$$

By the coupling inequality (8), we get

$$\mu^N(t,D_r^c) \leq \mu'^N(t,[0,r]),\ \forall r \in [0,\alpha].$$

As a consequence,

$$E_{m^N}\left(\mu^N(t,D_r^c)\right) \leq E_{m^N}\left(\mu'^N(t,[0,r])\right),\ \forall r \in [0,\alpha].$$

The family $\left(E_{m^N}\left(\mu'^N(t,.)\right)\right)_{N\geq 2}$ is uniformly tight for any arbitrarily chosen $t > 0$, since one can choose the constant Q in (7) equal to Q' for all $N \geq 2$, by (12) and (18). Then the family $\left(E_{m^N}\left(\mu^N(t,.)\right)\right)_{N\geq 2}$ is also uniformly tight for any given $t > 0$. This implies the uniform tightness of the family of random measures $(\mu^N(t,dx))_{N\geq 2}$ (see [20]). If we set m^N equal to the stationary distribution M^N, then we get by stationarity that $\mathcal{X}^N$ is distributed as $\mu^N(t,.)$, for all $N \geq 2$ and $t > 0$. Finally, the family of empirical stationary distributions $(\mathcal{X}^N)_{N\geq 2}$ is uniformly tight. $\square$

3 Yaglom limit's approximation

We consider now the particular case $\mathcal{J}^{(N)}(x_1,\dots,x_N) = \frac{1}{N-1}\sum_{k=1,k\neq i}^{N} \delta_{x_k}$: at each jump time, the particle which hits the boundary jumps to the position of a particle chosen uniformly between the $N-1$ remaining ones. We assume moreover that $q_i^{(N)} = q$ doesn't depend on i,N. In this framework, we are able to identify the limit of the empirical stationary distribution sequence, when the number of particles tends to infinity. This leads us to an approximation method of the Yaglom limits (4), including cases where the drift of the diffusion process isn't bounded and where the boundary ∂D_0 is neither of class C^2 nor bounded.

Let D_0 be an open domain of $\mathbb{R}^d$, with $d \geq 1$. We denote by $\mathbb{P}^0$ the law of the diffusion process defined on D_0 by

$$dX_t = dB_t - \nabla V(X_t)dt,\ X_0 = x \in D_0 \tag{19}$$

and absorbed at the boundary ∂D_0. Here B is a d-dimensional Brownian motion and $V \in C^2(D_0,\mathbb{R})$. We assume that Hypothesis 2 below is fulfilled, so that the Yaglom limit

$$\nu_0 = \lim_{t\to+\infty} \mathbb{P}^0_x \left(X_t \in .|t \leq \tau_\partial\right), \forall x \in D_0 \tag{20}$$

exists and doesn't depend on x, as proved by Cattiaux and Méléard in [6, Theorem B.2]. We emphasize the fact that this hypothesis allows the drift ∇V of the diffusion process (19) to be unbounded and the boundary ∂D_0 to be neither of class C^2 nor bounded. In particular, the results of the previous section aren't available in all generality for diffusion processes with law $\mathbb{P}^0$.

Hypothesis 2. *We assume that*

1. $\mathbb{P}^0_x(\tau_\partial < +\infty) = 1$,
2. $\exists C > 0$ *such that* $G(x) = |\nabla V|^2(x) - \Delta V(x) \geq -C > -\infty$, $\forall x \in D_0$,
3. $\overline{G}(R) \to +\infty$ *as* $R \to \infty$, *where*
$$\overline{G}(R) = \inf\left\{G(x); |x| \geq R \text{ and } x \in D_0\right\},$$
4. *For all* $R > 0$, *one can find an increasing sequence of open bounded sets* $K_n(R)$ *such that the boundary of* $K_n(R)\cap D_0$ *is of class* C^1 *and* $\bigcup_n \left(K_n(R) \cap D_0\right) = B(0,R)\cap\overline{D_0}$, *where* $B(0,R)$ *is the Euclidean ball of radius* R.
5. *There exists* $R > 0$ *such that*
$$\int_{D_0\cap\{d(x,\partial D_0)>R\}} e^{-2V(x)}dx < \infty \text{ and } \int_{D_0\cap\{d(x,\partial D_0)\leq R\}} \left(\int_D p_1^{D_0}(x,y)dy\right) e^{-V(x)}dx < \infty.$$
Here $p_1^{D_0}$ *is the transition density of the diffusion process* (19) *with respect to the Lebesgue measure.*

Remark 2. For example, it is proved in [6] that Hypothesis 2 is fulfilled by the Lotka-Volterra system studied numerically in Subsection 3.3.3. Up to a change of variable, this system is defined by the diffusion process with values in $D_0 = \mathbb{R}^2_+$, which satisfies

$$\begin{aligned}
dY^1_t &= dB^1_t + \left(\frac{r_1 Y^1_t}{2} - \frac{c_{11}\gamma_1\left(Y^1_t\right)^3}{8} - \frac{c_{12}\gamma_2 Y^1_t\left(Y^2_t\right)^2}{8} - \frac{1}{2Y^1_t}\right)dt \\
dY^2_t &= dB^2_t + \left(\frac{r_2 Y^2_t}{2} - \frac{c_{22}\gamma_2\left(Y^2_t\right)^3}{8} - \frac{c_{21}\gamma_1 Y^2_t\left(Y^1_t\right)^2}{8} - \frac{1}{2Y^1_t}\right)dt
\end{aligned} \tag{21}$$

and is killed at ∂D_0. Here B^1,B^2 are two independent one-dimensional Brownian motions and the parameters of the diffusion process fulfill condition (46).

In order to define the interacting particle process of the previous section, we introduce a cut-off of D_0 near its boundary. More precisely, let $(D_\epsilon)_{\epsilon\geq 0}$ be a family of bounded open subsets of D_0 of class C^2, which tends to D_0 in the sense that, for all compact subset $K \subset D_0$, $\exists \epsilon > 0$ such that $K \subset D_\epsilon$. For all $0 < \epsilon < \epsilon'$, we assume that $d(D_\epsilon, \partial D_0) > 0$, and $D_\epsilon \subsetneq D_{\epsilon'}$. For all $\epsilon > 0$, we denote by $\mathbb{P}^\epsilon$ the law of the diffusion process defined on D_ϵ by

$$dX_t^\epsilon = dB_t - \nabla V(X_t^\epsilon)dt,\ X_0^\epsilon = x \in D_\epsilon$$

and absorbed at the boundary ∂D_ϵ. Here B is a d-dimensional Brownian motion. For all $\epsilon > 0$, the diffusion process with law $\mathbb{P}^\epsilon$ clearly fulfills the conditions of Section 2 . For all $N \geq 2$, let $(X^{\epsilon,1}, ..., X^{\epsilon,N})$ be the interacting particle process defined by the law $\mathbb{P}^\epsilon$ between the jumps and by the jump measure $\mathcal{J}^{(\epsilon,N)}(x_1, ..., x_N) = \frac{1}{N-1}\sum_{k=1,k\neq i}^N \delta_{x_k}$. By Theorem 2.1, this process is well defined and exponentially ergodic.

For all $\epsilon > 0$ and all $N \geq 2$, we denote by $M^{\epsilon,N}$ the stationary distribution of $(X^{\epsilon,1}, ..., X^{\epsilon,N})$ and by $\mathcal{X}^{\epsilon,N}$ the associated empirical stationary distribution.

We are now able to state the main result of this section.

Theorem 3.1. *Assume that Hypothesis 2 is satisfied. Then*

$$\lim_{\epsilon\to 0}\lim_{N\to\infty} \mathcal{X}^{\epsilon,N} = \nu_0, \tag{22}$$

in the weak topology of random measures.

In Section 3.1, we fix $\epsilon > 0$ and we prove that the sequence $(\mathcal{X}^{\epsilon,N})_{N\geq 2}$ converges to a probability ν_ϵ when N goes to infinity. In particular, we prove that ν_ϵ is the Yaglom limit associated with $\mathbb{P}^\epsilon$, which exists by [16]. In Section 3.2, we conclude the proof, proceeding by a compactness/uniqueness argument: we prove that $(\nu_\epsilon)_{0<\epsilon<1/2}$ is a uniformly tight family and we show that each limiting probability of the family $(\nu_\epsilon)_{0<\epsilon<1/2}$ is equal to the Yaglom limit ν_0. The last Section 3.3.3 is devoted to numerical illustrations of Theorem 3.1.

3.1 Convergence of $(\mathcal{X}^{\epsilon,N})_{N\geq 2}$, when $\epsilon > 0$ is fixed

Proposition 3.2. *Let $\epsilon > 0$ be fixed. The sequence of empirical stationary distributions $(\mathcal{X}^{\epsilon,N})_{N\geq 2}$ converges to ν_ϵ in the weak topology of random measures when N goes to infinity, where ν_ϵ is the Yaglom limit associated with $\mathbb{P}^\epsilon$.*

Remark 3. The Yaglom limit ν_ϵ exists and is the unique quasi-stationary distribution associated with $\mathbb{P}^\epsilon$. Moreover it satisfies

$$\nu_\epsilon = \lim_{t\to\infty} \mathbb{P}_m^\epsilon \left(X_t^\epsilon \in .|X_t^\epsilon \in D_\epsilon\right),\ \forall m \in \mathcal{M}_1(D_\epsilon), \tag{23}$$

by [6, Proposition B.12].

Proof of Proposition 3.2. The initial distribution of the process $(X^{\epsilon,1}, ..., X^{\epsilon,N})$ is chosen equal to its stationary distribution $M^{\epsilon,N}$. For all $t \geq 0$, we denote by $\mu^{\epsilon,N}(t,dx)$ its empirical measure at time t (by stationarity, $\mu^{\epsilon,N}(t,dx)$ and $\mathcal{X}^{\epsilon,N}$ have the same law). We set

$$\nu^{\epsilon,N}(t,dx) = \left(\frac{N-1}{N}\right)^{A_t^N} \mu^{\epsilon,N}(t,dx),$$

where $A_t^N = \sum_{n=1}^{\infty} \mathbf{1}_{\tau_n \leq t}$ denotes the number of jumps before time t. Intuitively, we introduce a loss of $1/N$ of the total mass at each jump, in order to approximate the distribution of the diffusion process (19) without conditioning. We will come back to the study of $\mu^{\epsilon,N}$ and the conditioned diffusion process by normalizing $\nu^{\epsilon,N}$.

For all $\epsilon \geq 0$, we denote by $\mathcal{L}^\epsilon$ the infinitesimal generator of the diffusion process with law $\mathbb{P}^\epsilon$. From the Itô's formula applied to the semimartingale $\mu^{\epsilon,N}(t,\psi) = \frac{1}{N}\sum_{i=1}^N \psi(X_t^{\epsilon,i})$, where $\psi \in C^2(D_\epsilon,\mathbb{R}^d)$, we get

$$\mu^{\epsilon,N}(t,\psi) = \mu^{\epsilon,N}(0,\psi) + \int_0^t \mu^{\epsilon,N}(s,\mathcal{L}^\epsilon\psi)ds + \mathcal{M}^{c,\epsilon,N}(t,\psi) + \mathcal{M}^{j,\epsilon,N}(t,\psi) + \frac{1}{N-1}\sum_{0\leq\tau_n\leq t} \mu^{\epsilon,N}(\tau_n\text{-},\psi), \quad (24)$$

where $\mathcal{M}^{c,\epsilon,N}(t,\psi)$ is the continuous martingale

$$\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^d \int_0^t \frac{\partial\psi}{\partial x_j}(X_s^{\epsilon,i})dB_s^{i,j}$$

and $\mathcal{M}^{j,\epsilon,N}(t,\psi)$ is the pure jump martingale

$$\frac{1}{N}\sum_{i=1}^N \sum_{0\leq\tau_n^i\leq t}\left(\psi(X_{\tau_n^i}^{\epsilon,i}) - \frac{N}{N-1}\mu^{\epsilon,N}(\tau_n^i\text{-},\psi)\right).$$

Applying the Itô's formula to the semimartingale $\nu^{\epsilon,N}(t,\psi)$, we deduce from (24) that

$$\nu^{\epsilon,N}(t,\psi) = \nu^{\epsilon,N}(0,\psi) + \int_0^t \nu^{\epsilon,N}(s,\mathcal{L}^\epsilon\psi)ds + \int_0^t \left(\frac{N-1}{N}\right)^{A_s^N} d\mathcal{M}^{c,\epsilon,N}(s,\psi) + \sum_{0\leq\tau_n\leq t}\left(\nu^{\epsilon,N}(\tau_n,\psi) - \nu^{\epsilon,N}(\tau_n\text{-},\psi)\right).$$

Where we have

$$\nu^{\epsilon,N}(\tau_n,\psi) - \nu^{\epsilon,N}(\tau_n\text{-},\psi) = \left(\frac{N-1}{N}\right)^{A_{\tau_n}^N}\left(\mu^{\epsilon,N}(\tau_n,\psi) - \mu^{\epsilon,N}(\tau_n\text{-},\psi)\right) + \mu^{\epsilon,N}(\tau_n\text{-},\psi)\left(\left(\frac{N-1}{N}\right)^{A_{\tau_n}^N} - \left(\frac{N-1}{N}\right)^{A_{\tau_n\text{-}}^N}\right).$$

But

$$\mu^{\epsilon,N}(\tau_n,\psi) - \mu^{\epsilon,N}(\tau_n\text{-},\psi) = \frac{1}{N-1}\mu^{\epsilon,N}(\tau_n\text{-},\psi) + \mathcal{M}^{j,\epsilon,N}(\tau_n,\psi) - \mathcal{M}^{j,\epsilon,N}(\tau_n\text{-},\psi)$$

and

$$\left(\frac{N-1}{N}\right)^{A_{\tau_n}^N} - \left(\frac{N-1}{N}\right)^{A_{\tau_n\text{-}}^N} = -\frac{1}{N-1}\left(\frac{N-1}{N}\right)^{A_{\tau_n}^N}.$$

Then

$$\begin{aligned}\nu^{\epsilon,N}(\tau_n,\psi)-\nu^{\epsilon,N}(\tau_n\text{-},\psi) &= \left(\frac{N-1}{N}\right)^{A^N_{\tau_n}}\left(\mathcal{M}^{j,\epsilon,N}(\tau_n,\psi)-\mathcal{M}^{j,\epsilon,N}(\tau_n\text{-},\psi)\right).\\ &= \frac{N-1}{N}\left(\frac{N-1}{N}\right)^{A^N_{\tau_n\text{-}}}\left(\mathcal{M}^{j,\epsilon,N}(\tau_n,\psi)-\mathcal{M}^{j,\epsilon,N}(\tau_n\text{-},\psi)\right).\end{aligned}$$

That implies

$$\begin{aligned}\nu^{\epsilon,N}(t,\psi)-\nu^{\epsilon,N}(0,\psi) = \int_0^t \nu^{\epsilon,N}(s,\mathcal{L}^\epsilon\psi)ds+\int_0^t\left(\frac{N-1}{N}\right)^{A^N_s} d\mathcal{M}^{c,\epsilon,N}(s,\psi)\\ +\frac{N-1}{N}\sum_{0\leq\tau_n\leq t}\left(\frac{N-1}{N}\right)^{A^N_{\tau_n\text{-}}}\left(\mathcal{M}^{j,\epsilon,N}(\tau_n,\psi)-\mathcal{M}^{j,\epsilon,N}(\tau_n\text{-},\psi)\right).\end{aligned}$$

It yields that, for all smooth functions $\Psi(t,x)$ vanishing at the boundary of D_ϵ,

$$\begin{aligned}\nu^{\epsilon,N}(t,\Psi(t,.))-\nu^{\epsilon,N}(0,\Psi(0,.)) = \int_0^t \nu^{\epsilon,N}(s,\frac{\partial\Psi(s,.)}{\partial s}+\frac{\partial\Psi(s,.)}{\partial x}q+\frac{1}{2}\frac{\partial^2\Psi(s,.)}{\partial x^2})ds\\ +\mathcal{N}^{c,\epsilon,N}(t,\Psi)+\mathcal{N}^{j,\epsilon,N}(t,\Psi),\end{aligned} \tag{25}$$

where $\mathcal{N}^{c,\epsilon,N}(t,\Psi)$ is the continuous martingale

$$\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^d\int_0^t\left(\frac{N-1}{N}\right)^{A^N_s}\frac{\partial\Psi}{\partial x_j}(s,X^{\epsilon,i}_s)dB^{i,j}_s$$

and $\mathcal{N}^{j,\epsilon,N}(t,\Psi)$ is the pure jump martingale

$$\frac{1}{N}\sum_{i=1}^N\sum_{0\leq\tau^i_n\leq t}\left(\frac{N-1}{N}\right)^{A^N_{\tau^i_n\text{-}}}\left(\Psi(\tau^i_n,X^{\epsilon,i}_{\tau^i_n})-\frac{N}{N-1}\mu^{\epsilon,N}(\tau^i_n\text{-},\Psi(\tau^i_n\text{-},.))\right).$$

For all $\delta>0$, define $\Psi^\delta(t,x)=P^\epsilon_{T-t}P^\epsilon_\delta f(x)$, where $f\in C^2(D)$ vanishes on ∂D, and (P^ϵ_t) is the semigroup associated with $\mathbb{P}^\epsilon$. From Kolmogorov's equation (see [13, Proposition 1.5 p.9]),

$$\frac{\partial}{\partial s}\Psi^\delta(s,x)+\frac{1}{2}\Delta\Psi^\delta(s,x)+q(x)\nabla\Psi^\delta(s,x)=0.$$

It yields that

$$\nu^{\epsilon,N}(t,\Psi^\delta(t,.))-\nu^{\epsilon,N}(0,\Psi^\delta(0,.))=\mathcal{N}^{c,\epsilon,N}(t,\Psi^\delta)+\mathcal{N}^{j,\epsilon,N}(t,\Psi^\delta). \tag{26}$$

Since $\left(\frac{N-1}{N}\right)^{A^N_s}\leq 1$ a.s., we get

$$\begin{aligned}E\left(\mathcal{N}^{c,\epsilon,N}(T,\Psi^\delta)^2\right) &\leq \frac{T}{N}\|\nabla\Psi^\delta\|^2_\infty\\ &\leq \frac{T}{N}\frac{c_\epsilon}{\sqrt{(T-t+\delta)\wedge 1}}\|f\|_\infty\end{aligned} \tag{27}$$

where $c_\epsilon > 0$ is a positive constant. The last inequality comes from [28, Theorem 4.5] on gradient estimates in regular domains of $\mathbb{R}^d$. The jumps of the martingale $\mathcal{M}^{j,\epsilon,N}(t,\Psi^\delta)$ are smaller than $\frac{2}{N}\|\Psi^\delta\|_\infty$, then

$$E\left[\sum_{0\le\tau_n\le T}\left(\frac{N-1}{N}\right)^{2A_{\tau_n\text{-}}}\left(\mathcal{M}^{j,\epsilon,N}(\tau_n,\Psi^\delta(\tau_n,.))-\mathcal{M}^{j,\epsilon,N}(\tau_n\text{-},\Psi^\delta(\tau_n\text{-},.))\right)^2\right]$$
$$\le \frac{4}{N^2}\|\Psi^\delta\|_\infty^2 E\left[\sum_{0\le\tau_n\le T}\left(\frac{N-1}{N}\right)^{2A_{\tau_n\text{-}}}\right]$$
$$\le \frac{4}{N}\|\Psi^\delta\|_\infty^2.$$

Then

$$E\left(\mathcal{N}^{j,N}(\Psi,T)^2\right)\le\frac{4}{N}\|\Psi\|_\infty^2\le\frac{4}{N}\|f\|_\infty^2. \tag{28}$$

We get from (26), (27) and (28) that

$$\sqrt{E\left(\left|\nu^N(t,P^\epsilon_{T-t+\delta}f)-\nu^N(0,P^\epsilon_{T+\delta}f)\right|^2\right)}\le\frac{C_{\epsilon,\delta}}{\sqrt{N}}\|f\|_\infty$$

where $C_{\epsilon,\delta}$ is a positive constant which does not depend on f. In particular, one can find a strictly decreasing sequence $(\delta_N)_N$ which converges to 0 and such that

$$\sqrt{E\left(\left|\nu^{\epsilon,N}(T,P^\epsilon_{\delta_N}f)-\nu^{\epsilon,N}(0,P^\epsilon_{T+\delta_N}f)\right|^2\right)}\le\|f\|_\infty o(N).$$

But $\|P^\epsilon_{\delta_N}f-f\|_\infty$ tends to 0 when δ_N goes to 0, then

$$\sqrt{E\left(\left|\nu^{\epsilon,N}(T,f)-\nu^{\epsilon,N}(0,P^\epsilon_T f)\right|^2\right)}\to 0. \tag{29}$$

The family of random probabilities $(\mathcal{X}^{\epsilon,N})_{N\ge0}$ is uniformly tight, by Theorem 2.3. Let $\mathcal{X}^\epsilon$ be one of its limit probabilities. By definition, there exists a strictly increasing map $\varphi:\mathbb{N}\mapsto\mathbb{N}$, such that $\mathcal{X}^{\epsilon,\varphi(N)}$ converges in law to $\mathcal{X}^\epsilon$ when $N\to\infty$. Since $\nu^{\epsilon,N}(0,.)=\mu^{\epsilon,N}(0,.)$ has the same law as $\mathcal{X}^{\epsilon,N}$, we deduce from (29) that

$$E\left(\nu^{\epsilon,\varphi(N)}(T,f)\right)\xrightarrow[N\to\infty]{}E\left(\mathcal{X}^\epsilon(P^\epsilon_T f)\right) \tag{30}$$

for all continuous function f which vanishes at the boundary of D_ϵ. But the family $\left(\mu^{\epsilon,\varphi(N)}(T,.)\right)_N$ is uniformly tight, then $\left(\nu^{\epsilon,\varphi(N)}(T,.)\right)_N$ is also uniformly tight. By (30), its unique limit is then the measure $\mathcal{X}^\epsilon(P^\epsilon_T.)$ defined by $f\mapsto\mathcal{X}^\epsilon(P^\epsilon_T f)$. We finally get

$$\nu^{\epsilon,\varphi(N)}(T,.)\xrightarrow[N\to\infty]{law}\mathcal{X}^\epsilon(P^\epsilon_T.). \tag{31}$$

In particular,

$$\left(\nu^{\epsilon,\varphi(N)}(T,D_\epsilon),\nu^{\epsilon,\varphi(N)}(T,.)\right)\xrightarrow[N\to\infty]{law}\left(\mathcal{X}^\epsilon(P^\epsilon_T\mathbf{1}_{D_\epsilon}),\mathcal{X}^\epsilon(P^\epsilon_T.)\right).$$

But $\mathcal{X}^\epsilon(P^\epsilon_T \mathbf{1}_{D_\epsilon})$ never vanishes almost surely, so that

$$\mu^{\epsilon,\varphi(N)}(T,.) = \frac{\nu^{\epsilon,\varphi(N)}(T,.)}{\nu^{\epsilon,\varphi(N)}(T,D_\epsilon)} \xrightarrow[N\to\infty]{law} \frac{\mathcal{X}^\epsilon(P^\epsilon_T .)}{\mathcal{X}^\epsilon(P^\epsilon_T \mathbf{1}_{D_\epsilon})} = \mathbb{P}^\epsilon_{\mathcal{X}^\epsilon}(X^\epsilon_T \in .|X^\epsilon_T \in D_\epsilon)$$

By stationarity, $\mu^{\epsilon,\varphi(N)}(T,.)$ and $\mathcal{X}^{\epsilon,N}$ have the same law, and converge in law to $\mathcal{X}^\epsilon$ when $N \to \infty$. It yields that $\mathcal{X}^\epsilon$ and $\mathbb{P}^\epsilon_{\mathcal{X}^\epsilon}(X^\epsilon_T \in .|X^\epsilon_T \in D_\epsilon)$ have the same law. But $\mathbb{P}^\epsilon_{\mathcal{X}^\epsilon}(X^\epsilon_T \in .|X^\epsilon_T \in D_\epsilon)$ converges almost surely to ν_ϵ when $T \to \infty$, by (23). We deduce from it that $\mathcal{X}^\epsilon$ has the same law as ν_ϵ. As a consequence, the unique limit probability of the uniformly tight family $(\mathcal{X}^{\epsilon,N})_N$ is ν_ϵ, which allows us to conclude the proof of Proposition 3.2. □

3.2 Convergence of the family $(\nu_\epsilon)_{0<\epsilon<1}$

We show in Subsection 3.2.1 that the family $(\nu_\epsilon)_{0<\epsilon<1}$ is uniformly tight. In Subsection 3.2.2, we prove that its unique probability limit is ν_0, which concludes the proof of Theorem 3.1.

3.2.1 Uniform tightness of the family $(\nu_\epsilon)_{0<\epsilon<1}$

Proposition 3.3. *Assume that hypothesis 2 is fulfilled. Then the family $(\nu_\epsilon)_{0<\epsilon<1}$ is uniformly tight. Moreover, every limit point is absolutely continuous with respect to the Lebesgue measure, with a density bounded by ce^{-V}, where c is a positive constant.*

Proof of Proposition 3.3. Let us recall some results from [16] and [6] on the spectral theory of $\mathcal{L}^\epsilon$. It has a simple eigenvalue $\lambda_\epsilon > 0$ with minimal real part. The corresponding normalized eigenfunction η_ϵ is strictly positive on D_ϵ, belongs to $C^2(D_\epsilon,\mathbb{R})$ and fulfills

$$\mathcal{L}^\epsilon \eta_\epsilon = -\lambda_\epsilon \eta_\epsilon \text{ and } \int_{D_\epsilon} \eta_\epsilon(x)^2 \sigma(dx) = 1, \tag{32}$$

where

$$\sigma(dx) = e^{-2V(x)}dx.$$

Moreover, we have

$$d\nu_\epsilon = \frac{\eta_\epsilon d\sigma}{\int_{D_\epsilon} \eta_\epsilon(x) d\sigma(x)}, \forall \epsilon \geq 0. \tag{33}$$

In order to prove that $(\nu_\epsilon)_{0<\epsilon<1}$ is uniformly tight, we show that $\left(\int_{D_\epsilon} \eta_\epsilon(x)d\sigma(x)\right)_{0<\epsilon<1}$ is uniformly bounded below by a positive constant $A > 0$, and we conclude by proving that the family $(\eta_\epsilon d\sigma)_{0<\epsilon<1}$ is uniformly tight.

Let us prove that

$$A = \inf_{0<\epsilon<1} \int_{D_\epsilon} \eta_\epsilon(x) d\sigma(x) > 0. \tag{34}$$

In order to achieve this goal, assume the converse: one can find a sequence of positive numbers $(\epsilon_k)_{k\in\mathbb{N}}$ which converges to 0 and such that $\int_{D_{\epsilon_k}} v_{\epsilon_k}(x)e^{-V(x)}dx = \int_{D_{\epsilon_k}} \eta_{\epsilon_k}(x)d\sigma(x) \xrightarrow[k\to\infty]{} 0$, where we set $v_\epsilon = \eta_\epsilon e^{-V}$. Thanks to [6], there exists a constant $\kappa > 0$ such that

$$v_\epsilon(x) < \kappa, \forall \epsilon \geq 0, \forall x \in D_\epsilon. \tag{35}$$

In particular, we have

$$\int_{D_{\epsilon_k}} v_{\epsilon_k}(x)^2 e^{-V(x)}dx \le \kappa \int_{D_{\epsilon_k}} v_{\epsilon_k}(x) e^{-V(x)}dx \xrightarrow[k\to\infty]{} 0. \tag{36}$$

Let us show that $(v_\epsilon(x)^2dx)_{\epsilon>0}$ is uniformly tight. If D_0 is bounded, it is a direct consequence of the uniform bound (35) . Assume that D_0 isn't bounded, then

$$\int_{D_\epsilon\cap|x|\ge R} v_\epsilon^2(x)dx \le \frac{1}{\overline{G}(R)} \int_{D_\epsilon\cap|x|\ge R} v_\epsilon^2(x)G(x)dx, \tag{37}$$

where $\overline{G}(R) \to +\infty$ when $R \to +\infty$ (see Hypothesis 2). For all $x \in D_\epsilon$, (32) leads to

$$\frac{1}{2}\Delta v_\epsilon - \frac{1}{2}G(x)v_\epsilon(x) = -\lambda_\epsilon v_\epsilon(x) \text{ and } \int_{D_\epsilon} v_\epsilon(x)^2dx = 1.$$

Then

$$\begin{aligned}\int_{D_\epsilon} v_\epsilon^2(x)G(x)dx &= \lambda_\epsilon \int_{D_\epsilon} v_\epsilon(x)^2dx + \int_{D_\epsilon} v_\epsilon(x)\Delta v_\epsilon(x)dx\\ &= \lambda_\epsilon - \int_{D_\epsilon} |\nabla v_\epsilon(x)|^2dx\\ &\le \lambda_\epsilon,\end{aligned} \tag{38}$$

where the second equality is a consequence of the Green's formula (see [2, Corollary 3.2.4]). But the eigenvalue λ_ϵ of $-L^\epsilon$ is given by (see for instance [31, chapter XI, part 8])

$$\begin{aligned}\lambda_\epsilon &= \inf_{\phi\in C_0^\infty(D_\epsilon),\ \langle\phi,\phi\rangle_\sigma=1} \langle L^\epsilon\phi,\phi\rangle_\sigma,\\ &= \inf_{\phi\in C_0^\infty(D_\epsilon),\ \langle\phi,\phi\rangle_\sigma=1} \left\langle L^0\phi,\phi\right\rangle_\sigma,\end{aligned} \tag{39}$$

where $C_0^\infty(D_\epsilon)$ is the vector space of infinitely differentiable functions with compact support in D_ϵ and $\langle f,g\rangle_\sigma = \int_{D_0} f(u)g(u)d\sigma(u)$. We deduce from it that λ_ϵ increases with ϵ and is uniformly bounded above by λ_1. The uniform bound (38) and the inequality (37) allow us to conclude that the family $(v_\epsilon(x)^2dx)_{\epsilon>0}$ is uniformly tight.

As a consequence, one can find (after extracting a sub-sequence) a non-negative map $m : D_0 \to \mathbb{R}_+$ such that, for all continuous and bounded function $\phi : D_0 \to \mathbb{R}$,

$$\int_{D_{\epsilon_k}} v_{\epsilon_k}(y)^2\phi(y)dy \xrightarrow[k\to\infty]{} \int_{D_0} m(y)\phi(y)dy. \tag{40}$$

Indeed, (v_ϵ^2) being uniformly bounded, all limit measures are absolutely continuous with respect to the Lebesgue measure. In particular,

$$\int_{D_{\epsilon_k}} v_{\epsilon_k}(x)^2 \min(e^{-V(x)},1)dx \xrightarrow[k\to\infty]{} \int_{D_0} m(x)\min(e^{-V(x)},1)dx.$$

We deduce from (36) that

$$\int_{D_0} m(x)\min(e^{-V(x)},1)dx = 0.$$

But $\min(e^{-V(\cdot)},1)$ is continuous and positive on D_0, so that m vanishes almost every where. Finally, by the convergence property (40) applied to $\phi = 1$ almost everywhere, we have

$$1 = \int_{D_{\epsilon_k}} v_{\epsilon_k}(x)^2 dx \xrightarrow[k\to\infty]{} 0,$$

which is absurd. Finally A is strictly positive.

Fix an arbitrary positive constant $\alpha > 0$ and let us prove that one can find a compact set $K_\alpha \subset D_0$ such that

$$\int_{K_\alpha^c} \eta_\epsilon(x)d\sigma(x) \leq \alpha,\ \forall \epsilon \in]0,1[. \tag{41}$$

Let R be the positive constant of the fifth part of Hypothesis 2. For all compact set K, we have

$$\int_{K^c} \eta_\epsilon(x)d\sigma(x) = \int_{K^c\cap\{d(x,\partial D_0)>R\}} \eta_\epsilon(x)d\sigma(x) + \int_{K^c\cap\{d(x,\partial D_0)\leq R\}} \eta_\epsilon(x)d\sigma(x). \tag{42}$$

But, from the proof of [6, Proposition B.6],

$$\int_{K^c\cap\{d(x,\partial D_0)>R\}} \eta_\epsilon(x)d\sigma(x) \leq \sqrt{\int_{K^c\cap\{d(x,\partial D_0)>R\}} e^{-2V(x)}dx} \tag{43}$$

and

$$\int_{K^c\cap\{d(x,\partial D_0)\leq R\}} \eta_\epsilon(x)d\sigma(x) \leq e^{C/2}e^{\lambda_\epsilon}\|v_\epsilon\|_\infty \int_{K^c\cap\{d(x,\partial D_0)\leq R\}} \left(\int_D p_1^{D_0}(x,y)dy\right) dx. \tag{44}$$

On the one hand, $e^{\lambda_\epsilon}\|v_\epsilon\|_\infty$ is uniformly bounded above by $e^{\lambda_1}\kappa$. On the other hand, both integrals on the right hand side are well defined, thanks to Hypothesis 2. Finally, one can find a compact set K_α such that (43) and (44) are both bounded by $\alpha/2$. Since (41) is fulfilled for all $\alpha > 0$, the family $(\eta_\epsilon d\sigma)_{0<\epsilon<1}$ is uniformly tight.

Finally, it yields from equality (33) and the uniform bound A, that the family $(\nu_\epsilon)_{\epsilon>0}$ is uniformly tight. Moreover, ν_ϵ has a density which is bounded by $\kappa e^{-V}/A$, uniformly in $\epsilon > 0$. Then it is uniformly bounded on every compact set, so that every limiting distribution is absolutely continuous with respect to the Lebesgue measure, with a density bounded by $\kappa e^{-V}/A$. □

3.2.2 Uniqueness of the limiting probability

Proposition 3.4. *Assume that Hypothesis 2 holds. Let ν be a probability measure which is the limit of a sub-sequence $(\nu_{\epsilon_k})_{k\in\mathbb{N}}$, where $\epsilon_k \to 0$ when $k \to \infty$. Then ν is the Yaglom limit ν_0 associated with $\mathbb{P}^0$.*

Proof of Proposition 3.4. Thanks to Proposition 3.2, ν has a density η with respect to σ, and $\eta \leq \kappa e^V/A$. Let us prove that η belongs to $\mathbb{L}^2(d\sigma)$. Since $\nu_{\epsilon_k} \to \nu$, we have, for all $f \in C_0(D_0,\mathbb{R})$ (which denotes the set of continuous real functions with compact support on D_0),

$$\begin{aligned}\int_{D_0} f(x)\eta(x)^2 d\sigma(x) = \int_{D_0} f(x)\eta(x)d\nu(x) &= \lim_{k\to\infty}\int_{D_0} f(x)\eta(x)d\nu_{\epsilon_k}(x)\\ &= \lim_{k\to\infty}\int_{D_0} f(x)\frac{\eta_\epsilon(x)}{\langle\eta_{\epsilon_k},\mathbf{1}_{D_\epsilon}\rangle}d\nu(x)\end{aligned}$$

since $\frac{\eta_{\epsilon_k}(x)}{\left\langle \eta_{\epsilon_k}, \mathbf{1}_{D_\epsilon}\right\rangle}$ is the density of ν_{ϵ_k} with respect to σ, by (33). For the same reasons,

$$\int_{D_0} f(x)\eta(x)^2 d\sigma(x) = \lim_{k\to\infty}\lim_{k'\to\infty}\int_{D_0} f(x)\frac{\eta_{\epsilon_k}(x)\eta_{\epsilon_{k'}}(x)}{\left\langle \eta_{\epsilon_k}, \mathbf{1}_{D_{\epsilon_k}}\right\rangle_\sigma \left\langle \eta_{\epsilon_{k'}}, \mathbf{1}_{D_{\epsilon_{k'}}}\right\rangle_\sigma} d\sigma(x).$$

For all $\epsilon > 0$, we have $\int_{D_0}\eta_\epsilon^2 d\sigma(x) = 1$ by (32), and $\langle \eta_\epsilon, \mathbf{1}_{D_\epsilon}\rangle_\sigma > A$ by (34). Then, by the Cauchy-Schwarz inequality, we get

$$\int_{D_0} f(x)\eta(x)^2 d\sigma(x) \leq \|f\|_\infty / A^2.$$

It yields that $\eta \in \mathbb{L}^2(d\sigma)$.

We denote by E_0 the orthogonal space of η_0 in $\mathbb{L}^2(d\sigma)$. We prove that η is proportional to η_0 by showing that η is orthogonal to $E_0 \cap C_0(D_0)$. For all $f \in E_0\cap C_0(D_0)$ and all $x \in D_0$, $P_t^{\epsilon_k} f(x)$ converges to $P_t^0 f(x)$ when $k \to \infty$. But $\nu_{\epsilon_k} \to \nu$ when $k\to\infty$, then we have

$$\begin{aligned}
\left\langle P_t^0 f, \eta\right\rangle_\sigma &= \int_{D_0} P_t^0 f(x) d\nu(x)\\
&= \lim_{k\to\infty}\int_{D_0} P_t^{\epsilon_k} f(x) d\nu_{\epsilon_k}(x)\\
&= \lim_{k\to\infty} e^{-\lambda_{\epsilon_k} t}\int_{D_0} f(x) d\nu_{\epsilon_k}(x),
\end{aligned}$$

where the last equality comes from [16]. But $\lambda_k \to \lambda_0$ by (39) and $\int_{D_0} f(x)d\nu_{\epsilon_k}(x) \to \int_{D_0} f(x) d\nu(x)$ when $k\to\infty$. As a consequence,

$$\left\langle \eta, P_t^0 f\right\rangle_\sigma = e^{-\lambda_0 t}\left\langle \eta, f\right\rangle_\sigma, \ \forall f \in E_0\cap C_0(D_0),\ \forall t \geq 0.$$

But η belongs to $\mathbb{L}^2(d\sigma)$, then we have by [6, Theorem A.4]

$$\lim_{t\to\infty} e^{\lambda_0 t}\left\langle \eta, P_t^0 f\right\rangle = 0,\ \forall f \in E_0 \cap C_0(D_0).$$

We deduce that $\langle \eta, f\rangle_\sigma = 0$ for all $f \in E_0\cap C_0(D_0)$. This allows us to conclude that η is proportional to η_0. Finally, ν and ν_0 are two proportional probabilities, then $\nu = \nu_0$. $\square$

3.3 Numerical simulations

3.3.1 The Wright-Fisher case

The Wright-Fisher with values in]0,1[conditioned to be killed at 0 is the diffusion process driven by the SDE

$$dZ_t = \sqrt{Z_t(1-Z_t)}dB_t - Z_t dt,\ Z_0 = z \in]0,1[,$$

and killed when it hits 0 (1 is never reached). Huillet proved in [19] that the Yaglom limit of this process exists and has the density $2-2x$ with respect to the Lebesgue measure.

In order to apply Theorem 3.1, we define $\mathbb{P}^0$ as the law of $X_. = \arccos(1-2Z_.)$. Then $\mathbb{P}^0$ is the law of the diffusion process with values in $]0,\pi[$, driven by the SDE

$$dX_t = dB_t - \frac{1-2\cos X_t}{2\sin X_t}dt,\ X_0 = x \in]0,\pi[,$$

killed when it hits 0 (π is never reached). One can easily check that this diffusion process fulfills Hypothesis 2. We denote by ν_0 its Yaglom limit.

For all $\epsilon \in]0,\pi/2[$, we define $D_\epsilon =]\epsilon,\pi-\epsilon[$. Let $\mathbb{P}^\epsilon$ and ν_ϵ be as in Section 3. We proceed to the numerical simulation of the N-interacting particle system $(X^{\epsilon,1},...,X^{\epsilon,N})$ with $\epsilon = 0.001$ and $N = 1000$. This leads us to the computation of $E(\mathcal{X}^{N,\epsilon})$, which is an approximation of ν_0. After the change of variable $Z. = 2\cos(X.)$, we see on Figure 3 that the simulation is very close to the expected result $(2-2x)dx$, which shows the efficiency of the method.

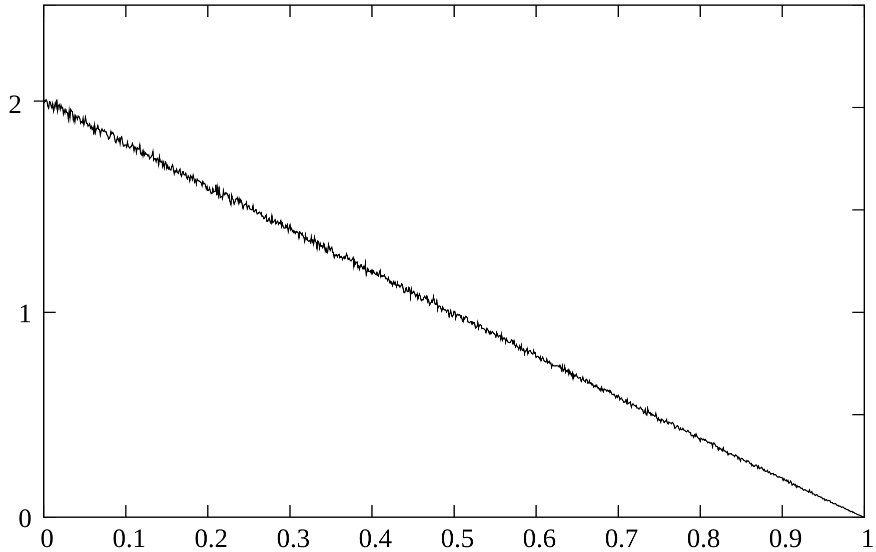

Figure 3: $E(\mathcal{X}^{\epsilon,N})$ in the Wright-Fisher case

3.3.2 The logistic case

The logistic Feller diffusion with values in $]0,+\infty[$ is defined by the stochastic differential equation

$$dZ_t = \sqrt{Z_t}dB_t + (rZ_t - cZ_t^2)dt,\ Z_0 = z > 0, \tag{45}$$

and killed when it hits 0. Here B is a 1-dimensional Brownian motion and r,c are two positive constants. In order to use Theorem 3.1, we make the change of variable $X. = 2\sqrt{Z.}$. This leads us to the study of the diffusion process with values in $D_0 =]0,+\infty[$, which is killed at 0 and satisfies the SDE

$$dX_t = dB_t - \left(\frac{1}{2X_t} - \frac{rX_t}{2} + \frac{cX_t^3}{4}\right)dt,\ X_0 = x \in]0,+\infty[.$$

We denote by $\mathbb{P}^0$ its law. Cattiaux et al. proved in [5] that Hypothesis 2 is fulfilled in this case. Then the Yaglom limit ν_0 associated with $\mathbb{P}^0$ exists and one can apply Theorem 3.1

with $D_\epsilon =]1/\epsilon,\epsilon[$ for all $\epsilon \in]0,1/2[$. As above and for all $N \geq 2$, we denote by $\mathbb{P}^\epsilon$ the law of the diffusion process restricted to D_ϵ and by $\mathcal{X}^{\epsilon,N}$ the empirical stationary distribution of the N-interacting particle process associated with $\mathbb{P}^\epsilon$.

We've proceeded to the numerical simulation of the interacting particle process for a large number of particles and a small value of ϵ. This allows us to compute $E(\mathcal{X}^{\epsilon,N})$, which gives us a numerical approximation of ν_0, thanks to Theorem 3.1.

In the numerical simulations below, we set ϵ equal to 0.0001 and $N = 10000$. We compute $E(\mathcal{X}^{\epsilon,N})$ for different values of the parameters r and c in (45). The results are grahically represented in Figure 4. As it could be wanted for, greater is c, closer is the support of the QSD to 0. We thus numerically describe the impact of the linear and quadratic terms on the Yaglom limit.

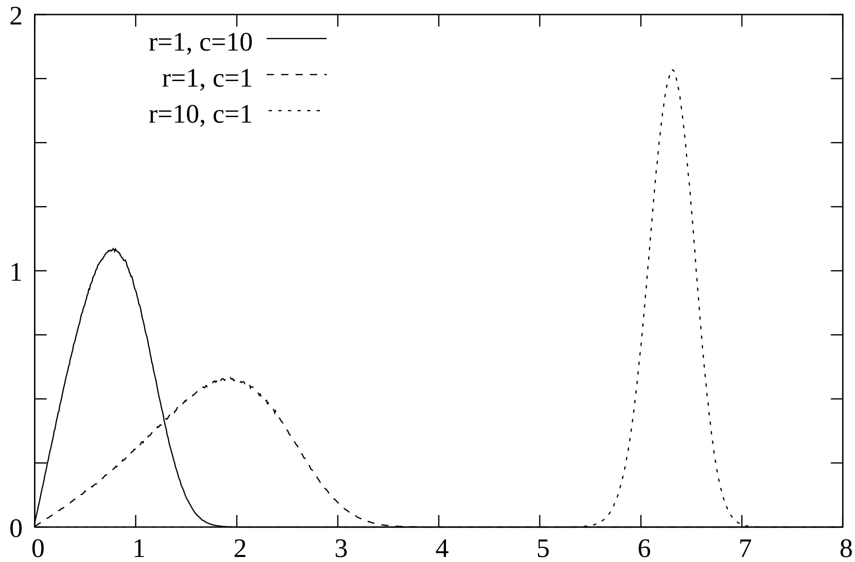

Figure 4: $E(\mathcal{X}^{\epsilon,N})$ for the diffusion process (45), with different values of r and c

3.3.3 Stochastic Lotka-Volterra Model

We apply our results to the stochastic Lotka-Volterra system with values in $D = \mathbb{R}_+^2$ studied in [6], which is defined by the following stochastic differential system

$$\begin{aligned} dZ_t^1 &= \sqrt{\gamma_1 Z_t^1} dB_t^1 + \left(r_1 Z_t^1 - c_{11}(Z_t^1)^2 - c_{12} Z_t^1 Z_t^2\right) dt, \\ dZ_t^2 &= \sqrt{\gamma_2 Z_t^2} dB_t^2 + \left(r_2 Z_t^2 - c_{21} Z_t^1 Z_t^2 - c_{22}(Z_t^2)^2\right) dt, \end{aligned}$$

where (B^1,B^2) is a bi-dimensional Brownian motion. We are interested in the process absorbed at ∂D.

More precisely, we study the process $(X^1,X^2) = (2\sqrt{Z_t^1/\gamma_1},2\sqrt{Z_t^2/\gamma_2})$, with values in $D_0 = \mathbb{R}_+^2$, which satisfies the SDE (21) and is killed at ∂D_0. We denote its law by $\mathbb{P}^0$. The coefficients are supposed to satisfy

$$c_{11},c_{21} > 0,\ c_{12}\gamma_2 = c_{21}\gamma_1 < 0 \text{ and } c_{11}c_{22} - c_{12}c_{21} > 0. \tag{46}$$

In [6], this case was called *the weak cooperative case* and the authors proved that it is a sufficient condition for Hypothesis 2 to be fulfilled. Then the Yaglom limit $\nu_0 = \lim_{t\to+\infty}\mathbb{P}^0_x\left((Y^1_t,Y^2_t)\in .|t<\tau_\partial\right)$ is well defined and we are able to apply Theorem 3.1. For each $\epsilon>0$, we define D_ϵ as it is described on Figure 5. With this definition, it is clear that all conditions of Theorems 2.1 and 3.1 are fulfilled.

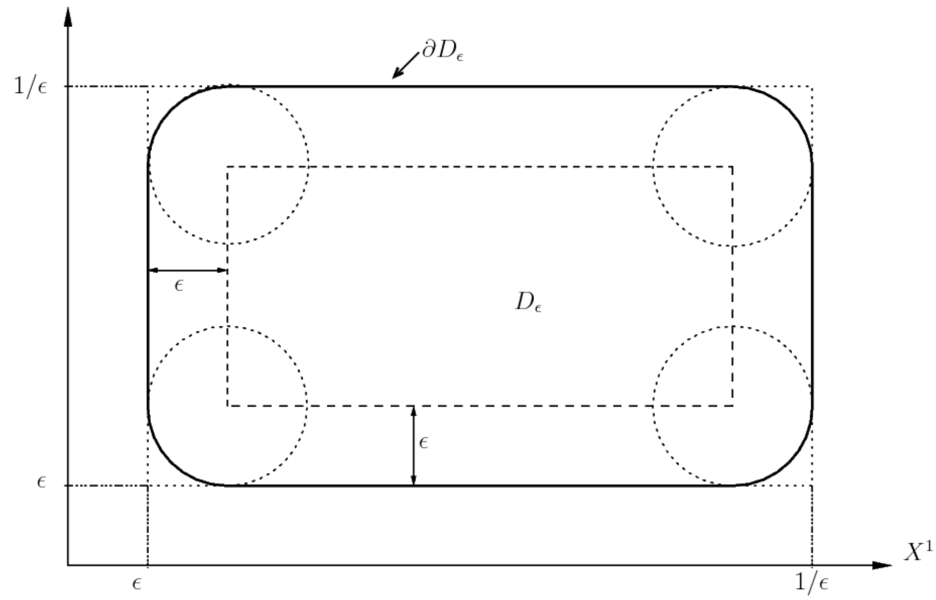

Figure 5: Definition of D_ϵ

We choose $\epsilon = 0.0001$ and we simulate the long time behavior of the interacting particle process with $N = 10000$ particles for different values of c_{12} and c_{21}. The values of the other parameters are $r_1 = 1 = r_2 = 1$, $c_{11} = c_{22} = 1$, $\gamma_1 = \gamma_2 = 1$. The results are illustrated on Figure 6. One can observe that a greater value of the cooperating coefficients $-c_{12} = -c_{21}$ leads to a Yaglom limit whose support is further from the boundary and covers a smaller area. In other words, the more the two populations cooperate, the bigger the surviving populations are.

Acknowledgments I am extremely grateful to my Ph.D. supervisor Sylvie Méléard for his careful and indispensable help on the form and the content of this paper. I would like to thank Pierre Collet and my colleagues Jean-Baptiste Bellet and Khalid Jalalzai for their advices on functional analysis.

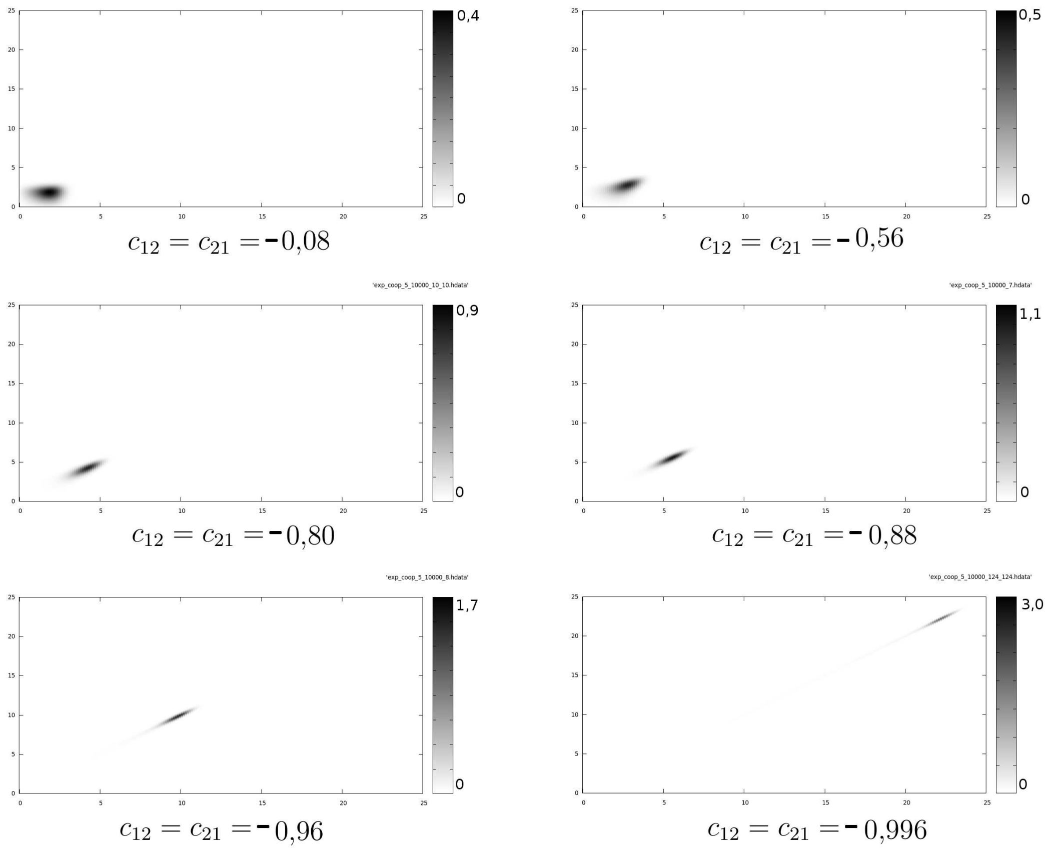

Figure 6: Empirical stationary distribution of the interacting particle process for different values of $c_{12} = c_{21}$

References

[1] O. O. Aalen and H. K. Gjessing. Understanding the shape of the hazard rate: a process point of view. *Statist. Sci.*, 16(1):1–22, 2001. With comments and a rejoinder by the authors.

[2] G. Allaire. *Analyse numérique et optimisation.* Éditions de l'École Polytechnique, 2005.

[3] K. Burdzy, R. Holyst, D. Ingerman, and P. March. Configurational transition in a fleming-viot-type model and probabilistic interpretation of laplacian eigenfunctions. *J. Phys. A*, 29(29):2633–2642, 1996.

[4] K. Burdzy, R. Hołyst, and P. March. A Fleming-Viot particle representation of the Dirichlet Laplacian. *Comm. Math. Phys.*, 214(3):679–703, 2000.

[5] P. Cattiaux, P. Collet, A. Lambert, S. Martínez, S. Méléard, and J. San Martín. Quasi-stationary distributions and diffusion models in population dynamics. *Ann. Probab.*, 37(5):1926–1969, 2009.

[6] P. Cattiaux and S. Méléard. Competitive or weak cooperative stochastic lotka-volterra systems conditioned to non-extinction. *J. Math. Biology*, (6):797–829, 2010.

[7] J. A. Cavender. Quasi-stationary distributions of birth-and-death processes. *Adv. Appl. Probab.*, 10(3):570–586, 1978.

[8] M. Chaleyat-Maurel and N. E. Karoui. Un problème de réflexion et ses applications au temps local et aux équations différentielles stochastiques sur $\mathbb{R}$. cas continu. *Astérisque*, 52-53:117–144, 1978.

[9] P. Collet, S. Martínez, and J. San Martín. Asymptotic laws for one-dimensional diffusions conditioned to nonabsorption. *Ann. Probab.*, 23(3):1300–1314, 1995.

[10] J. N. Darroch and E. Seneta. On quasi-stationary distributions in absorbing discrete-time finite Markov chains. *J. Appl. Probability*, 2:88–100, 1965.

[11] M. C. Delfour and J.-P. Zolésio. *Shapes and geometries*, volume 4 of *Advances in Design and Control.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001. Analysis, differential calculus, and optimization.

[12] D. Down, S. P. Meyn, and R. L. Tweedie. Exponential and uniform ergodicity of Markov processes. *Ann. Probab.*, 23(4):1671–1691, 1995.

[13] S. N. Ethier and T. G. Kurtz. *Markov processes.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986. Characterization and convergence.

[14] P. A. Ferrari, H. Kesten, S. Martinez, and P. Picco. Existence of quasi-stationary distributions. A renewal dynamical approach. *Ann. Probab.*, 23(2):501–521, 1995.

[15] P. A. Ferrari and N. Marić. Quasi stationary distributions and Fleming-Viot processes in countable spaces. *Electron. J. Probab.*, 12:no. 24, 684–702 (electronic), 2007.

[16] G. L. Gong, M. P. Qian, and Z. X. Zhao. Killed diffusions and their conditioning. *Probab. Theory Related Fields*, 80(1):151–167, 1988.

[17] I. Grigorescu and M. Kang. Hydrodynamic limit for a Fleming-Viot type system. *Stochastic Process. Appl.*, 110(1):111–143, 2004.

[18] I. Grigorescu and M. Kang. The Doeblin condition for a class of diffusions with jumps. Preprint, 2009.

[19] T. Huillet. On Wright Fisher diffusion and its relatives. *Journal of Statistical Mechanics: Theory and Experiment*, 11:6–+, Nov. 2007.

[20] A. Jakubowski. Tightness criteria for random measures with application to the principle of conditioning in Hilbert spaces. *Probab. Math. Statist.*, 9(1):95–114, 1988.

[21] M. Kolb and D. Steinsaltz. Quasilimiting behavior for one-dimensional diffusions with killing. *Available at http://arxiv.org/abs/1004.5044*, 2010.

[22] T. Li and J. J. Anderson. The vitality model: A way to understand population survival and demographic heterogeneity. *Theoretical Population Biology*, 76(2):118 – 131, 2009.

[23] M. Lladser and J. San Martín. Domain of attraction of the quasi-stationary distributions for the Ornstein-Uhlenbeck process. *J. Appl. Probab.*, 37(2):511–520, 2000.

[24] J.-U. Löbus. A stationary Fleming-Viot type Brownian particle system. *Math. Z.*, 263(3):541–581, 2009.

[25] S. Martinez, P. Picco, and J. San Martin. Domain of attraction of quasi-stationary distributions for the Brownian motion with drift. *Adv. in Appl. Probab.*, 30(2):385–408, 1998.

[26] I. Nåsell. Extinction and quasi-stationarity in the verhulst logistic model. *Journal of Theoretical Biology*, 211(1):11 – 27, 2001.

[27] P. Polett. Quasi-stationary distributions : a bibliography. http://www.maths.uq.edu.au/~pkp/papers/qsds/qsds.pdf.

[28] E. Priola and F.-Y. Wang. Gradient estimates for diffusion semigroups with singular coefficients. *J. Funct. Anal.*, 236(1):244–264, 2006.

[29] D. Steinsaltz and S. N. Evans. Markov mortality models: Implications of quasistationarity and varying initial conditions. *Theo. Pop. Bio.*, 65(65):319–337, 2004.

[30] A. M. Yaglom. Certain limit theorems of the theory of branching random processes. *Doklady Akad. Nauk SSSR (N.S.)*, 56:795–798, 1947.

[31] K. Yosida. *Functional Analysis, volume 123 of Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen.* Springer-Verlag, 1968.