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The bounded confidence model of opinion dynamics, introduced by Deffuant *et al.*, is a stochastic model for the evolution of [0,1]-valued opinions within a finite group of peers. We show that as time goes to infinity, the opinions evolve into a random non-interacting set of clusters, and subsequently the opinions in each cluster converge to their barycenter; the limit empirical distribution is called a partial consensus. Then, we prove a mean-field limit result: for i.i.d. initial opinions, as the number of peers increases and time is rescaled accordingly, the peers asymptotically behave as i.i.d. peers, each influenced by opinions drawn independently from the unique solution of a nonlinear integro-differential equation. As a consequence, the (random) empirical distribution process converges to this (deterministic) solution. We also show that as time goes to infinity, this solution converges to a partial consensus, and identify sufficient conditions for the limit not to depend on the initial condition, and for formation of total consensus. Finally, we show that if the equation has an initial condition with a density, then its solution has a density at all times, develop a numerical scheme to solve the corresponding functional equation of the Kac type, and show, using numerical examples, that bifurcations may occur.

Keywords: Social networks; reputation; opinion; mean-field limit; propagation of chaos; nonlinear integro-differential equation; kinetic equation; numerical experiments.

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1. Introduction

Some models about opinion dynamics are based on binary values,[15,3,32,24,38,34] and often lead to attractors that display uniformity of opinions. These models are not valid for scenarios such as the social network of truck drivers interested in the quality of food of a highway restaurant or the critics' ratings about the new opening movies, for which it is required to have a continuous spectrum of opinions. For example, a

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continuous model is widely used in politics when people are positioned according to how left- (or right-) wing their opinions are.[12]

One of the most popular models is the bounded confidence model introduced by Deffuant et al.,[9] in which repeatedly two peers are randomly selected and influence one another if their $[0, 1]$-valued opinions differ by less than a deviation threshold. This has been extensively studied: from the topological point of view, ranging from random graphs[13] to lattice topology,[39,40] and from the dimensional point of view, as it can be generalized to multidimensional vector opinions.[31,40] There are also studies[39] in which the deviation threshold depends on the peer (*i.e.*, some people are more tolerant than others) or in which the interaction is performed and averaged among all the potential pairs of peers that tolerate each other.[11,21]

Reputation systems have lately emerged due to the necessity to measure trust about users while doing transactions over the internet. Popular examples that use reputation systems are e-Bay[33] or Bizrate.[37] The model introduced in Refs 25 and 7 for the evolution of the trust and the potential effects that a group of liars might have while trying to attack the system is a generalization of the bounded confidence model, in particular when there are no liars nor direct observations and the evolution of the system is only carried by interaction throughout the different peers.

The mean-field approach is a deterministic approximation for a large number of peers. It has been used in many different contexts such as TCP connections,[36,4,20] HTTP flows,[5] bandwidth sharing between streaming and file transfers,[23] mobile networks,[8], robot swarms,[27] transportation systems,[2] and reputation systems.[25,30,29] For online reputation systems it is appealing to use such methods, as the number of users may be very large (over 400 million for Facebook[1]).

However, justifying the validity of the mean-field approach is not easy, and the proof methods in the cited papers do not apply here, as we discuss in Section 4.

In this paper, we consider a fully connected network, the same deviation test for all peers, and $[0, 1]$-valued opinions. The model (and mean-field approach) can be easily generalized to vector-valued opinions. We make the following contributions.

• We prove that in the model with finitely many peers, as time increases, opinions tend to group into clusters, the number of which remains constant after some random finite time. Subsequently, within every cluster all opinions converge to their barycenter. The distribution of opinions thus converges to a degenerate form, which we call a "partial consensus", in which there are only a small number of fixed opinions which differ too much to influence each other.

• We prove a mean-field limit result, propagation of chaos: in the limit when the number of peers goes to infinity and time is rescaled accordingly, if the peers are i.i.d. of arbitrary law m_0 at time zero, then the processes of their opinions become i.i.d., each with a nonlinear Markovian evolution corresponding to that of a peer being influenced by opinions drawn independently from the unique solution of a nonlinear integro-differential equation starting at m_0. If m_0 has a density, then this solution has a density at all times, which satisfies a functional formulation of this equation. This implies law of large numbers results: the (random) process empirical measures converge to the (deterministic) Markovian law described above, and their marginal processes converge to the solution of the integro-differential equation.

The probabilistic structure of the limit equation is the same as that of kinetic equations, such as the cutoff spatially-homogeneous Boltzmann or Kac equations, used in statistical mechanics to describe the limit of certain particle systems with binary interaction. The mean-field limit proofs in this paper use results elaborated in this setting by Graham-Méléard[19,18] and Desvillettes *et al.*[10]

From this perspective, the limit is as if each peer were influenced by an infinite supply of independent statistically similar peers, which have instantaneous opinions distributed according to the nonlinear integro-differential equation given by the consistency relations coming from the resulting feedback.

To the best of our knowledge, this is the first rigorous mean-field limit result for this opinion model. Similar integro-differential equations were used before,[9,6,26] but without formal justification. Our equation differs by a factor 2, and the equations in these references appear to be slightly incorrect. This illustrates the importance of being able to derive the macroscopic equation from a microscopic description, as we do in this paper. Our result is also more general in that we make no particular assumption on m_0 (other than being a probability distribution).

• The mean-field limit has similar long-time behavior as the model with finitely many peers, namely it converges to a partial consensus.

• We develop a numerical method for the integro-differential equation, and use it to explore the properties of the model. We observe phase transitions while varying the deviation threshold. We model the scenario of a company fusion, categorizing the workers into "undecided" and "extremists". We obtain that having 20% of the workers "undecided" is enough to unite the two factions of extremists and achieve consensus. Last, we establish a bound on the deviation threshold, in order to determine if there is total consensus or not, under the assumption of symmetric initial conditions.

The rest of the paper is organized as follows. Section 2 describes the finite model, and Section 3 studies some of its long time properties. Section 4 rigorously derives the mean-field limit, and Section 5 studies some of its long time properties. Section 6 is devoted to numerical results for the mean-field limit. All proofs are in Section 7.

2. Model and Notation

We use the model for $N \geq 2$ interacting peers introduced in Ref 9. The random variable (r.v.) $X_i^N(k)$ with values in $[0,1]$ denotes the reputation record kept at peer $i \in \{1,\dots,N\}$ at time $k \in \mathbb{N} = \{0,1,\dots\}$, representing its belief (or opinion) about a given subject, the same for all peers. The system state at time $k \in \mathbb{N}$ is given by the collection $X^N(k) = (X_i^N(k))_{1\leq i\leq N}$.

The discrete-time process $X^N = (X^N(k), k \in \mathbb{N})$ evolves in function of the *deviation threshold* $\Delta \in (0,1]$ and the *confidence factor* $w \in (0,1)$. At each instant k, two peers i and j are selected uniformly at random without replacement, and

- if $|X_i^N(k) - X_j^N(k)| > \Delta$ then $X^N(k+1) = X^N(k)$, the two peers' beliefs being too different for mutual influence,

- if $|X_i^N(k) - X_j^N(k)| \le \Delta$ then the values of peers i and j are updated to

$$X_i^N(k+1) = wX_i^N(k) + (1-w)X_j^N(k)\,,$$
$$X_j^N(k+1) = wX_j^N(k) + (1-w)X_i^N(k)\,,$$

  and the values of the other peers do not change at time $k+1$, the two peers having sufficiently close beliefs to influence each other.

Small values of Δ and large values of w mean that the peers trust very much their own beliefs in comparison to the new information given by the other interacting peer. If $w = \frac{1}{2}$ then both peers will have the average value after actual mutual influence. The extreme excluded values $\Delta = 0$ or $w = 1$ correspond to peers never changing belief, and $w = 0$ to peers exchanging beliefs if these are close enough.

Of interest is the reduced description given by the *empirical measure* Λ^N with samples in $\mathcal{P}([0,1]^{\mathbb{N}})$, and its marginal process $M^N = (M^N(k), k \in \mathbb{N})$ also called the *occupancy process* with sample paths in $\mathcal{P}([0,1])^{\mathbb{N}}$, given by

$$\Lambda^N = \frac{1}{N}\sum_{n=1}^{N} \delta_{X_i^N} = \frac{1}{N}\sum_{n=1}^{N} \delta_{(X_i^N(k), k\in\mathbb{N})}\,, \qquad M^N(k) = \frac{1}{N}\sum_{n=1}^{N} \delta_{X_i^N(k)}\,,$$

so that, for bounded measurable $g : [0,1]^{\mathbb{N}} \to \mathbb{R}$ and $h : [0,1] \to \mathbb{R}$,

$$\langle g, \Lambda^N \rangle = \frac{1}{N}\sum_{n=1}^{N} g(X_i^N)\,, \qquad \langle h, M^N(k)\rangle = \frac{1}{N}\sum_{n=1}^{N} h(X_i^N(k))\,.$$

We will also re-scale time as $t = \frac{k}{N}$ and use the rescaled occupancy process $\tilde{M}^N = (\tilde{M}^N(t), t \in \mathbb{R}_+)$, given by $\tilde{M}^N(t) = M^N(\lfloor Nt \rfloor)$. In Section 4, this process is shown to converge in probability to a deterministic process $(m(t), t \in \mathbb{R}_+)$, called the "mean field limit".

3. Long-time behavior of the finite N model

When the number of peers is fixed and finite, we prove that as k goes to infinity $M^N(k)$ converges almost surely to a *random* probability measure $M^N(\infty)$. We show that $M^N(\infty)$ is a combination of at most $\lceil \frac{1}{\Delta} \rceil$ Dirac measures at points separated by at least Δ. A key observation here is that if h is any convex function then $\langle h, M^N(k)\rangle$ is non-increasing in k. Dittmer and Krause [11,22] obtained similar results, but for a deterministic model.

Definition 3.1. We say that $\nu \in \mathcal{P}[0,1]$ is a *partial consensus with* m_0 *components* if $\nu = \sum_{m=1}^{m_0} \alpha_m \delta_{x_m}$ with $x_m \in [0,1]$, $|x_m - x_{m'}| > \Delta$ for $m \neq m'$, and $\alpha_m > 0$. Necessarily $m_0 \le \lceil \frac{1}{\Delta} \rceil$ and $\sum_{m=0}^{m_0} \alpha_m = 1$. If $m_0 = 1$, *i.e.*, if ν is a Dirac measure, we say that ν is a *total consensus*.

If $M^N(k)$ is a partial consensus, then peers are grouped in a number of components, within one component all peers have the same value, and components are too far to interact. A partial consensus is an absorbing state for M^N, and Theorem 3.9 below shows that $M^N(k)$ converges almost surely as $k \to \infty$ to one such state.

3.1. *Convexity and Moments*

We start with results about convexity and moments, which are needed to establish the convergence result, and are also of independent interest.

Proposition 3.2. *For any convex function $h : [0,1] \to \mathbb{R}$, any x, y, and w in $[0,1]$,*

$$h\left(wx + (1-w)y\right) + h\left(wy + (1-w)x\right) - h(x) - h(y) \leq 0$$

with equality when h is strictly convex possible only if $x = y$ or $w = 0$ or $w = 1$.

It follows immediately from the model definition that in any sample path, moments are non-decreasing with time, and the first moment is constant.

Corollary 3.3. *If $h : [0,1] \to \mathbb{R}$ is a convex function, then $\langle h, M^N(k)\rangle$ is a non-increasing function of k along any sample path.*

Applying Corollary 3.3 to $h(x) = x$, $h(x) = -x$ and $h(x) = x^n$ gives the following:

Corollary 3.4. *For $n = 1, 2, \ldots$ and $k \in \mathbb{N}$, let $\mu_n^N(k) = \frac{1}{N}\sum_{i=1}^N X_i^N(k)^n$ denote the n-th moment of $M^N(k)$ and $\sigma^N(k)$ the standard deviation, given by $\sigma^N(k)^2 = \mu_2^N(k) - \mu_1^N(k)^2$.*

(1) The mean $\mu_1^N(k)$ is stationary in k, i.e., *$\mu_1^N(k) = \mu_1^N(0)$.*
(2) The moments and the standard deviation are non-increasing in k, i.e., *if $k \leq k'$ then $\mu_n^N(k) \geq \mu_n^N(k')$ and $\sigma^N(k) \geq \sigma^N(k')$*

Next, we prove that stationarity of moments is equivalent to reaching partial consensus:

Proposition 3.5. *If $M^N(k)$ is a partial consensus, then $\mu_n^N(k') = \mu_n^N(k)$ for all $n \geq 1$ and $k' \geq k$. Conversely, if for some $n \geq 2$ there exists a (random) instant k such that $\mu_n^N(k') = \mu_n^N(k)$ for all $k' \geq k$, then $M^N(k') = M^N(k)$ for all $k' \geq k$ and $M^N(k)$ is a partial consensus, almost surely.*

3.2. *Almost Sure Convergence to Partial Consensus*

Definition 3.6. We say that two peers i and j are *connected* at time k if their values x and y satisfy $|y - x| \leq \Delta$. We say that $F \subset \{1, 2, \ldots, N\}$ is a *cluster* at time k if it is a maximal connected component.

In other words, a cluster is a maximal set of peers such that every peer can pass the deviation test with one neighbour in the cluster. The set of clusters at time k

is a random partition of the set of peers. The following proposition states that a cluster can either split or stay constant, but cannot grow.

Proposition 3.7. *Let $\mathcal{C}^N(k) = \{C_1, \dots, C_\ell\}$ be the set of clusters at time k. Then either $\mathcal{C}^N(k+1) = \mathcal{C}^N(k)$ or $\mathcal{C}^N(k+1) = \left(\mathcal{C}^N(k) \setminus C_{\ell_1}\right) \cup \mathcal{C}'$ where $\mathcal{C}'$ is a partition of C_{ℓ_1}, for some $\ell_1 \in \{1, \dots, \ell\}$.*

The number of clusters is thus non decreasing, and since it is bounded by $\lceil \frac{1}{\Delta} \rceil$, it must be constant after some time. The previous proposition implies that the clusters themselves remain unchanged, *i.e.*, we have shown the following corollary.

Corollary 3.8. *There exists a random time K^N such that*

$$\mathcal{C}^N(k) = \mathcal{C}^N(K^N) \text{ for } k \geq K^N .$$

Finally, we show that the occupancy measure converges to a partial consensus. Let L^N be the *final number of clusters*, *i.e.*, the cardinality of $\mathcal{C}^N(K^N)$.

Theorem 3.9. *As k goes to infinity, $M^N(k)$ converges almost surely, for the weak topology on $\mathcal{P}[0,1]$, to a random probability $M^N(\infty)$, which is a partial consensus with L^N components.*

We use the usual weak topology for probability measures, for which ν_n converges to ν if and only if $\langle f, \nu_n \rangle$ converges to $\langle f, \nu \rangle$ for any continuous (and hence bounded) $f : [0,1] \to \mathbb{R}$. Equivalently, the cumulative distribution function (CDF) of ν_n converges to the CDF of ν at all continuity points of the limit.

Theorem 3.9 notably implies that there is convergence to total consensus if and only if $L^N = 1$. The probability $p^* := \mathbb{P}(L^N = 1)$ of convergence to total consensus is not necessarily 0 or 1, but:

(1) If the diameter of $M^N(0)$ is less than Δ (*i.e.*, $\max_{i,j} \left|X_i^N(0) - X_j^N(0)\right| < \Delta$) then $p^* = 1$ (obvious);
(2) If there is more than 1 cluster in $M^N(0)$ then $p^* = 0$ (follows from Proposition 3.7).

4. Mean-field limit results when N goes to infinity

4.1. *Topological and measure-theoretic preliminaries*

Let $\mathcal{S}$ be a metric space with a σ-field (not necessarily the Borel σ-field), $\mathcal{P}(\mathcal{S})$ the space of probability measures on $\mathcal{S}$ (for this σ-field), and $D(\mathbb{R}_+, \mathcal{S})$ the Skorohod space of right-continuous paths with left-hand limits (for this metric).

When $\mathcal{S}$ is given the Borel σ-field, the weak topology of $\mathcal{P}(\mathcal{S})$ corresponds to the convergences

$$P_n \xrightarrow[n\to\infty]{\text{weak}} P \Leftrightarrow \langle f, P_n \rangle \xrightarrow[n\to\infty]{} \langle f, P \rangle \,, \ \forall f \in C_b(\mathcal{S}, \mathbb{R})$$

where $C_b(\mathcal{S},\mathbb{R})$ denotes the space of continuous bounded functions. Convergence in law of random elements, defined possibly on distinct probability spaces but having common sample space $\mathcal{S}$, is defined as weak convergence of their laws:

$$Y_n \xrightarrow[n\to\infty]{\text{law}} Y \Leftrightarrow \mathcal{L}(Y_n) \xrightarrow[n\to\infty]{\text{weak}} \mathcal{L}(Y) \Leftrightarrow \mathbb{E}(f(Y_n)) \xrightarrow[n\to\infty]{} \mathbb{E}(f(Y))\,,\ \forall f\in C_b(\mathcal{S},\mathbb{R})\,.$$

If $\mathcal{S}$ is separable and is given the Borel σ-field, then the weak topology is metrizable and $\mathcal{P}(\mathcal{S})$ is separable (Ethier-Kurtz,[14] Theorems 3.3.1 and 3.1.7).

If $\mathcal{S}$ is not separable, then the Borel σ-field is usually too strong to sustain reasonable probability measures, and $\mathcal{S}$ must be given a weaker, separable, σ-field. This causes problems between topological and measure-theoretic issues, and classic results such as the Portmanteau theorem (Ref. 14, Theorem 3.3.1) may fail to hold.

The natural σ-field on $D(\mathbb{R}_+,\mathcal{S})$ is the product (or projection) σ-field of the σ-field on $\mathcal{S}$, and will always be used in the sequel. The classical topology given $D(\mathbb{R}_+,\mathcal{S})$ is the Skorohod topology, which can be metrized by (3.5.2) or (3.5.21) in Ref. 14. If $\mathcal{S}$ is separable then $D(\mathbb{R}_+,\mathcal{S})$ is separable (Ref. 14, Theorem 3.5.6) and then, if $\mathcal{S}$ is given the Borel σ-field, the Borel σ-field of the Skorohod topology and the product σ-field coincide. For weak convergence with a continuous limit process, uniform convergence on bounded time intervals may be used with adequate measurability assumptions on the test functions (Ref. 14, Theorem 3.10.2).

4.2. *Mean-field regime, rescaled and auxiliary systems*

The number N of peers is typically large, and we let it go to infinity. At each time-step two peers are possibly updated, and the empirical measures have jumps of order $\frac{1}{N}$, hence time must be rescaled by a factor N. This is a mean-field limit, in which time is usually rescaled by physical considerations; here, we could say "the more peers, the more often they meet". It is also related to fluid limits.

A non-trivial continuous-time limit process is expected for the *rescaled* system

$$\widetilde{X}^N = (\widetilde{X}^N_i)_{1\le i\le N}\,,\qquad \widetilde{X}^N = (\widetilde{X}^N(t), t\in\mathbb{R}_+) = (X^N(\lfloor Nt\rfloor), t\in\mathbb{R}_+)\,, \tag{4.1}$$

with sample paths in $D(\mathbb{R}_+,[0,1]^N)$. The empirical measure $\widetilde{\Lambda}^N$ and its marginal process $\widetilde{M}^N = (\widetilde{M}^N(t), t\in\mathbb{R}_+)$ (satisfying $\widetilde{\Lambda}^N_t = \widetilde{M}^N(t)$ with classic notation for marginal laws for process laws) are given by

$$\widetilde{\Lambda}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\widetilde{X}^N_i} = \frac{1}{N}\sum_{i=1}^N \delta_{(\widetilde{X}^N_i(t), t\in\mathbb{R}_+)}\,,\qquad \widetilde{M}^N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{\widetilde{X}^N_i(t)}\,, \tag{4.2}$$

respectively with samples in $\mathcal{P}(D(\mathbb{R}_+,[0,1]))$ and sample paths in $D(\mathbb{R}_+,\mathcal{P}[0,1])$.

An *auxiliary* (rescaled) system is obtained by randomizing the jump instants of the original model by waiting i.i.d. exponential r.v. of mean $\frac{1}{N}$ between selections. A convenient construction uses a Poisson process $(A(t), t\in\mathbb{R}_+)$ of intensity 1 to set

$$\begin{aligned}
&\widehat{X}^N = (\widehat{X}^N_i)_{1\le i\le N}\,,\quad \widehat{X}^N = (\widehat{X}^N(t), t\in\mathbb{R}_+) = (X^N(A(Nt)), t\in\mathbb{R}_+)\,,\\
&\widehat{\Lambda}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}^N_i}\,,\quad \widehat{M}^N = (\widehat{M}^N(t), t\in\mathbb{R}_+)\,,\ \ \widehat{M}^N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}^N_i(t)}\,,
\end{aligned} \tag{4.3}$$

with sample spaces as above.

If T_k for $k \geq 0$ are given by $T_0 = 0$ and the jump instants of $(A(t), t \in \mathbb{R}_+)$, then

$$\widetilde{X}^N(t) = \widehat{X}^N(t') = X^N(k)\,, \qquad \frac{k}{N} \leq t < \frac{k+1}{N}\,, \quad \frac{T_k}{N} \leq t' < \frac{T_{k+1}}{N}\,. \tag{4.4}$$

Note that $\widetilde{M}^N(t) = M^N(\lfloor Nt \rfloor)$ and $\widehat{M}^N(t) = M^N(A(Nt))$, but that the relationship between $\widetilde{\Lambda}^N$ and $\widehat{\Lambda}^N$ and Λ^N is more involved.

The process $\widehat{X}^N$ is a pure-jump Markov process with rate bounded by N, at which two peers are chosen uniformly at random without replacement, say i and j at time t, and

- if $|\widehat{X}_i^N(t-) - \widehat{X}_j^N(t-)| > \Delta$ then $\widehat{X}^N(t) = \widehat{X}^N(t-)$,
- if $|\widehat{X}_i^N(t-) - \widehat{X}_j^N(t-)| \leq \Delta$ then only the values of peers i and j change to

$$\widehat{X}_i^N(t) = w\widehat{X}_i^N(t-) + (1-w)\widehat{X}_j^N(t-)\,,$$
$$\widehat{X}_j^N(t) = wX_j^N(t-) + (1-w)\widehat{X}_i^N(t-)\,.$$

Remark 4.1. Each of the $\frac{N(N-1)}{2}$ unordered pairs of peers is thus chosen at rate $\frac{2}{N-1}$, and then both peers undergo a *simultaneous* jump in their values if these are close enough. Each peer is thus affected at rate 2.

The generator $\mathcal{A}^N$ of $\widehat{X}^N = (\widehat{X}_n^N)_{1\leq n\leq N}$ acts on $f \in L^\infty([0,1]^N)$ as

$$\mathcal{A}^N f((x_n)_{1\leq n\leq N}) = \frac{2}{N-1} \sum_{1\leq i<j\leq N} [f((x_n)_{1\leq n\leq N}^{i,j}) - f((x_n)_{1\leq n\leq N})]1_{\{|x_i - x_j| \leq \Delta\}} \tag{4.5}$$

where $(x_n)_{1\leq n\leq N}^{i,j}$ is obtained from $(x_n)_{1\leq n\leq N}$ by replacing x_i and x_j with $wx_i + (1-w)x_j$ and $wx_j + (1-w)x_i$ and leaving the other coordinates fixed. Its operator norm is bounded by $2N$, and the law of the corresponding Markov process $\widehat{X}^N$ is well-defined in terms of the law of $\widehat{X}^N(0) = X^N(0)$. From a statistical mechanics perspective, $\widehat{X}^N$ is a particle system in *binary mean-field interaction.*

For $1 \leq k \leq N$, this generator acts on $h_k \in L^\infty([0,1]^N)$ which depend only on the k-th coordinate, of the form $h_k((x_n)_{1\leq n\leq N}) = h(x_k)$ for some $h \in L^\infty[0,1]$, as

$$\begin{aligned} &\frac{2}{N-1} \sum_{1\leq j\leq N: j\neq k} [h(wx_k + (1-w)x_j) - h(x_k)]1_{\{|x_k - x_j| \leq \Delta\}} \\ &\qquad := \mathcal{A}\left(\frac{1}{N-1} \sum_{1\leq j\leq N: j\neq k} \delta_{x_j}(dy)\right) h(x_k) \end{aligned} \tag{4.6}$$

where the generators $\mathcal{A}(\mu)$ act on $h \in L^\infty[0,1]$ as

$$\mathcal{A}(\mu)h(x) = 2\langle [h(wx + (1-w)y) - h(x)]1_{\{|x-y|\leq\Delta\}}\,, \mu(dy)\rangle\,, \quad \mu \in \mathcal{P}[0,1]\,. \tag{4.7}$$

Hence, if the $\widehat{X}_i^N(0)$ converge in law to i.i.d. r.v. of law m_0, then the $\widehat{X}_i^N$ are expected to converge in law to i.i.d. processes of law Q, the law of a time-inhomogeneous Markov process $\widehat{X}$ with initial law m_0 and generator $\mathcal{A}(m(t))$ at time $t \in \mathbb{R}_+$, where $m(t) = \mathcal{L}(\widehat{X}_t) = Q_t$ is the instantaneous law of this same process, the marginal of Q.

Considering the forward Kolmogorov equation for this Markov process, $(m(t), t \in \mathbb{R}_+)$ should satisfy the following weak (or distributional-sense) formulation of a nonlinear integro-differential equation.

Definition 4.1 (Problem 1). We say that $m = (m(t), t \in \mathbb{R}_+)$ with $m(t) \in \mathcal{P}[0,1]$ is solution to Problem 1 with initial value $m_0 \in \mathcal{P}[0,1]$ if $m(0) = m_0$ and

$$\begin{aligned}\langle h, m(t)\rangle - \langle h, m(0)\rangle &= \int_0^t \langle \mathcal{A}(m(s))h, m(s)\rangle\, ds \\ &:= \int_0^t 2\Big\langle [h(wx + (1-w)y) - h(x)]1_{\{|x-y|\le\Delta\}}\,, m(s)(dy)m(s)(dx)\Big\rangle\, ds \qquad (4.8)\end{aligned}$$

for all test functions $h \in L^\infty[0,1]$. This can be written more symmetrically as

$$\begin{aligned}\langle h, m(t)\rangle - \langle h, m(0)\rangle = \int_0^t &\Big\langle [h(wx + (1-w)y) + h(wy + (1-w)x) \\ &- h(x) - h(y)]1_{\{|x-y|\le\Delta\}}\,, m(s)(dy)m(s)(dx)\Big\rangle\, ds\,. \qquad (4.9)\end{aligned}$$

Remark 4.2. The basic probabilistic structure is the same as in the weak forms (2.1), (2.2), (2.4) (with $\mathcal{L} = 0$) of the spatially homogeneous version (without x-dependence) of the Boltzmann equation (1.1) in Graham-Méléard[19], the weak form (1.7) of the (cutoff) Kac equation (1.1)-(1.2) in Desvillettes *et al.*,[10] the nonlinear Kolmogorov equation (2.7) in Graham,[16] and the kinetic equation (9.4.4) in Graham.[17] The weak formulation involves explicitly the generator of the underlying Markovian dynamics, and allows to understand it more directly. We shall discuss the functional formulation (for probability density functions) of this integro-differential equation in Section 6, which involves an adjoint expression of this generator.

The distance in total variation norm of μ and μ' in $\mathcal{P}(\mathcal{S})$ is given by

$$|\mu - \mu'| = \sup_{\|\phi\|_\infty \le 1} \langle \phi, \mu - \mu'\rangle = 2\sup\{\mu(A) - \mu'(A) : \text{measurable } A \subset \mathcal{S}\}\,. \qquad (4.10)$$

Theorem 4.2. *Consider the generators $\mathcal{A}(\mu)$ given by (4.7), and m_0 in $\mathcal{P}[0,1]$.*

(1) There is a unique solution $m = (m(t), t \in \mathbb{R}_+)$ to Problem 1 starting at m_0. For the total variation norm on $\mathcal{P}[0,1]$, $t \mapsto m(t)$ is continuous, and $m_0 \mapsto (m(t), t \in \mathbb{R}_+)$ is continuous for uniform convergence on bounded time sets.

(2) There is a unique law $Q = \mathcal{L}(\widehat{X})$ on $D(\mathbb{R}_+, [0,1])$ for an inhomogeneous Markov process $\widehat{X} = (\widehat{X}(t), t \in \mathbb{R}_+)$ with generator $\mathcal{A}(m(t))$ at time t and initial law $\mathcal{L}(\widehat{X}(0)) = m_0$. Its marginal $Q_t = \mathcal{L}(\widehat{X}_t)$ is given by $m(t)$.

Note that the presence of indicator functions requires quite strong topologies. For instance, if $0 < a < b = a + \Delta < 1$ and $m_0 = \frac{1}{2}(\delta_a + \delta_b)$, then there exists $M_+^N(0)$ with support not intersecting $[a,b]$ and converging weakly to $m(0)$, and starting there $M_+^N(k)$ and $m_+^N(t)$ have at least two clusters and support outside

$[a,b]$. There exists also $M_-^N(0)$ with support inside (a,b) and converging weakly to $m(0)$, and $M_-^N(k)$ and $m_+^N(t)$ have one cluster and support inside (a,b) for any $k \in \mathbb{N}$, and will be a total consensus after some random time.

In statistical mechanics, convergence to an i.i.d. system is called *chaoticity*, and the fact that chaoticity at time 0 implies chaoticity at further times is called *propagation of chaos*, a terminology nowadays often restricted to process convergence. Compactness-uniqueness methods are classically used for proofs, see Sznitman,[35] and also Méléard,[28] Graham-Méléard[19] Section 4, and Graham,[16,17] but require weak topologies for compactness criteria, and continuity properties in order to pass to the limit; hence, the indicator functions prevent using them here.

Another difficulty for proofs is the presence of *simultaneous* jumps of pairs of particles. This prevents relating the interacting system in a simple way to an independent system (which cannot have such jumps), or writing an evolution formula for the empirical measure which is almost in closed form. Because of that, the coupling methods introduced by Sznitman,[35] see also Méléard[28] and Graham-Robert,[20] cannot be adapted here. Moreover, these use contraction techniques, and the metric used is too weak for the indicator functions.

4.3. *Rigorous mean-field limit results for the auxiliary system*

Systems of this type were studied in Graham-Méléard[18,19] (see also Ref. 10). The first paper studied a class of not necessarily Markovian multitype interacting systems, as a model for communication networks. The second studied Monte-Carlo methods for a class of Boltzmann models, and in particular expressed some notions and results of the first in this framework. Their results yield the following.

For $k \geq 1$ and $T \geq 0$ and laws P and P' on $D(\mathbb{R}_+, [0,1]^k)$, let $|P-P'|_T$ denote the distance in variation norm (4.10) of the restrictions of P and Q on $D([0,T],[0,1]^k)$. When clear, the processes will be restricted to $[0,T]$ without further mention.

Theorem 4.3. *Consider the auxiliary system (4.3) for $N \geq 2$. If the $\widehat{X}_i^N(0) := X_i^N(0)$ are i.i.d. of law m_0, then there is propagation of chaos. More precisely, let $m = (m(t), t \in \mathbb{R}_+)$ and Q be as in Theorem 4.2 for $m(0) = m_0$, and $T > 0$.*

(1) For $1 \leq k \leq N$,

$$|\mathcal{L}(\widehat{X}_1^N, \dots, \widehat{X}_k^N) - \mathcal{L}(\widehat{X}_1^N) \otimes \cdots \otimes \mathcal{L}(\widehat{X}_k^N)|_T \leq 2k(k-1)\frac{2T+4T^2}{N-1}\,,$$

and

$$\left| \frac{1}{N}\sum_{i=1}^N \mathcal{L}(\widehat{X}_i^N) - Q \right|_T \leq |\mathcal{L}(\widehat{X}_i^N) - Q|_T \leq 6\frac{\exp(2T)-1}{N+1}\,.$$

(2) For any $\phi : D([0,T],[0,1]) \to \mathbb{R}$ such that $\|\phi\|_\infty \leq 1$,

$$\mathbb{E}\left[\left\langle \phi, \widehat{\Lambda}^N - \frac{1}{N}\sum_{i=1}^N \mathcal{L}(\widehat{X}_i^N)\right\rangle^2\right] \leq \frac{4+8T+16T^2}{N}\,.$$

Moreover

$$\widehat{\Lambda}^N \xrightarrow[N\to\infty]{\text{in probab.}} Q\,, \qquad \widehat{M}^N \xrightarrow[N\to\infty]{\text{in probab.}} m\,,$$

respectively for the weak topology on $\mathcal{P}(D(\mathbb{R}_+,[0,1]))$ *with the Skorohod topology on* $D(\mathbb{R}_+,[0,1])$*, and for the topology of uniform convergence on bounded time intervals on* $D(\mathbb{R}_+,\mathcal{P}[0,1])$ *with the weak topology on* $\mathcal{P}[0,1]$*.*

Remark 4.3. The convergence result for $\widehat{\Lambda}^N$ is equivalent to convergence in law to Q (Ethier-Kurtz,[14] Corollary 3.3.3). The convergence result for $\widehat{M}^N$ implies convergence in law to m for test functions which are continuous, bounded, and measurable for the product σ-field (Ref. 14, Theorem 3.10.2). Separability issues restrict these results, and in fact convergence of $\widehat{\Lambda}^N$ holds for any convergence induced by a denumerable set of bounded measurable functions. See also Section 4.1.

4.4. *From the auxiliary to the rescaled system*

For $k \geq 1$, let a_k denote the Skorohod metric on $D(\mathbb{R}_+,[0,1]^k)$ given by (3.5.21) in Ethier-Kurtz[14] for the atomic metric $(x,y) \mapsto 1_{\{x\neq y\}}$ on $[0,1]^k$ (which induces the topology of all subsets of $[0,1]^k$, for which any function is continuous). Note that a_k is measurable with respect to the usual Borel σ-field on $[0,1]^k \times [0,1]^k$.

A *time-change* is an increasing homeomorphism of $\mathbb{R}_+$, *i.e.*, a continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ which is null at the origin and strictly increasing to infinity. Two paths are close for a_k if there is a time-change close to the identity such that the time-change of one path is equal to the other path.

Eq. (4.4) is the key to obtain the following quite general result showing that the rescaled system $\widetilde{X}^N$ is very close to the the auxiliary system $\widehat{X}^N$, up to a well-controlled (random) time-change.

Theorem 4.4. *Consider the rescaled system* (4.1) *and the auxiliary system* (4.3) *for* $N \geq 2$*. Then* $\lim_{N\to\infty} a_N(\widetilde{X}^N, \widehat{X}^N) = 0$ *in probability.*

This result and Theorem 4.3 now yield the main mean-field convergence result.

Theorem 4.5. *Consider the rescaled system* (4.1) *for* $N \geq 2$*. If the* $\widetilde{X}_i^N(0) := X_i^N(0)$ *are i.i.d. of law* m_0*, then there is propagation of chaos. More precisely, let* $m = (m(t), t \in \mathbb{R}_+)$ *and* Q *be as in Theorem 4.2 for* $m(0) = m_0$*.*

(1) For $1 \leq k \leq N$*,*

$$\lim_{N\to\infty} \mathcal{L}(\widetilde{X}_1^N, \dots, \widetilde{X}_k^N) = Q^{\otimes k}\,,$$

for the weak topology on $\mathcal{P}(D(\mathbb{R}_+,[0,1]^k))$ *induced by test functions which are either uniformly continuous for the Skorohod metric* a_k*, bounded, and measurable for the usual product* σ*-field (for the usual Borel* σ*-field on* $[0,1]^k$*), or continuous for the usual Skorohod topology (for the usual metric on* $[0,1]^k$*) and bounded.*

(2) For the usual topology of $[0,1]$,

$$\widetilde{\Lambda}^N \xrightarrow[N\to\infty]{\text{in probab.}} Q\,, \qquad \widetilde{M}^N \xrightarrow[N\to\infty]{\text{in probab.}} m\,,$$

respectively for the weak topology on $\mathcal{P}(D(\mathbb{R}_+,[0,1]))$ *with the Skorohod topology on* $D(\mathbb{R}_+,[0,1])$*, and for the topology of uniform convergence on bounded time intervals on* $D(\mathbb{R}_+,\mathcal{P}[0,1])$ *with the weak topology on* $\mathcal{P}[0,1]$.

For the second result, see Remark 4.3.

5. Infinite *N* Model

We now study the mean-field limit $m = (m(t), t \in \mathbb{R}_+)$ obtained in Section 5 when N goes to infinity. As for the finite N model, we find that there is convergence to a partial consensus, as time goes to infinity; the limit may depend on the initial conditions, which is the deterministic counterpart of the fact that the limit is random when N is finite. We are able to say more; in particular, we find tractable sufficient conditions for the limit to be a total consensus.

5.1. *Convexity and Moments*

Applying Proposition 3.2 to the equivalent definition of Problem 1 given by (4.9) yields the following:

Corollary 5.1. *Let* $m = (m(t), t \in \mathbb{R}_+)$ *be solution of Problem 1. If* $h : [0,1] \to \mathbb{R}$ *is convex, then* $\langle h, m(t)\rangle$ *is a non-increasing function of* t.

For $n = 1, 2, \dots$ *and* $t \in \mathbb{R}_+$*, let* $\mu_n(t) = \int_0^1 x^n\, m(t)(dx)$ *denote the* n*-th moment of* $m(t)$*, and* $\sigma(t)$ *denote its standard deviation, given by* $\sigma(t)^2 = \mu_2(t) - \mu_1(t)^2$.

(1) The mean $\mu_1(t)$ *is stationary,* i.e., $\mu_1(t) = \mu_1(0)$.
(2) The moments $\mu_n(t)$ *are non-increasing in* t*,* i.e., *if* $t_1 \le t_2$ *then* $\mu_n(t_1) \ge \mu_n(t_2)$.
(3) The standard deviation $\sigma(t)$ *is a non-increasing function of* t.

Furthermore, we have the bounds:

Proposition 5.2. *For all* $t \ge 0$*, we have* $\sigma(0) \ge \sigma(t) \ge \sigma(0)e^{-4w(1-w)t}$.

Note that Corollary 5.1 and Proposition 5.2 generalize results of Ref 6, which established similar results for the case $w = 1/2$. However, our bound in Proposition 5.2 is different, as the equation considered in Ref. 6 misses a factor 2.

5.2. *Convergence to Partial Consensus*

It is immediate that a partial consensus is a stationary point for Problem 1, *i.e.*, if $(m(t), t \in \mathbb{R}_+)$ is solution of Problem 1 with initial value a partial consensus m_0, then $m(t) = m_0$ for all t. Conversely, we show, in Theorem 3 below, that any trajectory $(m(t), t \in \mathbb{R}_+)$ converges to a partial consensus.

It is useful to consider the essential sup and inf of $m(t)$, defined as follows.

Definition 5.3. For $\nu \in \mathcal{P}[0,1]$, let $\operatorname{ess\,sup}(\nu) = \inf\{b \in [0,1], \nu((b,1]) = 0\}$ and $\operatorname{ess\,inf}(\nu) = \sup\{a \in [0,1], \nu([0,a)) = 0\}$.

Note that if $\operatorname{ess\,inf}(\nu) = a$ and $\operatorname{ess\,sup}(\nu) = b$, then the support of ν is included in $[a,b]$, *i.e.*, for any measurable $B \subset [0,1]$, $\nu(B) = \nu(B \cap [a,b])$.

Proposition 5.4. *Let $(m(t), t \in \mathbb{R}_+)$ be solution of Problem 1. Then* $\operatorname{ess\,sup}(m(t))$ *[resp.* $\operatorname{ess\,inf}(m(t))$*] is a non-increasing [resp. non-decreasing] function of t.*

Theorem 5.5. *Let $(m(t), t \in \mathbb{R}_+)$ be solution of Problem 1. As t goes to infinity, $m(t)$ converges, for the weak topology on $\mathcal{P}[0,1]$, to $m(\infty)$ which is a partial consensus for every $\Delta' < \Delta$ (i.e. $m(\infty) = \sum_{m=1}^{m_0} \alpha_m \delta_{x_m}$ with $x_m \in [0,1]$, $|x_m - x_{m'}| \geq \Delta$ for $m \neq m'$, and $\alpha_m > 0$.).*

Note that the limit $m(\infty)$ may depend on the initial condition m_0, and may or may not be a total consensus (as shown in the next section). We are in particular interested in finding initial conditions that guarantee that $m(\infty)$ is a total consensus. The following is an immediate consequence of Proposition 5.4:

Corollary 5.6. *If the diameter of m_0 is less than Δ,* i.e.*, if* $\operatorname{ess\,sup}(m_0) - \operatorname{ess\,inf}(m_0) < \Delta$*, then $m(\infty)$ is a total consensus.*

Note that the converse is not true: if the diameter of m_0 is larger or equal than Δ, there may be convergence to total consensus (see next section for an example).

5.3. *Convergence to Total Consensus*

We find sufficient criteria for guaranteeing some upper bounds on the number of components of $m(\infty)$, in particular, we find some sufficient conditions for convergence to total consensus. Although the bounds are suboptimal, to the best of our knowledge, they are the first of their kind. The bounds are based on Corollary 5.1.

First define, for $n \in \{1,2,3,\dots\}$ and $\mu_0 \in [0,1]$, the set $P_n(\mu_0)$ of partial consensus with n components and mean μ_0, *i.e.*, $\nu \in P_n(\mu)$ iff there is some sequence $0 \leq x_1 < \cdots < x_n \leq 1$, with $x_i + \Delta < x_{i+1}$, some sequence α_i for $i = 1,\dots,n$ with $0 < \alpha_i < 1$ and $\sum_{i=1}^n \alpha_i = 1$, such that $\nu = \frac{1}{n}\sum_{i=1}^n \alpha_i \delta_{x_i}$ and $\frac{1}{n}\sum_{i=1}^n \alpha_i x_i = \mu_0$. Second, for any convex, continuous $h : [0,1] \to \mathbb{R}_+$, let $Q_n(\mu_0, h)$ be the set of strict lower bounds of the image by the mapping $\nu \mapsto \langle h, \nu\rangle$ of $P_n(\mu_0)$, *i.e.*, $q \in Q_n(\mu_0, h)$ iff for any consensus ν with n components and mean μ_0, it holds that $\langle h, \nu\rangle > q$. If $P_n(\mu_0)$ is empty, let $Q_n(\mu_0, h) = \mathbb{R}_+$.

Note that $Q_n(\mu_0, h)$ is necessarily an interval, with lower bound 0. The following proposition states that Q_n is non decreasing with n.

Proposition 5.7. *For $n \in \{1,2,3,\dots\}$, $\mu_0 \in [0,1]$, and any convex, continuous $h : [0,1] \to \mathbb{R}_+$, it holds that $Q_n(\mu_0, h) \subset Q_{n+1}(\mu_0, h)$.*

Combining Proposition 5.7 with Corollary 5.1, we obtain:

Theorem 5.8. *Let $(m(t), t \in \mathbb{R}_+)$ be a solution of Problem 1 with initial condition m_0, and d be the number of components of the limiting partial consensus $m(\infty)$. Assume that, for some $n \in \{2, 3, \dots\}$, some convex continuous $h : [0, 1] \to \mathbb{R}_+$, and some $q \geq 0$, we have $q \in Q_n(\mu_0, h)$, where μ_0 is the mean of m_0.*

If $\langle h, m_0 \rangle \leq q$ then $d \leq n - 1$.

We now give an example of use of the theorem, obtained by taking $n = 2$ and $h(x) = |x - \mu_0|$.

Corollary 5.9. *Let $(m(t), t \in \mathbb{R}_+)$ be a solution of Problem 1 with initial condition m_0, and let μ_0 be the mean of m_0. Assume that $\Delta \geq \frac{1}{2}$ and $1 - \Delta \leq \mu_0 \leq \Delta$. If*

$$\int_0^1 |x - \mu_0| \, m_0(dx) < \frac{2}{\Delta} \min \{\mu_0(\Delta - \mu_0),\ (1 - \mu_0)(\Delta - 1 + \mu_0)\}$$

then $m(t)$ converges to total consensus.

If we apply this to m_0 equal to the uniform distribution, we find the sufficient condition $\Delta > \frac{2}{3}$ for convergence to total consensus. In Corollary 5.15 we find a better result, obtained by exploiting symmetry properties.

Definition 5.10. We say that $\nu \in \mathcal{P}[0, 1]$ is symmetric if the image measure of ν by $x \mapsto 1 - x$ is ν itself.

Note that if ν has a density f, this simply means that $f(x) = f(1 - x)$. Necessarily, if ν is symmetric, the mean of ν is $\frac{1}{2}$. If a partial consensus $\nu = \frac{1}{n} \sum_{i=1}^n \alpha_i \delta_{x_i}$ (with $x_i < x_{i+1}$) is symmetric, then $x_{n+1-i} = 1 - x_i$ and $\alpha_{n+1-i} = \alpha_i$; in particular, if n is odd, $x_{\frac{n+1}{2}} = \frac{1}{2}$.

Proposition 5.11. *Let $(m(t), t \in \mathbb{R}_+)$ be a solution of Problem 1 with initial value m_0. If m_0 is symmetric, then $m(t)$ is symmetric for all $t \geq 0$.*

We can extend the previous method to the symmetric case, as follows. Define SP_n as the set of *symmetric* partial consensus with n components and $q \in SQ_n(h)$ iff for any symmetric consensus ν with n components $\langle h, \nu \rangle > q$. If SP_n is empty, then $SQ_n(h) = \mathbb{R}_+$. We have similarly:

Proposition 5.12. *For $n \in \{1, 2, 3, \dots\}$, and any convex, continuous $h : [0, 1] \to \mathbb{R}_+$, $SQ_n(h) \subset SQ_{n+1}(h)$.*

Theorem 5.13. *Let $(m(t), t \in \mathbb{R}_+)$ be a solution of Problem 1 with* symmetric *initial condition m_0, and d be the number of components of the limiting partial consensus $m(\infty)$. Assume that, for some $n \in \{2, 3, \dots\}$, some convex continuous $h : [0, 1] \to \mathbb{R}_+$, and some $q \geq 0$, we have $q \in SQ_n(h)$.*

If $\langle h, m_0 \rangle \leq q$ then $d \leq n - 1$

We apply Theorem 5.13 with $h(x) = \left| x - \frac{1}{2} \right|$. It is easy to see that for $\nu \in SP_2$ we have $\langle h, \nu \rangle \geq \frac{\Delta}{2}$, which shows the following:

Corollary 5.14. *Let $(m(t), t \in \mathbb{R}_+)$ be a solution of Problem 1 with* symmetric *initial condition m_0. If $\Delta > 2\int_0^1 \left|x - \frac{1}{2}\right| \, m_0(dx)$ then $m(t)$ converges either to total consensus or to partial consensus with 3 or more components.*

In particular, if m_0 is the uniform distribution on $[0,1]$, then $\int_0^1 \left|x - \frac{1}{2}\right| m_0(dx) = \frac{1}{4}$ and the condition in the previous corollary is $\Delta > \frac{1}{2}$, thus we have shown:

Corollary 5.15. *Let $(m(t), t \in \mathbb{R}_+)$ be a solution of Problem 1 with initial condition the uniform distribution on $[0,1]$. If $\Delta > \frac{1}{2}$ then $m(t)$ converges to total consensus.*

Corollary 5.16. *Let $(m(t), t \in \mathbb{R}_+)$ be a solution of Problem 1 with initial condition $m_0 = \left(\frac{1-\alpha}{2}\right)\delta_0 + \alpha\delta_{\frac{1}{2}} + \left(\frac{1-\alpha}{2}\right)\delta_1$. There is convergence to total consensus for*

$$\Delta > 1 - \alpha \text{ if } \alpha \leq \tfrac{1}{2}\,, \text{ or } \Delta > \tfrac{1}{2} \text{ if } \alpha \geq \tfrac{1}{2}\,.$$

6. Numerical Approach

In the mean-field limit, the dynamical behavior of the system of peers can be described by the integro-differential equation given in weak form in Definition 4.1 (Problem 1). This equation has no closed solution to our knowledge, and we have developed a numerical method for it.

We describe the algorithm, and analyze its precision and complexity. An important fact is that this algorithm requires considerably less running time than the probabilistic methods used in Neau[31] when N is large (which is not surprising in dimension 1). The program consists in 600 lines of C++ code, and the parsing and plotting of the results was done using Matlab.

6.1. *Functional formulation of Problem 1*

The numerical method is based on the functional formulation for probability density functions (PDFs) obtained by duality from the weak formulation in Definition 4.1 of Problem 1. The following result is fundamental in this aspect.

Theorem 6.1. *Let $(m(t), t \geq 0)$ be a solution of Problem 1. If the initial condition m_0 is absolutely continuous with respect to Lebesgue measure, then so is $m(t)$ for every $t \geq 0$, and moreover the densities $f(\cdot, t)$ of $m(t)$ satisfy the integro-differential equation*

$$\frac{\partial f(x,t)}{\partial t} = \frac{2}{w}\int_{x-\Delta w}^{x+\Delta w} f\left(\frac{x-(1-w)y}{w}, t\right) f(y,t)\, dy - 2f(x,t)\int_{x-\Delta}^{x+\Delta} f(y,t)\, dy\,. \tag{6.1}$$

Conversely, if $f : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ is a solution of Eq.(6.1) such that $f(\cdot,t)$ is a PDF with support $[0,1]$ for every $t \geq 0$, then the probability measures $m(t)(dx) = f(x,t)\, dx$ solve Problem 1.

This result and Theorem 4.2 yield an existence and uniqueness result for Eq.(6.1). This equation can be derived in statistical mechanics fashion by balance considerations. For the gain term, a particle in state $\frac{x-(1-w)y}{w}$ interacts at rate 2 (see Remark 4.1) with a particle in state y to end up in state x, and the joint density for this pre-interaction configuration at time t is $\frac{1}{w}f\big(\frac{x-(1-w)y}{w},t\big)f(y,t)$ (particles are "independent before interacting"). The loss term is derived similarly.

Remark 6.1. As noted in Remark 4.2, this is a Boltzmann-like equation. This is more obvious making the change of variables leading to post-interaction states x and y, which is possible for $w \neq 1/2$ and yields the equivalent formulation

$$\frac{\partial f(x,t)}{\partial t} = \frac{2}{2w-1}\int_{x-\Delta(2w-1)}^{x+\Delta(2w-1)} f\left(\frac{wx-(1-w)y}{2w-1}\right) f\left(\frac{wy-(1-w)x}{2w-1}\right) dy \\ - 2f(x,t)\int_{x-\Delta}^{x+\Delta} f(y,t)\,dy \quad (6.2)$$

more reminiscent of Boltzmann or Kac equations such as (1.1) in Graham-Méléard[19] or (1.1)-(1.2) in Desvillettes *et al.*[10] In these, the fact that the gain term involves pre-collisional velocities is obscured by the symmetries between pre-collisional and post-collisional velocites.

In the rest of this section we assume that the hypothesis of the above theorem holds. We show next that if the PDF $f(\cdot,0)$ is bounded then so is $f(\cdot,t)$ and we can control its growth over time.

Proposition 6.2. *Let* $|f(\cdot,t)|_\infty \overset{\text{def}}{=} \sup_{x\in[0,1]} |f(x,t)|$. *Assume* $|f(\cdot,0)|_\infty < \infty$. *Then* $|f(\cdot,T)|_\infty \le e^{\left(\frac{2}{w}+\frac{2}{1-w}\right)T}(M(0)+4)-4, \quad \forall\, T.$

It follows that $f(\cdot,t)$ is bounded for all t, and iteratively, using, (6.1), f is C^∞ on its second variable.

Having controlled the growth of $f(x,t)$ it's easy to control the growth of its derivatives:

Proposition 6.3. $\left|\frac{\partial}{\partial t}f(\cdot,t)\right|_\infty \le \left(\frac{2}{w}+\frac{2}{1-w}\right)(M(t)+4)$.

Thus, we have the following corollary:

Corollary 6.4. *If* $|f(\cdot,0)|_\infty < \infty$, *then* $\left|\frac{\partial^n}{\partial t^n}f(\cdot,T)\right|_\infty < \infty, \quad \forall\, n,T<\infty.$

6.2. *Numerical Solution of Eq.(6.1)*

Facing the impossibility to solve the equation analytically, we simulate numerically equation (6.1) discretizing in time and keeping track of the regularized (constant-spline approximated) approximation of the solution $f^r(x,t)$ at time t. Note that at time $t=0$, $f^r(x,0)$ is the given initial condition.

6.2.1. *Algorithm*

The algorithm used takes as input an initial condition $f^r(x,0)$, which is a piecewise constant function of I intervals, a time T after which we want to calculate an approximate solution and a maximum error ε and outputs an approximation of the solution $f^r(x,T)$. It works as follows:

First, we perform a discretization in t. In steps of Δt we approximate $f^r(x,t+\Delta t)$ by using a forward Euler method. In other words, we say that:

$$f^r(x,t+\Delta t) \approx f^r(x,t) + \Delta t \partial_t f^r(x,t) = f^e(x,t+\Delta t)$$

Here we exploit the fact that $f^r(x,t)$ is a piecewise constant function, so that we can calculate analytically the derivative which is a piecewise linear function. The deduction of the formula for the derivative is explained later. Hence, $f^e(x,t+\Delta t)$ is also piecewise linear, as it is the sum of a piecewise linear and a piecewise constant function. Then, we approximate $f^e(x,t+\Delta t)$ with another piecewise constant function (which we will call $f^r(x,t+\Delta t)$ for simplicity) of $I_{t+\Delta t}$ intervals, so that we can reuse the same scheme and we can compute explicitly the expression for the derivative. The constants are chosen in order to minimize the L^1 norm of the error (or, equivalently, the total variation norm of its associated measure).

We perform this loop until we calculate $f^r(x,T)$ in steps of Δt.

Knowing beforehand the complexity, we can choose the parameters Δt and I_t so that the total error is less than the specified. We have two ways of selecting them, either in a fixed or in an adaptative way:

The first way consists on having a constant number of intervals throughout the algorithm. Although the internal loop is executed faster (only once), we might overestimate the number of intervals at some time, where the equation is not stiff enough or Δt is very small. In contrast, if we decide to adapt the number of intervals at each step so that we bound the maximum error per iteration, we are sure that we won't have more than the necessary intervals, but at the cost of possibly having to recalculate $f^r(x,t)$ several times, when errors are big. In any case, the asymptotic cost of both algorithms is the same, as the calculation of $f^r(x,t)$ is not the bottleneck, which is the calculation of $f^e(x,t)$.

Both algorithms are given next.

Algorithm 1 Fixed I_t

Input $f^r(x,0), T, \varepsilon_{max}$
Output $f^r(x,T)$
Pick Δt and I according to ε_{max}
for $t \leftarrow 0$ **to** T **step** Δt **do**
  $f^e(x,t+\Delta t) \leftarrow f^r(x,t) + \Delta t \partial_t f^r(x,t)$
  $f^r(x,t+\Delta t) \leftarrow$ `PiecewiseConstantApproximation`$(f^e(x,t+\Delta t), I)$
end for

Algorithm 2 Adaptive I_t

Input $f^r(x.0), T, \varepsilon_{max}, \Delta t$
Output $f^r(x, T)$
for $t \leftarrow 0$ **to** T **step** Δt **do**
  $f^e(x, t + \Delta t) \leftarrow f^r(x, t) + \Delta t \partial_t f^r(x, t)$
  $I \leftarrow 1$
  repeat
    $f^r(x, t + \Delta t) \leftarrow$ `PiecewiseConstantApproximation`$(f^e(x, t + \Delta t), I)$
    $\varepsilon_{curr} \leftarrow$ `GetError`$(f^r(x, t + \Delta t), f^e(x, t + \Delta t))$
    $I \leftarrow 2I$
  until $\varepsilon_{curr} < \varepsilon_{max}$
end for

The method `PiecewiseConstantApproximation` returns the best piecewise constant approximation for a piecewise linear function in terms of minimizing the L^1 norm of the functions (or the total variation norm of the associated measures) while the method `GetError` returns the error made by such approximation.

6.2.2. *Optimal $f^r(x, t)$*

We want to determine which is the optimal piecewise constant approximation for $f^e(x, t)$ and have the following proposition:

Proposition 6.5. *The optimal constant which minimizes the error on any interval $X = [x_s, x_e]$ is given by $f^e\left(\frac{x_s+x_e}{2}, t\right)$*

We also need the following lemma, which will be used for the bounding of the method's error:

Lemma 6.6. *$f^r(x, t)$ has mass 1 for any t.*

6.2.3. *Analytical expression of $\partial_t f^r(x, t)$*

Now we will give an exact expression for the derivative, given that $f^r(x, t)$ is piecewise constant. This helps to understand how the calculation of the derivative is implemented and its asymptotic cost. We can write, for any t:

$$f^r(x, t) = \sum_{i=1}^{I} a_i [H(x - x_{i+1}) - H(x - x_i)]$$

where $H(x)$ is the Heaviside step function. Defining for any x_i and x_j:

$$I_1^{i,j}(x) \stackrel{\text{def}}{=} \int_{x-\Delta}^{x+\Delta} H(x - x_i)H(z - x_j)dz = \int_{-\Delta}^{\Delta} H(x - x_i)H(x + u - x_j)du$$

$$I_2^{i,j}(x) \stackrel{\text{def}}{=} \frac{1}{w}\int_{x-w\Delta}^{x+w\Delta} H(z-x_i)H\left(\frac{x-(1-w)z-wx_j}{w}\right)dz$$
$$= \int_{-\Delta}^{\Delta} H(x+wu-x_i)H(x-(1-w)u-x_j)du$$

The expression of $I_1^{i,j}(x)$ and $I_2^{i,j}(x)$ depends on the relative order between x_i, x_j and $m = \max\{(1-w)x_i + wx_j, x_i - w\Delta\}$ and is summarized in tables 1 and 2. Finally, we can calculate $\partial_t f^r(x,t)$ as:

$$\partial_t f^r(x,t) = -2\sum_{i,j} a_i a_j (I_1^{i,j}(x) + I_1^{i+1,j+1}(x) - I_1^{i,j+1}(x) - I_1^{i+1,j}(x))$$
$$+2\sum_{i,j} a_i a_j (I_2^{i,j}(x) + I_2^{i+1,j+1}(x) - I_2^{i,j+1}(x) - I_2^{i+1,j}(x)).$$

Case	$I_1^{i,j}(x)$
$x_i \le x_j - \Delta \le x_j + \Delta$	$\begin{cases} 0 & \text{if } x \le x_j - \Delta \\ x-(x_j-\Delta) & \text{if } x_j - \Delta \le x \le x_j + \Delta \\ 2\Delta & \text{if } x_j + \Delta \le x \end{cases}$
$x_j - \Delta \le x_i \le x_j + \Delta$	$\begin{cases} 0 & \text{if } x \le x_i \\ x-(x_j-\Delta) & \text{if } x_i \le x \le x_j + \Delta \\ 2\Delta & \text{if } x_j + \Delta \le x \end{cases}$
$x_j - \Delta \le x_j + \Delta \le x_i$	$\begin{cases} 0 & \text{if } x \le x_i \\ 2\Delta & \text{if } x_i \le x \end{cases}$

Table 1. $I_1^{i,j}(x)$

6.2.4. *Error Bound*

To calculate the error made by our approximation, define

$$g^s(x,t) \stackrel{\text{def}}{=} f(x,t) \quad \text{if } t \ge s \ge 0, \quad g^s(x,t) \stackrel{\text{def}}{=} f^r(x,t) \text{ if } 0 \le t < s,$$

and let $\nu_e^t(dx)$, $\nu_r^t(dx)$ and $\mu_s^t(dx)$ be the measures associated to $f^e(x,t)$, $f^r(x,t)$ and $g^s(x,t)$ respectively. Note that $\nu_r^t(dx) = \mu_t^t(dx)$. Thus, we want to calculate:

$$\varepsilon_{tot} = |\mu_0^T(dx) - \nu_r^T(dx)|_T = \left|\sum_{k=1}^{T/(\Delta t)} \mu_{(k-1)\Delta t}^T(dx) - \mu_{k\Delta t}^T(dx)\right|_T \le \sum_{k=1}^{T/(\Delta t)} \left|\mu_{(k-1)\Delta t}^T(dx) - \mu_{k\Delta t}^T(dx)\right|_T.$$

We can decompose the error done in each iteration of the loop as:

$$|\mu_{k\Delta t}^{k\Delta t}(dx) - \mu_{(k-1)\Delta t}^{k\Delta t}(dx)|_T \le |\nu_r^{k\Delta t}(dx) - \nu_e^{k\Delta t}(dx)|_T + |\nu_e^{k\Delta t}(dx) - \mu_{(k-1)\Delta t}^{k\Delta t}(dx)|_T = \varepsilon_{c.s} + \varepsilon_{eu}.$$

Let I_0 be the smallest I such that $w\Delta$, $(1-w)\Delta$ and Δ are multiples of $\frac{1}{I}$. Assuming that I is large enough to be a multiple of I_0, we can prove the following:

Case	$I_2^{i,j}(x)$
$m \le x_i + w\Delta \le x_j - (1-w)\Delta \le x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \le x_j - (1-w)\Delta \\ \frac{x-x_j}{1-w} + \Delta & \text{if } x_j - (1-w)\Delta \le x \le x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \le x \end{cases}$
$x_i + w\Delta \le m \le x_j - (1-w)\Delta \le x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \le x_j - (1-w)\Delta \\ \frac{x-x_j}{1-w} + \Delta & \text{if } x_j - (1-w)\Delta \le x \le x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \le x \end{cases}$
$m \le x_j - (1-w)\Delta \le x_i + w\Delta \le x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \le x_j - (1-w)\Delta \\ \frac{x-x_j}{1-w} - \frac{x_i-x}{w} & \text{if } x_j - (1-w)\Delta \le x \le x_i + w\Delta \\ \frac{x-x_j}{1-w} + \Delta & \text{if } x_i + w\Delta \le x \le x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \le x \end{cases}$
$x_i + w\Delta \le x_j - (1-w)\Delta \le m \le x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \le m \\ \frac{x-x_j}{1-w} + \Delta & \text{if } m \le x \le x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \le x \end{cases}$
$m \le x_j - (1-w)\Delta \le x_j + (1-w)\Delta \le x_i + w\Delta$	$\begin{cases} 0 & \text{if } x \le x_j - (1-w)\Delta \\ \frac{x-x_j}{1-w} - \frac{x_i-x}{w} & \text{if } x_j - (1-w)\Delta \le x \le x_j + (1-w)\Delta \\ \Delta - \frac{x_i-x}{w} & \text{if } x_j + (1-w)\Delta \le x \le x_i + w\Delta \\ 2\Delta & \text{if } x_i + w\Delta \le x \end{cases}$
$x_i + w\Delta \le x_j - (1-w)\Delta \le x_j + (1-w)\Delta \le m$	$\begin{cases} 0 & \text{if } x \le m \\ 2\Delta & \text{if } m \le x \end{cases}$
$x_j - (1-w)\Delta \le m \le x_i + w\Delta \le x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \le m \\ \frac{x-x_j}{1-w} - \frac{x_i-x}{w} & \text{if } m \le x \le x_i + w\Delta \\ \frac{x-x_j}{1-w} + \Delta & \text{if } x_i + w\Delta \le x \le x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \le x \end{cases}$
$x_j - (1-w)\Delta \le x_i + w\Delta \le m \le x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \le m \\ \frac{x-x_j}{1-w} + \Delta & \text{if } m \le x \le x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \le x \end{cases}$
$x_j - (1-w)\Delta \le m \le x_j + (1-w)\Delta \le x_i + w\Delta$	$\begin{cases} 0 & \text{if } x \le m \\ \frac{x-x_j}{1-w} + \frac{x_i-x}{w} & \text{if } m \le x \le x_j + (1-w)\Delta \\ \Delta - \frac{x_i-x}{w} & \text{if } x_j + (1-w)\Delta \le x \le x_i + w\Delta \\ 2\Delta & \text{if } x_i + w\Delta \le x \end{cases}$
$x_j - (1-w)\Delta \le x_i + w\Delta \le x_j + (1-w)\Delta \le m$	$\begin{cases} 0 & \text{if } x \le m \\ 2\Delta & \text{if } m \le x \end{cases}$
$x_j - (1-w)\Delta \le x_j + (1-w)\Delta \le m \le x_i + w\Delta$	$\begin{cases} 0 & \text{if } x \le m \\ \Delta - \frac{x_i-x}{w} & \text{if } m \le x \le x_i + w\Delta \\ 2\Delta & \text{if } x_i + w\Delta \le x \end{cases}$
$x_j - (1-w)\Delta \le x_j + (1-w)\Delta \le x_i + w\Delta \le m$	$\begin{cases} 0 & \text{if } x \le m \\ 2\Delta & \text{if } m \le x \end{cases}$

Table 2. $I_2^{i,j}(x)$

Proposition 6.7.

$$\varepsilon_{c.s}(I) = \varepsilon_{c.s}(I_0)\frac{I_0}{I} \le \frac{\Delta t}{2}|\partial_t f^r(x,(k-1)\Delta t)|_\infty \frac{I_0}{I}. \qquad (6.3)$$

We will now bound $|\partial_t f^r(x,(k-1)\Delta t)|_\infty$.

Proposition 6.8. *Let* $M(t) = |f^r(x,t)|_\infty$*. Assuming that* $|f^r(x,0)|_\infty = M(0) =$

$M<\infty$ we have the following uniform bound:

$$|\partial_t f^r(x,k\Delta t)|_\infty \le C_1(M,T) \quad \forall\, 0\le k\le \frac{T}{\Delta t}-1.$$

Substituting in (6.3), we get that:

$$\varepsilon_{c.s} \le \frac{C_1 I_0}{2}\frac{\Delta t}{I} = O\left(\frac{\Delta t}{I}\right). \tag{6.4}$$

Proposition 6.9. *We have*

$$\varepsilon_{eu} = O((\Delta t)^2). \tag{6.5}$$

Adding equations (6.4) and (6.5) we get that:

$$|\mu^{k\Delta t}_{k\Delta t}(dx) - \mu^{k\Delta t}_{(k-1)\Delta t}(dx)|_T \le \varepsilon_{c.s} + \varepsilon_{eu} = O\left((\Delta t)^2 + \frac{\Delta t}{I}\right).$$

Finally, we will bound $|\mu^T_{(k-1)\Delta t}(dx) - \mu^T_{k\Delta t}(dx)|_T$ in terms of $|\mu^{k\Delta t}_{k\Delta t}(dx) - \mu^{k\Delta t}_{(k-1)\Delta t}(dx)|_T$:

Proposition 6.10. *For all $1\le k\le \frac{T}{\Delta t}$ and for all $t\ge k\Delta t$ we have:*

$$\left|\mu^t_{k\Delta t}(dx) - \mu^t_{(k-1)\Delta t}(dx)\right|_T \le e^{8(t-k\Delta t)}\left|\mu^{k\Delta t}_{k\Delta t}(dx) - \mu^{k\Delta t}_{(k-1)\Delta t}(dx)\right|_T.$$

Combining the previous propositions, we can conclude the following:

Theorem 6.11. *For any fixed T, the error of the method is $O(\frac{1}{I}+\Delta t)$.*

6.2.5. *Complexity*

We will now give the complexity analysis of both algorithms. For simplicity of the analysis, we will assume that I is large enough so that $w\Delta$, $(1-w)\Delta$ and Δ are multiples of $\frac{1}{I}$.

For the first algorithm we have that the computation of the derivative takes $O(I^2)$, since we have a double sum over I intervals. Also, this produces $O(I^2)$ splines because every $I^{i,j}_k(x), k=1,2$ is composed of at most 4 splines. Since the splines are not produced in increasing order of x, we need to sort them, which takes $O(I^2\log I)$ time. Taking into account the expression of the derivative and the assumption on I, the support of every spline is the union of some of the intervals, i.e, there isn't any spline such that its support doesn't fully cover some interval. Therefore, we can compress our $O(I^2)$ splines into $O(I)$ splines in one pass ($O(I^2)$ time). Finally, we only need one pass to make the piecewise constant spline approximation since now everything is sorted and compressed. This takes $O(I)$ time.

Since all this loop is executed $\frac{T}{\Delta t}$ times, the running time has complexity $O\left(\frac{1}{\Delta t} I^2 \log I\right)$.

For the second algorithm, the procedure (and the cost) is the same until the piecewise constant approximation. In this case, we double the number of intervals until we are below some error ε_{max}. Therefore, the total cost is $O\left(\sum_{i=0}^{k} 2^i\right) = O(2^{k+1})$ for some k because both the error calculation and the piecewise constant approximation are linear in the number of intervals. As we know from the previous subsection that the error per iteration is $O\left(\frac{1}{I}\right)$ once fixed Δt, k is $O(-\log(\varepsilon_{max}))$ and therefore the complexity is $O\left(\frac{1}{\varepsilon_{max}}\right)$. Adding this for the $\frac{T}{\Delta t}$ executions of the loop, we get that the total running time is $O\left(\frac{1}{\Delta t} \frac{1}{\varepsilon_{max}^2} \log\left(\frac{1}{\varepsilon_{max}}\right)\right)$.

6.3. *Numerical Results*

In this section, we present the results got obtained by simulating using the above described algorithm. We study different scenarios for the initial distribution: uniform, extremist and undecided and beta. We plot different bifurcations (in terms of how many components we have at the end) depending on Δ. Moreover, we compare the experimental results with the bounds obtained in section 5 and the probabilistic Monte Carlo simulations presented in Ref 9.

6.4. *Evolution of the system: different settings*

In order to illustrate the behavior of the system as time passes, we show how the system evolves from a uniform distribution to one (or more) components, depending on the deviation threshold Δ. We run those sets of experiments for 3 different values of w, specifically $0.5, 0.75$, and 0.9 and plot the probability function at times $t = 0$, $t = 20$ and $t = 100$. The simulations have been done with the parameters $I = 200, \Delta t = 0.1, T = 100$. Although the set of parameters might theoretically yield a big error, in practice this error is much smaller.

From the images, we see that w does not seem to impact the number of components of $m(\infty)$, but the weights do depend on w.

6.5. *Extremists and Undecided*

We now present some common scenarios: imagine a company fusion and the opinion of the employees about the new company, or a rough categorization of voters in an election. We can characterize these opinions as extremists (either 0 or 1) or undecided (0.5). The density of the opinions is α for the undecided and $\frac{1-\alpha}{2}$ for each of the extremist classes. To simulate this, we have approximated the initial conditions (Diracs) to constant splines of value $I\alpha$ and $I\frac{1-\alpha}{2}$ respectively, centered at their corresponding points, such that the initial condition has mass 1. We plot the result (1 component, i.e. total consensus, or 2 components) for each pair (α, Δ) in $[0, 1] \times \left[\frac{1}{2}, 1\right]$ in Figure 4. We know from Corollary 5.16 that total consensus must

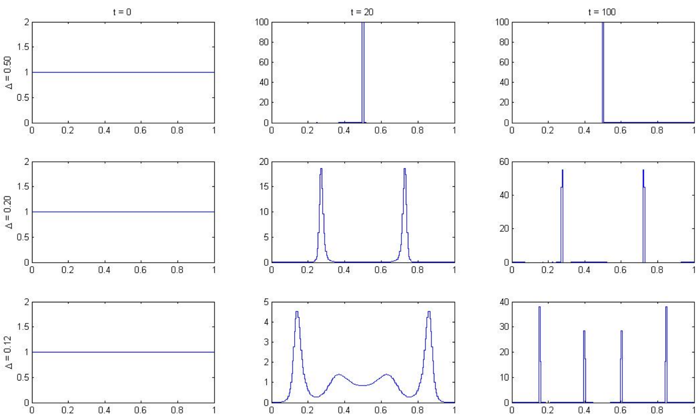


Fig. 1. $w = 0.5$. Evolution of $m(t)$ at times $t = 0, 20, 100$.

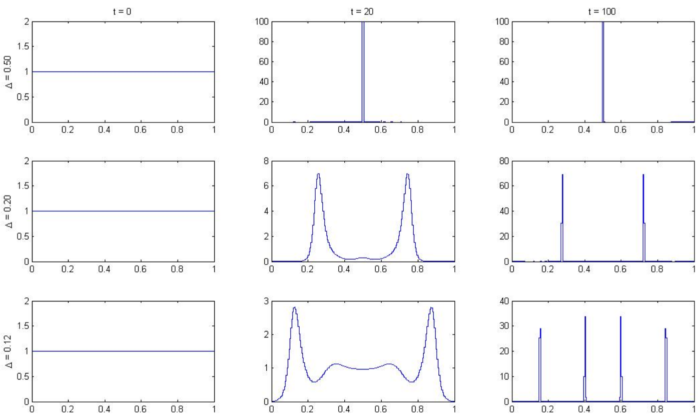


Fig. 2. $w = 0.75$. Evolution of $m(t)$ at times $t = 0, 20, 100$.

occur for $\Delta \geq \alpha$ and we see that the region of convergence to total consensus is a bit larger, and slightly depends on w.

Note that values of Δ smaller than $\frac{1}{2}$ would result in no motion at all. We do this for the previous set of values for w and find that in every case, the fraction of undecided people necessary to achieve consensus is much smaller than what one would expect.

We also plot the center of masses of the first half of the distribution to show that it is not a smooth function of α and that close to the critical value $\Delta_c(\alpha)$ there is a jump. We did this for the previous 3 values of w but show only one result for

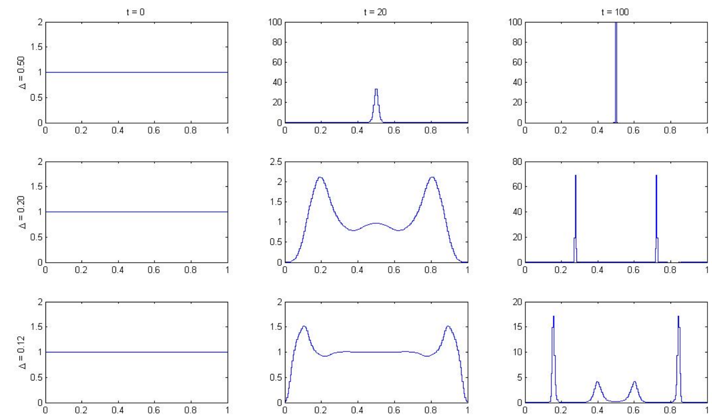
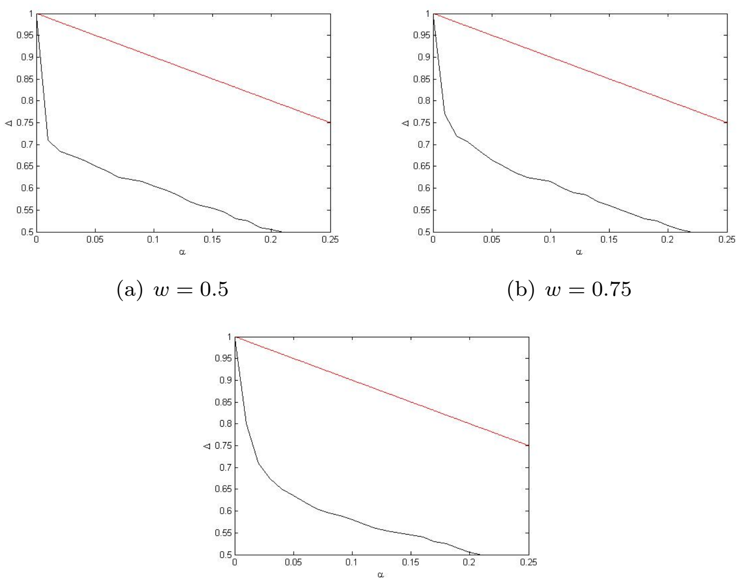


Fig. 3. $w = 0.9$. Evolution of $m(t)$ at times $t = 0, 20, 100$.



(a) $w = 0.5$

(b) $w = 0.75$

(c) $w = 0.9$

Fig. 4. Bifurcation diagram for extremists and undecided. The curly line separates the region of convergence to total consensus (above) from convergence to a partial consensus with two components. The straight line is the sufficient condition in Corollary 5.16.

brevity.

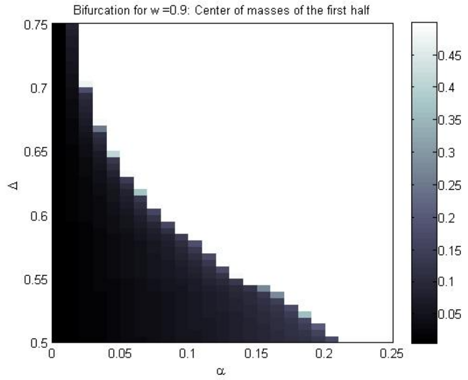


Fig. 5. $w = 0.9$. Center of masses of the first half, showing that the transition is abrupt.

6.6. *Initial uniform conditions in terms of delta*

We present here the evolution of the number of components with respect to Δ, using as initial condition a uniform distribution. Note that we have capped the situations with more than 7 components into the category "7 or more", which are represented by 7 in the graph. For a component to be considered as such, we require that it has at least 1% of the total mass. Otherwise we consider it as a zero. Again, the results are plotted for the 3 different values of w.

We observe that the results are almost independent of w, as there is almost no difference between the 3 curves (see Figure 6 for the combined plot of all 3 functions). Another interesting thing to remark is that if we compare our results for $w = 0.5$ with the deterministic model with the ones in Ref 9 with the probabilistic model, the intervals of Δ in which they have a high probability of convergence to n components correspond to the same intervals in which we have convergence to n components. This suggests that the approximation for $N = \infty$ is good enough to preserve properties such as the final state.

6.7. *Beta distribution as initial condition*

Here we study the evolution of the number of components with respect to Δ, using as initial condition a Beta(1,6) distribution. The functions that have 5 or more components have been put into the category represented with a 5. Again, we consider a component if it has 1% of the total mass or more. We present the results for the 3 different values of w.

We can observe again the same phenomenon as in the uniform case, namely that the influence of w is negligible. If we compare the results from the ones in Subsection 7.3, we can conclude that the final result depends on the initial condition, even for the same parameters w and Δ. Moreover, we can see that for a fixed (w, Δ), if we start with a Beta distribution the number of components will be smaller or equal

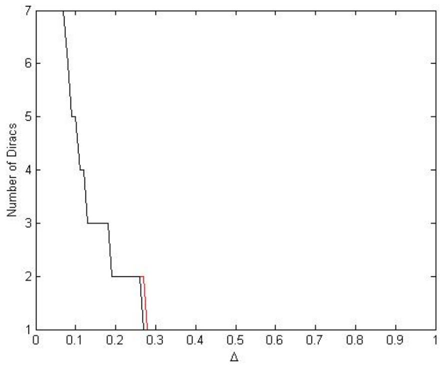
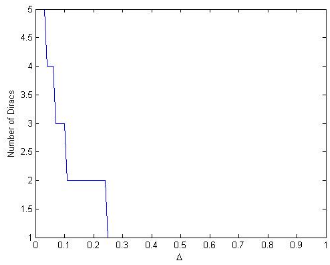
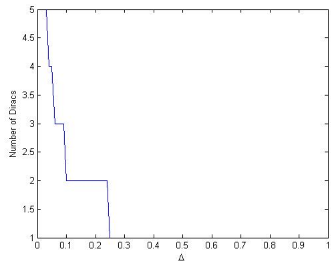


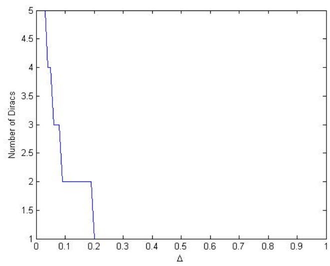
Fig. 6. Δ vs Number of components of $m(\infty)$. Uniform initial conditions. Blue - $w = 0.5$ (below black), Red - $w = 0.75$, Black - $w = 0.9$



(a) $w = 0.5$



(b) $w = 0.75$



(c) $w = 0.9$

Fig. 7. Δ vs Number of components of $m(\infty)$. Initial condition Beta(1,6).

than if we start with a uniform one. This is explained by the fact that with the Beta distribution the mass is more concentrated than with the Uniform distribution (in our case: to the left) and therefore it should be harder (i.e, Δ should be smaller) to split in the same number of components.

7. Proofs

Proof of Proposition 3.2

By definition, since h is convex,

$$h(wx + (1-w)y) \leq wh(x) + (1-w)h(y)\,,$$
$$h(wy + (1-w)x) \leq wh(y) + (1-w)h(x)\,,$$

with strict inequalities if h is strictly convex except when $x = y$ or $w \in \{0, 1\}$, and summing these two inequalities yields the result.

Proof of Proposition 3.5

The first statement is obvious, since a partial consensus is an absorbing state.

We prove the second statement. It follows from the second statement in Lemma 1 that, if the two peers, say (i, j) chosen at any time slot k' are such that $\left|X_i^N(k') - X_j^N(k')\right| \leq \Delta$ and $X_i^N(k') \neq X_j^N(k')$, then $\mu_n^N(k'+1) < \mu_n^N(k')$. Assume now that the hypothesis of the second statement holds. It follows that all peers chosen for interaction at times $k' \geq k$ have reputation values that either differ by more than Δ, or are equal, thus, at any time slot $k' \geq k$, the interaction has no effect. It follows that $M^N(k) = M^N(k')$ for $k' \geq k$.

Further, assume that $M^N(k)$ is not a partial consensus. Thus, there exists a pair of peers (i, j) such that $\left|X_i^N(k) - X_j^N(k)\right| \leq \Delta$ and $X_i^N(k) \neq X_j^N(k)$. The pair (i, j) is never chosen in a interaction at times $k' \geq k$, for otherwise this would contradict the fact that $M^N(k')$ is stationary. But this occurs with probability 0.

Proof of Proposition 3.7

Let i, j be the peers selected for interaction at time k. If they are in different clusters, then there is no change to the process and the proposition holds. Assume now that i, j are in the same cluster, say $\ell_1 = \ell$. After interaction, the distance between i to any peer, say i', not in C_ℓ is increased; since i and i' are not connected at time k, they are not either at time $k+1$. The same holds between j and i'. Therefore, the only difference between connections at time k and $k+1$ concern pairs of peers that that are both in C_ℓ. Thus $\mathcal{C}(k+1) = \{C_1, ..., C_{\ell-1}\} \cup \mathcal{C}'$ where $\mathcal{C}'$ is a partition of C_ℓ.

Proof of Theorem 3.9

Let $\sigma^2(k)$ be the variance of $M^N(k)$ (we drop superscript N in the notation local to this proof). By Corollary 3.4, $\sigma(k)$ is non decreasing and nonnegative, and thus converges to some $\sigma(\infty)$.

For $k \geq K^N$ the set of clusters remains the same, $\mathcal{C}^N(k) = \{C_1, .., C_\ell\}$, and we can thus define the diameter of cluster ℓ_1 by

$$\delta_{\ell_1}(k) = \max_{i,j \in C_{\ell_1}} \left|X_i^N(k) - X_j^N(k)\right| \tag{7.1}$$

and let, for all $\ell_1 \in \{1, ..., L^N\}$:

$$\delta_{\ell_1} = \limsup_{k \geq K^N} \delta_{\ell_1}(k)$$

Assume that $\delta_{\ell_1} > 0$ for some ℓ_1. Since $\sigma^2(k)$ is a Cauchy sequence, there exists some random time $K_1 \geq K^N$ such that for all $k > K_1$ and $k' > K_1$:

$$\left|\sigma^2(k') - \sigma^2(k)\right| < \frac{2w(1-w)}{N} \left(\frac{\delta_{\ell_1}}{2}\right)^2 \tag{7.2}$$

Thus there is an infinite subsequence of time slots $K_2(n)$, with $n \in \mathbb{N}$, such that $K_2(n) \geq K^N$, $K_2(n) \geq K_1$ and

$$\delta_{\ell_1}(K_2(n)) > \frac{\delta_{\ell_1}}{2} > 0$$

For $k \geq K^N$, let $(I(k), J(k))$ be a pair of peers that achieves the maximum in Eq.(7.1) and let E_k be the event "the pair of peers selected for interaction at time k is $(I(k), J(k))$". The probability of E_k, conditional to all past up to time slot k, is $\frac{2}{N(N-1)}$, thus is constant and positive. Thus the probability that E_k occurs infinitely often is 1, i.e. with probability 1 we can extract an infinite subsequence of time slots $K_3(n)$ of $K_2(n)$ such that $E_{K_3(n)}$ is true. By Lemma 7.1, we have

$$\sigma^2\left(K_3(n)+1\right) - \sigma^2\left(K_3(n)\right) > \frac{2w(1-w)}{N} \left(\frac{\delta_{\ell_1}}{2}\right)^2$$

which contradicts Eq.(7.2). This proves that $\delta_{\ell_1} = 0$ for all cluster ℓ_1.

Let $\mu_{\ell_1}(k)$ be the empirical mean of cluster ℓ_1 at time $k \geq K^N$. Since interactions that modify the state of the process at times $k \geq K^N$ are all intra-cluster, it follows that $\mu_{\ell_1}(k) = \mu_{\ell_1}(K^N) := \mu_{\ell_1}(\infty)$ for all $k \geq K^N$. For $i \in C_{\ell_1}$, $\left|X_i^N(k) - \mu_{\ell_1}(k)\right| \leq \delta_{\ell_1}(k) \to 0$, it follows that $X_i^N(k) \to \mu_{\ell_1}(\infty)$ as $k \to \infty$. Thus, for any continuous $f : [0,1] \to \mathbb{R}$:

$$\lim_{k \to \infty} \langle f, M^N(k) \rangle = \frac{1}{N} \sum_{\ell_1=1}^{L^N} N_{\ell_1} f\left(\mu_{\ell_1}(\infty)\right)$$

where N_{ℓ_1} is the cardinality of C_{ℓ_1}. This shows that, with probability 1, $M^N(k)$ converges to $M^N(\infty) = \frac{1}{N} \sum_{\ell_1=1}^{L^N} N_{\ell_1} \delta_{\mu_{\ell_1}(\infty)}$.

It remains to show that $M^N(\infty)$ is a partial consensus. This follows from the fact that if i and j are not in the same cluster at time slot k, then $\left|X_i^N(k) - X_j^N(k)\right| > \Delta$, which implies that $|\mu_{\ell_1}(k) - \mu_{\ell_2}(k)| > \Delta$ if $\ell_1 \neq \ell_2$ and, since, $\mu_{\ell_1}(k)$ is stationary for k large enough, that $|\mu_{\ell_1}(\infty) - \mu_{\ell_2}(\infty)| > \Delta$.

Lemma 7.1. *Let (i,j) be the pair of peers chosen for interaction at time slot k. Assume that $\left|X_i^N(k) - X_j^N(k)\right| \leq \Delta$. Then the reduction in variance is $\sigma^2(k+1) - \sigma^2(k) = \frac{2w(1-w)}{N}\left(X_i^N(k) - X_j^N(k)\right)^2$.*

Proof. By direct computation. □

Proof of Theorem 4.2

We write (4.7) and (4.8) in the notation of Section 2.2 in Graham[16], in which the corresponding equations are (2.5) and (2.7), and

$$\begin{aligned}\mathcal{A}(\mu)h(x) &= 2\big\langle [h(wx+(1-w)y)-h(x)]1_{\{|x-y|\le\Delta\}}\,,\mu(dy)\big\rangle\\ &= \int (h(z)-h(x))J(\mu,x,dz)\end{aligned}$$

for $J(\mu,x,dz)$ the image measure of $1_{\{|x-y|\le\Delta\}}2\mu(dy)$ by $y\mapsto wx+(1-w)y$. Since $|J(\mu,x,\cdot)|\le 2$ and $|J(\mu,x,\cdot)-J(\nu,x,\cdot)|\le 2|\mu-\nu|$, the assumptions of Proposition 2.3 in Ref. 16 are satisfied, yielding the results. The family (4.7) is uniformly bounded by 4 in operator norm, and thus there is a well-defined inhomogeneous Markov process with generator $\mathcal{A}(m(t))$ at time t and arbitrary initial law.

Proof of Theorem 4.3

First, the proof of *(1)*. The generator $\mathcal{A}^N$ corresponds to the "binary mean-field model" (2.6) in Graham-Méléard[19] with N instead of n and $\mathcal{L}_i=0$, and (using $\sum_{1\le i\ne j\le N}=2\sum_{1\le i<j\le N}$) "jump kernel"

$$\widehat{\mu}(x,y,dh,dk)=1_{\{|x-y|\le\Delta\}}2\delta_{\{(w-1)x+(1-w)y,(w-1)y+(1-w)x\}}(dh,dk)$$

which is uniformly bounded in total mass by $\Lambda=2$. We conclude with Theorem 3.1 in Ref. 19 and the triangular inequality $|\frac{1}{N}\sum_{i=1}^N\mathcal{L}(\widehat{X}_i^N)-Q|_T\le|\mathcal{L}(\widehat{X}_i^N)-Q|_T$ (the $\widehat{X}_i^N$ are exchangeable).

Now, the proof of *(2)*. As in the proof of Theorem 3.1 in Ref. 19,

$$\left\langle \phi,\widehat{\Lambda}^N-\frac{1}{N}\sum_{i=1}^N\mathcal{L}(\widehat{X}_i^N)\right\rangle^2=\frac{1}{N^2}\left[\sum_{i=1}^N(\phi(\widehat{X}_i^N)-\mathbb{E}[\phi(\widehat{X}_i^N)])\right]^2$$

in which

$$\begin{aligned}\left[\sum_{i=1}^N(\phi(\widehat{X}_i^N)-\mathbb{E}[\phi(\widehat{X}_i^N)])\right]^2 &= \sum_{i=1}^N(\phi(\widehat{X}_i^N)-\mathbb{E}[\phi(\widehat{X}_i^N)])^2\\ &\quad+\sum_{1\le i\ne j\le N}(\phi(\widehat{X}_i^N)-\mathbb{E}[\phi(\widehat{X}_i^N)])(\phi(\widehat{X}_j^N)-\mathbb{E}[\phi(\widehat{X}_j^N)])\end{aligned}$$

where the first sum on the r.h.s. has N terms, the second $N(N-1)$, and

$$\begin{aligned}&\mathbb{E}\big[(\phi(\widehat{X}_i^N)-\mathbb{E}[\phi(\widehat{X}_i^N)])(\phi(\widehat{X}_j^N)-\mathbb{E}[\phi(\widehat{X}_j^N)])\big]\\ &\qquad=\mathbb{E}[\phi(\widehat{X}_i^N)\phi(\widehat{X}_j^N)]-\mathbb{E}[\phi(\widehat{X}_i^N)]\mathbb{E}[\phi(\widehat{X}_j^N)]\,,\end{aligned}$$

and we conclude to the first formula in *(2)* using *(1)* for $k=2$.

Classically, the weak topology in the Polish space $\mathcal{P}(D(\mathbb{R}_+,[0,1]))$ has a convergence-determining sequence $(g_m)_{m\ge1}$ of continuous functions bounded by 1 (such a sequence is constructed in the proof of Proposition 3.4.4 in Ethier-Kurtz[14]), and can thus be metrized by $d(P,Q) = \left(\sum_{i\geq 1} 2^{-i}\langle g_m, P-Q\rangle^2\right)^{1/2}$. Moreover, the first formula in *(2)* and the second in *(1)* imply that $\mathbb{E}(d(\widehat{\Lambda}^N, Q)^2)$ goes to 0, which proves convergence in probability for $\widehat{\Lambda}^N$.

The result for $\widehat{\Lambda}^N$ implies the result for its marginal process $\widehat{M}^N$ as a quite general topological fact, since the limit marginal process m is continuous and the spaces are Polish (Theorem 4.6 in Graham-Méléard,[19] Section 4.3 in Méléard[28]); proofs first use the Skorohod topology, and then Theorem 3.10.2 in Ref. 14.

Proof of Theorem 4.4

Let $\lambda_N : \mathbb{R}_+ \to \mathbb{R}_+$ be the (random) time-change given by the linear interpolation of $\lambda_N(\frac{k}{N}) = \frac{T_k}{N}$, *i.e.*, by

$$t \in \left[\frac{k}{N}, \frac{k+1}{N}\right] \mapsto \lambda_N(t) = (k+1-tN)\frac{T_k}{N} + (tN-k)\frac{T_{k+1}}{N}\,, \qquad k \in \mathbb{N}\,.$$

Then (4.4) implies that

$$\widetilde{X}^N(t) = \widehat{X}^N(\lambda_N(t))\,, \qquad t \in \mathbb{R}_+\,,$$

so that their atomic distance is null. The triangular inequality yields, for $k \in \mathbb{N}$,

$$|\lambda_N(t) - t| \leq \left|\frac{T_k}{N} - \frac{k}{N}\right| + \frac{1}{N}(T_{k+1} - T_k) + \frac{1}{N}\,, \qquad t \in \left[\frac{k}{N}, \frac{k+1}{N}\right],$$

and hence, for any $T > 0$,

$$\sup_{0\leq t\leq T} |\lambda_N(t) - t| \leq \frac{1}{N}\sup_{0\leq k\leq \lfloor NT\rfloor} |T_k - k| + \frac{1}{N}\sup_{0\leq k\leq \lfloor NT\rfloor}(T_{k+1} - T_k) + \frac{1}{N}\,.$$

For $\varepsilon > 0$, Kolmogorov's maximal inequality implies that

$$\mathbb{P}\left(\frac{1}{N}\sup_{0\leq k\leq \lfloor NT\rfloor} |T_k - k| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2 N^2}\sum_{i=1}^{\lfloor NT\rfloor} \mathrm{var}(T_i - T_{i-1}) = \frac{\lfloor NT\rfloor}{\varepsilon^2 N^2}\,,$$

and classically

$$\mathbb{P}\left(\frac{1}{N}\sup_{0\leq k\leq \lfloor NT\rfloor}(T_{k+1} - T_k) \geq \varepsilon\right) = 1 - (1 - \mathrm{e}^{-N\varepsilon})^{\lfloor NT\rfloor+1} \leq (\lfloor NT\rfloor + 1)\mathrm{e}^{-N\varepsilon}\,.$$

Hence, for all $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{0\leq t\leq T} |\lambda_N(t) - t| \geq \delta\right) = 0\,,$$

from which the result follows.

Proof of Theorem 4.5

Result *(1)* follows from the previous convergence in probability result and Theorem 4.3, using either the uniform continuity of the test functions (for the atomic metric) or Corollary 3.3.3 in Ethier-Kurtz[14] (for the usual metric). Result *(2)*, which involves Polish spaces, follows as for Theorem 4.3.

Proof of Proposition 5.2

For $0 < b$ and $t \in [0, b]$ define $u(t) := \sigma^2(b-t) - \sigma^2(b)$. Note that $\mu_1(t)$ is a constant thus $u(t) = \mu_2(b-t) - \mu_2(b)$. By the alternative definition of Problem 1

$$u(t) = -\int_{b-t}^{b} \int_{[0,1]^2} \left[(wx + (1-w)y)^2 + (wy + (1-w)x)^2 - x^2 - y^2\right] 1_{\{|x-y|\leq\Delta\}} m(s)(dx)m(s)(dy)ds$$

By Proposition 3.2, the bracket is nonpositive, and the indicator function is upper bounded by 1 thus

$$\begin{aligned} u(t) &\leq -\int_{b-t}^{b} \int_{[0,1]^2} \left[(wx + (1-w)y)^2 + (wy + (1-w)x)^2 - x^2 - y^2\right] m(s)(dx)m(s)(dy)ds \\ &= K \int_{b-t}^{t} \sigma^2(s)ds = K\left(\sigma^2(b) + \int_0^t u(s)ds\right) \end{aligned}$$

with $K = 4w(1-w)$. By Grönwall's lemma:

$$u(t) \leq K\sigma^2(b)t + K^2\sigma^2(b)e^{Kt} \int_0^t se^{-Ks}ds = \sigma^2(b)\left(e^{Kt} - 1\right)$$

Let $t = b$ and the proposition follows.

Proof of Proposition 5.4

Fix some $t_0 \geq 0$; we will show that $\operatorname{ess\,inf}(m(t)) \geq \operatorname{ess\,inf}(m(t_0))$ for every $t \geq t_0$. Clearly, it is sufficient to consider the case $\operatorname{ess\,inf}(m(t_0)) > 0$. Take some arbitrary $a < \operatorname{ess\,inf}(m(t_0))$. Let $h(x) = 1_{\{x \leq a\}}$ and $\varphi(t) = \langle h, m(t)\rangle$. We have $\varphi(t_0) = 0$ and, by definition of Problem 1:

$$\varphi(t) \leq 2\int_{t_0}^{t} \langle |h(wx + (1-w)y) - h(x)|\,, m(s)(dx)m(s)(dy)\rangle ds$$

Note that $|h(wx + (1-w)y) - h(x)| \leq 1$ and that $h(wx + (1-w)y) - h(x) \neq 0$ requires either $x \leq a, y > a$ or $x > a, y \leq a$. Thus

$$\varphi(t) \leq 2\int_{t_0}^{t} 2\varphi(s)(1-\varphi(s))ds \leq 4\int_{t_0}^{t} \varphi(s)ds$$

By Grönwall's lemma, this shows that $\varphi(t) = 0$ for $t \geq t_0$. Thus $m(t)[0, a] = 0$ for all $t \geq t_0$ and this is true for any $a < \operatorname{ess\,inf}(m(t_0))$ thus $\operatorname{ess\,inf}(m(t)) \geq \operatorname{ess\,inf}(m(t_0))$. This shows $\operatorname{ess\,inf}(m(t))$ is non decreasing; the proof is similar by analogy for the $\operatorname{ess\,sup}$.

Proof of Theorem 5.5

1. We show that $m(t)$ converges to some probability $m(\infty)$. This follows from Proposition 3.2 applied for example to the family of functions $h_\omega : x \to e^{-\omega x}$ indexed by $\omega \in [0, \infty)$. For any fixed ω, $\langle h_\omega, m(t)\rangle$ is a nondecreasing function of t and is non-negative, thus converges as $t \to \infty$. The limit is a probability (apply convergence to the constant equal to 1).
2. We would like to conclude that $m(\infty)$ is a stationary point, i.e. $\langle \mathcal{A}(m(\infty))h, m(\infty)\rangle = 0$ for any $h \in L^\infty[0,1]$, however there is a technical difficulty since the definition of $\mathcal{A}$ involves the non continuous function $1_{\{|x-y|\leq\Delta\}}$. We circumvent the difficulty as follows. For $\varepsilon > 0$ and smaller than Δ, let $\ell_\varepsilon(x)$ be the continuous function of $x \in \mathbb{R}^+$ equal to 1 for $x \leq \Delta - \varepsilon$, 0 for $x \geq \Delta$, and the linear interpolation in-between. We have $1_{\{x\leq\Delta-\varepsilon\}} \leq \ell_\varepsilon(x) \leq 1_{\{x\leq\Delta\}}$ for all $x \geq 0$. Let $h(x) = x^2$. By the alternative definition of Problem 1, for t and $u \geq 0$:

$$\begin{aligned}&\langle h, m(t+u)\rangle - \langle h, m(t)\rangle \\ &\leq -2w(1-w)\int_t^{t+u} \langle (x-y)^2 \ell_\varepsilon(|x-y|), m(s)(dx)m(s)dy\rangle ds\end{aligned}$$

Fix $u \geq 0$ and let $t \to \infty$. By weak convergence of the product measure $m(t) \otimes m(t)$ it follows that

$$0 \leq -2w(1-w)u\langle (x-y)^2 \ell_\varepsilon(|x-y|), m(\infty)(dx)m(\infty)dy\rangle$$

and thus $\langle (x-y)^2 \ell_\varepsilon(|x-y|), m(\infty)(dx)m(\infty)dy\rangle = 0$ from where we conclude that

$$\langle (x-y)^2 1_{\{|x-y|\leq\Delta-\varepsilon\}}, m(\infty)(dx)m(\infty)dy\rangle = 0 \tag{7.3}$$

for all $\varepsilon \in (0, \Delta)$.
3. Fix some $\varepsilon > 0$ and integrate the previous equation with respect to y; it comes that $\langle r(x), m(\infty)(dx)\rangle = 0$ with $r(x) \overset{\text{def}}{=} \langle (y-x)^2 1_{\{|y-x|\leq\Delta-\varepsilon\}}, m(\infty)(dy)\rangle$, thus there is a set $\Omega_1 \subset [0,1]$ with $m(\infty)(\Omega_1) = 1$ and $r(x) = 0$ for every $x \in \Omega_1$. Let x_1 be an element of Ω_1 (which is not empty since $m(\infty)(\Omega_1) = 1$). Then $r(x_1) = 0$ and thus $m(\infty)\left([(x_1 - \Delta + \varepsilon, x_1) \cup (x_1, x_1 + \Delta - \varepsilon)] \cap [0,1]\right) = 0$ and the restriction of $m(\infty)$ to $(x_1 - \Delta + \varepsilon, x_1 + \Delta - \varepsilon) \cap [0,1]$ is a dirac mass at x_1. Apply the same reasoning to the complement of $(x_1 - \Delta + \varepsilon, x_1 + \Delta - \varepsilon)$, this shows recursively that $m(\infty)$ is a finite sum of Dirac masses, i.e. $m(\infty) = \sum_{i=1}^I \alpha_i \delta_{x_i}$ for some $I \in \mathbb{N}$, $\alpha_i > 0$, $\sum_{i=1}^I \alpha_i = 1$ and $x_i \in [0,1]$.

Assume that $|x_i - x_j| < \Delta$ for some $i \neq j$. Apply Eq.(7.3) with $\varepsilon = \frac{\Delta - |x_i - x_j|}{2}$. The right-handside of Eq.(7.3) is lower bounded by $\alpha_i\alpha_j(x_i - x_j)^2 > 0$, which is a contradiction. Therefore $|x_i - x_j| \geq \Delta$ for all $i \neq j$.

Proof of Proposition 5.7

First we show that if $\nu \in P_{n+1}(\mu_0)$ then there exists some $\nu' \in P_n(\mu_0)$ with $\langle h, \nu'\rangle \leq \langle h, \nu\rangle$, which will clearly show the proposition.

We are given $\nu = \sum_{i=1}^{n+1} \alpha_i \delta_{x_i} \in P_{n+1}(\mu_0)$. Let $x_n' = \frac{\alpha_n x_n + \alpha_{n+1} x_{n+1}}{\alpha_n + \alpha_{n+1}}$ and

$$\nu' = \sum_{i=1}^{n-1} \alpha_i \delta_{x_i} + \left(\left(\alpha_n + \alpha_{n+1} \right) \delta_{x_n'} \right)$$

We have $\nu' \in P_n(\mu)$ and by convexity of h:

$$\left(\alpha_n + \alpha_{n+1} \right) h(x_n') \leq \alpha_n h(x_n) + \alpha_{n+1} h(x_{n+1})$$

thus $\langle h, \nu' \rangle \leq \langle h, \nu \rangle$ as required.

Proof of Theorem 5.8

By hypothesis $\langle h, m_0 \rangle \leq q$ and since h is continuous, by Theorem 5.5, $\langle h, m(\infty) \rangle \leq q$. Since the mean of $m(\infty)$ is also μ_0 (again by Theorem 5.5 applied to $h(x) = x$), it follows that q is not in $Q_d(h, \mu_0)$. Together with the hypothesis $q \in Q_n(h, \mu_0)$, Proposition 5.7 implies that $d < n$.

Proof of Proposition 5.11

Let $m'(t)$ be the image measure of $m(t)$ by $x \mapsto 1 - x$. By direct computation and the alternative form of Problem 1, it follows that $m'(t)$ is solution to Problem 1 with initial condition $m'(0) = m(0)$. By uniqueness, $m'(t) = m(t)$.

Proof of Proposition 5.12

Let ν be a symmetric partial consensus with $n + 1$ components. We do as in the proof of Proposition 5.7: If $n + 1$ is even, we replace the two middle components by their weighted averages. If $n + 1$ is odd, we replace the three middle components $x_{m-1}, x_m = 0.5, x_{m+1}$ (with $m = n/2 + 1$) by two components $(\alpha_{m-1} x_{m-1} + 0.5 \alpha_m x_m)/(\alpha_{m-1} + 0.5 \alpha_m)$ and $(0.5 \alpha_m x_m + \alpha_{m+1} x_{m+1})/(0.5 \alpha_m + \alpha_{m+1})$ with weights $\alpha_{m-1} + 0.5 \alpha_m$ and $0.5 \alpha_m + \alpha_{m+1}$. We obtain some $\nu' \in SP_n$ and $\langle h, \nu' \rangle \leq \langle h, \nu \rangle$ for any convex h, thus if $q \in SQ_n(h)$ we must also have $q \in SQ_{n+1}(h)$.

Proof of Theorem 5.13

The proof is similar to Theorem 5.8.

Proof of Theorem 6.1

Assuming that m_0 is absolutely continuous, the fact that $m(t)$ is absolutely continuous can be proved by probabilistic arguments which use representations by inhomogeneous Markov processes with uniformly bounded jump rates.

More precisely, the proof of Theorem 2.1 in Desvillettes *et al.*,[10] for a class of equations (the generalized cutoff Kac equation) with the same probabilistic structure as ours, extends immediately to the present situation. It is an extension of Theorem 4.2 proved using only its hypotheses.

If $m = (m(t), t \in \mathbb{R}_+)$ is a solution of Problem 1 and $m(t)(dx) = f(x,t)\,dx$ then, for any bounded h, an elementary change of variables yields

$$\begin{aligned}
&\int h(x) f(x,t)\,dx - \int h(x) f(x,0)\,dx \\
&\quad = 2\int_0^t \iint h(wx + (1-w)y) 1_{\{|x-y|\le\Delta\}} f(x,s) f(y,s)\,dxdy\,ds \\
&\qquad - 2\int_0^t \iint h(x) 1_{\{|x-y|\le\Delta\}} f(x,s) f(y,s)\,dxdy\,ds \\
&\quad = \frac{2}{w}\int_0^t \int h(x') \left[\int_{x'-\Delta w}^{x'+\Delta w} f\left(\frac{x'-(1-w)y}{w}, s\right) f(y,s)\,dy\right] dx'\,ds \\
&\qquad - 2\int_0^t \int h(x) f(x,s) \left[\int_{x-\Delta}^{x+\Delta} f(y,s)\,dy\right] dx\,ds
\end{aligned}$$

from which (6.1) readily follows.

The converse statement follows by integrating Eq.(6.1) by $h(x)\,dx$, which after the reverse change of variables yields Problem 1 as a weak formulation.

Eq.(6.2) is obtained similarly using the change of variables $x' = \frac{wx-(1-w)y}{2w-1}$ and $y' = \frac{wy-(1-w)x}{2w-1}$.

Proof of Proposition 6.2

Because of the non-negativeness of $f(x,t)$ for all t, we have:

$$\frac{\partial f(x,t)}{\partial t} \le \frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t) f\left(\frac{x-(1-w)y}{w}, t\right) dy.$$

For a fixed arbitrary t, let $A_i = \{x \in \ \mathrm{Supp}(f(x,t)) | i-1 < f(x,t) \le i\},\ i > 0$ be the level sets. Note that $A_j = \emptyset$ for all $j > \lceil M(t) \rceil$ and that the A_i are disjoint. For any x, we have that:

$$\frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t) f\left(\frac{x-(1-w)y}{w}, t\right) dy$$

$$\le \frac{2}{w}\sum_{i,j} \mu\left(\left\{y \,\middle|\, y \in A_i, \frac{x-(1-w)y}{w} \in A_j\right\}\right) \max\{i,j\}^2.$$

Using the fact that the A_i are disjoint we can get that:

$$\frac{2}{w}\sum_{i,j}\mu\left(\left\{y\,\middle|\,y\in A_i,\frac{x-(1-w)y}{w}\in A_j\right\}\right)\max\{i,j\}^2$$
$$=\frac{2}{w}\sum_{i}\mu\left(\left\{y\,\middle|\,y\in A_i,\frac{x-(1-w)y}{w}\in\bigcup_{k\le i}A_k\right\}\right)i^2$$
$$+\frac{2}{w}\sum_{i}\mu\left(\left\{y\,\middle|\,y\in\bigcup_{k<i}A_k,\frac{x-(1-w)y}{w}\in A_i\right\}\right)i^2=I_1+I_2.$$

We can bound I_1 and I_2 now as:

$$I_1\le\frac{2}{w}\sum_i\mu(A_i)i^2,\qquad I_2\le\frac{2}{1-w}\sum_i\mu(A_i)i^2,$$

subject to the following restrictions:

$$\sum_i\mu(A_i)\le 1,\qquad \sum_i(i-1)\mu(A_i)\le\int_0^1 f(x,t)dx=1.$$

Plugging the second restriction into the bound of I_1 and I_2, we get that:

$$\sum_{i=1}^{\lceil M(t)\rceil}\mu(A_i)i^2\le\frac{\lceil M(t)\rceil^2}{\lceil M(t)\rceil-1}+\sum_{i=1}^{\lceil M(t)\rceil-1}\mu(A_i)\left(i^2-\frac{\lceil M(t)\rceil^2}{\lceil M(t)\rceil-1}(i-1)\right)$$

$$=\frac{\lceil M(t)\rceil^2}{\lceil M(t)\rceil-1}+\frac{1}{\lceil M(t)\rceil-1}\sum_{i=1}^{\lceil M(t)\rceil-1}\mu(A_i)\left(\lceil M(t)\rceil i-\lceil M(t)\rceil-i\right)\left(i-\lceil M(t)\rceil\right).$$

The maximum of the RHS is attained when $\mu(A_i)=0\quad\forall\, i>1$ and $\mu(A_1)$ is as big as possible. By the first restriction, $\mu(A_1)=1$. In that case, we have that:

$$\sum_{i=1}^{\lceil M(t)\rceil}\mu(A_i)i^2\le\frac{\lceil M(t)\rceil^2}{\lceil M(t)\rceil-1}+1\le\lceil M(t)\rceil+3\le M(t)+4.$$

Therefore:

$$\sup_{A_i}\left\{\sum_i\mu(A_i)i^2\right\}\le M(t)+4.$$

Finally, for any x we have:

$$\frac{2}{w}\int_{x-w\Delta}^{x+w\Delta}f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy\le I_1+I_2\le\left(\frac{2}{w}+\frac{2}{1-w}\right)(M(t)+4),$$

which means that:

$$M'(t)\le\left(\frac{2}{w}+\frac{2}{1-w}\right)(M(t)+4).$$

Integrating, we get the result.

Proof of Proposition 6.3

Again, because of the non-negativeness of $f(x,t)$ we have, for all x:

$$\left|\frac{\partial}{\partial t} f(x,t)\right| \\ \leq \max\left\{\frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t) f\left(\frac{x-(1-w)y}{w}, t\right) dy,\ 2f(x,t)\left(\int_{x-\Delta}^{x+\Delta} f(y,t)dy\right)\right\}.$$

On the one hand, we have that:

$$2f(x,t)\left(\int_{x-\Delta}^{x+\Delta} f(y,t)dy\right) \leq 2M(t)\int_0^1 f(y,t)dy \leq 2M(t),$$

on the other, using Proposition 6.2:

$$\frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t) f\left(\frac{x-(1-w)y}{w}, t\right) dy \leq \left(\frac{2}{w} + \frac{2}{1-w}\right)(M(t)+4),$$

therefore:

$$\left|\frac{\partial}{\partial t} f(\cdot,t)\right|_\infty \leq \left(\frac{2}{w} + \frac{2}{1-w}\right)(M(t)+4).$$

Proof of Proposition 6.5

As $f^e(x, t+\Delta t)$ is piecewise linear, we can treat each interval independently. Given a $\nu_e(x)$ associated to $f^e(x) = ax+b$ we want to find:

$$\min_{\nu_r}\int_X |d\nu_e(x) - d\nu_r(x)| = \min_M \int_{x_s}^{x_e} |ax+b-M|dx.$$

If $a=0$, then $M=b$ has zero error. Let's suppose $a\neq 0$. If M lies between ax_s+b and ax_e+b, then:

$$\min_M \int_{x_s}^{x_e} |ax+b-M|dx = \min_M \frac{1}{2a}[(ax_e+b-M)^2 + (ax_s+b-M)^2] = \frac{a}{4}(x_e-x_s)^2.$$

The minimum is attained for

$$M_{min} = \frac{\displaystyle\int_{x_s}^{x_e}(ax+b)dx}{x_e-x_s} = \frac{a}{2}(x_s+x_e)+b,$$

which is the value of the function at the midpoint of the interval. If M lies outside ax_s+b and ax_e+b, then the error is greater than the previous case as we could minimize it by setting M to one of the extremal values of f^e in X.

Proof of Lemma 6.6

Let $f^r(x,t)$ be defined piecewise in the intervals $X_i = [x_i, x_{i+1}]$ and let $M_{min,i}$ be the value of M that minimizes the error for the interval X_i. We have that, independently of t:

$$\int_0^1 f^r(x,t)dx = \int_0^1 \sum_{i=1}^{I} M_{min,i} 1_{X_i} dx = \sum_{i=1}^{I} \int_{X_i} \frac{\int_{x_i}^{x_{i+1}} f^e(y,t)dy}{x_{i+1}-x_i} dx = \int_0^1 f^e(y,t)dy.$$

Proof of Proposition 6.7

We first calculate the error when $I = I_0$. Keeping in mind that for any interval, the slope of $f^e(x,k\Delta t)$ is bounded by $\frac{2\Delta t|\partial_t f^r(x,(k-1)\Delta t)|_\infty}{1/I_0}$, yielding:

$$\varepsilon_{c.s.}(I_0) \leq I_0 \frac{\text{Max. Slope}}{4}\left(\frac{1}{I_0}\right)^2 = \frac{\Delta t}{2}|\partial_t f^r(x,(k-1)\Delta t)|_\infty. \tag{7.4}$$

However, if we divide each interval in two, the error is halved, because the error with two intervals equals $2\frac{\text{Slope}}{4}\left(\frac{1}{1/2I_0}\right)^2$, where with one is equal to $\frac{\text{Slope}}{4}\left(\frac{1}{1/I_0}\right)^2$. Therefore, for sufficiently large I we can write:

$$\varepsilon_{c.s}(I) = \varepsilon_{c.s}(I_0)\frac{I_0}{I} \leq \frac{\Delta t}{2}|\partial_t f^r(x,(k-1)\Delta t)|_\infty \frac{I_0}{I}. \tag{7.5}$$

Proof of Proposition 6.8

$$M(\Delta t) = |f^r(x,\Delta t)|_\infty \leq |f^e(x,\Delta t)|_\infty \leq |f^r(x,0)|_\infty + \Delta t|\partial_t f^r(x,0)|_\infty$$

$$\leq M + \Delta t K_1 M + \Delta t K_2 = (1+\Delta t K_1)M + \Delta t K_2,$$

where $K_1 = \frac{2}{w} + \frac{2}{1-w}, K_2 = \frac{8}{w} + \frac{8}{1-w}$. The first inequality is true because when we approximate by piecewise constant splines, the maximum of the function decreases and the third is true by Proposition 6.2. Note that in order to be able to apply it we are implicitly using Lemma 6.6 as the total mass is conserved. By induction:

$$M\left(\frac{T}{\Delta t}\Delta t\right) \leq (1+\Delta t K_1)^{\frac{T}{\Delta t}} M + \Delta t K_2 \sum_{i=0}^{T/\Delta t - 1} (1+\Delta t K_1)^i$$

$$= (1+\Delta t K_1)^{\frac{T}{\Delta t}} M + \frac{K_2}{K_1}((1+\Delta t K_1)^{\frac{T}{\Delta t}} - 1) \leq \frac{K_2}{K_1}(1+\Delta t K_1)^{\frac{T}{\Delta t}}\left(M + \frac{K_2}{K_1}\right).$$

We can now bound $M(k\Delta t)$ in the following way. As K_1 and K_2 are positive, taking into account that $(1+K_1\Delta t)^{\frac{T}{\Delta t}}$ is decreasing with Δt, we have for any k:

$$M(k\Delta t) \leq (1+\Delta t K_1)^{\frac{T}{\Delta t}} M + \frac{K_2}{K_1}(1+\Delta t K_1)^{\frac{T}{\Delta t}} \leq e^{K_1 T}\left(M + \frac{K_2}{K_1}\right).$$

Using Proposition 6.3:

$$|\partial_t f^r(x,(k-1)\Delta t)|_\infty \leq K_1 M((k-1)\Delta t) + K_2 \leq K_1 e^{K_1 T}\left(M + \frac{K_2}{K_1}\right) + K_2 = C_1.$$

Proof of Proposition 6.9

We have that:

$$\begin{aligned}\varepsilon_{eu} &= |\nu_e^{k\Delta t}(dx) - \mu_{(k-1)\Delta t}^{k\Delta t}(dx)|_T \\ &= \int_0^1 |g^{(k-1)\Delta t}(x,k\Delta t) - g^{(k-1)\Delta t}(x,(k-1)\Delta t) - \Delta t \partial_t g^{(k-1)\Delta t}(x,(k-1)\Delta t)|dx \\ &\leq \frac{1}{2}(\Delta t)^2 |\partial_{tt}^2 g^{(k-1)\Delta t}(x,(k-1)\Delta t)|_\infty + O\left((\Delta t)^3\right).\end{aligned}$$

By Corollary 6.4, we can bound, for any k:

$$\begin{aligned}|\partial_{tt}^2 g^{(k-1)\Delta t}(x,(k-1)\Delta t)|_\infty &\leq 16\Delta |\partial_t g^{(k-1)\Delta t}(x,(k-1)\Delta t)|_\infty |g^{(k-1)\Delta t}(x,(k-1)\Delta t)|_\infty \\ &\leq 16\Delta\left(K_1 e^{K_1 T}\left(M + \frac{K_2}{K_1}\right) + K_2\right) e^{K_1 T}\left(M + \frac{K_2}{K_1}\right) = C_2,\end{aligned}$$

therefore:

$$\varepsilon_{eu} \leq \frac{C_2}{2}(\Delta t)^2 + O((\Delta t)^3) = O((\Delta t)^2). \tag{7.6}$$

Proof of Proposition 6.10

$$\begin{aligned}&\frac{\partial}{\partial t}\int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)|dx \leq \int_0^1 |\partial_t g^{k\Delta t}(x,t) - \partial_t g^{(k-1)\Delta t}(x,t)| \\ &\leq \int_0^1 2\left|-g^{k\Delta t}(x,t)\int_{x-\Delta}^{x+\Delta} g^{k\Delta t}(y,t)dy + g^{(k-1)\Delta t}(x,t)\int_{x-\Delta}^{x+\Delta} g^{(k-1)\Delta t}(y,t)dy\right| \\ &\quad + \int_0^1 \frac{2}{w}\left|\int_{x-w\Delta}^{x+w\Delta} g^{k\Delta t}(y,t) g^{k\Delta t}\left(\frac{x-(1-w)y}{w},t\right)dy\right. \\ &\quad \left. - \int_{x-w\Delta}^{x+w\Delta} g^{(k-1)\Delta t}(y,t) g^{(k-1)\Delta t}\left(\frac{x-(1-w)y}{w},t\right)dy\right| = I + J.\end{aligned}$$

We will first bound I. We have that:

$$I \leq 2\int_0^1 |g^{(k-1)\Delta t}(x,t) - g^{k\Delta t}(x,t)| \int_{x-\Delta}^{x+\Delta} g^{(k-1)\Delta t}(y,t)dzdx$$

$$+2\int_0^1 g^{k\Delta t}(x,t)\int_{x-\Delta}^{x+\Delta} |g^{(k-1)\Delta t}(y,t) - g^{k\Delta t}(y,t)|dzdx = I_1 + I_2.$$

On the one hand:

$$I_1 \leq 2\int_0^1 |g^{(k-1)\Delta t}(x,t) - g^{k\Delta t}(x,t)|dx,$$

on the other:

$$I_2 \leq 2\int_0^1 g^{k\Delta t}(x,t)\int_0^1 |g^{(k-1)\Delta t}(y,t) - g^{k\Delta t}(y,t)|dzdx \leq 2\int_0^1 |g^{(k-1)\Delta t}(x,t) - g^{k\Delta t}(x,t)|dx.$$

Now we will bound J:

$$J \leq \frac{2}{w}\int_0^1 \int_{x-w\Delta}^{x+w\Delta} g^{k\Delta t}(y,t)\left| g^{k\Delta t}\left(\frac{x-(1-w)y}{w}, t\right) - g^{(k-1)\Delta t}\left(\frac{x-(1-w)y}{w}, t\right)\right| dzdx$$

$$+\frac{2}{w}\int_0^1 \int_{x-w\Delta}^{x+w\Delta} \left| g^{k\Delta t}(y,t) - g^{(k-1)\Delta t}(y,t)\right| g^{(k-1)\Delta t}\left(\frac{x-(1-w)y}{w}, t\right) dzdx = J_1 + J_2$$

$$J_1 = 2\int_0^1 \int_{x-\Delta}^{x+\Delta} g^{k\Delta t}(x,t)\left| g^{k\Delta t}(y,t) - g^{(k-1)\Delta t}(y,t)\right| dzdx$$

$$\leq 2\int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)|dx$$

$$J_2 = 2\int_0^1 \int_{x-\Delta}^{x+\Delta} \left| g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)\right| g^{(k-1)\Delta t}(y,t)\, dzdx$$

$$\leq 2\int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)|dx.$$

Adding all the equations together we get that:

$$\frac{\partial}{\partial t}\int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)|dx \leq I + J \leq I_1 + I_2 + J_1 + J_2$$

$$\leq 8\int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)|dx.$$

Integrating:

$$\left|\mu^t_{k\Delta t}(dx) - \mu^t_{(k-1)\Delta t}(dx)\right|_T = \int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)|dx$$

$$\leq e^{8(t-k\Delta t)}\int_0^1 |g^{k\Delta t}(x,k\Delta t) - g^{(k-1)\Delta t}(x,k\Delta t)|dx$$

$$= e^{8(t-k\Delta t)}|\mu^{k\Delta t}_{k\Delta t}(dx) - \mu^{k\Delta t}_{(k-1)\Delta t}(dx)|_T,$$

as we wanted to prove.

Proof of Theorem 6.11

$$\varepsilon_{tot} \leq \sum_{k=1}^{T/(\Delta t)} \left|\mu^{T}_{(k-1)\Delta t}(dx) - \mu^{T}_{k\Delta t}(dx)\right|_{T} \leq e^{8T} \sum_{k=1}^{T/(\Delta t)} \left|\mu^{k\Delta t}_{(k-1)\Delta t}(dx) - \mu^{k\Delta t}_{k\Delta t}(dx)\right|_{T}$$
$$= e^{8T}\frac{T}{\Delta t} O\left((\Delta t)^2 + \frac{\Delta t}{I}\right) = O\left(\Delta t + \frac{1}{I}\right).$$

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