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Optimal Design of Low-Contrast Two Phase Structures for the Wave Equation

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Optimal Design of Low-Contrast Two Phase Composites for Wave Propagation *

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Abstract

This paper is concerned with the following optimal design problem: find the distribution of two phases in a given domain that minimizes an objective function evaluated through the solution of a wave equation. This type of optimization problem is known to be ill-posed in the sense that it generically does not admit a minimizer among classical admissible designs. Its relaxation could be found, in principle, through homogenization theory but, unfortunately, it is not always explicit, in particular for objective functions depending on the solution gradient. To circumvent this difficulty we make the simplifying assumption that the two phases have a low constrast. Then, a second order asymptotic expansion with respect to the small amplitude of the phase coefficients yields a simplified optimal design problem which is amenable to relaxation by means of H-measures. We prove a general existence theorem in a larger class of composite materials and propose a numerical algorithm to compute minimizers in this context. As in the case of an elliptic state equation, the optimal composites are shown to be rank one laminates. However the proof that relaxation and small amplitude limit commute is more delicate than in the elliptic case.

Keywords: optimal design, H-measures, homogenization

1 Introduction

The homogenization method is one of the most successful approaches in shape and topology optimization. Most of the literature on the subject is devoted to problems where the state equation is stationary [1], [4], [18]. We implicitly include in this body of literature the many works on the optimal design of structures submitted to forced vibrations (see section 2.1.2 in

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[4]), for which the state equation is of Helmholtz type, i.e. the wave equation in the frequency domain. Very few papers are concerned with a time dependent state equation, be it the heat equation [16] or the wave equation (in the time domain) [11], [13], [15]. One possible reason for this lack of contributions is the additional difficulties which arise in this context. From a theoretical point of view, we see at least two of them. First, there are no simple situations, like single state equations in the conductivity setting or compliance minimization in the elasticity setting, where the optimality condition helps in reducing the complexity of the optimal microstructure. For example, optimal microstructures are unknown for any type of objective function in the elastodynamics setting. Second, the relaxation of a gradient-based objective function relies on a corrector result which is not available for the wave equation except for well-prepared initial data [6]. Of course, these theoretical difficulties have numerical counterparts and, even when the relaxed formulation is available, the optimal microstructures are complicated, typically laminates of high rank. Therefore there is room for a simplified setting, allowing for a complete theoretical and numerical treatment. Following the lead of [3] we suggest to consider a second order small-amplitude approximation of the problem and to relax it by using the theory of H-measures, due to Gérard [8] and Tartar [17]. The use of H-measures for studying small-amplitude composite materials was previously initiated by Tartar [17]: the main advantage is the induced simplification in the analysis since the necessary tools of homogenization theory are replaced by the simpler notion of H-measures (see Remark 13 below).

Let us present our model problem which, for simplicity, is expressed for a scalar-valued unknown, like in a conductivity model. We hasten to say that all our results are also valid in the elasticity setting or in any other multiphysics or multiple loads setting (see Remark 16). In particular, all our numerical computations will be made for the linearized elasticity system. We consider a smooth bounded open set Ω in $\mathbb{R}^N$ filled by two isotropic materials of nearly equal conductivity or elasticity tensors. Specifically we consider a region with characteristic function χ to contain a material with conductivity (or elasticity) tensor A_1, the complementary region in Ω contains a second material of conductivity (or elasticity) tensor A_0. The two tensors are assumed to be symmetric, coercive and related by the contrast parameter η,

$$A_1 = (1+\eta)\,A_0,$$

yielding the overall tensor

$$A_\chi(x) = A_0\,(1-\chi(x)) + A_1\chi(x) = A_0\,(1+\eta\chi(x))\,.$$

We assume the same contrast relation for the positive material densities, i.e., $\rho_1 = (1+\eta)\,\rho_0$ and

$$\rho_\chi(x) = \rho_0\,(1-\chi(x)) + \rho_1\chi(x) = \rho_0\,(1+\eta\chi(x))\,.$$

For a given final time $0 < T < +\infty$, we consider waves propagating in the domain Ω. In other words, we look at the wave equation:

$$\begin{cases} \rho_0 (1+\eta\chi) \dfrac{\partial^2 u}{\partial t^2} - \operatorname{div}(A_0 (1+\eta\chi) \nabla u) = f & \text{in } \Omega \times (0,T) \\ u(x,0) = u_{init}(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial t}(x,0) = v_{init}(x) & \text{in } \Omega \\ u(x,t) = 0 & \text{on } \Gamma_d \times (0,T) \\ A_0 (1+\eta\chi) \nabla u(x,t) \cdot \hat{n} = 0 & \text{on } \Gamma_n \times (0,T), \end{cases} \tag{1}$$

where Γ_d, Γ_n is a smooth partition of the boundary $\partial\Omega$ (with Γ_d of positive $(N-1)$-dimensional measure). Introducing the function space V, defined by

$$V = \{\phi \in H^1(\Omega) \text{ such that } \phi = 0 \text{ on } \Gamma_d\}, \tag{2}$$

we assume that $u_{init} \in V$ and $v_{init} \in L^2(\Omega)$ are initial data and $f \in L^2((0,T) \times \Omega)$ is an applied force. As is well-known there exists a unique solution u of (1) in the space $C^0([0,T]; V) \cap C^1([0,T]; L^2(\Omega))$. Actually we shall assume that the initial data are smoother for an additional regularity of solutions (see Lemma 1 and Remark 12 for details and comments).

Remark 1. *There is no conceptual difficulty in replacing the homogeneous boundary data in (1) by non homogeneous Dirichlet and/or Neumann ones, having sufficient smoothness. For the sake of clarity in the exposition we do not treat the case of inhomogeneous boundary data.*

An optimal design problem associated to the wave equation (1) is the minimization of an objective function

$$\inf_{\chi \in L^\infty(\Omega;\{0,1\})} J(\chi) \tag{3}$$

where $J(\chi)$ depends implicitly on χ through the solution u. Two typical examples of objective function are

$$J(\chi) = \int_0^T \int_\Omega j(x,u)\, dxdt, \tag{4}$$

and

$$J(\chi) = \int_0^T \int_\Omega j(x,\nabla u)\, dxdt. \tag{5}$$

In both cases we assume that the integrand $j(x,\lambda)$ is a Carathéodory function, of class C^2 with adequate growth conditions with respect to its second argument. Typically, we assume that there exists a constant $C > 0$ such that, for any $x \in \Omega$ and $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{R}^N$),

$$|j(x,\lambda)| \le C(|\lambda|^2 + 1), \quad |j'(x,\lambda)| \le C(|\lambda| + 1), \quad |j''(x,\lambda)| \le C, \tag{6}$$

where the notation $'$ means derivation with respect to the second argument λ. Of course, more subtle and less restrictive assumptions are possible. In the sequel, for the ease of notations we shall drop the dependence on x in the definition of $j(x,\lambda)$.

Remark 2. *Without loss of generality it is possible to add to the objective functions (4) and (5) a similar cost at the final time T. On the same token we could consider an objective function carried by a boundary integral*

$$J(\chi) = \int_0^T \int_{\partial\Omega} j(x,u)\, dxdt.$$

Note however that (4) and (5) do not depend on χ. There are other difficulties in this latter case and we refer to Remark 10 for comments on this issue. It is also common practice to add a volume constraint on χ in the minimization (3): there is no additional difficulty in this case.

The next section is devoted to the so-called small-amplitude approximation of (3) which amounts to making a second-order Taylor expansion with respect to η of the state equation (1) and of the objective functions (4) and (5). The rest of the paper is then a theoretical and numerical study of this small-amplitude approximation. The contents of the paper is described at the end of Section 2.

2 Small-Amplitude Approximation

The main idea of the small-amplitude approximation [17], [3], assuming that the parameter η is small, consists of making a (formal) second order expansion in η of the solution

$$u(x,t) = u_0(x,t) + \eta u_1(x,t) + \eta^2 u_2(x,t) + \mathcal{O}\left(\eta^3\right). \tag{7}$$

We shall come back later in Section 6 to the justification of this expansion and the precise meaning of the remainder term. Plugging this expansion of u into (1) and collecting terms of the same order of η yields the series of equations in order 1, η, and η^2:

$$\left\{\begin{array}{r} \rho_0 \dfrac{\partial^2 u_0}{\partial t^2} - \operatorname{div}\left(A_0 \nabla u_0\right) = f \text{ in } \Omega \times (0,T) \\ u_0(x,0) = u_{init}(x) \text{ in } \Omega \\ \dfrac{\partial u_0}{\partial t}(x,0) = v_{init}(x) \text{ in } \Omega \\ u_0(x,t) = 0 \text{ on } \Gamma_d \times (0,T) \\ A_0 \nabla u_0(x,t) \cdot \hat{n} = 0 \text{ on } \Gamma_n \times (0,T), \end{array}\right. \tag{8}$$

$$\left\{\begin{array}{r} \rho_0 \dfrac{\partial^2 u_1}{\partial t^2} - \operatorname{div}\left(A_0 \nabla u_1\right) = -\rho_0 \chi \dfrac{\partial^2 u_0}{\partial t^2} + \operatorname{div}\left(A_0 \chi \nabla u_0\right) \text{ in } \Omega \times (0,T) \\ u_1(x,0) = 0 \text{ in } \Omega \\ \dfrac{\partial u_1}{\partial t}(x,0) = 0 \text{ in } \Omega \\ u_1(x,t) = 0 \text{ on } \Gamma_d \times (0,T) \\ A_0 \nabla u_1(x,t) \cdot \hat{n} = -\chi A_0 \nabla u_0(x,t) \cdot \hat{n} \text{ on } \Gamma_n \times (0,T), \end{array}\right. \tag{9}$$

$$\begin{cases} \rho_0 \dfrac{\partial^2 u_2}{\partial t^2} - \operatorname{div}(A_0 \nabla u_2) = -\rho_0 \chi \dfrac{\partial^2 u_1}{\partial t^2} + \operatorname{div}(A_0 \chi \nabla u_1) \text{ in } \Omega \times (0,T) \\ u_2(x,0) = 0 \text{ in } \Omega \\ \dfrac{\partial u_2}{\partial t}(x,0) = 0 \text{ in } \Omega \\ u_2(x,t) = 0 \text{ on } \Gamma_d \times (0,T) \\ A_0 \nabla u_2(x,t) \cdot \hat{n} = -\chi A_0 \nabla u_1(x,t) \cdot \hat{n} \text{ on } \Gamma_n \times (0,T). \end{cases} \tag{10}$$

Obviously (8) contains no dependency upon the characteristic function χ and it admits a unique solution u_0 in the space $C^0([0,T];V) \cap C^1([0,T];L^2(\Omega))$. However problems (9) and (10) do depend upon χ and, since their right hand sides are not smooth a priori, existence of the solutions u_1 and u_2 needs to be established. We postpone this matter for the moment and refer to Lemma 3 below.

We then plug the ansatz (7) into the objective function $J(\chi)$ which we want to minimize. We define the small-amplitude objective function $J_{sa}(\chi)$ as its second order truncation, namely

$$J(\chi) = J_{sa}(\chi) + \mathcal{O}(\eta^3),$$

where, for the objective function (4), we have

$$J_{sa}(\chi) = \int_0^T \int_\Omega \left(j(u_0) + \eta j'(u_0) u_1 + \eta^2 \left(j'(u_0) u_2 + \frac{1}{2} j''(u_0)(u_1)^2 \right) \right) dxdt, \tag{11}$$

while, for the other objective function (5), we obtain instead

$$J_{sa}(\chi) = \int_0^T \int_\Omega \Big(j(\nabla u_0) + \eta j'(\nabla u_0) \cdot \nabla u_1 + \eta^2 \Big(j'(\nabla u_0) \cdot \nabla u_2 + \frac{1}{2} j''(\nabla u_0) \nabla u_1 \cdot \nabla u_1 \Big) \Big) dxdt. \tag{12}$$

Again we have to prove that formula (11) or (12) makes sense for the solutions u_0, u_1, u_2 of (8), (9) and (10) (see Lemma 3 below). Note that we have dropped the dependence on x for the integrand j and its derivative for the sake of simplicity in the presentation.

We call the following minimization the *small-amplitude* optimization problem,

$$\inf_{\chi \in L^\infty(\Omega;\{0,1\})} J_{sa}(\chi). \tag{13}$$

Here J_{sa} is defined by (11) or (12). Although (13) is a simplified approximation of (3) it is still not a well-posed problem, namely it does not admit minimizer. Indeed, minimizing sequences of (13) do not usually converge to another characteristic function, taking only values 0 and 1 on Ω, but rather converge (weakly) to a density, taking values in the entire interval $[0,1]$. It is thus necessary to relax the small amplitude problem (13). In the case of elliptic PDE's, this relaxation has already been carried out in [3] using the theory of H-measures. Section 4 is precisely devoted to a short presentation of this necessary tool which is simpler than the full theory of homogenization. Before this Section 3 is devoted to various necessary a priori estimates which, in particular, will justify the existence of u_1 and u_2, as well as the fact that the small-amplitude objective function J_{sa} is well defined. Section 5 will then be devoted to the relaxation of the small-amplitude optimization problem (13). The justification that (13) is an approximation of the original problem (3) at order $\mathcal{O}(\eta^3)$ is the topic of Section 6. Compared to the elliptic case, new difficulties arise by lack of analytic dependence of the solution $u(t, x)$ with respect to the small-amplitude parameter η (see Remark 5). In particular the regularity of each term u_i in the ansatz (7) depends on that of the time derivative of the previous term $\partial u_{i-1}/\partial t$ which makes the convergence of the ansatz (7) tricky and requires us to introduce various a priori estimates and smoothness assumptions for the data in Section 3. Section 7 will establish optimality conditions which prove that optimal microstructures can always be found in the class of rank-one, or simple, laminates. Eventually Section 8 gives a numerical algorithm for computing relaxed minimizers of (13) which is applied to some test cases in two space dimensions.

3 A Priori Estimates

We begin with classical existence and smoothness results for the solution u_0 of (8). As is well known, the regularity of the solution increases with that of the initial data and source term.

Lemma 1. *Recall that the space V is defined by (2). Under the assumptions*

$$u_{init} \in V, \quad v_{init} \in L^2(\Omega), \quad f \in L^2\left((0,T) \times \Omega\right), \tag{14}$$

there exists a unique solution u_0 of (8) in the space $C^0\left([0,T]; V\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$.

Under the assumptions

$$u_{init} \in H^2(\Omega) \cap V, \quad v_{init} \in V, \quad f \in H^1\left((0,T); L^2(\Omega)\right), \tag{15}$$

the solution u_0 belongs to the space $C\left([0,T]; H^2(\Omega)\right) \cap C^1\left([0,T]; H^1(\Omega)\right) \cap C^2\left([0,T]; L^2(\Omega)\right)$.

Under the assumptions

$$u_{init} \equiv 0, \quad v_{init} \in H^2(\Omega) \cap V, \quad f \in H^2\left((0,T); L^2(\Omega)\right) \text{ and } f(x,0) \in V, \tag{16}$$

the solution u_0 belongs to the space $C^2\left([0,T]; H^1(\Omega)\right) \cap C^3\left([0,T]; L^2(\Omega)\right)$.

Under the assumptions

$$\begin{gathered} u_{init} \equiv 0, \quad v_{init} \equiv 0, \quad f \in H^3\left((0,T); L^2(\Omega)\right) \\ f(x,0) \equiv 0 \text{ and } \frac{\partial f}{\partial t}(x,0) \in V, \end{gathered} \tag{17}$$

the solution u_0 belongs to the space $C^3\left([0,T]; H^1(\Omega)\right) \cap C^4\left([0,T]; L^2(\Omega)\right)$.

Remark 3. *The assumptions (16) and (17) are slightly non optimal for Lemma 1 but are motivated by their later use in Lemma 4. Since we wish to avoid multiplying the number of different assumptions, we decide to have the same smoothness assumptions (15), (16) and (17) throughout the paper. Further comments on the use of these smoothness assumptions are made later in Remark 12.*

Proof. These results are classical (see e.g. chapter 5 in volume 2 of [12] or section 7.2 of [7]) and we simply indicate the main ideas behind them. The existence of a solution u_0 of (8) in the usual energy space is of course well known under assumption (14). The result obtained under assumption (15) is derived by writing that $\frac{\partial u_0}{\partial t}$ is solution in the energy space of the wave equation obtained by time derivation of (8). Similarly, assumption (16) corresponds to $\frac{\partial^2 u_0}{\partial t^2}$ being solution of a wave equation, and (17) to $\frac{\partial^3 u_0}{\partial t^3}$. □

The next step is to prove a priori estimates for u_1 and u_2 that will be uniform with respect to the characteristic function χ. To this end, we prove a lemma on a priori estimates for the solution of a generic wave equation, similar to (9) and (10), with an integer $i \geq 1$,

$$\begin{cases} \rho_0 \dfrac{\partial^2 u_i}{\partial t^2} - \operatorname{div}(A_0 \nabla u_i) = -\rho_0 \chi \dfrac{\partial^2 u_{i-1}}{\partial t^2} + \operatorname{div}(A_0 \chi \nabla u_{i-1}) \text{ in } \Omega \times (0,T) \\ u_i(x,0) = 0 \text{ in } \Omega \\ \dfrac{\partial u_i}{\partial t}(x,0) = 0 \text{ in } \Omega \\ u_i(x,t) = 0 \text{ on } \Gamma_d \times (0,T) \\ A_0 \nabla u_i(x,t) \cdot \hat{n} = -\chi A_0 \nabla u_{i-1}(x,t) \cdot \hat{n} \text{ on } \Gamma_n \times (0,T). \end{cases} \tag{18}$$

We introduce the energy space E_T defined by

$$E_T = \{\varphi \text{ such that } \frac{\partial \varphi}{\partial t} \in L^\infty\left((0,T); L^2(\Omega)\right), \text{ and } \nabla\varphi \in L^\infty\left((0,T); L^2(\Omega)^N\right)\}$$

with the norm

$$\|\phi\|_{E_T} = \left\| \frac{\partial \phi}{\partial t} \right\|_{L^\infty((0,T);L^2(\Omega))} + \|\nabla \phi\|_{L^\infty((0,T);L^2(\Omega)^N)}.$$

Lemma 2. *If u_{i-1} belongs to E_T, then there exists a unique solution u_i of (18) in the space $C^0([0,T]; L^2(\Omega)) \cap C^1([0,T]; V')$ where V' is the dual space of V defined in (2). Furthermore, there exists a constant $C(T)$, which does not depend on the characteristic function χ, such that the solution of (18) satisfies*

$$\|u_i\|_{L^\infty((0,T);L^2(\Omega))} \leq C(T)\|u_{i-1}\|_{E_T}. \tag{19}$$

If furthermore $\partial u_{i-1}/\partial t$ belongs to E_T, then there exists a unique solution u_i of (18) in the space $C^0([0,T]; V) \cap C^1([0,T]; L^2(\Omega))$ and there exists a constant $C(T)$, which does not depend on the characteristic function χ, such that the solution of (18) satisfies

$$\|u_i\|_{E_T} \leq C(T)\left(\|u_{i-1}\|_{E_T} + \|\frac{\partial u_{i-1}}{\partial t}\|_{E_T}\right). \tag{20}$$

Proof. The existence of solutions to (18) in the proposed spaces is classical [7], [12]. Multiplying equation (18) by $\partial u_i/\partial t$ and integrating by parts yields the usual energy equality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\rho_0 \left|\frac{\partial u_i}{\partial t}\right|^2 + A_0 \nabla u_i \cdot \nabla u_i\right) dx = \int_\Omega \left(-\rho_0 \chi \frac{\partial^2 u_{i-1}}{\partial t^2}\frac{\partial u_i}{\partial t} - A_0 \chi \nabla u_{i-1} \cdot \nabla \frac{\partial u_i}{\partial t}\right) dx.$$

Integrating by parts in time the last term in the above equality leads to

$$-\int_0^T \int_\Omega A_0 \chi \nabla u_{i-1} \cdot \nabla \frac{\partial u_i}{\partial t} dt dx = \int_0^T \int_\Omega A_0 \chi \nabla \frac{\partial u_{i-1}}{\partial t} \cdot \nabla u_i dt dx - \int_\Omega A_0 \chi \nabla u_{i-1}(T) \cdot \nabla u_i(T) dx.$$

By standard arguments, and using the smoothness of u_{i-1}, we deduce from this energy equality the estimate (20) in the energy space E_T.

If u_{i-1} is less smooth, namely merely belonging to E_T, we need to introduce a time regularization, defined by

$$v_i\,(x,t) = \int_0^t u_i(x,s)\,ds\,.$$

The equation satisfied by v_i is

$$\left\{\begin{array}{r} \rho_0 \dfrac{\partial^2 v_i}{\partial t^2} - \operatorname{div}\left(A_0 \nabla v_i\right) = -\rho_0 \chi \dfrac{\partial u_{i-1}}{\partial t} + \operatorname{div}\left(A_0 \chi \nabla v_{i-1}\right) \text{ in } \Omega \times (0,T) \\ v_i(x,0) = 0 \text{ in } \Omega \\ \dfrac{\partial v_i}{\partial t}\,(x,0) = 0 \text{ in } \Omega \\ v_i(x,t) = 0 \text{ on } \Gamma_d \times (0,T) \\ A_0 \nabla v_i(x,t) \cdot \hat{n} = -A_0 \nabla v_{i-1}(x,t) \cdot \hat{n} \text{ on } \Gamma_n \times (0,T). \end{array}\right. \tag{21}$$

The energy estimate for (21) is obtained by multiplying it by $\frac{\partial v_i}{\partial t}$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\rho_0 \left| \frac{\partial v_i}{\partial t} \right|^2 + A_0 \nabla v_i \cdot \nabla v_i \right) dx = \int_\Omega \left(-\rho_0 \chi \frac{\partial u_{i-1}}{\partial t}\frac{\partial v_i}{\partial t} - A_0 \chi \nabla v_{i-1} \cdot \nabla \frac{\partial v_i}{\partial t} \right) dx.$$

The first term in the right hand side causes no problem since $-\rho_0 \chi \frac{\partial u_{i-1}}{\partial t}$ is bounded in $L^\infty\left((0,T); L^2(\Omega)\right)$. For the second one we perform a time integration by parts to get

$$-\int_0^T \int_\Omega A_0 \chi \nabla v_{i-1} \cdot \nabla \frac{\partial v_i}{\partial t} dt dx = \int_0^T \int_\Omega A_0 \chi \nabla u_{i-1} \cdot \nabla v_i dt dx - \int_\Omega A_0 \chi \nabla v_{i-1}(T) \cdot \nabla v_i(T) dx$$

which can easily be bounded since u_{i-1} belongs to E_T. Therefore we deduce estimate (19). □

As a consequence of Lemma 2 we obtain the following justification of all terms involved in our small amplitude problem.

Lemma 3. *Under the assumptions (15) for the data, the solution u_1 of (9) belongs to the energy space E_T and the solution u_2 of (10) belongs to $L^\infty\left((0,T); L^2(\Omega)\right)$. Eventually, the small amplitude objective function (11) is well defined and has finite value. The same is true for the other objective function (12) if we add a condition on the integrand j on the boundary Γ_n, namely*

$$j'(x,\lambda) = g(x,\lambda) A_0 \lambda \quad \forall\, x \in \Gamma_n, \lambda \in \mathbb{R}^N, \tag{22}$$

for some real valued function $g(x,\lambda)$.

Proof. By our assumptions on the data, Lemma 1 implies that the solution u_0 of (8) is such that $\frac{\partial u_0}{\partial t} \in E_T$. Applying estimate (20) of Lemma 2 implies that u_1 belongs to E_T. Subsequently, estimate (19) of Lemma 2 yields that u_2 belongs to $L^\infty\left((0,T); L^2(\Omega)\right)$. In view of assumption (6) it implies that the small amplitude objective function (11) is a finite integral. Concerning the gradient-based objective function (12), the only difficult term is

$$\int_0^T \int_\Omega \eta^2 j'(\nabla u_0) \cdot \nabla u_2 dx dt$$

because ∇u_2 does not belong to $L^\infty\left((0,T);L^2(\Omega)^N\right)$. However, since $u_0 \in C([0,T];H^2(\Omega))$, then $j'(\nabla u_0)$ belongs to $C([0,T];H^1(\Omega))$ and the above integral makes sense by an integration by parts

$$\int_0^T \int_\Omega \eta^2 j'(\nabla u_0)\cdot\nabla u_2 dxdt = -\int_0^T \int_\Omega \eta^2 \mathrm{div}\left(j'(\nabla u_0)\right) u_2 dxdt \qquad (23)$$

because of the boundary conditions for u_0, u_2 and (22). Therefore (12) is well defined and finite. □

Remark 4. *One can avoid the technical assumption (22) for the gradient-based objective function in Lemma 3 if we replace the smoothness assumptions (15) for the data by (16). Then, the result (28) in Lemma 4 implies directly that ∇u_2 belongs to $L^\infty((0,T);L^2(\Omega))$ and there is no need to perform the integration by parts (23).*

Remark 5. *Lemma 2 suggests a lack of analyticity for the solution u of the wave equation (1) with respect to the parameter η, at least in the energy space E_T. Indeed, writing u as a series in η,*

$$u(x,t) = \sum_{i\geq 0} \eta^i u_i(x,t),$$

estimate (20) indicates that each term u_i can be controled in E_T merely by $\frac{\partial u_{i-1}}{\partial t}$, so no convergence in E_T can be expected. Let us point out that, even if (20) is not optimal (for example, the upper bound can be evaluated in the L^1-norm in time), one cannot avoid to "lose" one derivative in the norm of u_{i-1} controlling that of u_i. This is in sharp contrast with the elliptic case, where the solution depends analytically on the parameter η [17], and explains the additional difficulties in the sequel.

As a convincing example, we now show that this lack of analyticity is obvious, for any reasonable Sobolev-type norm, on the explicit solution for a one-dimensional wave equation with constant coefficients on the entire line $\mathbb{R}$ without any source term. Indeed, in such a case the explicit solution is given as the superposition of two waves travelling in opposite directions

$$u(x,t) = a^+(x-ct) + a^-(x+ct)$$

where the functions $a^\pm$ are determined by the initial data and $c = \sqrt{A/\rho}$ is the sound speed. Clearly, the derivatives of u with respect to the parameter c involves derivatives of $a^\pm$, which are equivalent to time derivatives of u. Thus one cannot obtain a convergent Taylor series of u with respect to c if u merely belongs to a functional space involving a finite number of derivatives (as the energy space) and is not at least inifinitely differentiable with respect to (x,t).

For the reasons detailed in Remark 5 we shall need further smoothness of the solution of (18), beyond that provided by Lemma 2. A remarkable feature of the boundary value problem (18) is that the time derivative of its solution $w_i = \partial u_i/\partial t$ satisfies a system of the same type, except with different initial data. This is of course a consequence of the fact that the characteristic function χ does not depend on time t. More precisely, for $i \geq 1$, $w_i = \frac{\partial u_i}{\partial t}$

is formally a solution of

$$\begin{cases} \rho_0 \dfrac{\partial^2 w_i}{\partial t^2} - \operatorname{div}\left(A_0 \nabla w_i\right) = -\rho_0 \chi \dfrac{\partial^2 w_{i-1}}{\partial t^2} + \operatorname{div}\left(A_0 \chi \nabla w_{i-1}\right) \text{ in } \Omega \times (0,T) \\ w_i(x,0) = 0 \text{ in } \Omega \\ \dfrac{\partial w_i}{\partial t}(x,0) = \dfrac{\partial^2 u_i}{\partial t^2}(x,0) \text{ in } \Omega \\ w_i(x,t) = 0 \text{ on } \Gamma_d \times (0,T) \\ A_0 \nabla w_i(x,t) \cdot \hat{n} = -\chi A_0 \nabla w_{i-1}(x,t) \cdot \hat{n} \text{ on } \Gamma_n \times (0,T), \end{cases} \tag{24}$$

with the initial velocity

$$\frac{\partial w_i}{\partial t}(x,0) = \frac{\partial^2 u_i}{\partial t^2}(x,0) = -\chi \frac{\partial^2 u_{i-1}}{\partial t^2} + \frac{1}{\rho_0}\left(\operatorname{div}\left(A_0 \nabla u_i\right) + \operatorname{div}\left(A_0 \chi \nabla u_{i-1}\right)\right). \tag{25}$$

Similarly, the second-order time derivative $z_i = \frac{\partial^2 u_i}{\partial t^2}$ formally satisifes

$$\begin{cases} \rho_0 \dfrac{\partial^2 z_i}{\partial t^2} - \operatorname{div}\left(A_0 \nabla z_i\right) = -\rho_0 \chi \dfrac{\partial^2 z_{i-1}}{\partial t^2} + \operatorname{div}\left(A_0 \chi \nabla z_{i-1}\right) \text{ in } \Omega \times (0,T) \\ z_i(x,0) = \dfrac{\partial^2 u_i}{\partial t^2}(x,0) \text{ in } \Omega \\ \dfrac{\partial z_i}{\partial t}(x,0) = \dfrac{\partial^2 w_i}{\partial t^2}(x,0) \text{ in } \Omega \\ z_i(x,t) = 0 \text{ on } \Gamma_d \times (0,T) \\ A_0 \nabla z_i(x,t) \cdot \hat{n} = -\chi A_0 \nabla z_{i-1}(x,t) \cdot \hat{n} \text{ on } \Gamma_n \times (0,T) \end{cases} \tag{26}$$

with initial position given by (25) and initial velocity

$$\frac{\partial z_i}{\partial t}(x,0) = \frac{\partial^2 w_i}{\partial t^2}(x,0) = -\chi \frac{\partial^2 w_{i-1}}{\partial t^2} + \frac{1}{\rho_0}\left(\operatorname{div}\left(A_0 \nabla w_i\right) + \operatorname{div}\left(A_0 \chi \nabla w_{i-1}\right)\right). \tag{27}$$

Fortunately, in the sequel we need only a priori estimates for w_1 and z_1, which thus depends on the smoothness of u_0. We therefore require additional smoothness of the data.

Lemma 4. *Under the assumptions (16) for the data, we have*

$$\|w_1\|_{E_T} \leq C(T) \quad \text{where } w_1 = \frac{\partial u_1}{\partial t}. \tag{28}$$

Under the assumptions (17) for the data, we have

$$\|z_1\|_{E_T} \leq C(T) \quad \text{where } z_1 = \frac{\partial^2 u_1}{\partial t^2}. \tag{29}$$

In both (28) and (29) the constant $C(T)$ does not depend on the characteristic function χ.

Proof. We first prove (28). By Lemma 1 the assumptions (16) imply that $\frac{\partial w_0}{\partial t} = \frac{\partial^2 u_0}{\partial t^2} \in E_T$ so the source term in (24), for $i = 1$, belongs to the dual of E_T and causes no problem. The main difficulty is to evaluate the smoothness of the initial velocity (25). Using equation (9), and since $u_1(x,0)=0$, we compute

$$\frac{\partial w_1}{\partial t}(x,0)=\frac{1}{\rho_0}\Big(-\chi f(x,0)+A_0\nabla\chi\cdot\nabla u_{init}(x)\Big). \tag{30}$$

To obtain that w_1 belongs to the energy space E_T we must have $\frac{\partial w_1}{\partial t}(x,0)\in L^2(\Omega)$ and since χ is discontinuous and unknown, the only possibility is to assume that u_{init} vanishes. This finishes the proof of (28).

We then prove (29). The source term in (26), for $i=1$, belongs to the dual of E_T if $\frac{\partial z_0}{\partial t}=\frac{\partial^3 u_0}{\partial t^3}\in E_T$. This is the case in view of Lemma 1 and our assumptions (17). The initial position $z_1(x,0)=\frac{\partial w_1}{\partial t}(x,0)$ has already been computed in (30): it further belongs to $H^1(\Omega)$ if $f(x,0)\equiv 0$ because χ is discontinuous. The initial velocity is computed through equation (24) for $i=1$:

$$\frac{\partial z_1}{\partial t}(x,0)=\frac{\partial^2 w_1}{\partial t^2}(x,0)=\frac{1}{\rho_0}\Big(\mathrm{div}\,(A_0\nabla w_1)-\rho_0\chi\frac{\partial^2 w_0}{\partial t^2}+\mathrm{div}\,(A_0\chi\nabla w_0)\Big)(x,0)\,.$$

Since $w_1(x,0)=0$ and using the time derivative of equation (8) we deduce

$$\frac{\partial z_1}{\partial t}(x,0)=\frac{1}{\rho_0}\Big(-\chi\frac{\partial f}{\partial t}(x,0)+A_0\nabla\chi\cdot\nabla v_{init}(x)\Big). \tag{31}$$

To obtain that z_1 belongs to the energy space E_T we must have $\frac{\partial z_1}{\partial t}(x,0)\in L^2(\Omega)$ and since χ is discontinuous and unknown, the only possibility is to assume that v_{init} vanishes. This finishes the proof of (29). □

4 A brief review of H-measure theory

We briefly recall the definition of H-measures, introduced by Gérard [8] and Tartar [17]. An H-measure is a default measure which quantifies the lack of compactness of weakly converging sequences in $L^2(\mathbb{R}^N)$. More precisely, it indicates where in the physical space, and at which frequency in the Fourier space, are the obstructions to strong convergence. Since their inception H-measures have been the right tool for studying small amplitude homogenization [17] and related optimal design problems [3]. All results below are due to [8] and [17], to which we refer for complete proofs.

We denote by $\mathbb{S}_{N-1}$ the unit sphere in $\mathbb{R}^N$, $C(\mathbb{S}_{N-1})$ is the space of continuous complex-valued functions on $\mathbb{S}_{N-1}$, and $C_0(\mathbb{R}^N)$ is that of continuous complex-valued functions decreasing to 0 at infinity in $\mathbb{R}^N$. As usual $\overline{z}$ denotes the complex conjugate of the complex number z. The Fourier transform operator in $L^2(\mathbb{R}^N)$, denoted by $\mathcal{F}$, is defined by

$$(\mathcal{F}\phi)(\xi)=\int_{\mathbb{R}^N}\phi(x)e^{-2i\pi x\cdot\xi}dx \quad \forall\phi\in L^2(\mathbb{R}^N).$$

Theorem 1. *Let $u_\varepsilon=(u^i_\varepsilon)_{1\le i\le p}$ be a sequence of functions defined in $\mathbb{R}^N$ with values in $\mathbb{R}^p$ which converges weakly to 0 in $L^2(\mathbb{R}^N)^p$. There exists a subsequence (still denoted by ε) and* *a family of complex-valued Radon measures $(\mu_{ij}(x,\xi))_{1\le i,j\le p}$ on $\mathbb{R}^N \times \mathbb{S}_{N-1}$ such that, for any functions $\phi_1(x), \phi_2(x) \in C_0(\mathbb{R}^N)$ and $\psi(\xi) \in C(\mathbb{S}_{N-1})$, it satisfies*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \mathcal{F}\left(\phi_1 u_\varepsilon^i\right)(\xi)\overline{\mathcal{F}\left(\phi_2 u_\varepsilon^j\right)}(\xi)\psi\left(\frac{\xi}{|\xi|}\right) d\xi = \int_{\mathbb{R}^N}\int_{\mathbb{S}_{N-1}} \phi_1(x)\overline{\phi}_2(x)\psi(\xi)\mu_{ij}(dx,d\xi)\,.$$

The matrix of measures $\mu = (\mu_{ij})_{1\le i,j\le p}$ is called the H-measure of the subsequence u_ε. It is hermitian and non-negative, i.e.

$$\mu_{ij} = \overline{\mu}_{ji}, \quad \sum_{i,j=1}^{p} \lambda_i\overline{\lambda}_j\mu_{ij} \ge 0\ \forall\lambda \in \mathbb{C}^p.$$

If we consider a sequence u_ε which converges weakly in $L^2(\mathbb{R}^N)^p$ to a limit u (instead of 0), then, applying Theorem 1 to $(u_\varepsilon - u)$, and taking $\psi \equiv 1$, we obtain a representation formula for the limit of quadratic expressions of u_ε

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \phi_1\overline{\phi_2}u_\varepsilon^i u_\varepsilon^j\,dx = \int_{\mathbb{R}^N}\phi_1\overline{\phi_2}u^i u^j\,dx + \int_{\mathbb{R}^N}\int_{\mathbb{S}_{N-1}}\phi_1(x)\overline{\phi}_2(x)\mu_{ij}(dx,d\xi)\,. \tag{32}$$

Therefore the H-measure appears as a default measure which gives a precise representation of the compactness default, taking into account the directions of the oscillation.

If some information is known on the derivatives of the sequence u_ε, then more can be said on the H-measure: this is a localization principle for the support of the H-measure.

Theorem 2. *Let $u_\varepsilon = (u_\varepsilon^i)_{1\le i\le p}$ be a sequence of functions defined in $\mathbb{R}^N$ with values in $\mathbb{R}^p$ which converges weakly to 0 in $L^2(\mathbb{R}^N)^p$ and defines an H-measure $\mu(x,\xi) = (\mu_{ij}(x,\xi))_{1\le i,j\le p}$. If, furthermore, u_ε satisfies the constraint*

$$\sum_{j=1}^{p}\sum_{k=1}^{N}\frac{\partial}{\partial x_k}\Big(C_{jk}(x)u_\varepsilon^j\Big) \to 0 \text{ in } H^{-1}_{loc}(\Omega) \text{ strongly,}$$

where the coefficients $C_{jk}(x)$ are continuous functions in $\Omega \subset \mathbb{R}^N$, then

$$\sum_{j=1}^{p}\sum_{k=1}^{N}\xi_k C_{jk}(x)\mu_{jm}(x,\xi) = 0 \text{ in } \Omega\times\mathbb{S}_{N-1}, \text{ for any } 1\le m\le p.$$

We now recall the particular case of characteristic functions [10], [17].

Lemma 5. *Let $\chi_\varepsilon(x)$ be a sequence of characteristic functions that weakly-* converges to a limit $\theta(x)$ in $L^\infty(\Omega;[0,1])$. Then the corresponding H-measure μ for the sequence $(\chi_\varepsilon - \theta)$ is necessarily of the type*

$$\mu(dx,d\xi) = \theta(x)\Big(1-\theta(x)\Big)\nu(dx,d\xi)$$

where, for given x, the measure $\nu(dx,d\xi)$ is a probability measure with respect to ξ, i.e. $\nu \in \mathcal{P}(\Omega,\mathbb{S}_{N-1})$ with

$$\mathcal{P}(\Omega,\mathbb{S}_{N-1}) = \left\{\begin{array}{l} \nu(x,\xi) \text{ Radon measure on } \Omega\times\mathbb{S}_{N-1} \text{ such that:} \\ \nu\ge 0,\ \displaystyle\int_{\mathbb{S}_{N-1}}\nu(x,\xi)\,d\xi = 1 \quad a.e.\ x\in\Omega \end{array}\right\}. \tag{33}$$

Conversely, for any such probability measure $\nu \in \mathcal{P}(\Omega, \mathbb{S}_{N-1})$ there exists a sequence χ_ε, which weakly- converges to θ in $L^\infty(\Omega; [0,1])$, such that $\theta(1-\theta)\nu$ is the H-measure of $(\chi_\varepsilon - \theta)$.*

Remark 6. *In the periodic setting the notion of H-measure has a very simple interpretation and it is often called two-point correlation function in the context of composite materials [14]. Indeed, let $u(x,y)$ be a smooth function defined on $\Omega \times Y$, with $Y = (0,1)^N$, such that $y \to u(x,y)$ is Y-periodic. Assuming that $\int_Y u(x,y)dy = 0$, it is easily seen that $u_\varepsilon(x) = u(x, x/\varepsilon)$ converges weakly to 0 in $L^2(\Omega)$. By using the Fourier series decomposition in Y, the H-measure μ of u_ε is simple to compute. Introducing*

$$u(x,y) = \sum_{k \in \mathbb{Z}^N} \hat{u}(x,k) e^{2i\pi k \cdot y},$$

we deduce

$$\mu(x,\xi) = \sum_{k \neq 0 \in \mathbb{Z}^N} |\hat{u}(x,k)|^2 \delta\left(\xi - \frac{k}{|k|}\right),$$

where δ is the Dirac mass.

5 Relaxed Formulation

The optimization problem (13) is not well-posed in the sense that it usually does not admit a minimizer. Indeed, a minimizing sequence of characteristic functions χ^ε does not necessarily converge to a characteristic function χ^0, but rather to some limit density θ. In this section we give the relaxed formulation of (13) using the theory of H-measures. In other words we compute the limit, as ε goes to 0, of the state equations (9), (10) and of the objective functions (11), (12), evaluated for the characteristic function χ^ε.

We shall pass to the limit first in the state equations, which requires little smoothness of the data, and second in the objective functions, which is more demanding on the regularity of the data. We begin with a lemma on a priori estimates for the solutions of (9) and (10).

Lemma 6. *For any sequence of characteristic functions χ^ε we denote by u_1^ε and u_2^ε the respective solutions of (9) and (10). Under the assumptions (15) for the data, there exists a constant $C(T)$, which does not depend on ε, such that*

$$\|u_1^\varepsilon\|_{E_T} \leq C(T), \tag{34}$$

and

$$\|u_2^\varepsilon\|_{L^\infty((0,T);L^2(\Omega))} \leq C(T). \tag{35}$$

Proof. The present lemma is just a combination of Lemmas 3 and 2. In particular, estimates (34) and (35) are simple consequences of Lemma 2. □

Lemma 7. *Assume that the data satisfy the smoothness assumption (15). For any sequence of characteristic functions χ^ε there exist a subsequence and limits $\theta \in L^\infty(\Omega; [0,1])$ and $\nu \in \mathcal{P}(\Omega, \mathbb{S}_{N-1})$, defined by (33), such that:*

$$\chi^\varepsilon \rightharpoonup \theta \text{ weakly * in } L^\infty(\Omega; [0,1]) \quad \text{and} \quad \theta(1-\theta)\nu \text{ is the H-measure of } (\chi^\varepsilon - \theta).$$

Furthermore, for the same subsequence,

$$u_1^\varepsilon \rightharpoonup u_1 \text{ weakly in } E_T \quad \text{and} \quad u_2^\varepsilon \rightharpoonup u_2 \text{ weakly in } L^\infty\left((0,T);L^2(\Omega)\right),$$

where u_1 and u_2 are the solutions of

$$\left\{\begin{array}{r}\rho_0\dfrac{\partial^2 u_1}{\partial t^2}-\operatorname{div}\left(A_0\nabla u_1\right)=-\rho_0\theta\dfrac{\partial^2 u_0}{\partial t^2}+\operatorname{div}\left(A_0\theta\nabla u_0\right)\ \text{ in } \Omega\times(0,T)\\ u_1(x,0)=0 \text{ in } \Omega\\ \dfrac{\partial u_1}{\partial t}(x,0)=0 \text{ in } \Omega\\ u_1(x,t)=0 \text{ on } \Gamma_d\times(0,T)\\ A_0\nabla u_1(x,t)\cdot\hat{n}=-\theta A_0\nabla u_0(x,t)\cdot\hat{n} \text{ on } \Gamma_n\times(0,T),\end{array}\right. \tag{36}$$

$$\left\{\begin{array}{r}\rho_0\dfrac{\partial^2 u_2}{\partial t^2}-\operatorname{div}\left(A_0\nabla u_2\right)=-\rho_0\theta\dfrac{\partial^2 u_1}{\partial t^2}+\operatorname{div}\left(\theta A_0\nabla u_1\right)\\ -\operatorname{div}\left(\theta\left(1-\theta\right)A_0MA_0\nabla u_0\right)\ \text{ in } \Omega\times(0,T)\\ u_2(x,0)=0 \text{ in } \Omega\\ \dfrac{\partial u_2}{\partial t}(x,0)=0 \text{ in } \Omega\\ u_2(x,t)=0 \text{ on } \Gamma_d\times(0,T)\\ A_0\nabla u_2\cdot\hat{n}=-\theta A_0\nabla u_1\cdot\hat{n}+\theta\left(1-\theta\right)A_0MA_0\nabla u_0\cdot\hat{n} \text{ on } \Gamma_n\times(0,T).\end{array}\right. \tag{37}$$

Here M is the second-order moment matrix of the H-measure ν, defined by

$$M=\int_{\mathbb{S}_{N-1}}\frac{\xi\otimes\xi}{\xi\cdot A_0\xi}d\nu\left(\xi\right). \tag{38}$$

Remark 7. *The matrix M is the "trace" of the microstructure built by the sequence χ^ε. It is what remains from the homogenized or effective tensor after making a small-amplitude expansion (see (66) and Remark 13 for more details).*

Proof. The zero-order equation (8) does not involve χ^ε and it is obvious to pass to the limit, by weak convergence, in (9) to obtain (36). With the bounds established in Lemma 6, we focus our attention on the limit of the second-order equation (10). We decompose the solution $u_2^\varepsilon=\hat{u}_2^\varepsilon+\check{u}_2^\varepsilon$ in two terms which are respectively solutions of

$$\left\{\begin{array}{r}\rho_0\dfrac{\partial^2 \check{u}_2^\varepsilon}{\partial t^2}-\operatorname{div}\left(A_0\nabla\check{u}_2^\varepsilon\right)=-\rho_0\chi^\varepsilon\dfrac{\partial^2 u_1^\varepsilon}{\partial t^2}\ \text{in } \Omega\times(0,T)\\ \check{u}_2^\varepsilon(x,0)=0 \text{ in } \Omega\\ \dfrac{\partial\check{u}_2^\varepsilon}{\partial t}(x,0)=0 \text{ in } \Omega\\ \check{u}_2^\varepsilon(x,t)=0 \text{ on } \Gamma_d\times(0,T)\\ A_0\nabla\check{u}_2^\varepsilon(x,t)\cdot\hat{n}=0 \text{ on } \Gamma_n\times(0,T),\end{array}\right. \tag{39}$$

and

$$\left\{\begin{array}{r}\rho_0\dfrac{\partial^2 \hat{u}_2^\varepsilon}{\partial t^2}-\operatorname{div}\left(A_0\nabla\hat{u}_2^\varepsilon\right)=\operatorname{div}\left(A_0\chi^\varepsilon\nabla u_1^\varepsilon\right)\ \text{in } \Omega\times(0,T)\\ \hat{u}_2^\varepsilon(x,0)=0 \text{ in } \Omega\\ \dfrac{\partial\hat{u}_2^\varepsilon}{\partial t}(x,0)=0 \text{ in } \Omega\\ \hat{u}_2^\varepsilon(x,t)=0 \text{ on } \Gamma_d\times(0,T)\\ A_0\nabla\hat{u}_2^\varepsilon(x,t)\cdot\hat{n}=-A_0\chi^\varepsilon\nabla u_1^\varepsilon\cdot\hat{n} \text{ on } \Gamma_n\times(0,T).\end{array}\right. \tag{40}$$

It is clear from the proof of Lemma 2 that $\hat{u}_2^\varepsilon$ and $\check{u}_2^\varepsilon$ satisfy the same a priori estimate as u_2^ε in Lemma 6.

It is easy to pass to the limit by standard weak convergence in (39) since, for any smooth test function $\psi(t,x)$ with compact support in $\mathbf{R}^+ \times \Omega$, its source term satisfies

$$\int_0^\infty \int_\Omega \rho_0 \chi^\varepsilon \frac{\partial^2 u_1^\varepsilon}{\partial t^2} \psi \, dtdx = \int_0^\infty \int_\Omega \rho_0 \chi^\varepsilon u_1^\varepsilon \frac{\partial^2 \psi}{\partial t^2} \, dtdx$$

and we can pass to the limit since u_1^ε, being bounded in E_T, is compact in $L^2\left((0,T)\times\Omega\right)$.

To pass to the limit in (40) we introduce time averages, defined for any $\varphi(t) \in C_0^\infty\left(\mathbf{R}^+\right)$, by

$$\hat{U}_2^\varepsilon = \int_0^\infty \hat{u}_2^\varepsilon \varphi \, dt \, , \quad U_1^\varepsilon = \int_0^\infty u_1^\varepsilon \varphi \, dt.$$

From (40) we deduce the following elliptic equation for $\hat{U}_2^\varepsilon$

$$\left\{ \begin{array}{r} -\mathrm{div}\left(A_0 \nabla \hat{U}_2^\varepsilon\right) = \mathrm{div}\left(A_0 \chi^\varepsilon \nabla U_1^\varepsilon\right) - \rho_0 \displaystyle\int_0^\infty \frac{\partial^2 \varphi}{\partial t^2} \hat{u}_2^\varepsilon dt \text{ in } \Omega \\ \hat{U}_2^\varepsilon = 0 \text{ on } \Gamma_d \\ A_0 \nabla \hat{U}_2^\varepsilon \cdot \hat{n} = -A_0 \chi^\varepsilon \nabla U_1^\varepsilon \cdot \hat{n} \text{ on } \Gamma_n. \end{array} \right. \tag{41}$$

Lemma 6 implies that the sequence U_1^ε is bounded in $H^1(\Omega)$ while $\hat{U}_2^\varepsilon$ seems to be just bounded in $L^2(\Omega)$. However, thanks to the usual elliptic a priori estimate for (41), we obtain an improved estimate which is that $\hat{U}_2^\varepsilon$ is bounded in $H^1(\Omega)$.

By Lemma 6 the second source term, $\rho_0 \int_0^\infty \frac{\partial^2 \varphi}{\partial t^2} \hat{u}_2^\varepsilon dt$, is bounded in $L^2(\Omega)$ and thus compact in V' (the dual of V defined in (2)). The first source term in (41) is more delicate since it is the product of two weakly converging sequences.

From (9) we also deduce an elliptic equation for U_1^ε which is

$$\left\{ \begin{array}{r} -\mathrm{div}\left(A_0 \nabla U_1^\varepsilon\right) = \mathrm{div}\left(A_0 \chi^\varepsilon \nabla U_0\right) - \rho_0 \chi^\varepsilon \displaystyle\int_0^\infty \frac{\partial^2 u_0}{\partial t^2} \varphi dt - \rho_0 \int_0^\infty \frac{\partial^2 \varphi}{\partial t^2} u_1^\varepsilon dt \text{ in } \Omega \\ U_1^\varepsilon = 0 \text{ on } \Gamma_d \\ A_0 \nabla U_1^\varepsilon \cdot \hat{n} = -A_0 \chi^\varepsilon \nabla U_0 \cdot \hat{n} \text{ on } \Gamma_n. \end{array} \right. \tag{42}$$

We again decompose $U_1^\varepsilon = \hat{U}_1^\varepsilon + \check{U}_1^\varepsilon$ where $\check{U}_1^\varepsilon$ is compact in $H^1(\Omega)$ and $\hat{U}_1^\varepsilon$ depends linearly on χ^ε through

$$\left\{ \begin{array}{r} -\mathrm{div}\left(A_0 \nabla \hat{U}_1^\varepsilon\right) = \mathrm{div}\left(A_0 \chi^\varepsilon \nabla U_0\right) \text{ in } \Omega \\ \hat{U}_1^\varepsilon = 0 \text{ on } \Gamma_d \\ A_0 \nabla \hat{U}_1^\varepsilon \cdot \hat{n} = -A_0 \chi^\varepsilon \nabla U_0 \cdot \hat{n} \text{ on } \Gamma_n. \end{array} \right. \tag{43}$$

By using the theory of H-measures we can link the oscillations of $\nabla \hat{U}_1^\varepsilon$ and χ^ε through a zero-order pseudo-differential operator. According to Lemma 3.5 in [3] (which was devoted to the same problem in the elliptic case), we claim that the weak L^2-limit of $\chi^\varepsilon \nabla U_1^\varepsilon$ is given by

$$\chi^\varepsilon \nabla U_1^\varepsilon \rightharpoonup \theta \nabla U_1 - \theta(1-\theta) M A_0 \nabla U_0 \text{ weakly in } L^2(\Omega)^N,$$

where the matrix M is the second order moment of the H-measure of χ^ε, as defined by (38). We remark that M is independent of the time averaging function $\varphi(t)$. It implies that the weak limit of (41) is

$$\begin{cases} -\text{div}\left(A_0\nabla\hat{U}_2\right) = \text{div}\left(A_0\theta\nabla U_1\right) - \text{div}\left(\theta(1-\theta)A_0MA_0\nabla U_0\right) - \rho_0\int_0^\infty \frac{\partial^2\varphi}{\partial t^2}u_2dt \text{ in } \Omega \\ \hat{U}_2 = 0 \text{ on } \Gamma_d \\ A_0\nabla\hat{U}_2\cdot\hat{n} = -A_0\theta\nabla U_1\cdot\hat{n} + \theta(1-\theta)A_0MA_0\nabla U_0\cdot\hat{n} \text{ on } \Gamma_n. \end{cases} \tag{44}$$

Recombining $\hat{U}_2$ with $\check{U}_2$ and eliminating the test function $\varphi(t)$ we recover the second-order limit system (37) above. □

Remark 8. *The above analysis could also be accomplished by taking the Laplace transform of the series of PDEs (8-10) and passing to the limit as ε goes to zero in the frequency domain.*

We now pass to the limit in the objective functions and consider first the case of an objective function depending on the state u itself and not on its gradient (the opposite case follows).

Lemma 8. *Assume that the data satisfy the smoothness assumption (15). Take a sequence $\{\chi^\varepsilon\}_{\varepsilon>0}$ such that $\chi^\varepsilon \rightharpoonup \theta$ weakly * in $L^\infty(\Omega)$, and the H-measure of $(\chi^\varepsilon-\theta)$ is $\theta(1-\theta)\nu$. Then, for the objective function (11), we have*

$$\lim_{\varepsilon\to 0} J_{sa}(\chi^\varepsilon) = J^*_{sa}(\theta,\nu),$$

where

$$J^*_{sa}(\theta,\nu) = \int_0^T\int_\Omega \left(j(u_0) + \eta j'(u_0)u_1 + \eta^2\left(j'(u_0)u_2 + \frac{1}{2}j''(u_0)(u_1)^2\right)\right)dxdt, \tag{45}$$

and u_0, u_1, and u_2 are the unique solutions to (8), (36), and (37) respectively.

Proof. For a sequence of characteristic functions χ^ε we denote by u_1^ε and u_2^ε the respective solutions of (9) and (10). The objective function (11) reads

$$J_{sa}(\chi^\varepsilon) = \int_0^T\int_\Omega \left(j(u_0) + \eta j'(u_0)u_1^\varepsilon + \eta^2\left(j'(u_0)u_2^\varepsilon + \frac{1}{2}j''(u_0)(u_1^\varepsilon)^2\right)\right)dxdt. \tag{46}$$

Thanks to Lemma 7 we can pass to the limit in (46) by weak convergence for u_2^ε in $L^\infty((0,T);L^2(\Omega))$ and by strong convergence of u_1^ε in $L^2((0,T)\times\Omega)$ (because of the compact embedding of the energy space E_T in which u_1^ε is bounded) to obtain the relaxed objective function (45). □

Lemma 9. *Assume that the data satisfy the smoothness assumption (16). Take a sequence $\{\chi^\varepsilon\}_{\varepsilon>0}$ such that $\chi^\varepsilon \rightharpoonup \theta$ weakly * in $L^\infty(\Omega)$, and the H-measure of $(\chi^\varepsilon-\theta)$ is $\theta(1-\theta)\nu$. Assume that the integrand j satisfies assumption (22) on the boundary Γ_n. Then, for the objective function (12) , we have*

$$\lim_{\varepsilon\to 0} J_{sa}(\chi^\varepsilon) = J^*_{sa}(\theta,\nu),$$

where

$$J^*_{sa}(\theta,\nu) = \int_0^T \int_\Omega \Big(j(\nabla u_0) + \eta j'(\nabla u_0)\cdot\nabla u_1 + \eta^2 j'(\nabla u_0)\cdot\nabla u_2 + \frac{1}{2}\eta^2 \left(j''(\nabla u_0)\nabla u_1\cdot\nabla u_1 + \theta(1-\theta)A_0NA_0\nabla u_0\cdot\nabla u_0\right)\Big)\,dxdt, \tag{47}$$

where $N(x)$ is a matrix defined by

$$N = \int_{\mathbb{S}^{N-1}} \frac{j''(\nabla u_0)\xi\cdot\xi}{(A_0\xi\cdot\xi)^2}\xi\otimes\xi\,d\nu(\xi) \tag{48}$$

and u_0, u_1, and u_2 are the unique solutions to (8), (36), and (37) respectively.

Remark 9. *The matrix N is the "trace" of the amplification factor in the gradient caused by the microstructure built by the sequence χ^ε. It is what remains from the notion of corrector in homogenization theory after making a small-amplitude expansion. Recall that correctors are necessary to get a strong convergence of the solution gradient which otherwise is merely weak (see [1], [18] if necessary).*

Proof. For a sequence of characteristic functions χ^ε, denoting by u_1^ε and u_2^ε the respective solutions of (9) and (10), the objective function (12) reads

$$J_{sa}(\chi^\varepsilon) = \int_0^T \int_\Omega \left(j(\nabla u_0)+\eta j'(\nabla u_0)\cdot\nabla u_1^\varepsilon+\eta^2\left(j'(\nabla u_0)\cdot\nabla u_2^\varepsilon + \frac{1}{2}j''(\nabla u_0)\nabla u_1^\varepsilon\cdot\nabla u_1^\varepsilon\right)\right)dxdt. \tag{49}$$

To pass to the limit in the third term of (49) we perform an integration by parts, like in Lemma 3 under the technical assumption (22),

$$\int_0^T\int_\Omega j'(\nabla u_0)\cdot\nabla u_2^\varepsilon\,dxdt = -\int_0^T\int_\Omega \operatorname{div}\left(j'(\nabla u_0)\right)u_2^\varepsilon\,dxdt$$

and we use the weak convergence of u_2^ε as given by Lemma 7. To pass to the limit in the fourth term of (49) we use again H-measure theory but, contrary to the simple proof of Lemma 7, we need to compute the H-measure of ∇u_1^ε in terms of that of χ^ε and not merely the H-measure of a time average of ∇u_1^ε. The argument is thus a little bit more involved and requires the additional smoothness provided by assumption (16).

We introduce the vector-valued sequence $g^\varepsilon(t,x)$ of the partial derivatives of u_1^ε plus the characteristic function χ^ε

$$g^\varepsilon = \left(\frac{\partial u_1^\varepsilon}{\partial t},\frac{\partial u_1^\varepsilon}{\partial x_1},\cdots,\frac{\partial u_1^\varepsilon}{\partial x_N},\chi^\varepsilon\right)$$

and for the ease of notations we shall denote the time t by x_0. Similarly the Fourier dual variable of t will be denoted by ξ_0. Although $g^\varepsilon(t,x)$ is defined on $\mathbb{R}^+\times\Omega$ we extend it by 0 outside Ω and by solving backward the wave equation (9) for negative time, so we may consider it as a bounded sequence in $L^2(\mathbb{R}^{N+1})^{N+2}$. In truth, g^ε is bounded in $L^\infty(\mathbb{R};L^2(\mathbb{R}^N))^{N+2}$, but multiplying it by a cut-off function $\varphi(t)\in C_c^\infty(\mathbb{R})$ yields the required L^2 bound and, since (49) is an integral on a finite time interval, this cut-off trick is enough to pass to the limit (we do not give all the details to simplify the exposition). We apply the definition of H-measures to this sequence g^ε (with $p = N+2$ and replacing $\mathbb{R}^N$ by $\mathbb{R}^{N+1}$ in Theorem 1) and it yields, after substraction of its weak limit and up to a subsequence, a H-measure $\mu = (\mu_{ij}(\tilde{x},\tilde{\xi}))_{1\le i,j\le N+2}$ with $\tilde{x} = (t,x) = (x_0, x_1, \cdots, x_N) \in \mathbb{R}^{N+1}$ and $\tilde{\xi} = (\xi_0, \xi) = (\xi_0, \xi_1, \cdots, \xi_N) \in \mathbb{S}^N$.

Recall that $\chi^\varepsilon(x)$ does not depend on time and, by Lemma 4, our assumption (16) implies that the sequence $w_1^\varepsilon = \frac{\partial u_1^\varepsilon}{\partial t}$ is uniformly bounded in the energy space E_T. This implies in particular that the sequence $\frac{\partial g^\varepsilon}{\partial t}$ is bounded in $L^2(\mathbb{R}^{N+1})^{N+2}$. Applying the localization principle of Theorem 2 we deduce that

$$\xi_0 \mu(\tilde{x},\tilde{\xi}) = 0 \text{ in } \mathbb{R}^{N+1} \times \mathbb{S}^N,$$

which implies that the support of the H-measure μ is concentrated on the hyperplane $\{\xi_0 = 0\}$ (in other words, there are no oscillations in the time variable x_0). We now adapt the proof of Lemma 3.10 in [17] to our wave equation (9) where, contrary to the case of Lemma 3.10 in [17], the source term is converging weakly (and not strongly) in $H^{-1}_{loc}(\mathbb{R}^{N+1})$. First, the compatibility conditions between the first $(N+1)$ components of g^ε, namely

$$\frac{\partial g_i^\varepsilon}{\partial x_k} = \frac{\partial g_k^\varepsilon}{\partial x_i} \quad 0 \le i,k \le N,$$

imply by virtue of Theorem 2 that $\xi_k \mu_{ij} = \xi_i \mu_{kj}$ for any $j \in \{1, 2, \cdots, N+2\}$. By a standard algebra (if necessary, see the proof of Lemma 3.10 in [17]) we deduce the following form for the hermitian measure μ

$$\mu = \begin{pmatrix} \xi \otimes \xi\, \kappa & \xi \alpha \\ \xi^T \overline{\alpha} & \theta(1-\theta)\nu \end{pmatrix} \tag{50}$$

where $\kappa(\tilde{x},\tilde{\xi})$ is a scalar real non-negative H-measure, $\alpha(\tilde{x},\tilde{\xi})$ is a possibly complex-valued scalar H-measure, and $\mu_{N+2,N+2} = \theta(1-\theta)\nu$ is just the H-measure of χ^ε. Second, we apply again Theorem 2 to the conservation equation deduced from (9)

$$\rho_0 \frac{\partial g_0^\varepsilon}{\partial x_0} - \operatorname{div}\left(A_0 (g_1^\varepsilon, \cdots, g_N^\varepsilon)^T\right) = -\rho_0 \chi^\varepsilon \frac{\partial^2 u_0}{\partial t^2} + \operatorname{div}\left(A_0 \chi^\varepsilon \nabla u_0\right).$$

After substraction of its weak limit, remarking that χ^ε converges strongly in $H^{-1}(\mathbb{R}^{N+1})$, we obtain, for any $k \in \{1, 2, \cdots, N+2\}$,

$$\rho_0 \xi_0 \mu_{0k} - \sum_{i,j=1}^{N} (A_0)_{ij} \xi_i \mu_{jk} = \sum_{i,j=1}^{N} (A_0)_{ij} \xi_i \frac{\partial u_0}{\partial x_j} \mu_{N+2,k}. \tag{51}$$

Taking into account the structure (50) of μ, we deduce from (51), for $1 \le k \le N+1$,

$$\left(\rho_0 (\xi_0)^2 - A_0 \xi \cdot \xi\right) \xi_k \kappa = A_0 \nabla u_0 \cdot \xi\, \xi_k \overline{\alpha}, \tag{52}$$

while for $k = N+2$ we obtain

$$\left(\rho_0 (\xi_0)^2 - A_0 \xi \cdot \xi\right) \alpha = A_0 \nabla u_0 \cdot \xi\, \theta(1-\theta)\nu. \tag{53}$$

Since the support of μ, and thus of κ and α, are restricted to the hyperplane $\{\xi_0 = 0\}$, we can simply cancel the term $(\xi_0)^2$ in (52) and (53). We also check that α is a real-valued measure and combining (52) and (53) we deduce the following relation between κ and ν

$$\kappa(\tilde{x}, \tilde{\xi}) = \theta(1-\theta) \left(\frac{A_0 \nabla u_0 \cdot \xi}{A_0 \xi \cdot \xi} \right)^2 \nu(x, \xi) \delta(\xi_0) \tag{54}$$

where δ is the usual Dirac mass. From (54) we thus obtain the H-measure of $\nabla u_1^\varepsilon = (g_1^\varepsilon, \cdots, g_N^\varepsilon)^T$ which is

$$\xi \otimes \xi \kappa = \theta(1-\theta) \left(\frac{A_0 \nabla u_0 \cdot \xi}{A_0 \xi \cdot \xi} \right)^2 \xi \otimes \xi \, \nu(x, \xi) \delta(\xi_0).$$

Therefore, the limit of

$$\int_0^T \int_\Omega j''(\nabla u_0) \nabla u_1^\varepsilon \cdot \nabla u_1^\varepsilon \, dx dt$$

is

$$\int_0^T \int_\Omega j''(\nabla u_0) \nabla u_1 \cdot \nabla u_1 \, dx dt + \int_0^T \int_\Omega \int_{\mathbb{S}^{N-1}} \theta(1-\theta) j''(\nabla u_0) \xi \cdot \xi \left(\frac{A_0 \nabla u_0 \cdot \xi}{A_0 \xi \cdot \xi} \right)^2 d\nu(x, \xi) \, dt$$

which is precisely the last line of (47) with formula (48). □

Theorem 3. *Under the respective assumptions of Lemmas 8 and 9 (depending on our choice of objective function), the relaxation of (13) is*

$$\min_{(\theta, \nu) \in \mathcal{U}_{ad}^*} J_{sa}^*(\theta, \nu) \tag{55}$$

where J_{sa}^* is defined by (45), or (47), and $\mathcal{U}_{ad}^*$ is defined by

$$\mathcal{U}_{ad}^* = \{(\theta, \nu) \in L^\infty(\Omega; [0, 1]) \times \mathcal{P}(\Omega, \mathbb{S}_{N-1})\}, \tag{56}$$

where the set of probability measures $\mathcal{P}(\Omega, \mathbb{S}_{N-1})$ is defined in (33). More precisely,

1. *there exists at least one minimizer (θ, ν) of (55),*
2. *any minimizer (θ, ν) of (55) is attained by a minimizing sequence χ_ε of (13) in the sense that χ_ε converges weakly-* to θ in $L^\infty(\Omega)$, $\theta(1-\theta)\nu$ is the H-measure of $(\chi_\varepsilon - \theta)$, and $\lim_{n \to +\infty} J_{sa}(\chi_\varepsilon) = J_{sa}^*(\theta, \nu)$,*
3. *any minimizing sequence χ_ε of (13) converges in the previous sense to a minimizer (θ, ν) of (55).*

Proof. It is a direct consequence of the previous Lemmas. Existence of a minimizer for (55) is obtained by taking a minimizing sequence in the original small amplitude problem (13) and passing to the limit thanks to Lemmas 8 or 9. The fact that any minimizer of (55) is attained by a minimizing sequence of (13) stems from Lemma 5 which states that any probability measure, upon multiplication by $\theta(1-\theta)$ is the H-measure of a sequence of characteristic functions χ_ε weakly converging to a limit density θ. □

Remark 10. *In the definitions (11) and (12) of the objective functions we assumed that the integrand $j(x, \lambda)$, with $\lambda = u(x)$ or $\lambda = \nabla u(x)$, does not directly depend on the characteristic function χ (but that this dependence is implicit, through the solution of the state equation). Actually, as already remarked in [3], our approach does not apply directly to an objective function where the integrand depends on χ as, for example,*

$$J(\chi) = \int_0^T \int_\Omega \Big((1-\chi) j_0(u) + \chi j_1(u) \Big) dx dt.$$

Indeed, in the second-order term of (11) we would have difficulties passing to the limit, as ε goes to zero, in the integral

$$\int_0^T \int_\Omega \Big((1-\chi^\varepsilon) j_0'(u_0) + \chi^\varepsilon j_1'(u_0) \Big) u_2^\varepsilon \, dx dt \tag{57}$$

because u_2^ε is merely weakly converging, as well as χ^ε. It would thus be impossible to characterize the relaxed small amplitude objective function, at least in terms of H-measures. However, if we assume that the two integrands also have a small contrast of order η, i.e.

$$j_1(\lambda) = j_0(\lambda) + \eta k(\lambda) \quad \forall \lambda \in \mathbb{R},$$

then, the second order expansion yields

$$\begin{aligned} J_{sa}(u_0, u_1, u_2) &= \int_0^T \int_\Omega j_0(u_0) \, dx dt + \eta \int_0^T \int_\Omega \left(j_0'(u_0) u_1 + \chi \, k(u_0) \right) \, dx dt \\ &+ \eta^2 \int_0^T \int_\Omega \left(j'(u_0) u_2 + \frac{1}{2} j''(u_0) (u_1)^2 + \chi \, k'(u_0) u_1 \right) \, dx dt \end{aligned}$$

in which the highest order terms in χ are quadratic. We can thus pass to the limit by using H-measures as before and obtain a relaxation result that we do not detail here.

6 Error estimate

The previous section was devoted to the relaxation of the small amplitude optimization problem (13) which is a second-order approximation of the original problem (3). However, it is not clear if the relaxed small amplitude problem (55) is still close, up to second-order, of the original problem (3). The purpose of the present section is thus to obtain an estimate of the remainder between the true solution of (1) and its second-order ansatz, which has to be uniform with respect to the characteristic function χ so it will still hold true after relaxation. In turn, it will yield an error estimate between the original objective function and its small amplitude approximation.

Lemma 10. *Define the remainder, $r = u - u_0 - \eta u_1 - \eta^2 u_2$, where u, u_0, u_1, u_2 are the solutions of (1), (8), (9), (10), respectively. Under the assumptions (16) for the data, there exists a constant $C(T)$, which depends neither on the characteristic function χ nor on the contrast parameter η, such that*

$$\|r\|_{L^\infty((0,T);L^2(\Omega))} \leq C\,(T)\,\eta^3. \tag{58}$$

Under the assumptions (17) for the data, we further have

$$\|r\|_{E_T} \leq C\,(T)\,\eta^3. \tag{59}$$

Proof. Plugging the definition of r into a partial differential equation of the type of (1) yields

$$\rho_0(1+\eta\chi)\frac{\partial^2 r}{\partial t^2} - \mathrm{div}\,(A_0(1+\eta\chi)\nabla r) = -\eta^3 \left[\rho_0\chi\frac{\partial^2 u_2}{\partial t^2} - \mathrm{div}\,(A_0\chi\nabla u_2)\right] \tag{60}$$

with homogeneous boundary and initial conditions. Equation (60) is similar to (18) so that Lemma 2 still applies and we deduce that

$$\|r\|_{L^\infty((0,T);L^2(\Omega))} \leq C(T)\eta^3\|u_2\|_{E_T},$$

and

$$\|r\|_{E_T} \leq C(T)\eta^3\|\frac{\partial u_2}{\partial t}\|_{E_T}.$$

Applying again Lemma 2 for $i = 2$ we deduce that

$$\|r\|_{L^\infty((0,T);L^2(\Omega))} \leq C(T)\eta^3\|\frac{\partial u_1}{\partial t}\|_{E_T}, \tag{61}$$

and

$$\|r\|_{E_T} \leq C(T)\eta^3\|\frac{\partial^2 u_1}{\partial t^2}\|_{E_T}. \tag{62}$$

Lemma 4 furnishes a priori estimates on the time derivatives of u_1, which are independent of χ, under appropriate smoothness assumptions on the initial data. Combining them with (61) and (62) yields the desired result. □

Theorem 4. *Assume that the integrand $j(\lambda)$ of the objective function is a quadratic function of λ. Under assumption (16) for the displacement-based objective function (4) and under assumption (17) for the gradient-based objective function (5), there exists a constant $C > 0$ such that, for any characteristic function χ,*

$$|J\,(\chi) - J_{sa}\,(\chi)\,| \leq C\eta^3. \tag{63}$$

In particular, it implies that

$$\left|\inf_{\chi\in L^\infty(\Omega;\{0,1\})} J_{sa}\,(\chi) - \min_{(\theta,\nu)\in\mathcal{U}^*_{ad}} J^*_{sa}(\theta,\nu)\right| \leq C\eta^3.$$

Proof. Let us consider the case of the displacement-based objective function (4) (the proof for the gradient-based objective function is similar). Since the integrand j is quadratic we write a second order Taylor expansion for which there is no remainder

$$j(u) = j(u_0) + j'(u_0)(u-u_0) + \frac{1}{2}j''(u_0)(u-u_0)^2.$$

Furthermore we have $u - u_0 = \eta u_1 + \eta^2 u_2 + r$, which implies

$$J\,(\chi) = J_{sa}\,(\chi) + \int_0^T\int_\Omega \left(j'(u_0)r + \frac{1}{2}j''(u_0)\left(2\eta^3 u_1 u_2 + \eta^4(u_2)^2 + 2\eta u_1 r + 2\eta^2 u_2 r + r^2\right)\right) dt\,dx.$$

By using assumption (6) on the integrand j, assumption (16) on the data and the result (58), we easily bound the last above integral by $C\eta^3$ which yields (63). □

Remark 11. *Our assumption of a quadratic integrand j is quite restrictive but all our numerical examples will be of this type. With some extra assumptions it is possible to address the case of non-quadratic integrand as well. To avoid unnecessary technicalities we content ourselves to indicate how Theorem 4 can be generalized with the following non-optimal hypotheses for the displacement-based objective function (4). Assume that the third derivative of j exists and is uniformly bounded, and take assumption (17). We write a third order Taylor expansion with exact remainder*

$$j(u) = j(u_0) + j'(u_0)(u - u_0) + \frac{1}{2}j''(u_0)(u - u_0)^2 + \frac{1}{6}j'''(u_m)(u - u_0)^3,$$

where $u_m(x, t)$ is a function taking values in the non-ordered interval $\Big(u(x, t), u_0(x, t)\Big)$. We bound the new remainder term by

$$\left| \int_0^T \int_\Omega j'''(u_m)(u - u_0)^3 dt\, dx \right| \le C\|u - u_0\|^3_{L^\infty((0,T);L^3(\Omega))} \le C\|\nabla(u - u_0)\|^3_{L^\infty((0,T);L^2(\Omega)^N)}$$

by Sobolev embedding which is valid, at least, for the space dimensions $N \le 6$. Then, since $u - u_0 = \eta u_1 + \eta^2 u_2 + r$, we obtain

$$\|\nabla(u - u_0)\|^3_{L^\infty((0,T);L^2(\Omega)^N)} \le C \left(\eta + \|r\|_{E_T}\right)^3$$

which yields the desired result by virtue of Lemma 10. There is certainly room for improving the hypotheses, but we do not want to dwell on that issue.

Remark 12. *The attentive reader has certainly already noticed that we used a graduation of three different smoothness assumptions on the data (initial position and velocity, applied load). Let us draw a global picture of their respective applications so far. The minimal hypothesis is (15) which is enough to give a meaning to the small-amplitude optimization problem (see Lemma 3), to compute the relaxed state equations (see Lemma 7) and the relaxed displacement-based objective function (see Lemma 8). A stronger assumption is (16) (that unfortunately enforces a zero initial position) which is used to compute the relaxed gradient-based objective function (see Lemma 9) and to estimate the error made in relaxing the displacement-based objective function (see Theorem 4). The strongest assumption (17) (that, very unfortunately, enforces both zero initial position and zero initial velocity) is used merely for the error estimate in the relaxation of the gradient-based objective function (see again Theorem 4).*

Remark 13. *As in the elliptic case (see section 3.3 in [3]), if the large amplitude optimization problem (3) is amenable to homogenization, then we can prove that the processes of relaxation and small-amplitude approximation are commutable. Indeed, our approach in the present paper is to, first, make a small-amplitude expansion and, second, relax by using H-measures. A different strategy is, first, to relax by using homogenization theory (which is not always possible, unfortunately), and, second, to make a small-amplitude expansion. Let us briefly indicate how this second method (if available) would lead to the same result. The* *homogenized version of the wave equation (1) is*

$$\begin{cases} \rho_{eff} \dfrac{\partial^2 \bar{u}}{\partial t^2} - \operatorname{div}\left(A_{eff} \nabla \bar{u}\right) = f & \text{in } \Omega \times (0,T) \\ \bar{u}(x,0) = u_{init}(x) & \text{in } \Omega \\ \dfrac{\partial \bar{u}}{\partial t}(x,0) = v_{init}(x) & \text{in } \Omega \\ \bar{u}(x,t) = 0 & \text{on } \Gamma_d \times (0,T) \\ A_{eff} \nabla \bar{u}(x,t) \cdot \hat{n} = 0 & \text{on } \Gamma_n \times (0,T). \end{cases} \tag{64}$$

Following [17] one can compute the small-amplitude approximation of the homogenized coefficients. For the density we exactly find

$$\rho_{eff} = \rho_0 \left(1 + \eta\theta\right), \tag{65}$$

while Tartar has proved in [17] that

$$A_{eff} = A_0 + \eta\theta A_0 - \theta\left(1-\theta\right)\eta^2 A_0 \left(\int_{\mathbb{S}_{N-1}} \frac{\xi \otimes \xi}{\xi \cdot A_0 \xi} d\nu \right) A_0 + \mathcal{O}\left(\eta^3\right), \tag{66}$$

where ν is the H-measure associated to the microstructure of A_{eff}. In turn, it implies the following small-amplitude expansion of the solution, $\bar{u}$, of (64)

$$\bar{u} = u_0 + \eta u_1 + \eta^2 u_2 + \mathcal{O}\left(\eta^3\right)$$

where u_0 is a solution of (8), u_1 is a solution to (36), and u_2 is a solution to (37). A similar expansion has to be made in the relaxed objective function (which unfortunately is rarely known !): it would yield our previous formulas (45) and (47). We skip the details and refer to section 3.3 in [3] for the elliptic case.

7 Optimality Conditions

After establishing a relaxed formulation of our small-amplitude optimization problem, proving that it is well-posed and establishing an error estimate with the original problem, it makes sense to find optimality conditions which hopefully will simplify the problem by characterizing optimal microstructures. In Section 8 it will be an essential ingredient for numerical gradient-based optimization methods.

We first consider the objective function (4), or (45), depending only on the state u and not on its gradient. The relaxed objective function $J^*_{sa}(\theta,\nu)$ depends implicitly of the H-measure ν through the term u_2 in (45). To eliminate u_2 and make the dependence on ν explicit in $J^*_{sa}(\theta,\nu)$, we introduce a first adjoint state p_0, defined as the solution of

$$\begin{cases} \rho_0 \dfrac{\partial^2 p_0}{\partial t^2} - \operatorname{div}\left(A_0 \nabla p_0\right) = j'\left(u_0\right) & \text{in } \Omega \times (0,T) \\ p_0\left(T\right) = \dfrac{\partial p_0}{\partial t}\left(T\right) = 0 & \text{in } \Omega \\ p_0 = 0 & \text{on } \Gamma_d \times (0,T) \\ \left(A_0 \nabla p_0\right) \cdot \hat{n} = 0 & \text{on } \Gamma_n \times (0,T). \end{cases} \tag{67}$$

Lemma 11. *The relaxed objective function simplifies to*

$$J_{sa}^{*}(\theta,\nu) = \int_0^T \int_\Omega \Big(j(u_0) + \eta j'(u_0)\, u_1 + \frac{1}{2}\eta^2 j''(u_0)(u_1)^2 \Big) dxdt \tag{68}$$
$$+\eta^2 \int_0^T \int_\Omega \left(-\rho_0 \theta \frac{\partial^2 p_0}{\partial t^2} u_1 - \theta A_0 \nabla u_1 \cdot \nabla p_0 + \theta(1-\theta) M A_0 \nabla u_0 \cdot A_0 \nabla p_0 \right) dxdt,$$

where M is, as before, defined by (38) as the second order moment of the H-measure ν. Furthermore, there exists a function $x \to \xi^*(x)$ from Ω to the unit sphere $\mathbb{S}_{N-1}$, which depends solely on ∇u_0 and ∇p_0 (and not on θ or u_1) such that, for any density θ, an optimal H-measure is the Dirac mass δ_{ξ^*}, i.e.,

$$J_{sa}^{*}(\theta, \delta_{\xi^*}) = \min_{\nu} J_{sa}^{*}(\theta,\nu).$$

Remark 14. *The precise definition of δ_{ξ^*} is $\delta_{\xi^*}(x,\xi) = \delta\big(\xi - \xi^*(x)\big)$. As a consequence of Lemma 11, in the hyperbolic case as in the elliptic one, the minimizing microstructure can be chosen as a rank-one laminate. The lamination direction of this microstructure may vary at each point, independently of the phase fraction field θ. As such, the relaxed objective function may be optimized with respect to the lamination direction separately from θ. Note that there is no uniqueness of the optimal microstructure in general.*

The fact that rank-one laminates are optimal is shared with the high porosity regime of shape optimization studied in [5].

Proof. The only term to modify in definition (45) of J_{sa}^* is

$$\int_0^T \int_\Omega j'(u_0) u_2 dxdt. \tag{69}$$

We multiply the adjoint equation (67) by u_2 and multiply equation (37) by p_0, proceed to integrate by parts and make a comparison. This classical computation yields that (69) is equal to

$$\int_0^T \int_\Omega \left(-\rho_0 \theta \frac{\partial^2 u_1}{\partial t^2} p_0 - \theta A_0 \nabla u_1 \cdot \nabla p_0 + \theta(1-\theta) M A_0 \nabla u_0 \cdot A_0 \nabla p_0 \right) dxdt.$$

In this last term we further perform another integration by parts in time and exploiting the initial and final conditions $p_0(T) = \frac{\partial p_0}{\partial t}(T) = 0$ and $u_1(0) = \frac{\partial u_1}{\partial t}(0) = 0$, we obtain

$$\int_0^T \int_\Omega \rho_0 \theta \frac{\partial^2 u_1}{\partial t^2} p_0 dxdt = \int_0^T \int_\Omega \rho_0 \theta \frac{\partial^2 p_0}{\partial t^2} u_1 dxdt$$

which finishes the proof of formula (68). This last integration by parts is useful only for numerical considerations in order to avoid calculation of the second time derivative of the first-order displacement field u_1 which has to be evaluated at each iteration of the optimization algorithm (see Section 8).

It is remarkable at this point to notice that the relaxed objective function J^*_{sa} is affine in M, which is the only term containing the H-measure ν. To minimize $J^*_{sa}(\theta,\nu)$ with respect to ν it is enough to minimize at each point $x \in \Omega$ the integrand

$$\int_0^T MA_0\nabla u_0 \cdot A_0\nabla p_0 \, dt = \left(\int_{\mathbb{S}_{N-1}} \frac{\xi \otimes \xi}{\xi \cdot A_0\xi} d\nu(\xi)\right) \cdot \left(\int_0^T A_0\nabla p_0 \otimes A_0\nabla u_0 dt\right).$$

By linearity in ν a possible minimizer is a Dirac mass in the direction $\xi^*(x)$ given by

$$\xi^*(x) = \mathrm{argmin}_{\xi\in\mathbb{S}_{N-1}} \int_0^T \frac{(A_0\nabla p_0(x,t)\cdot\xi)(A_0\nabla u_0(x,t)\cdot\xi)}{\xi \cdot A_0\xi} dt. \tag{70}$$

We readily check from (70) that the optimal H-measure δ_{ξ^*} does not depend on θ. $\square$

Remark 15. *It is possible to eliminate u_1 from the $\mathcal{O}(\eta)$ term in (68) by using again the adjoint state p_0. This will simplify a bit the computation of the gradient of the objective function. We find*

$$J^*_{sa}(\theta,\nu) = \int_0^T \int_\Omega \left(j(u_0) - \eta\left(\rho_0\theta\frac{\partial^2 u_0}{\partial t^2}p_0 + \theta A_0\nabla u_0\cdot\nabla p_0\right)\right) dxdt \tag{71}$$
$$+\eta^2\int_0^T\int_\Omega \left(\frac{1}{2}j''(u_0)(u_1)^2 - \rho_0\theta\frac{\partial^2 p_0}{\partial t^2}u_1 - \theta A_0\nabla u_1\cdot\nabla p_0 + \theta(1-\theta)MA_0\nabla u_0\cdot A_0\nabla p_0\right)dxdt.$$

We now introduce a second adjoint state to compute the derivative of the objective function with respect to θ. We define p_1, which is the solution to:

$$\begin{cases} \rho_0\dfrac{\partial^2 p_1}{\partial t^2} - \mathrm{div}(A_0\nabla p_1) = -\theta\rho_0\dfrac{\partial^2 p_0}{\partial t^2} + \mathrm{div}(\theta A_0\nabla p_0) + j''(u_0)u_1 & \text{in } \Omega\times(0,T), \\ p_1(T) = \dfrac{\partial p_1}{\partial t}(T) = 0 & \text{in } \Omega, \\ p_1 = 0 & \text{on } \Gamma_d\times(0,T), \\ (A_0\nabla p_1)\cdot\hat{n} = -(\theta A_0\nabla p_0)\cdot\hat{n} & \text{on } \Gamma_n\times(0,T). \end{cases} \tag{72}$$

Lemma 12. *The relaxed objective function (45) is Fréchet differentiable with respect to θ and its derivative is*

$$\nabla_\theta J^*_{sa}(\theta,\nu) = -\eta\int_0^T\left(\rho_0\frac{\partial^2 u_0}{\partial t^2}p_0 + A_0\nabla u_0\cdot\nabla p_0\right)dt \tag{73}$$
$$-\eta^2\int_0^T\left(\rho_0\frac{\partial^2 p_0}{\partial t^2}u_1 + A_0\nabla u_1\cdot\nabla p_0 - (1-2\theta)MA_0\nabla u_0\cdot A_0\nabla p_0\right)dt$$
$$-\eta^2\int_0^T\left(\rho_0\frac{\partial^2 u_0}{\partial t^2}p_1 + A_0\nabla u_0\cdot\nabla p_1\right)dt.$$

Proof. The fact that J^*_{sa} is Fréchet differentiable with respect to θ is classical and follows from the fact that J^*_{sa}, defined by (71), is obviously differentiable with respect to $\theta \in L^\infty(\Omega)$ and $u_1 \in E_T$, taken as independent variables, and further that $u_1 \in E_T$ is also differentiable in terms of $\theta \in L^\infty(\Omega)$ (see [2] if necessary). We denote by $z = \langle \frac{\partial u_1}{\partial \theta}, s \rangle$ the derivative of u_1 in the direction $s \in L^\infty(\Omega)$ which satisfies

$$\begin{cases} \rho_0 \dfrac{\partial^2 z}{\partial t^2} - \operatorname{div}(A_0 \nabla z) = -s\rho_0 \dfrac{\partial^2 u_0}{\partial t^2} + \operatorname{div}(sA_0 \nabla u_0) \text{ in } \Omega \times (0,T), \\ z(0) = \dfrac{\partial z}{\partial t}(0) = 0 \text{ in } \Omega, \\ z = 0 \text{ on } \Gamma_d \times (0,T), \\ A_0 \nabla z \cdot \hat{n} = -sA_0 \nabla u_0 \cdot \hat{n} \text{ on } \Gamma_n \times (0,T). \end{cases} \tag{74}$$

The directional derivative of the cost function (71) is then

$$\begin{aligned} \langle \nabla_\theta J_{sa}, s \rangle = -\eta \int_0^T \int_\Omega \left(s\rho_0 \frac{\partial^2 u_0}{\partial t^2} p_0 + sA_0 \nabla u_0 \cdot \nabla p_0 \right) dxdt \\ + \eta^2 \int_0^T \int_\Omega \Big(j''(u_0) u_1 z - s\rho_0 \frac{\partial^2 p_0}{\partial t^2} u_1 - \theta \rho_0 \frac{\partial^2 p_0}{\partial t^2} z \\ - sA_0 \nabla u_1 \cdot \nabla p_0 - \theta A_0 \nabla z \cdot \nabla p_0 + s(1-2\theta) A_0 M A_0 \nabla u_0 \cdot \nabla p_0 \Big) dxdt. \end{aligned} \tag{75}$$

To eliminate z, we use the adjoint state p_1. This classical computation, similar to the one made in the proof of Lemma 11, gives the desired result (73). □

We now turn to objective functions depending on the gradient like (5) and (47). We introduce an alternate version of the zero-order adjoint equation

$$\begin{cases} \rho_0 \dfrac{\partial^2 p_0}{\partial t^2} - \operatorname{div}(A_0 \nabla p_0) = -\operatorname{div}(j'(\nabla u_0)) & \text{in } \Omega \times (0,T) \\ p_0(T) = \dfrac{\partial p_0}{\partial t}(T) = 0 & \text{in } \Omega \\ p_0 = 0 & \text{on } \Gamma_d \times (0,T) \\ A_0 \nabla p_0 \cdot \hat{n} = 0 & \text{on } \Gamma_n \times (0,T). \end{cases} \tag{76}$$

Lemma 13. *The relaxed objective function simplifies to*

$$\begin{aligned} J^*_{sa}(\theta,\nu) = \int_0^T \int_\Omega \Big(j(\nabla u_0) + \eta j'(\nabla u_0) \cdot \nabla u_1 + \frac{1}{2}\eta^2 j''(\nabla u_0) \nabla u_1 \cdot \nabla u_1 \Big) dxdt \\ + \eta^2 \int_0^T \int_\Omega \Big(-\rho_0 \theta \frac{\partial^2 p_0}{\partial t^2} u_1 - \theta A_0 \nabla u_1 \cdot \nabla p_0 + \theta(1-\theta) M A_0 \nabla u_0 \cdot A_0 \nabla p_0 \\ + \frac{1}{2}\theta(1-\theta) N A_0 \nabla u_0 \cdot A_0 \nabla u_0 \Big) dxdt, \end{aligned} \tag{77}$$

where M and N are defined by (38) and (48) as second order moments of the H-measure ν. Furthermore, there exists a function $x \to \xi^*(x)$ from Ω to the unit sphere $\mathbb{S}_{N-1}$, which depends solely on ∇u_0 and ∇p_0 (and not on θ or u_1) such that, for any density θ, an optimal H-measure is the Dirac mass δ_{ξ^*}, i.e.,

$$J^*_{sa}(\theta, \delta_{\xi^*}) = \min_\nu J^*_{sa}(\theta,\nu).$$

Proof. The argument to obtain (77) follows exactly the proof of Lemma 11. To prove the second part of the lemma, we notice that the function

$$\int_0^T \int_\Omega \theta\,(1-\theta)\,(\nabla u_0 \cdot A_0 M A_0 \nabla p_0 + \nabla u_0 \cdot A_0 N A_0 \nabla u_0)\,dxdt \tag{78}$$

is still linear with respect to ν and can thus be minimized by selection of a minimizing Dirac mass, δ_{ξ^*} dependent only upon ∇u_0 and ∇p_0, and independent of θ. □

To calculate the directional derivative of the objective, we need to introduce another first-order adjoint state equation

$$\begin{cases} \rho_0 \dfrac{\partial^2 p_1}{\partial t^2} - \operatorname{div}(A_0 \nabla p_1) = -\theta \rho_0 \dfrac{\partial^2 p_0}{\partial t^2} + \operatorname{div}(\theta A_0 \nabla p_0) - \operatorname{div}(j''(\nabla u_0)\nabla u_1) \ \text{ in } \Omega \times (0,T), \\ p_1(T) = \dfrac{\partial p_1}{\partial t}(T) = 0 \text{ in } \Omega, \\ p_1 = 0 \text{ on } \Gamma_d \times (0,T), \\ A_0 \nabla p_1 \cdot \hat{n} = -\left(\theta A_0 \nabla p_0 - j''(\nabla u_0)\nabla u_1\right) \cdot \hat{n} \text{ on } \Gamma_n \times (0,T). \end{cases} \tag{79}$$

Lemma 14. *The relaxed objective function (47) is Fréchet differentiable with respect to θ and its derivative is*

$$\nabla_\theta J^*_{sa}(\theta,\nu) = -\eta \int_0^T \left(\rho_0 \frac{\partial^2 u_0}{\partial t^2} p_0 + A_0 \nabla u_0 \cdot \nabla p_0 \right) dt \tag{80}$$
$$-\eta^2 \int_0^T \left(\rho_0 \frac{\partial^2 p_0}{\partial t^2} u_1 + A_0 \nabla u_1 \cdot \nabla p_0 - (1-2\theta)\, M A_0 \nabla u_0 \cdot A_0 \nabla p_0 \right) dt$$
$$-\eta^2 \int_0^T \left(\rho_0 \frac{\partial^2 u_0}{\partial t^2} p_1 + A_0 \nabla u_0 \cdot \nabla p_1 - \frac{1}{2}(1-2\theta)\, N A_0 \nabla u_0 \cdot A_0 \nabla u_0 \right) dt.$$

We safely leave to the reader the proof of Lemma 14 which is parallel to that of Lemma 12.

Remark 16. *For simplicity we stated all our results so far in the case of a scalar wave equation, but clearly we never used the scalar character of the equation. Thus the same results hold true for the elastodynamic system of equations, including the result of Lemmas 12 and 13 that the optimal microstructure is a rank-one laminate. The same comment applies to any multi-physics or multiple-loads problem (see [3] for details if necessary).*

8 Numerical Simulations

8.1 Descent Algorithm

We now turn to the numerical minimization of the relaxed objective functions (45) and (47) studied in the previous sections. As we have demonstrated in Section 7, this optimization can be accomplished through the adjustment of two design parameters: the local lamination direction $\xi(x)$, and the local phase fraction $\theta(x)$. The independence of the lamination direction field ξ from the phase fraction field θ allows for the exact solution of the ξ field before optimization of the phase fraction. As we can see from (70) and (78), only the zero-order displacement $u_0\,(x,t)$, solution of (8), and the zero-order adjoint field $p_0\,(x,t)$, solution of (67) or (76), are required to calculate the optimal lamination direction. Fortunately, these two fields, u_0 and p_0 are also seen to be independent of the local phase fraction θ. We can thus compute once and and for all the optimal lamination direction ξ at the beginning of our algorithm. To solve the argmin problem (70) in order to find the optimal lamination direction, we use a simple iterative optimization algorithm such as Conjugate Gradient.

After finding the the optimal lamination direction, we iteratively minimize the objective function with respect to the sole design parameter θ. We use a simple gradient descent method based on formulas (73) or (80) for the derivative of the objective function. Volume (weight) constraints on the design can easily be taken into account by incorporating a Lagrange multiplier into the objective function gradient. Overall the algorithm writes, at each iteration n,

$$\theta^{n+1} = \mathcal{P}\Big(\theta^n - \ell\nabla_\theta J^*_{sa}(\theta^n, \delta_\xi) + \lambda^n\Big),$$

where $\ell > 0$ is the descent step, λ^n is the volume Lagrange multiplier and $\mathcal{P}$ is the projection operator on the range $[0,1]$ of admissible density values. The Lagrange multiplier λ^n is solved for at every step through dichotomy as in, for example, the optimization examples in [1]. We initialize the algorithm with a constant θ_0, i.e. a uniform distribution of the two phases. At each iteration the evaluation of the gradient $\nabla_\theta J^*_{sa}(\theta^n, \delta_\xi)$ requires the first order field u_1^n and its adjoint p_1^n which, unlike u_0 and p_0, are dependent upon the phase fraction θ^n. Therefore, two PDE's have to be solved at each iteration. Remark however that the stiffness and mass matrix are always the same since the density θ^n appears only in the right hand side. Thus they can be factorized once, say by Cholesky method, at the first iteration and stored for the rest of the iterations.

In order to insure that the step is indeed a descent, u_1^{n+1} and p_1^{n+1} are evaluated at the proposed θ^{n+1}. If $J_{sa}^{n+1} < J_{sa}^n$, the step is accepted and the descent step is possibly increased by a factor, say 1.1. If not, the step size is reduced, say by a factor 2, the updated θ^{n+1} is rejected and the iteration is repeated. The algorithm terminates when further reduction of the value of $J_{sa}\,(\theta)$ is impossible, either because the gradient $\nabla_\theta J^*_{sa}(\theta^n, \delta_\xi)$ is very small or because the value of the step size, ℓ, is reduced to beneath some threshold (e.g. $\ell \leq 10^{-8}$). After the termination of the gradient descent algorithm we are often left with an optimal distribution of phase fractions that contains values of θ between 0 and 1. Since these intermediate values reflect pointwise mixtures of the two phases which are not always meaningful from the applied perspective, we then commence a penalization procedure on the result of the gradient-descent with the aim of pushing the phase fraction field toward values of 0 and 1. Specifically, denoting by θ_{opt}^{n+1} the optimal phase fraction defined by

$$\theta_{opt}^{n+1} = \mathcal{P}\Big(\theta^n - \ell\nabla_\theta J^*_{sa}(\theta^n, \delta_\xi)\Big),$$

we modify it to favor values close to 0 and 1

$$\theta_{pen}^{n+1} = \mathcal{P}\Big(\frac{1-\cos\left(\pi\theta_{pen}^{n+1}\right)}{2} + \lambda_{pen}^n\Big).$$

Again the Lagrange multiplier λ^n_{pen} is the Lagrange multiplier for the volume constraint and is solved, as before, by dichotomy for each penalty iteration. For the examples in this paper the results are penalized for 5 iterations. Naturally this penalization procedure stands to perturb the design slightly from the optimal distribution of the phases achieved in the gradient descent.

All computations of u^n_0, p^n_0, u^n_1,p^n_1, and θ are done by the finite element method, using the FreeFEM++ package [9]. The domain Ω is meshed by triangles. For each simulation the displacement fields (u_0 and u_1) and their adjoints (p_0 and p_1) are interpolated on P2 finite elements. The volume fraction field θ is interpolated on P0 finite elements. The time discretization is implicit of second order.

In the sequel we plot the distribution of the stiffer phase. Since η shall be taken positive, we thus plot θ: white corresponds to $\theta = 0$ (weak phase A_0), and black stands for $\theta = 1$ (stiff phase A_1).

8.2 Elasticity setting

Although the theoretical results of the present paper have been presented in a scalar setting, all our numerical simulations are done in the elasticity setting. We emphasize again that our approach works in this vector-valued case too and we refer to [3] for details if necessary. We briefly recall the notations and defined the test problem under consideration in the next sections.

The elastic displacement is a function $u(x,t)$ from $\Omega \times (0,T)$ into $\mathbb{R}^N$ which is a solution of the elastodynamic equations

$$\left\{ \begin{array}{rl} \rho_\chi \dfrac{\partial^2 u}{\partial t^2} - \operatorname{div}\left(A_\chi e(u)\right) = 0 & \text{in } \Omega \times (0,T) \\ u(x,0) = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial t}(x,0) = 0 & \text{in } \Omega \\ u(x,t) = 0 & \text{on } \Gamma_d \times (0,T) \\ A_\chi e(u)(x,t)\cdot \hat{n} = f(x,t) & \text{on } \Gamma_n \times (0,T), \end{array} \right. \tag{81}$$

where $f(x,t)$ is some given applied load, a function from $\Gamma_n \times (0,T)$ into $\mathbb{R}^N$. The initial data are zero. The strain tensor is

$$e(u) = \frac{1}{2}\Big(\nabla u + (\nabla u)^T\Big),$$

and the stress tensor is $\sigma = A_\chi e(u)$. We assume that both phases are isotropic, namely for $i = 0,1$

$$A_i e(u) = 2\mu_i e(u) + \lambda_i (\operatorname{div} u) I_2,$$

where I_2 is the identity matrix and μ_i, λ_i are the Lamé coefficients.

8.3 Compliance minimization or dissipation maximization

In the steady-state case, a common example of many shape optimization algorithms is the minimization of the work done by the applied load, or compliance,

$$\int_{\Gamma_n} f \cdot u \, ds = \int_\Omega e(u) \cdot A_\chi e(u) \, dx.$$

In the time-dependent case, we wish to alter this formulation slightly. Specifically, we are interested in minimizing not the work done by the applied load, but instead the objective of interest is its power,

$$J(\chi) = \int_0^T \int_{\Gamma_n} f \cdot \frac{\partial u}{\partial t} \, ds dt.$$

This integrand somehow seems more natural as it is the time derivative of the total energy (kinetic plus potential) of the system, or energy dissipation,

$$\int_{\Gamma_n} f \cdot \frac{\partial u}{\partial t} \, ds = \frac{d}{dt}\left(\frac{1}{2} \int_\Omega \left(\rho_\chi \left(\frac{\partial u}{\partial t} \right)^2 + A_\chi e(u) \cdot e(u) \right) dx \right),$$

which implies in view of (81)

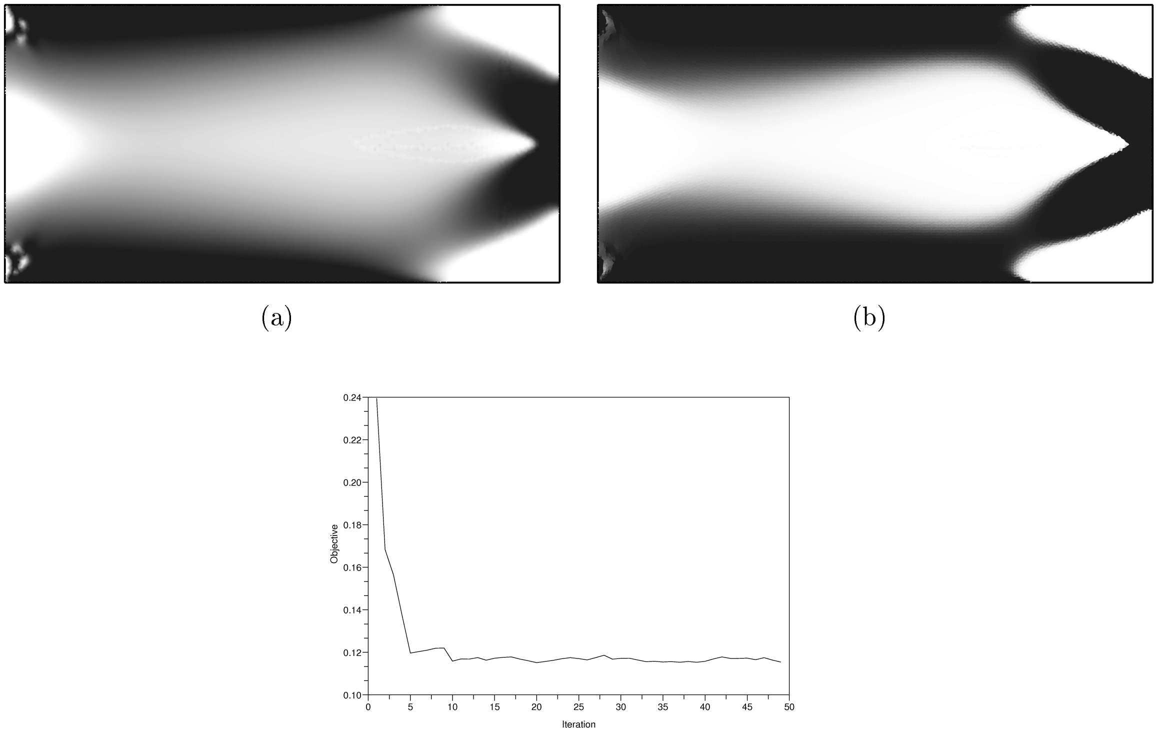
$$J(\chi) = \frac{1}{2} \int_\Omega \left(\rho_\chi \left(\frac{\partial u}{\partial t} \right)^2 + A_\chi e(u) \cdot e(u) \right) (T) dx.$$

In other words we want to minimize the final energy at time T or, equivalently, to maximize the dissipation. The test domain, $\Omega = [0,2] \times [-0.5, 0.5]$, is a long elastic cantilever beam fixed on the left side, $\Gamma_d = \{x = 0, y \in [-0.5, 0.5]\}$, and loaded with a vertical force, $f_0(x) = (0,1)^T$, applied at the center of the right side, $\Gamma_n = \{x = 2, y \in [-0.05, 0.05]\}$. The Lamé moduli are $\lambda_0 = 0.86$, $\mu_0 = 0.37$ and the material density is $\rho_0 = 1$. The contrast parameter is $\eta = 0.90$ and the total volume fraction of the stiffer phase is maintained at 0.5. Since this set of physical parameters admits a characteristic wave speed slightly larger than 1.0 (in the base material), we select the integration period $[0, T = 10]$ in order to allow any elastic waves to traverse the cantilever several times during optimization. The discretization of the domain Ω is on a very fine mesh consisting of 23091 vertices and 45580 triangles. The time discretization takes 40 uniformly spaced steps with $\delta t = 0.25$.

In the first case (Figure 1) the applied load is $f(t,x) = f_0(x)a(t)$, where the amplitude $a(t) = \sin\left(\frac{\pi}{20}t\right)$ is varying sinusoidally in time from 0 to 1 over the period of integration. The resulting structure is very similar to what is obtained in the steady-state setting [3].

Introducing a variation in the frequency of the applied load naturally varies the optimal structure of the cantilever. This can be seen, for instance, in the case (Figure 2) in which the frequency of load is four times that of the initial case, i.e., $a(t) = \sin\left(\frac{\pi}{5}t\right)$. The stiffer material is then largely realligned towards the fixed base, abandoning the formation of structure at the point of application of the load.

Nothing in the development of the algorithm dictates that the domain of integration must be all of Ω. For a given subdomain Σ of Ω we can, for example, seek to maximize the



(a) (b)

(c)

Figure 1: Volume fraction θ of the stiff phase for the dissipation maximizing cantilever (a), penalized configuration (b), and convergence history (c) for the load history $a(t) = \sin\left(\frac{\pi}{20}t\right)$ with $\eta = 0.90$ and a volume fraction of 0.50.

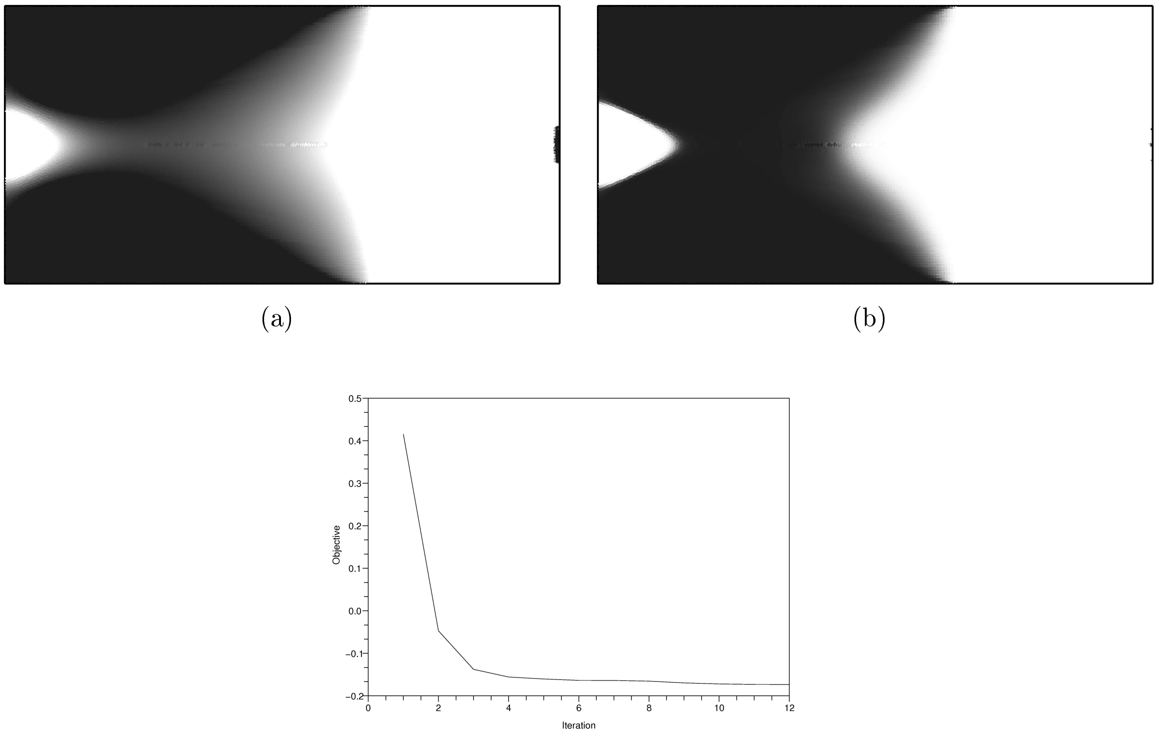
dissipation through that subdomain by adjusting the objective function.

$$J(\chi) = \int_0^T \frac{d}{dt}\left(\frac{1}{2}\int_\Sigma \left(\rho_\chi \left(\frac{\partial u}{\partial t}\right)^2 + A_\chi e(u)\cdot e(u)\right) dx\right) dt.$$

Remark here that we do not require that the boundary of Σ contain Γ_n. A first example (Figure 3) of a subdomain over which we choose to maximize the dissipation is the quarter of the cantilever nearest the applied load, $\Sigma = \{x \geq 1.5\}$. Since the constraint on the volume fraction is still an average of 0.5, we anticipate the formation of structures exterior to Σ. Somewhat surprisingly in this case, no structure forms that connect the far right quarter of the cantilever to its base. Another, somewhat more artificial, example (Figure 4) is included for a smaller, non-convex subdomain $\{x \geq y, x \leq -y, x \leq 0.4\} \cup \{x \leq y, x \geq -y, x \geq 0.4\}$. The area of this "bowtie" (0.32) is also much smaller than the desired volume fraction. In this case however, the majority of the stiff material goes to form structures connecting the desired subdomain to the base of the cantilever.

8.4 Strain and stress minimization

As we have seen, in the case where the objective is a function of the gradient (or strain) the formulation of our algorithm must change slightly in order to incorporate the additional H-



(a) (b)

(c)

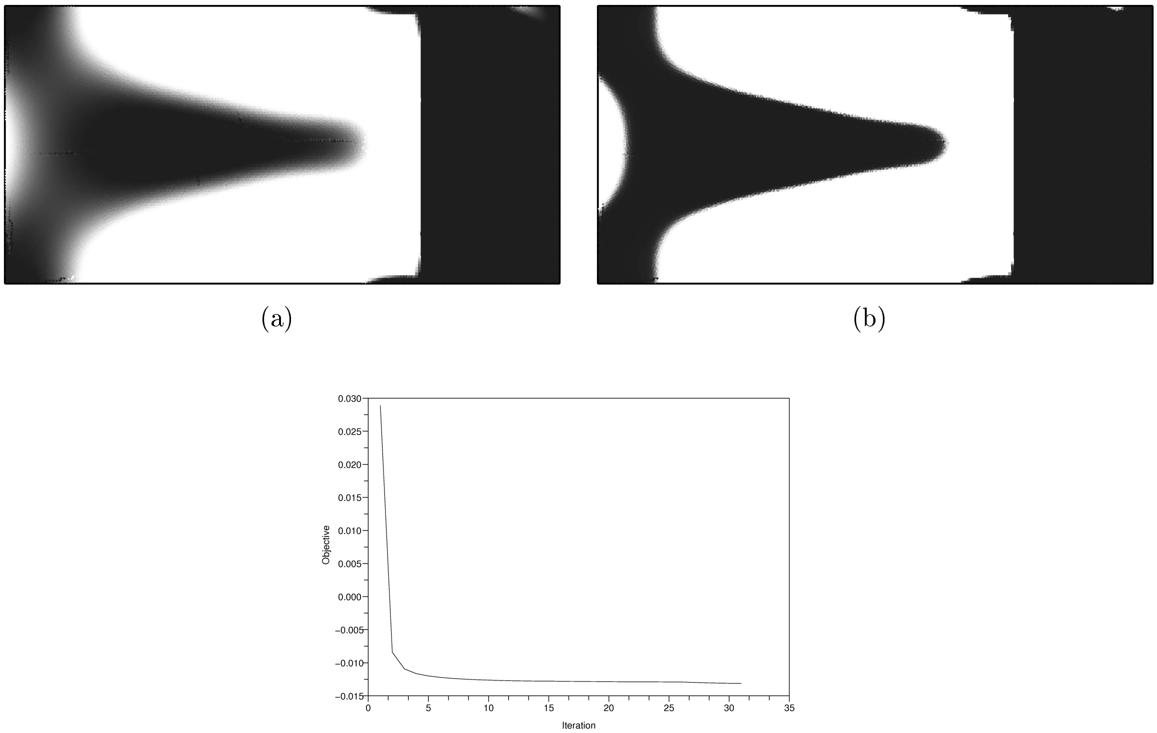
Figure 2: Volume fraction θ of the stiff phase for the dissipation maximizing cantilever (a), penalized configuration (b), and convergence history (c) for the load history $a(t) = \sin\left(\frac{\pi}{5}t\right)$ with $\eta = 0.90$ and a volume fraction of 0.50.

measure term. Numerically, since this only introduces a substantial change in the calculation of the optimal lamination direction field, the iterative stage of optimizing the field θ is not complicated at all. Two natural objective functions that result in an interesting contrast are stress (e.g the time integral of the L^2 norm of the stress) and strain (similarly the L^2 norm of the strain) optimization.

For the stress optimization example (Figure 5) we take a square $\Omega = [0,1] \times [0,1]$, fixed at the bottom $\Gamma_d = \{x \in [0,1], y = 0\}$, and apply a uniformly distributed load at the top $\Gamma_n = \{x \in [0,1], y = 1\}$. The load is kept constant and vertical throughout, $f(t,x) = (0,1)^T$. The objective function subject to optimization is

$$J(\chi) = \frac{1}{2} \int_0^T \int_\Omega |A_\chi e(u)|^2 \, dxdt.$$

The physical parameters remain the same as above as does the period of integration, $T = 10$, in order to allow the waves to traverse the domain several times during the optimization. The optimization is performed on a mesh consisting of 12070 vertices and 23738 triangles. The computation of the derivative (including the definition of the adjoint) is slightly different for a stress-based objective function than for a strain-based one (see [3] for the details).



(a) (b)

(c)

Figure 3: Dissipation is maximized in the far right quarter of the domain without penalty (a), and penalized after convergence (b). The convergence history is shown (c). The load history, contrast, and volume are the same as in the example in Figure 1.

The strain optimization (Figure 6) is performed in the same manner. With,

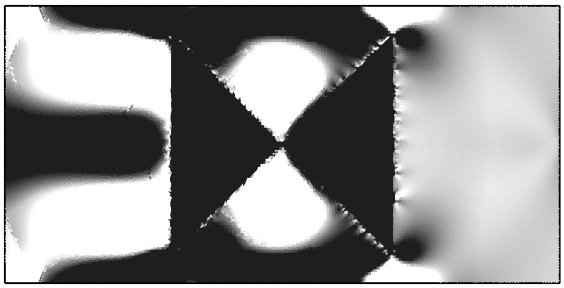
$$J(\chi) = \frac{1}{2}\int_0^T \int_\Omega |e(u)|^2 dx dt.$$

The marked difference in the optimal structures is not surprising compared to a similar analysis done in the elliptic case in [3].

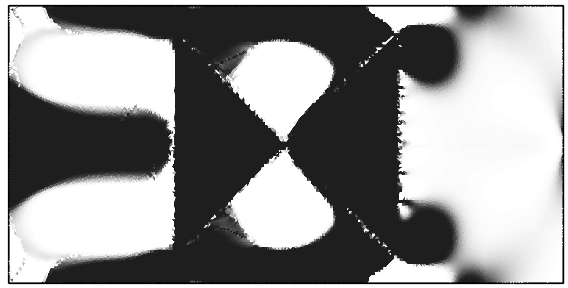
8.5 Dynamic wheel

In the interest of presenting an example that more closely resembles an application of the theory to an interesting physical problem we examine the case of an elastic wheel. On the annular domain, $\Omega = \{1.0 \geq x^2 + y^2 \geq 0.0025\}$, we fix the inner circle $\Gamma_d = \{x^2 + y^2 = 0.0025\}$ to model the attachment of the wheel to a rigid axle, and model the rolling of the wheel by applying an inward radial point force along the wheel's edge $\Gamma_n = \{x^2 + y^2 = 1.0\}$, varying the point of application of the force continuously in time. In other words, $f(x,t)$ is a point force applied to the point $x = (\cos(t), \sin(t))^T$ and taking the value $f(x,t) = (-\cos(t), -\sin(t))^T$.

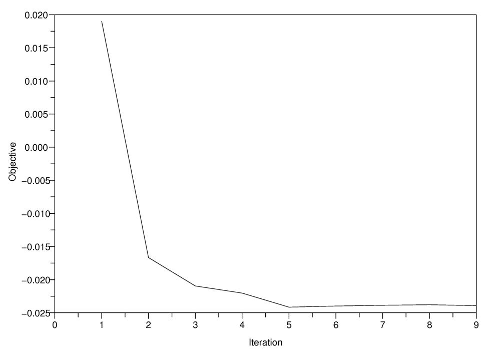
The period of integration is one rotation, namely $T = 2\pi$. Instead of starting from rest (which concentrates all of the stiffener in the center of the wheel), the initial conditions for the zero-order wave equation (8) are set equal to the final state after one hundred rotations of



(a)



(b)



(c)

Figure 4: Dissipation is maximized in the "bowtie" subdomain without penalty (a), and penalized after convergence (b). The convergence history is shown (c). The load history, contrast, and volume are the same as in the example in Figure 1.

the wheel (so it is almost a time periodic solution). Thus optimization begins with non-zero initial data, contrary to the previous examples. The volume fraction of the stiffer phase is maintained at 0.5.

We examine two objective functions: maximization of the dissipation (Figure 7),

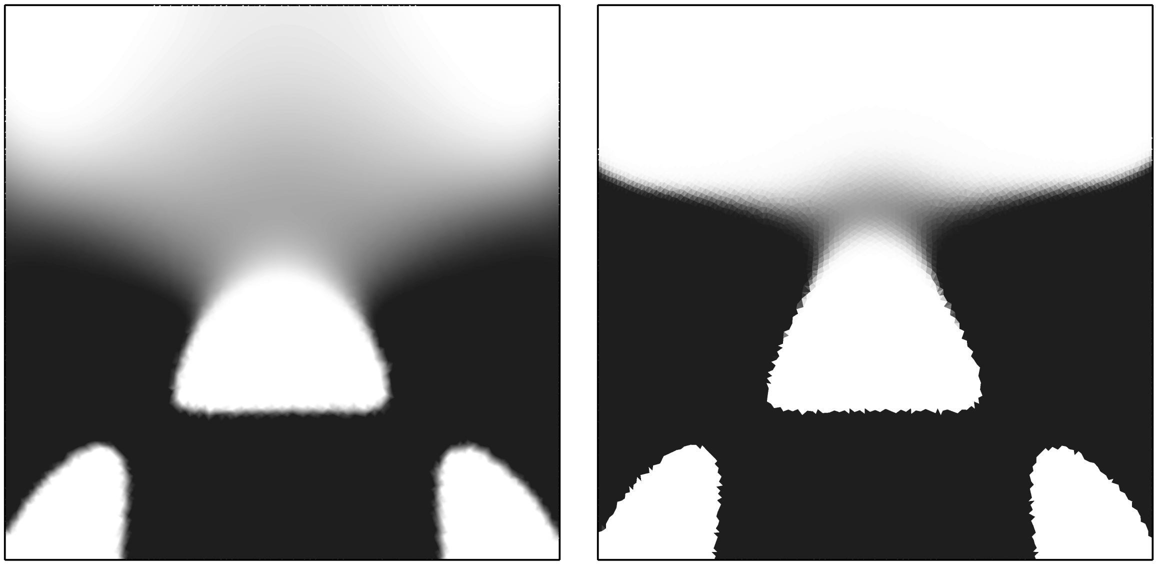
$$J(\chi) = \int_0^T \int_\Omega \Big(\rho_\chi \frac{\partial^2 u}{\partial t^2} \cdot \frac{\partial u}{\partial t} - \operatorname{div}\left(A_\chi e(u)\right) \cdot \frac{\partial u}{\partial t} \Big) dx dt,$$

and minimization of the shear stress (Figure 8),

$$J(\chi) = \int_0^T \int_\Omega \frac{1}{2} \sigma_{12}(u) : \sigma_{12}(u)\, dx dt,$$

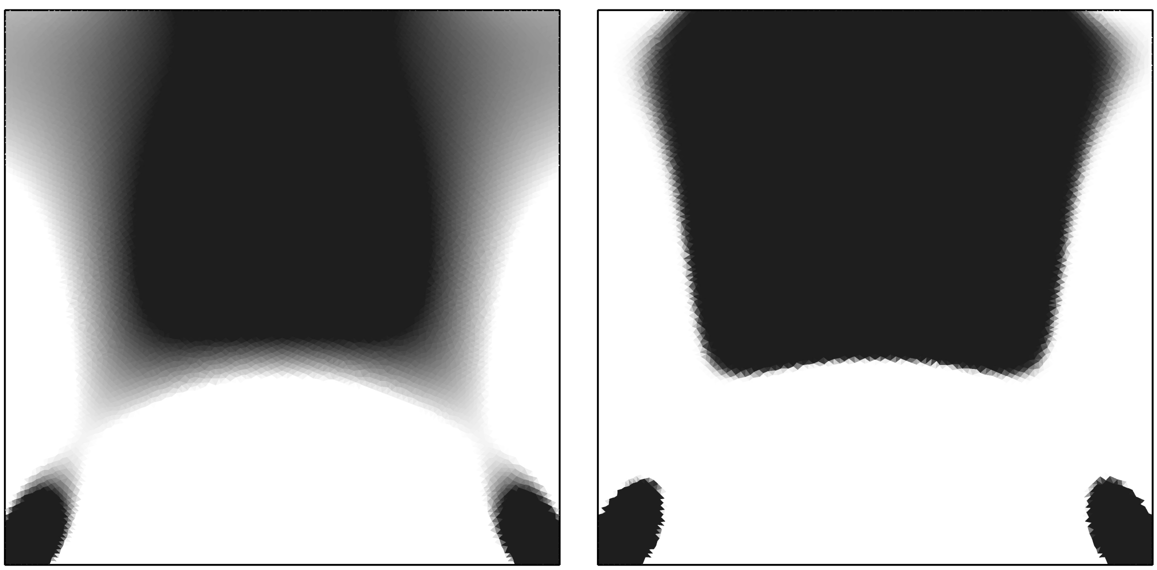
on a fine mesh of 14306 vertices and 28192 triangles.

The results of Figures 7 and 8 are very similar, up to a 90 degrees rotation. Because of our initial conditions which somehow approximate time periodic boundary conditions, it is expected that a non radial optimal design can not be unique since any rotation of it will yield a new optimal design.



(a) (b)

Figure 5: Optimal structure minimizing the L^2 norm of stress under constant distributed top load in the square without penalty (a), and after penalization (b).



(a) (b)

Figure 6: Optimal structure minimizing the L^2 norm of strain under constant distributed top load in the square without penalty (a), and after penalization (b).

Acknowledgment

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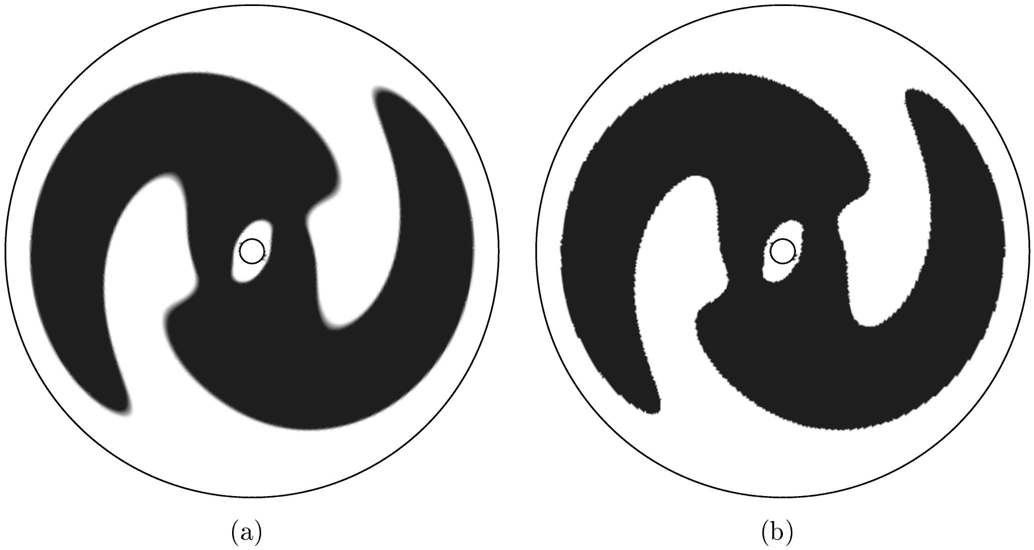

Figure 7: A dynamic elastic wheel to maximize the energy dissipation in a single rotation (a) is subsequently penalized (b).

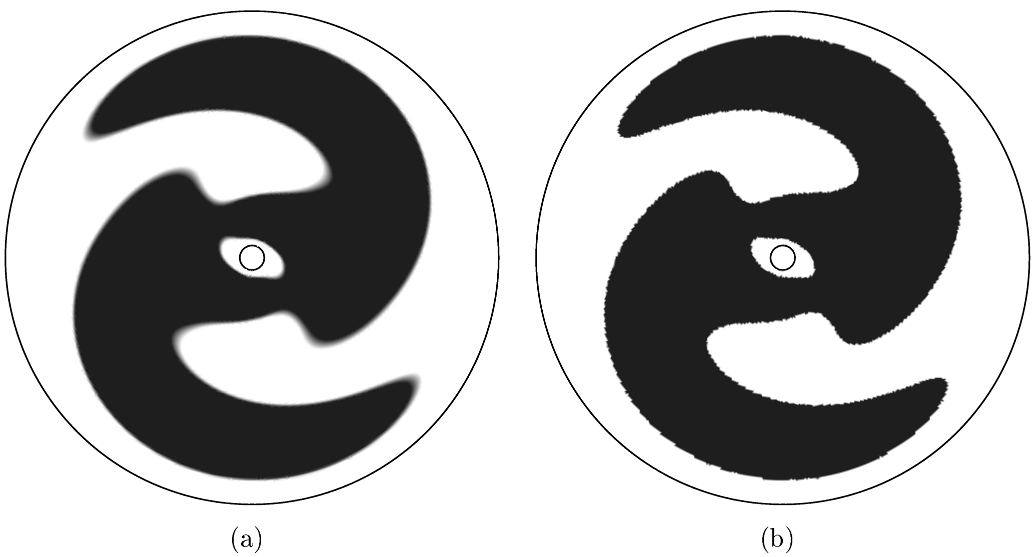

Figure 8: A dynamic elastic wheel to minimize the average shear stress in the wheel in a single rotation (a) is subsequently penalized (b).

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