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Entropy method for line-energies

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Abstract

In this paper we analyze energy functionals concentrated on the discontinuity lines of unit-length, divergence-free $2D$ vector fields. The motivation comes from thin-film micromagnetics where these functionals correspond to mesoscopic wall-energies. A natural issue consists in characterizing the line-energy densities for which the functionals are lower semicontinuous for a relevant topology. In fact, this is a necessary condition for being the $\Gamma-$limit of a family of energies. A key point in our study is the use of the notion of entropy production borrowed from the field of conservation laws. With this tool, we build a large class of lower semicontinuous line-energies. In particular, we prove that certain power functions lead to such line-energy functionals as conjectured in [2]. We also deduce compactness properties for these functionals leading to the existence of minimizers for their lower semicontinuous envelopes. Another natural question is whether the viscosity solution is a minimizing configuration. We show that the answer is in general negative by exhibiting some special nonconvex domains as counterexamples. However, we establish positive results for some special domains (stadium, ellipse and union of two discs). The case of general convex domains is still open.

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1 Introduction

1.1 Model

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with piecewise Lipschitz boundary that is oriented by the outer unit normal vector n. We focus on the set $\mathcal{S}(\Omega)$ of unit-length divergence-free $2D$ vector fields of bounded variation

$$\mathcal{S}(\Omega) \ := \ \left\{ m \in BV(\Omega, \mathbf{R}^2) \ : \ |m| = 1, \, \nabla \cdot m = 0 \ \text{ in } \Omega \right\}.$$

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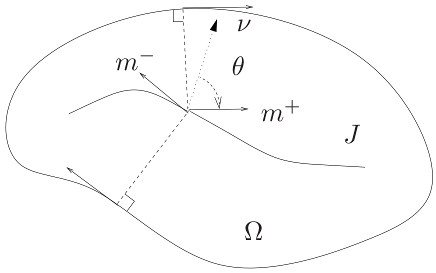

For a vector field $m \in BV(\Omega, S^1)$, the jump set $J(m)$ is a $\mathcal{H}^1-$rectifiable set oriented by a unit vector field $\nu : J(m) \to S^1$ and we will denote by $m^\pm : J(m) \to S^1$ the traces of m on $J(m)$ with respect to ν. We then introduce energy functionals that concentrate on the jump set of $m \in \mathcal{S}(\Omega)$:

$$\mathcal{I}_f(m) \ := \int_{J(m)} f(|m^+ - m^-|)\, d\mathcal{H}^1.$$

We only consider energy densities that depend on the jump size $|m^+ - m^-|$ via a *cost function* $f : [0, 2] \to \mathbf{R}_+$ which satisfies $f(0) = 0$ and is assumed to be *lower semicontinuous*. Notice that the divergence-free hypothesis on m ensures that the normal component $m \cdot \nu$ is continuous across the jump set $J(m)$. So, for $\mathcal{H}^1$-almost every $x \in J(m)$, we can characterize the jump of m by a so called "wall angle" $\theta(x)$ such that $m^\pm(x) = \cos\theta(x)\nu(x) \pm \sin\theta(x)\nu^\perp(x)$. In particular, $|m^+(x) - m^-(x)| = 2|\sin\theta(x)|$. For the same reason, the trace of the normal component $m \cdot n$ is well defined on $\partial\Omega$ and we can consider the minimization problem in the subset

$$\mathcal{S}_0(\Omega) = \left\{ m \in BV(\Omega, \mathbf{R}^2) \,:\, |m| = 1,\, \nabla \cdot m = 0 \text{ in } \Omega \text{ and } \quad m \cdot n = 0 \text{ on } \partial\Omega \right\}.$$

Our problem can be equivalently interpreted in terms of the stream function $\psi : \Omega \to \mathbf{R}$ associated to $m = \nabla^\perp \psi \in \mathcal{S}_0(\Omega)$. Then the above variational principle turns in analyzing the following energy functional

$$\int_{J(\nabla\psi)} f(|(\nabla\psi)^+ - (\nabla\psi)^-|)\, d\mathcal{H}^1 \tag{1}$$

over the set of solutions of the Dirichlet problem associated to the eikonal equation

$$|\nabla\psi| = 1 \text{ in } \Omega \quad \text{and} \quad \psi = 0 \text{ on } \partial\Omega.$$

The method of characteristics shows that for a simply connected bounded domain there is no smooth solution of the eikonal equation $|\nabla\psi| = 1$ in Ω satisfying the constaint $\psi = 0$ on $\partial\Omega$. Typical singularities are jump discontinuities of $\nabla\psi$ (equivalently of m) through line-singularities or vortices.

1.2 Motivation

Line-energy functionals $\mathcal{I}_f$ appear as natural candidates for the asymptotic energy of family of singularly perturbed functionals $\{G_\varepsilon\}_{\varepsilon\downarrow 0}$,

$$G_\varepsilon(m_\varepsilon) = \frac{1}{2\varepsilon}\int_\Omega g(\left|1 - |m_\varepsilon|^2\right|) + \frac{\varepsilon}{2}\int_\Omega |\nabla m_\varepsilon|^2, \tag{2}$$

defined for $m_\varepsilon \in H^1(\Omega, \mathbf{R}^2)$ satisfying the constraints $\nabla \cdot m_\varepsilon = 0$ in Ω and $m_\varepsilon \cdot n = 0$ on $\partial\Omega$. Here, $\varepsilon > 0$ is a small parameter and $g : \mathbf{R}_+ \to \mathbf{R}_+$ is some lower semicontinuous function such that $g(0) = 0$ and $g(t) > 0$ for $t > 0$. Variational models (2) arise in several physical applications such as smectic liquid crystals, film blisters or convective pattern formation (see e.g. [4], [22], [20], [17]).

Observe that the vector fields m_ε in (2) are not S^1-valued but their distance to S^1 is penalized by the first term of G_ε. As ε tends to 0, we expect that families $\{m_\varepsilon\}$ of uniformly bounded energies (2) will converge (up to extraction and in a certain topology, see below) to some limit m_0 satisfying the constraints

$$|m_0| = 1\ ,\ \nabla \cdot m_0 = 0 \ \text{ in } \Omega \quad \text{ and } \quad m_0 \cdot n = 0 \text{ on } \partial\Omega. \tag{3}$$

A natural question arises: if $\mathcal{I}_f$ is indeed the asymptotic energy of $\{G_\varepsilon\}$ as $\varepsilon \to 0$, what is the relation between the energy density f and the function g? The ansatz consists in reducing the $2D$ variational problem to a $1D$ asymptotic analysis: Assume that m_0 is of bounded variation and satisfies (3), i.e., $m_0 \in \mathcal{S}_0(\Omega)$. Then it is expected that at level $\varepsilon > 0$ the energy $G_\varepsilon(m_\varepsilon)$ concentrates on $1D$ transition layers of length scale ε through the line-singularities of m_0. With the above notations, let x_0 be a jump point of m_0, θ_0 be the angle defining the jump $m_0^\pm(x_0)$ and ν_0 be the orientation of the jump set at x_0 (see Figure 1). At level $\varepsilon > 0$, a $1D$ transition layer in the direction ν_0 has the form

$$m_\varepsilon(x_0 + t\nu_0) = \cos\theta_0 \nu_0 + u(t/\varepsilon)\nu_0^\perp,$$

where $u : \mathbf{R} \to \mathbf{R}$ is the rescaled profile of the tangential component of the layer satisfying $u(s) \overset{\pm s\uparrow\infty}{\longrightarrow} \pm\sin\theta_0$. (Observe that a divergence-free $1D$ transition layer has a constant normal component.) Using this ansatz, we obtain that the limit energy is given by $\mathcal{I}_f$ with the cost

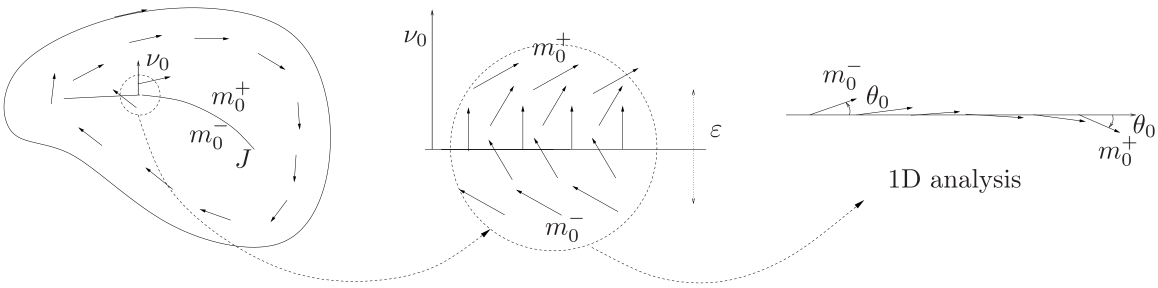

Figure 1: $1D$ anzatz : A line-singularity of a limit configuration m_0 (left picture) is regularized by a smooth $1D$ transition layer at the level $\varepsilon > 0$ connecting two limit states $m_0^\pm$ (middle picture). The full transition occurs in the normal direction ν_0 as represented in the right picture.

function computed as follows:

$$\begin{aligned} f(|2\sin\theta_0|) &= \min\left\{ \frac{1}{2}\int_{\mathbf{R}} \left\{ g(|\sin^2\theta_0 - u^2(s)|) + \left|\frac{du}{ds}(s)\right|^2 \right\} ds \ :\ u : \mathbf{R} \to \mathbf{R},\ u \overset{\pm s\uparrow\infty}{\longrightarrow} \pm\sin\theta_0 \right\} \\ &= 2\int_0^{\sin\theta_0} \sqrt{g(\sin^2\theta_0 - u^2)}\, du, \quad \theta_0 \in [0, \frac{\pi}{2}], \end{aligned} \tag{4}$$

which yields the connection between f and g. In particular, every power function $f(t) = t^p$ corresponds to $g(t) = ct^{p-1}$ in (4) where the constant c depends only on p.

For $g(t) = t^2$, the above ansatz is known to be relevant. The corresponding functional (2) has been introduced by Aviles and Giga [4]. The motivation comes either from solid mechanics, liquid crystals or micromagnetic models (see [7, 14]). It gave rise to a series of articles [20, 6, 2, 12, 8, 23] that justify that $\mathcal{I}_f$ with $f(t) = t^3/6$ (given by (4)) is indeed the asymptotic energy of $\{G_\varepsilon\}$. However, the $\Gamma-$convergence program is not completely solved: The main obstacle is the fact that the natural set of limit finite-energy configurations in not a subset of BV. Indeed, in [2] the authors construct a family of divergence-free vector fields $\{m_k\} \subset BV(\Omega, S^1)$ of finite energy, i.e., $\sup_k \mathcal{I}_f(m_k) < \infty$ that converges strongly in L^1 to some limit configuration $m \in L^\infty(\Omega, S^1)\setminus BV_{loc}$. The crucial point is that the cubic cost of small jumps of m in $\mathcal{I}_f(m)$ cannot control the linear cost of the jump part of ∇m. However, some regularity properties of BV-maps do hold for finite-energy limit configurations: in [11], it is shown that for such limit m it is still possible to define a $\mathcal{H}^1$-rectifiable jump set $J(m)$ so that the definition of $\mathcal{I}_f(m)$ makes sense. The situation is better if we focus on either zero-energy configurations (see [19]) or dilation invariant configurations ([13]).

Let us stress that for a general function g, the above $1D$ ansatz may be wrong. Indeed, in some cases, it is possible to decrease strictly the energy by substituting $2D$ mesoscopic structures for $1D$ transition layers. In these cases, the $1D$ asymptotic energy $\mathcal{I}_f$ (with f given by (4)) does not match the $2D$ Γ-limit energy of (2). Such counterexamples are obtained with non lower semicontinuous functionals $\mathcal{I}_f$ (see below Definition 1). Indeed, a Γ-limit functional over a metric space must be lower semicontinuous with respect to the induced topology. A first counterexample is given in [2]: it is shown that power functions $f(t) = t^p$ lead to non lower semicontinuous functional $\mathcal{I}_f$ for $p > 3$. A second counterexample is described in [1]: the cost function $f_{ARS}(2\sin\theta) = 2(\sin\theta - \theta\cos\theta)$ for $0 \le \theta \le \pi/2$ stemmed from the energy of $1D$ transition layers associated to a particular asymptotics of the micromagnetic energy. It turns out that $\mathcal{I}_{f_{ARS}}$ is not lower semicontinuous. In both cases it is possible to build a $2D$ mesoscopic structure with length-scale $\eta \ll 1$ between two limit states m^- and m^+ with an energetic cost strictly smaller than the cost of a direct $1D$ jump. An example of such $2D$ structure is described in [1] (see Figure 2) and stands for the cross-tie wall pattern in micromagnetics.

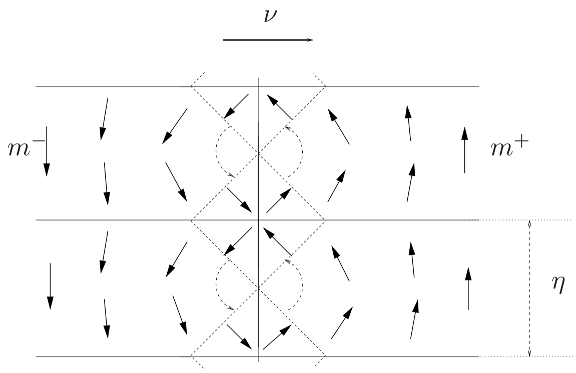

Figure 2: A cross tie wall. As $\eta \downarrow 0$, the $2D$ microstructure tends to a jump configuration (m^-, m^+) in direction ν and has less energy than the initial cost $f_{ARS}(2)$ corresponding to the $1D$ jump $m^\pm$ of angle $\theta = 90°$.

1.3 Lower semicontinuity

As explained above, lower semicontinuity implies the optimality of the $1D$ structure, so it is important to characterize cost functions f such that the line-energy $\mathcal{I}_f$ is lower semicontinuous in a relevant functional space.
Let us first specify this space. The case of cubic power function $f(t) = t^3$ has revealed that the weak BV-topology is too strong. Then it is natural to weaken the regularity by using the topology of L^1. Of course, in order for the constraint $|m_0| = 1$ to be stable under convergence, we need to use the strong L^1-topology. Then, let us extend the function $\mathcal{I}_f$ in $L^1(\Omega, \mathbf{R}^2)$ by $+\infty$, i.e.,

$$\mathcal{I}_f(m) = +\infty \quad \text{if} \quad m \in L^1(\Omega, \mathbf{R}^2) \setminus \mathcal{S}(\Omega),$$

and let us introduce a relaxed functional $\overline{\mathcal{I}_f}$ as the lower semicontinuous envelope of $\mathcal{I}_f$ with respect to the strong L^1-topology: $\overline{\mathcal{I}_f} : L^1(\Omega, \mathbf{R}^2) \to \mathbf{R} \cup \{+\infty\}$ is defined as

$$\overline{\mathcal{I}_f}(m) = \inf \left\{ \liminf_{k\to\infty} \mathcal{I}_f(m_k) \,:\, m_k \to m \text{ strongly in } L^1 \right\}, \quad \forall m \in L^1(\Omega, \mathbf{R}^2).$$

Obviously, $\overline{\mathcal{I}_f} \leq \mathcal{I}_f$ and all configurations of finite relaxed energy $\overline{\mathcal{I}_f}(m) < +\infty$ belong to

$$\mathcal{L}(\Omega) = \{m \in L^1(\Omega, \mathbf{R}^2) \,:\, |m| = 1 \text{ and } \nabla \cdot m = 0 \quad \text{in} \quad \Omega\}$$

which is a closed set in L^1. Recall that the normal component of $m \in \mathcal{L}(\Omega)$ at the boundary $\partial\Omega$ is well defined. In particular,

$$\mathcal{L}_0(\Omega) = \{m \in \mathcal{L}(\Omega) \,:\, m \cdot n = 0 \quad \text{on} \quad \partial\Omega\} \tag{5}$$

is a closed subset of $\mathcal{L}(\Omega)$.

Definition 1 *We say that the line-energy $\mathcal{I}_f : L^1(\Omega, \mathbf{R}^2) \to \mathbf{R} \cup \{+\infty\}$ is lower semicontinuous (l.s.c.) if $\mathcal{I}_f(m) = \overline{\mathcal{I}_f}(m)$ for every $m \in \mathcal{S}(\Omega)$.*

Remark 1 The above definition is weaker than asking for $\mathcal{I}_f$ to be lower semicontinuous in L^1 (i.e. $\mathcal{I}_f = \overline{\mathcal{I}_f}$ in $L^1(\Omega, \mathbf{R}^2)$). Indeed, as described in Section 1.2 for the Aviles-Giga model with cubic jump costs, it is proved in [2] that $\mathcal{I}_{t\mapsto t^3}(m) = \overline{\mathcal{I}}_{t\mapsto t^3}(m)$ for every $m \in \mathcal{S}(\Omega)$ (so, $\mathcal{I}_{t\mapsto t^3}$ is lower semicontinuous after Definition 1), but one can construct a limit configuration $m_0 \in L^1 \setminus BV$ with finite relaxed energy $\overline{\mathcal{I}}_{t\mapsto t^3}(m_0) < +\infty = \mathcal{I}_{t\mapsto t^3}(m_0)$.

A first result asserts a necessary condition: in order for the line-energy functional $\mathcal{I}_f$ to be lower semicontinuous, the cost function f should be also lower semicontinuous.

Proposition 1 *Let $f : [0, 2] \to \mathbf{R}_+$ be a measurable function. If $\mathcal{I}_f$ is lower semicontinuous, then f is lower semicontinuous on $[0, 2]$.*

We address the following question raised by Ambrosio, De Lellis and Mantegazza in [2].

Conjecture 1 $\mathcal{I}_f$ is lower semicontinuous for power cost functions $f(t) = t^p$ if $1 \leq p < 3$.

First of all, we give a partial positive answer to this question: the behavior as a power function t^p for $1 \leq p \leq 3$ at the origin is a natural one for appropriate cost function.

Theorem 1 *For every $p \in [1,3]$, there exists an appropriate cost function $f : [0,2] \to \mathbf{R}_+$ such that $f(t) = t^p$ for $t \in [0, \sqrt{2}]$ and $\mathcal{I}_f$ is lower semicontinuous.*

The power $t \mapsto t^3$ is critical for the method used in the present paper (see Remark 12).

Next, we will establish a positive answer to Conjecture 1 for $p = 2$. Our interest for this case has a physical motivation: In our previous work [17] we studied the energetic behavior of Bloch walls in micromagnetics. More precisely, we study the asymptotics of family of functionals

$$F_\varepsilon(m_\varepsilon) = \frac{1}{2\varepsilon} \int_\Omega m_{3,\varepsilon}^2 + \frac{\varepsilon}{2} \int_\Omega |\nabla m_\varepsilon|^2 \tag{6}$$

for $m_\varepsilon = (m'_\varepsilon, m_{3,\varepsilon}) \in H^1(\Omega, \mathbf{R}^3)$ (with $m'_\varepsilon = (m_{1,\varepsilon}, m_{2,\varepsilon})$) subject to the constraints $|m_\varepsilon| = 1$ in $\Omega \subset \mathbf{R}^2$, $\nabla \cdot m'_\varepsilon = 0$ in Ω and $m'_\varepsilon \cdot n = 0$ on $\partial\Omega$. In micromagnetics, the vanishing divergence constraint on m_ε (called magnetization) corresponds to vanishing magnetic charge density due to principle of pole avoidance (in analogy with electrostatics) while the first term in (6) stands for the crystalline anisotropy which is a function of m favoring certain easy axis (the planar directions in our case). Notice that F_ε is slightly different from the functional G_ε in (2) with $g(t) = t$ (corresponding to $f(t) = t^2/2$ by (4)): indeed, the first term coincides for both functionals (since $m_{3,\varepsilon}^2 = 1 - |m'_\varepsilon|^2$), while the second term in F_ε controls the one in G_ε since $|\nabla m_\varepsilon| \geq |\nabla m'_\varepsilon|$. The natural question is whether the Γ-limit of $\{F_\varepsilon\}_{\varepsilon\to 0}$ (or $\{G_\varepsilon\}_{\varepsilon \to 0}$) coincides with $\mathcal{I}_f$ which represents the energetic cost of $1D$ transitions layers (called Bloch walls). In [17], we establish partial results in this direction, in particular we proved that there exists a universal constant $C > 1$ such that

$$\mathcal{I}_f(m) \leq \; C \liminf_{\varepsilon \downarrow 0} F_\varepsilon(m_\varepsilon),$$

for any family $m_\varepsilon \in H^1(\Omega, S^2)$ satisfying $\nabla \cdot [(m_{\varepsilon,1}, m_{\varepsilon,2})\mathbf{1}_\Omega] = 0$ and converging in $L^1(\Omega)$ to some $m \in \mathcal{S}_0(\Omega)$. Unfortunately, we could not rule out the existence of lower energy $2D$ structures by proving that this inequality also holds with $C = 1$. However, the following result supports the conjecture that $\mathcal{I}_f$ is the limiting energy of the family F_ε (and G_ε) as $\varepsilon \to 0$. Indeed, Theorem 2 implies that it is not possible to significantly decrease the energy by substituting for a 1D transition layer a 2D mesoscopic structure obtained by assembling together 1D transition layers. (This does not rule out the possibility of 2D microscopic structures at scale ε or below inside the transition layers).

Theorem 2 *If $f(t) = t^2$, then $\mathcal{I}_f$ is lower semicontinuous.*

In fact the quadratic cost function stated in Theorem 2 is a particular case of a large family of cost functions that we will introduce in the following and which induce lower semicontinuous line-energies.

1.4 Entropies and cost functions

In our context, the notion of entropy has been introduced in [12] and we will use it extensively below. In [6] Aviles and Giga establish a representation formula for $\mathcal{I}_{t \mapsto t^3}$ for proving its lower semicontinuity, their construction is based on a single particular entropy. Here we will prove Theorem 2 by generalizing their method to any (possibly large) set of entropies. In this sense, the present paper is the opportunity to investigate further the relations between entropies and line-energies.

The starting point consists in regarding the set of constraints (3) satisfied by our configurations as a scalar conservation law. Indeed, writing $m_0 = (u, h(u))$ for the flux $h(u) = \pm\sqrt{1-u^2}$, the divergence-free condition on m_0 turns into

$$\partial_t u + \partial_s h(u) = 0, \tag{7}$$

where $(t, s) := (x_1, x_2)$ correspond to (time, space) variables. The notion of entropy as introduced in [12] and defined below corresponds to entropy-entropy flux pairs for this scalar conservation law.

Definition 2 ([12]) *We will say that $\Phi \in C^\infty(S^1, \mathbf{R}^2)$ is an entropy if for every $z = e^{i\theta} = (\cos\theta, \sin\theta) \in S^1$, we have*

$$\frac{d}{d\theta}\Phi(z) \cdot z \;=\; 0, \tag{8}$$

where $\frac{d}{d\theta}\Phi(z)$ stands for the angular derivative $\frac{d}{d\theta}[\Phi(e^{i\theta})]$. The set of entropies is denoted by ENT.

This notion is consistent with the property that a smooth vector field m satisfying (3) induces vanishing entropy production $\nabla \cdot [\Phi(m)] = 0$. As expected, for $m \in \mathcal{S}(\Omega)$, the *entropy production*

$$\mu_\Phi(m) := \nabla \cdot [\Phi(m)]$$

is a measure supported on the jump set of m.

Proposition 2 *Let $\Phi \in ENT$ be an entropy and $m \in \mathcal{S}(\Omega)$. Then we have*

$$\mu_\Phi(m) \;=\; \big\{\Phi(m^+) - \Phi(m^-)\big\} \cdot \nu\, \mathcal{H}^1 \llcorner J(m), \tag{9}$$

where $J(m)$ is the $\mathcal{H}^1$-rectifiable jump set of m oriented by ν and $m^\pm$ are the traces of m on $J(m)$.

The main contribution of the paper is to associate an appropriate cost function to every subset of entropies:

Definition 3 *For a subset $S \subset ENT$, we define the cost function $c_S : [0,2] \to \mathbf{R}_+$ by*

$$c_S(t) \;:=\; \sup\big\{\big[\Phi(z^+) - \Phi(z^-)\big] \cdot \nu \;:\; \Phi \in S,\; (z^-, z^+, \nu) \in \mathcal{T},\; |z^+ - z^-| = t\big\},$$

where $\mathcal{T}$ defines the set of admissible jump discontinuities:

$$\mathcal{T} \;:=\; \big\{(z^-, z^+, \nu) \in (S^1)^3 \;:\; (z^+ - z^-) \cdot \nu = 0\big\}.$$

Remark 2 The set $\mathcal{T}$ is motivated by the structure of jump discontinuities of divergence-free vector fields $m \in \mathcal{S}(\Omega)$. Indeed, one has $(m^+ - m^-) \cdot \nu = 0$ $\mathcal{H}^1$-a.e. on the jump set $J(m)$ oriented by the normal ν. The cost function c_S is nonnegative since one can switch ν to $-\nu$ so that $[\Phi(z^+) - \Phi(z^-)] \cdot \nu \geq 0$.

Observe that these cost functions depend only on the jump size. To be consistent with this isotropy, we will impose the following geometric constraints on our sets of entropies.

Definition 4 *A subset $S \subset ENT$ is symmetric if $S = -S$ and it is said to be equivariant if $R^{-1}SR = S$ for every rotation $R \in \mathrm{SO}(2)$. For any subset of entropies $S \subset ENT$, we will denote*

$$\langle S \rangle \;:=\; \left\{ \pm R^{-1}\Phi R \,:\, \Phi \in S,\, R \in \mathrm{SO}(2) \right\}$$

the smallest symmetric and equivariant subset of entropies which contains S.

For proving that $\mathcal{I}_{c_S}$ is lower semicontinuous for nonempty symmetric equivariant subsets $S \subset ENT$, we introduce the following functionals (inspired by (9)) which generalize Theorem 2.1 in [6].

Definition 5 *Let $S \subset ENT$. We define $\mathcal{E}_S : L^1(\Omega, \mathbf{R}^2) \to \bar{\mathbf{R}}$ by*

$$\mathcal{E}_S(m) \;:=\; \sup \left\{ \sum_{i=1}^{n} \langle \mu_{\Phi_i}(m), \alpha_i \rangle \;:\; n \geq 0,\; (\Phi_i, \alpha_i) \subset S \times \mathcal{D}(\Omega, \mathbf{R}_+),\; \sum_{i=1}^{n} \alpha_i \leq 1 \right\} \text{ if } m \in \mathcal{L}(\Omega);$$

otherwise, we set $\mathcal{E}_S(m) = +\infty$ for $m \in L^1(\Omega, \mathbf{R}^2) \setminus \mathcal{L}(\Omega)$.

As a supremum of continuous functionals over L^1, this new energy is lower semicontinuous with respect to the strong L^1 topology. In the above definition we use a partition of unity to localize the entropy production. In particular, in the neighborhood of a jump discontinuity x, we can choose a sequence of entropies maximizing the local entropy production as in the definition of $c_S(|m^+(x) - m^-(x)|)$. Using this property, we will prove that $\mathcal{E}_S$ coincides with $\mathcal{I}_{c_S}$ on $\mathcal{S}(\Omega)$:

Theorem 3 *Let $S \subset ENT$ be nonempty, symmetric and equivariant. For every $m \in \mathcal{S}(\Omega)$, we have*

$$\mathcal{E}_S(m) \;=\; \mathcal{I}_{c_S}(m) \;=\; \overline{\mathcal{I}_{c_S}}(m).$$

In particular, $\mathcal{I}_{c_S}$ is lower semicontinuous and $\mathcal{E}_S \leq \overline{\mathcal{I}_{c_S}}$ in $L^1(\Omega, \mathbf{R}^2)$.

We deduce that the class of cost functions in Definition 3 leads to lower semicontinuous line-energy functionals. Finally, for proving Theorem 2 we will construct a subset $S \subset ENT$ so that $f = c_S$. As expected, this method is also valid for the cost functions mentioned in Examples 1 and 2 below, corresponding to the Aviles-Giga and "cross-tie wall" models.

Example 1 (Aviles-Giga cost function) There exists a subset $S_1 \subset ENT$ generated by one entropy Φ_1 (i.e., $S_1 = \langle \{\Phi_1\} \rangle$) such that $c_{S_1}(t) = t^3$ for $t \in [0, 2]$.

Example 2 ("Cross-tie wall" cost function) There exists a subset $S_2 \subset ENT$ generated by one entropy $\Phi_2 \in C^{1,1}(S^1, \mathbf{R}^2)$ such that

$$c_{S_2}(2\sin\theta) = \begin{cases} \sin\theta - \theta\cos\theta & \text{if } 0 \leq \theta \leq \pi/4, \\ \sqrt{2} - \left(\dfrac{\pi}{2} - \theta\right)\cos\theta - \sin\theta & \text{if } \pi/4 < \theta \leq \pi/2. \end{cases}$$

For these examples, the corresponding entropies have been introduced in [20] and [1] respectively. Obviously, not all appropriate cost functions can be associated to subsets of entropies generated by only one entropy. For example, if $c_S(t) = t^2$ for every $t \in [0, 2]$, we are compelled to construct a subset S generated by an infinite family of entropies (see the proof of Theorem 2).

Conjecture 2 Is it true that every lower semicontinuous line-energy $\mathcal{I}_f$ has the form $\mathcal{I}_{c_S}$ for some subset of entropies $S \subset ENT$?

Remark 3 One can address problem (1) for higher dimensions $N \geq 3$. In this case, DeLellis proved in [10] that the power function $f(t) = t^3$ (in the Aviles-Giga model) does not lead anymore to a lower semicontinuous hypersurface-energy as in the two-dimensional case. The microscopic structure breaking the one-dimensional ansatz considered in [10] can be adapted to other power functions $f(t) = t^p$. We highlight the fact that our approach for treating lower semicontinuous line-energies via entropy method cannot be extended to hypersurface-energy functionals if $N \geq 3$. Indeed, for $N = 3$, standard computations show that the only entropies associated to the system of conservation laws generated by

$$u : \Omega \subset \mathbf{R}^3 \to \mathbf{R}^3, \quad |u| = 1 \quad \text{and} \quad \nabla \times u = 0 \quad \text{in } \Omega$$

are the trivial entropies.

1.5 Existence of minimizers for the relaxed energy $\overline{\mathcal{I}_f}$

Now we deal with a second issue: the existence of minimizers of the relaxed energy functional $\overline{\mathcal{I}_f}$ under certain boundary conditions. (Without imposed boundary conditions, the problem is trivial, $\overline{\mathcal{I}_f}$ has vanishing minimal value and every constant unit vector field is a minimizer.) Firstly, we impose the following boundary condition $m \cdot n = 0$ on $\partial\Omega$ to our configurations m, so we are looking for minimizers in $\mathcal{L}_0(\Omega)$ (see (5)).

Suppose that the cost function f is equal to c_S for some subset $S \subset ENT$. Then the relative compactness in L^1 of the sublevel sets of $\overline{\mathcal{I}_f}$ would imply the existence of minimizers of the relaxed functional $\overline{\mathcal{I}_f}$ in $\mathcal{L}_0(\Omega)$. For that, one should be able to rule out oscillations for configurations of uniformly bounded energy. It turns out that it is possible if the set,

$$S_f := \left\{ \Phi \in ENT \ : \ [\Phi(z^+) - \Phi(z^-)] \cdot \nu \leq f(|z^+ - z^-|), \ \forall (z^-, z^+, \nu) \in \mathcal{T} \right\},$$

composed of the admissible entropies associated with f, is large enough. More precisely, we will obtain compactness if $t^3 \lesssim f(t)$ in $[0, 2]$ (see Theorem 4 below).

Remark 4 Observe that S_f is symmetric and equivariant. Moreover,

$$c_{S_f} \leq f \quad \text{in} \quad [0, 2] \tag{10}$$

and S_f is the maximal subset of ENT such that inequality (10) holds.

Theorem 4 *Let f be a cost function such that $\inf_{t \in (0,2]} \frac{f(t)}{t^3} > 0$ and $c_{S_f} = f$. Then $\overline{\mathcal{I}_f}$ (respectively, $\mathcal{E}_{S_f}$) admits at least one minimizer over $\mathcal{L}_0(\Omega)$.*

It means that a minimizer $m \in \mathcal{L}_0(\Omega)$ of $\overline{\mathcal{I}_f}$ can be written as a limit of a sequence $\{m_k\}$ in $\mathcal{S}_0(\Omega)$ such that $\overline{\mathcal{I}_f}(m) = \lim_k \mathcal{I}_f(m_k)$. However, we do not know whether the minimizer m belongs to $\mathcal{S}_0(\Omega)$.

Remark 5 The existence result in Theorem 4 is still valid if we replace $\mathcal{L}_0(\Omega)$ by any closed subset of $\mathcal{L}(\Omega)$. But this does not cover the case of general Dirichlet boundary conditions. However, the following strategy can be adopted for Dirichlet boundary condition $m = u_{bd}$ on $\partial\Omega$. If we can extend $u_{bd} : \partial\Omega \to S^1$ as a divergence-free vector field $u \in BV(O, S^1)$ for some smooth open set $O \supset \bar{\Omega}$, then the argument in Theorem 4 shows the existence of minimizers of the functional $F(m) := \mathcal{I}_f(m; O) - \mathcal{I}_f(u; O \setminus \bar{\Omega})$ in the closed set

$$\{m \in \mathcal{L}(O) \ : \ m \equiv u \text{ a.e. in } O \setminus \bar{\Omega}\}.$$

Observe that finite energy configurations $F(m) < \infty$ satisfy $m \in BV(O, S^1)$, $m \cdot n = u_{bd} \cdot n \, \mathcal{H}^1$−a.e. on $\partial\Omega$ (since m is of vanishing divergence), the jump of the tangential component $[m \cdot n^\perp]$ on $\partial\Omega$ is penalized through $F(m)$ by the boundary term:

$$\int_{\partial\Omega} f(|m^+ - m^-|) \, d\mathcal{H}^1,$$

where $m^\pm$ denote the inner and outer traces of m on $\partial\Omega$ with respect to n (here, $m^+ = u_{bd}$ on $\partial\Omega$). The minimizing problem does not depend on the extended domain O or on the extension vector field u.

1.6 Viscosity solution

We are also interested in the minimization problem under the more restrictive boundary condition $m = n^\perp$ on $\partial\Omega$. This condition makes sense for $m \in BV$ and defines a new subset of $\mathcal{S}_0(\Omega)$,

$$\mathcal{S}_\perp(\Omega) \ := \ \left\{m \in \mathcal{S}_0(\Omega) \ : \ m = n^\perp \text{ on } \partial\Omega\right\}.$$

For configurations in this set, no change of orientation is allowed along the boundary. The motivation comes from micromagnetics where the boundary vortices are strongly penalized in certain asymptotic regimes (see [16, 18]).

The natural question in this context is whether the minimizer of $\mathcal{I}_f$ over $\mathcal{S}_\perp(\Omega)$ exists and is associated to the viscosity solution of the Dirichlet problem for the eikonal equation, i.e., letting ψ_0 be the distance function to the boundary

$$\psi_0 = \text{dist}\,(x, \partial\Omega),$$

we will always denote the corresponding map in $\mathcal{S}_\perp(\Omega)$ by

$$m_0 = \nabla^\perp \psi_0.$$

We will still call m_0 the *viscosity solution* on Ω. In relation with (1), this amounts to considering stream functions ψ satisfying $m = \nabla^\perp\psi \in BV(\Omega, S^1)$ and the boundary conditions $\psi = 0$ and $\frac{\partial\psi}{\partial n} = -1 \ \mathcal{H}^1$−a.e on $\partial\Omega$. The viscosity solution m_0 is known in the micromagnetic jargon under the name of Landau state.

It is conjectured that if the domain Ω is convex, then the viscosity solution minimizes $\mathcal{I}_f$ in $\mathcal{S}_\perp(\Omega)$ for $f(t) = t^p$, $1 \le p \le 3$. The result is proved for $p = 3$ when Ω is an ellipse in [20]. For $p = 1$ and if Ω a convex polygon, it is proved in [5] that m_0 minimizes $\mathcal{I}_f$ over the set $\{m \in \mathcal{S}_\perp(\Omega) \ : \ \nabla m \text{ is piecewise constant}\}$. We first give a positive answer in the case of a stadium for general appropriate cost functions:

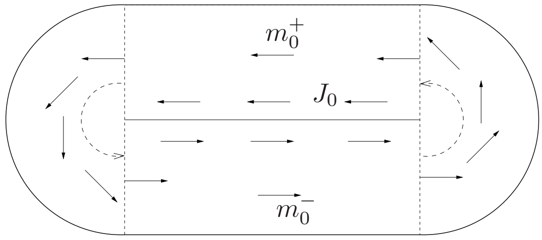

Figure 3: Stadium shaped domain and the corresponding viscosity solution.

Theorem 5 *Let $S \subset ENT$ be nonempty, symmetric and equivariant. We consider the stadium-shaped domain Ω (see Figure 3)*

$$\Omega = (-L, L) \times (-1, 1) \cup B((-L, 0), 1) \cup B((L, 0), 1),$$

for some $L \geq 0$. Then the viscosity solution m_0 minimizes $\mathcal{I}_{c_S}$ over $\mathcal{S}_\perp(\Omega)$.

We also prove positive results for some other special domains non necessarily convex (in particular, ellipse and union of two discs) in the case of some particular appropriate cost functions (see Proposition 18 and Corollary 2).

For nonconvex domains, it is proved in [5] that for power cost functions $f(t) = t^p$ with $p \leq 4/3$, there exists a nonconvex polygonal domain Ω such that m_0 does not minimize $\mathcal{I}_f$ over $\mathcal{S}_\perp(\Omega)$. Moreover, the same counterexamples indicate that for every power cost function with $p > 0$, m_0 does not minimize $\mathcal{I}_f$ in $\mathcal{S}_0(\Omega)$. In [20], the authors exhibit a nonconvex Lipschitz domain (a union of two intersecting discs) such that m_0 is not a minimizer of $\mathcal{I}_f$ in $\mathcal{S}_\perp(\Omega)$ for every $f(t) = t^p$ with $p \neq 3$; in the case $f(t) = t^3$, m_0 is a minimizer of $\mathcal{I}_f$, but it is not unique. It was conjectured that for some other nonconvex domains, m_0 is not a minimizer of $\mathcal{I}_{t\mapsto t^3}$. In the following, we prove this conjecture. In fact, we show a more general fact: there exists a nonconvex domain such that *for any* fixed positive cost function f, the viscosity solution is not optimal in $\mathcal{S}_\perp(\Omega)$.

Theorem 6 *There exists a nonconvex piecewise Lipschitz domain Ω such that the viscosity solution is not a minimizer of $\mathcal{I}_f$ over $\mathcal{S}_\perp(\Omega)$ for every lower semicontinuous function $f : [0, 2] \to \mathbf{R}_+$ such that $\int_{\sqrt{2}}^2 f(t)\, dt > 0$.*

The above domain Ω (a union of four disks and a square) is non smooth, but universal for every positive cost function f. Moreover, by slightly modifying the boundary of Ω, we can show that the result is not restricted to nonsmooth domains. Still, the modified smooth domain is no longer universal with respect to the cost function.

Theorem 7 *For every bounded lower semicontinuous function $f : [0, 2] \to \mathbf{R}_+$ such that $\int_{\sqrt{2}}^2 f(t)\, dt > 0$, there exists a nonconvex $C^{1,1}$ domain Ω such that the viscosity solution is not a minimizer of $\mathcal{I}_f$ over $\mathcal{S}_\perp(\Omega)$.*

The outline of the paper is as follows: In Section 2 we describe some properties of entropies and lower semicontinuous line-energies and prove Theorem 3. The compactness issue is addressed in

Section 3 where we prove Theorem 4. In Section 4 we introduce some tools for computing the cost functions c_S when the set S is generated by a single entropy and extend our methods to the case of *non-smooth* entropies. These situations cover Examples 1 and 2. The Conjecture 1 is addressed in Section 5 where we focus on the line-energy $\mathcal{I}_{t\mapsto t^2}$ and prove Theorem 2 and in Section 6 where we prove Theorem 1. Finally, in Section 7, we first present some situations where the viscosity solution is a minimizer, in particular we prove Theorem 5 . Then we exhibit the counterexamples leading to Theorems 6 and 7.

2 Properties of entropies and semicontinuous line-energies

We begin by presenting some properties of entropies as introduced in [12]. An alternative and equivalent definition for Definition 2 is given by the following property:

Proposition 3 *Let $\Phi \in C^\infty(S^1, \mathbf{R}^2)$. Then $\Phi \in ENT$ is an entropy if and only if for every $m \in C^\infty(\Omega, S^1)$ and $x \in \Omega$,*

$$\nabla \cdot m(x) = 0 \implies \nabla \cdot [\Phi(m)](x) = 0. \tag{11}$$

Proof. Assume that $\Phi \in ENT$. Let $m \in C^\infty(\Omega, S^1)$ and $x \in \Omega$. Then m has a smooth lifting Θ on a small ball $B \subset \Omega$ centered in x, i.e., $m = e^{i\Theta}$ in B. If $\nabla \cdot m(x) = 0$, then $\nabla\Theta(x) \cdot m^\perp(x) = 0$ which implies that $\nabla\Theta(x)$ and $m(x)$ are collinear, i.e., there exists a real constant $\kappa(x)$ such that $\nabla\Theta(x) = \kappa(x)m(x)$. Therefore,

$$\nabla \cdot [\Phi(m)](x) = \nabla \cdot [\Phi(e^{i\Theta})](x) = \frac{d}{d\theta}\Phi(m(x)) \cdot \nabla\Theta(x) = \kappa(x)\frac{d}{d\theta}\Phi(m(x)) \cdot m(x) \overset{(8)}{=} 0 \tag{12}$$

and the property (11) follows.

Conversely, let $\Phi \in C^\infty(S^1, \mathbf{R}^2)$ such that (11) holds. Set $z \in S^1$. W.l.o.g. we may assume that Ω contains the origin 0 and moreover, that the ball $B(0, \frac{1}{2}) \subset \Omega$. Motivated by (12), we want to construct a map $m \in C^\infty(\Omega, S^1)$ such that $m(0) = z$ and $\nabla \cdot m(0) = 0$ so that z and $\nabla\Theta(0) \neq 0$ are collinear. The idea is to consider m as the vortex map centered at $z^\perp$ defined on $B(0, \frac{1}{2})$ by

$$m(x) \;:=\; \left(\frac{x - z^\perp}{|x - z^\perp|}\right)^\perp \qquad \text{in } B(0, \frac{1}{2})$$

and m to be smooth in $\Omega \setminus B(0, \frac{1}{2})$. Then one computes $\nabla\Theta(0) = z$. Since $\nabla \cdot m(0) = 0$, (11) implies $\nabla \cdot [\Phi(m)](0) = 0$ and therefore, as in (12), we obtain

$$\frac{d}{d\theta}\Phi(z) \cdot z = \frac{d}{d\theta}\Phi(m(0)) \cdot \nabla\Theta(0) = \nabla \cdot [\Phi(m)](0) = 0.$$

$\square$

In the next Sections, we will frequently need to compute explicitly entropy productions. For this we will always use a second characterization given in Proposition 4 (see Lemma 2.4 in [12]) jointly with the convolution formula of Proposition 5 below.

Proposition 4 *Let $\Phi \in C^\infty(S^1, \mathbf{R}^2)$. Then $\Phi \in ENT$ is an entropy if and only if there exists a (unique) $2\pi-$periodic $\varphi \in C^\infty(\mathbf{R})$ such that for every $z = e^{i\theta} \in S^1$,*

$$\Phi(z) = \varphi(\theta) z + \frac{d\varphi}{d\theta}(\theta) z^\perp. \tag{13}$$

In this case,

$$\frac{d}{d\theta}\Phi(z) = \lambda(\theta) z^\perp, \tag{14}$$

where $\lambda \in C^\infty(\mathbf{R})$ is the $2\pi-$periodic function defined by $\lambda = -\Lambda\varphi := \varphi + \frac{d^2}{d\theta^2}\varphi$ in $\mathbf{R}$.

Remark 6 There exists a unique extension of $\Lambda : C^\infty(S^1 \approx \mathbf{R}/2\pi\mathbb{Z}) \to C^\infty(S^1)$ as a linear nonbounded operator $\Lambda : L^2(S^1) \to L^2(S^1)$ with the domain $D(\Lambda) = H^2(S^1)$. Moreover, the kernel of Λ is given by $\ker(\Lambda) = \mathbf{R}\sin \oplus \mathbf{R}\cos$, the spectrum of Λ is $\sigma(\Lambda) = \{k^2 - 1 : k \in \mathbb{N}^*\}$ and the range of Λ is $R(\Lambda) = \ker(\Lambda)^\perp$. Consequently, for every $\lambda \in \ker(\Lambda)^\perp$, there exists a unique $\varphi \in L^2(S^1)/_{\ker(\Lambda)}$ such that $-\Lambda\varphi = \lambda$ and the corresponding entropy Φ given by (13) is uniquely defined by λ up to a constant.

Proof. For $z = e^{i\theta} \in S^1$, we decompose $\Phi(z)$ in the orthonormal basis $(z, z^\perp)$ as follows:

$$\Phi(z) = \varphi(\theta) z + \psi(\theta) z^\perp$$

with $\varphi(\theta) = \Phi(z) \cdot z$ and $\psi(\theta) = \Phi(z) \cdot z^\perp$. Obviously, $\varphi, \psi \in C^\infty(\mathbf{R})$ are $2\pi-$periodic functions. Differentiating the above equality, we obtain

$$\frac{d}{d\theta}\Phi(z) = \left(\frac{d\varphi}{d\theta}(\theta) - \psi(\theta)\right) z + \left(\frac{d\psi}{d\theta}(\theta) + \varphi(\theta)\right) z^\perp.$$

By (8), one concludes that the condition $\Phi \in ENT$ is equivalent to asking that the coordinate in the direction z vanishes in the above expression, i.e., $\frac{d\varphi}{d\theta} = \psi$ in $\mathbf{R}$. As a consequence, we obtain $\frac{d}{d\theta}\Phi(z) = \lambda z^\perp$ with $\lambda = \varphi + \frac{d^2}{d\theta^2}\varphi$. □

Proposition 5 *Using the notations of Proposition 4, for $(z^+, z^-, \nu) \in \mathcal{T}$ with $\nu = e^{ix}$, $x \in \mathbf{R}$ and $z^\pm = e^{i(x \pm \beta)}$ with $\beta \in [-\pi, \pi]$, we have the convolution formula*

$$[\Phi(z^+) - \Phi(z^-)] \cdot \nu \ = \ (\lambda \star \sin_\beta)(x), \qquad x \in \mathbf{R}, \tag{15}$$

where we write

$$\sin_\beta(x) \ = \ \begin{cases} \operatorname{sgn}(\beta)\sin x & \text{for } |x| \le |\beta|, \\ 0 & \text{for } |x| > |\beta|. \end{cases}$$

Proof. Let $\Phi \in ENT$ and let $z^\pm$ and ν be as in the hypotheses. Using the Notations of Proposition 4, we compute

$$[\Phi(z^+) - \Phi(z^-)] \cdot \nu \ = \int_{x-\beta}^{x+\beta} \frac{d}{d\theta}\Phi(e^{i\theta}) \cdot e^{ix}\, d\theta \ = \int_{x-\beta}^{x+\beta} \lambda(\theta)\sin(x - \theta)\, d\theta,$$

which yields (15). □

Remark 7 For further simplifications, we record here some symmetries. First we have the obvious identity,

$$\lambda \star \sin_\beta \;=\; -\lambda \star \sin_{-\beta}\,. \tag{16}$$

Next, the entropy production is invariant under a change in the direction of the orientation $(z^+, z^-, \nu) \mapsto (z^-, z^+, -\nu)$. Plugging this change of orientation in (15), we obtain, in the case $\beta \in [0, \pi]$,

$$\lambda \star \sin_\beta \;=\; (\lambda \star \sin_{\pi-\beta})(\pi + \cdot). \tag{17}$$

Notice that in fact, this identity is equivalent to $\lambda \star \sin \equiv 0$ (here, $\sin = \sin_\pi$) which is a consequence of $\lambda \in (\ker \Lambda)^\perp$.

For $m \in L^1(\Omega, S^1)$ and $\Phi \in \, ENT$, we define the entropy production

$$\mu_\Phi(m) \;:=\; \nabla \cdot [\Phi(m)] \;\in \mathcal{D}'(\Omega).$$

Proposition 3 suggests that the entropies are a useful tool for detecting jump discontinuities of vanishing divergence maps with values in S^1. Indeed, for such BV-vector fields m, it is stated in Proposition 2 that the entropy production concentrates on the jump set of m.

Proof of Proposition 2. For the BV-map m, the gradient measure ∇m splits into two terms

$$\nabla m = \nabla_d m + (m^+ - m^-) \otimes \nu \mathcal{H}^1 \llcorner J(m)$$

where the diffuse measure $\nabla_d m = (\partial_{j,d} m_i)$ is a $\mathcal{M}_2(\mathbf{R})$ valued measure that does not charge $\mathcal{H}^1$-rectifiable sets and we denote the traces of m by $m^\pm(x) = \lim_{\varepsilon \downarrow 0} m(x \pm \varepsilon \nu(x))$ in $L^1_{loc}(J(m), S^1)$. It is known that there exists a BV-lifting Θ of m, i.e., $m = e^{i\Theta}$ a.e. in Ω (see e.g. [9], [15] or [21]). Since $\nabla \cdot m = 0$, one has $\nabla_d \cdot m = 0$ and the chain rule applied to $m = e^{i\Theta}$ leads to

$$m^\perp \cdot \nabla_d \Theta = 0 \quad |\nabla_d m| - \text{a.e. in } \; \Omega,$$

where $\nabla_d \Theta$ stands for the diffuse part of the measure $\nabla \Theta$ (throughout the paper, we always identify m and Θ by their precise representative that is defined $\mathcal{H}^1$-a.e. on $\Omega \setminus J(m)$). Therefore, there exists a diffuse real measure $\kappa << |\nabla_d m|$ such that $\nabla_d \Theta = m\,\kappa$. Applying now the chain rule to $\Phi(e^{i(\cdot)}) \circ \Theta(\cdot)$, we deduce

$$\begin{aligned} \mu_\Phi(m) \;=\; \nabla \cdot [\Phi(e^{i\Theta})] &= \frac{d}{d\theta}\Phi(m) \cdot \nabla_d \Theta + [\Phi(m^+) - \Phi(m^-)] \cdot \nu\, \mathcal{H}^1 \llcorner J(m) \\ &= \frac{d}{d\theta}\Phi(m) \cdot m\,\kappa + [\Phi(m^+) - \Phi(m^-)] \cdot \nu\, \mathcal{H}^1 \llcorner J(m) \end{aligned}$$

and by (8), the conclusion is straightforward. □

Formula (9) inspires a new way for defining line-energies as in Definition 5. The new energies $\mathcal{E}_S$ associated to a set of entropies S are lower semicontinuous in L^1 and there are useful for proving that the associated line-energies $\mathcal{I}_{c_S}$ are lower semicontinuous (after Definition 1).

Proposition 6 *For every subset $S \subset ENT$, the energy $\mathcal{E}_S : L^1(\Omega, \mathbf{R}^2) \to \bar{\mathbf{R}}$ is lower semicontinuous with respect to the strong L^1-topology.*

Proof. For every $n \geq 0$ and $(\Phi_i, \alpha_i) \subset S \times \mathcal{D}(\Omega, \mathbf{R}_+)$ such that $\sum_{i=1}^n \alpha_i \leq 1$, the functional

$$m \mapsto \sum_{i=1}^n \langle \mu_{\Phi_i}(m), \alpha_i \rangle = -\sum_{i=1}^n \int_\Omega \Phi_i(m) \cdot \nabla \alpha_i \, dx$$

is continuous over the closed subset $\mathcal{L}(\Omega) \subset L^1(\Omega, \mathbf{R}^2)$ endowed with the strong L^1-topology. Extended by $+\infty$ on $L^1(\Omega, \mathbf{R}^2) \setminus \mathcal{L}(\Omega)$, it becomes a lower semicontinuous functional $L^1(\Omega, \mathbf{R}^2) \to \bar{\mathbf{R}}$. Since a supremum of such functionals is also lower semicontinuous, we get the conclusion. $\square$

For every $S \subset ENT$, we associate a cost function c_S as in Definition 3. These cost functions have the following properties:

Proposition 7 *For every $S \subset ENT$, the cost function $c_S : [0, 2] \to \mathbf{R}$ is a nonnegative lower semicontinuous function with $c_S(0) = 0$.*

Proof. The first property is trivial since one can always replace ν by $-\nu$ so that $[\Phi(z^+) - \Phi(z^-)] \cdot \nu \geq 0$. For the second property, one can write

$$c_S = \sup_{\Phi \in S} c_{\{\Phi\}} \quad \text{with} \quad c_{\{\Phi\}}(t) = \sup \left\{ \left[\Phi(z^+) - \Phi(z^-) \right] \cdot \nu \, : \, (z^-, z^+, \nu) \in \mathcal{T}, \ |z^+ - z^-| = t \right\}.$$

Since the $c_{\{\Phi\}} : [0, 2] \to \mathbf{R}$ are continuous, their supremum c_S is lower semicontinuous. $\square$

The lower semicontinuity of a cost function is a necessary condition so that the associated line-energy functional is l.s.c. as stated in Proposition 1:

Proof of Proposition 1. Let $x_0 \in \Omega$ and $\{\Omega^-, \Omega^+\}$ be the open partition of Ω after cutting Ω by the vertical line passing through x_0. For $t \in [0, 1]$, we define the piecewise constant vector field $m_t \in \mathcal{S}(\Omega)$ by $m_t(x) = (\sqrt{1 - t^2}, \pm t)$ for $x \in \Omega^\pm$. The map $t \mapsto m_t$ is continuous from $[0, 1]$ into $\mathcal{S}(\Omega)$ endowed with strong L^1-topology and since

$$\mathcal{I}_f(m_t) = \int_{\overline{\Omega}^- \cap \overline{\Omega}^+} f(|m_t^+ - m_t^-|) \, d\mathcal{H}^1 = \mathcal{H}^1 \left(\overline{\Omega}^- \cap \overline{\Omega}^+ \right) f(2t),$$

the lower semicontinuity of f follows from the lower semicontinuity of $\mathcal{I}_f$. $\square$

We now prove Theorem 3: the energy $\mathcal{E}_S$ is an extension of a line-energy functional of type $\mathcal{I}_{c_S}$ for the cost function c_S. Observe that we assume the equivariance of S because we are interested in cost functions c_S depending only on the size of the jump $t = |z^+ - z^-|$. Without these restrictions, the definition

$$\tilde{c}_S(z^-, z^+, \nu) := \sup_{\Phi \in S} \left[\Phi(z^+) - \Phi(z^-) \right] \cdot \nu, \quad (z^-, z^+, \nu) \in \mathcal{T} \tag{18}$$

would lead to possibly nonisotropic cost functions $\tilde{c}_S$. However, $\tilde{c}_S : \mathcal{T} \to \mathbf{R}$ remains lower semicontinuous on the compact set $\mathcal{T} \subset (S^1)^3$ and the symmetry of S still implies the nonnegativity of $\tilde{c}_S$.

Proof of Theorem 3. We first show that $\mathcal{E}_S(m) = \mathcal{I}_{c_S}(m)$ for $m \in \mathcal{S}(\Omega)$. The proof will proceed in several steps.

Step 1. $\mathcal{E}_S(m) \le \mathcal{I}_{c_S}(m)$. Let $n \ge 0$ and $\{(\Phi_i, \alpha_i) \subset S \times \mathcal{D}(\Omega, \mathbf{R}_+)\}_{1 \le i \le n}$ such that $\sum_{i=1}^n \alpha_i \le 1$. By (9), one has:

$$\sum_{i=1}^n \langle \mu_{\Phi_i}(m), \alpha_i \rangle = \sum_{i=1}^n \int_{J(m)} \alpha_i \left[\Phi_i(m^+) - \Phi_i(m^-) \right] \cdot \nu \, d\mathcal{H}^1$$
$$\le \int_{J(m)} c_S(|m^+ - m^-|) \left(\sum_{i=1}^n \alpha_i \right) d\mathcal{H}^1 \le \int_{J(m)} c_S(|m^+ - m^-|) \, d\mathcal{H}^1.$$

Therefore,

$$\mathcal{E}_S(m) \ \le \ \mathcal{I}_{c_S}(m).$$

Step 2. The case of a finite set $S \subset ENT$. For proving the converse inequality, we first assume that $S = \{\Phi_1, \cdots, \Phi_p\}$ is a finite and symmetric subset of ENT (so, for this issue, S is not equivariant); we will obtain the general case by using a density argument. For every triplet $T = (z^-, z^+, \nu) \in \mathcal{T}$, we set as in (18):

$$\tilde{c}_S(T) = \max \left\{ \left[\Phi_i(z^+) - \Phi_i(z^-) \right] \cdot \nu \, : \, i = 1, \ldots, p \right\} \ge 0.$$

Observe that the cost function $\tilde{c}_S$ is defined for each triplet T while c_S depends only on the jump size $|z^+ - z^-|$. We want to prove that

$$\int_{J(m)} \tilde{c}_S(m^-(x), m^+(x), \nu(x)) \, d\mathcal{H}^1(x) \le \mathcal{E}_S(m). \tag{19}$$

For that, we denote for each $T = (z^-, z^+, \nu) \in \mathcal{T}$,

$$i_T := \min \left\{ i \in \{1, \cdots, p\} \ : \ [\Phi_i(z^+) - \Phi_i(z^-)] \cdot \nu = \tilde{c}_S(T) \right\}$$

the smallest index for which the corresponding entropy reaches the maximum cost. In particular, it induces a disjoint measurable covering of $\mathcal{T}$ given by the sets of triplets

$$s_k := \{ T \in \mathcal{T} \ : \ i_T = k \} \quad \text{for} \quad k = 1, \ldots, p.$$

Notice that this cost function $\tilde{c}_S : \mathcal{T} \to \mathbf{R}$ is continuous by equi-continuity of S. It is also nonnegative thanks to the symmetry $S = -S$. Since m is BV and $\{\Phi_k\}$ are Lipschitz maps, we have that $\tilde{c}_S(m^-, m^+, \nu) \in L^1(J(m))$ on the $\mathcal{H}^1$−rectifiable set $J(m)$; thus, for every $\varepsilon > 0$, there exists a finite union of disjoint closed C^1 curves $J_\varepsilon \subset J(m)$ such that $J_\varepsilon \cap \partial\Omega = \emptyset$ and

$$\int_{J(m) \setminus J_\varepsilon} \tilde{c}_S(m^-, m^+, \nu) \, d\mathcal{H}^1 \ < \ \varepsilon. \tag{20}$$

The idea for proving (19) consists in constructing a suitable partition of unity that splits J_ε in disjoint components $J_{\varepsilon,k}$ where the entropy Φ_k leads to the maximal cost, for every $1 \le k \le p$. More precisely, for $k \in \{1, \cdots, p\}$, we set

$$J_{\varepsilon,k} := \{ y \in J_\varepsilon \ : \ (m^-(y), m^+(y), \nu(y)) \in s_k \};$$

hence, $J_\varepsilon = \cup_{k=1}^p J_{\varepsilon,k}$ and

$$\tilde{c}_S(m^-, m^+, \nu) \mathbf{1}_{s_k}(m^-, m^+, \nu) = [\Phi_k(m^+) - \Phi_k(m^-)] \cdot \nu \mathbf{1}_{J_{\varepsilon,k}} \quad \mathcal{H}^1 - \text{a.e.} \quad \text{in } J_\varepsilon. \tag{21}$$

Step 3. Mollifiers. Consider $\varepsilon > 0$ to be fixed for the moment. We introduce a mollifier $\rho \in C_c^\infty((-1,1),\mathbf{R}_+)$ satisfying $\int_{-1}^1 \rho(t)\,dt = 1$ and ρ is even. Then for $\eta > 0$, we define the $1D-$mollification of the measure $\mathcal{H}^1 \llcorner J_{\varepsilon,k}$, $k \in \{1,\dots,p\}$:

$$\alpha_{k,\eta}(x) \;:=\; \frac{1}{\eta}\int_{J_{\varepsilon,k}} \rho\left(\frac{|x-y|}{\eta}\right) d\mathcal{H}^1(y) \qquad \text{for every } x \in \mathbf{R}^2.$$

Observe that $\alpha_{k,\eta}$ is a nonnegative smooth function supported in Ω for η small enough since $J_\varepsilon \cap \partial\Omega = \emptyset$ and J_ε is a closed set. We show that $\{\alpha_{k,\eta}\}_{k\in\{1,\dots,p\}}$ is a partition of unity as $\eta \to 0$ in the sense that

$$M \;:=\; \lim_{\eta\downarrow 0}\, \sup_{x\in\mathbf{R}^2} \sum_{k=1}^p \alpha_{k,\eta}(x) \le 1. \tag{22}$$

First of all, let us check that

$$\lim_{\eta\downarrow 0} \sum_{k=1}^p \alpha_{k,\eta}(x) = \begin{cases} 1 & \text{if } x \in Int(J_\varepsilon), \\ \frac{1}{2} & \text{if } x \in J_\varepsilon \setminus Int(J_\varepsilon), \\ 0 & \text{if } x \in \Omega \setminus J_\varepsilon. \end{cases} \tag{23}$$

Indeed, if $x \in Int(J_\varepsilon)$, we parametrize J_ε around x by the arc-length parameterization γ such that $\gamma(0) = x$ and $|\gamma'(t)| = 1$ on a small interval centered in 0. Then for small η, one has

$$\sum_{k=1}^p \alpha_{k,\eta}(x) = \frac{1}{\eta}\int_{J_\varepsilon} \rho\left(\frac{|x-y|}{\eta}\right) d\mathcal{H}^1(y) = \int_{-10}^{10} \rho\left(\frac{|\gamma(0)-\gamma(\eta t)|}{\eta}\right) dt \overset{\eta\downarrow 0}{\longrightarrow} \int_{\mathbf{R}} \rho(t)\,dt = 1.$$

If x belongs the boundary of J_ε, then similarly one has

$$\sum_{k=1}^p \alpha_{k,\eta}(x) = \int_0^{10} \rho\left(\frac{|\gamma(0)-\gamma(\eta t)|}{\eta}\right) dt \to \int_0^\infty \rho(t)\,dt = \frac{1}{2}.$$

If $x \in \mathbf{R}^2 \setminus J_\varepsilon$, then for $\eta < \operatorname{dist}(x, J_\varepsilon)/2$ we have $\displaystyle\sum_{k=1}^p \alpha_{k,\eta}(x) = 0$ which gives (23). Now we want to prove (22). For that, one can choose $\{\eta_n\}_n$ to be a decreasing sequence converging to 0 and a sequence of points $\{x_n\}_n \subset \Omega$ such that

$$M_n \;:=\; \sum_{k=1}^p \alpha_{k,\eta_n}(x_n) = \;\frac{1}{\eta_n}\int_{J_\varepsilon} \rho\left(\frac{|x_n-y|}{\eta_n}\right) d\mathcal{H}^1(y) \overset{n\uparrow\infty}{\longrightarrow} M.$$

Without loss of generality, we may assume that $\{x_n\}$ converges to some $x \in \bar{\Omega}$. Clearly, if $x \notin J_\varepsilon$, then for n large enough we have $J_\varepsilon \cap B(x_n,\eta_n) = \emptyset$ and $M_n = 0$, therefore (22) holds. Assume now that $x \in J_\varepsilon$ and let γ be an arc-length parameterization of J_ε in the neighborhood of x such that $\gamma(0) = x$. Since J_ε is a finite union of C^1 curves, we have $\gamma'(t)\cdot\gamma'(0) > 1/2$ in some neighborhood of $t = 0$. Consequently, there exists $\alpha > 0$ such that for any ball $B(y,r) \subset B(x,\alpha)$ the diameter of the set $\{t \;:\; \gamma(t) \in B(y,r)\}$ is smaller that $4r$. In particular, for n large enough there exists $t_n \to 0$ such that $\gamma(t_n) \in B(x_n,\eta_n)$ and

$$M_n \;=\; \frac{1}{\eta_n}\int_{t_n-2\eta_n}^{t_n+2\eta_n} \rho\left(\frac{|x_n-\gamma(t)|}{\eta_n}\right) dt.$$

Using the change of variables $t = t_n + \eta_n \ell$, we write

$$M_n \;=\; \int_{-2}^{2} \rho\left(\frac{|x_n - \gamma(t_n + \eta_n \ell)|}{\eta_n}\right)\, d\ell.$$

Extracting a subsequence if necessary, we may assume that $\left\{\frac{1}{\eta_n}(x_n - \gamma(t_n))\right\}_n$ converges to some $\sigma \in \bar{B}(0,1)$. Thus, the sequence of functions $\left\{\ell \mapsto \frac{x_n - \gamma(t_n+\eta_n\ell)}{\eta_n}\right\}_n$ converges pointwise to $\ell \mapsto \sigma - \ell\gamma'(0)$. Letting n tend to $+\infty$, we obtain by dominated convergence theorem

$$M \;=\; \int_{-2}^{2} \rho\left(|\sigma - \ell\gamma'(0)|\right)\, d\ell \;\le\; 1,$$

which proves (22).

Step 4. We show that for every Lebesgue point x of $(m^-, m^+, \nu) : J_\varepsilon \to \mathcal{T}$ one has

$$\lim_{\eta \downarrow 0} \sum_{k=1}^{p} \alpha_{k,\eta}(x)[\Phi_k(m^+(x)) - \Phi_k(m^-(x))] \cdot \nu(x) = \tilde{c}_S(m^-(x), m^+(x), \nu(x)).$$

Indeed, we have

$$\sum_{k=1}^{p} \alpha_{k,\eta}(x)[\Phi_k(m^+(x)) - \Phi_k(m^-(x))] \cdot \nu(x) = I \pm II^{\pm} \pm III^{\pm}$$

where

$$\begin{aligned}
I &= \sum_{k=1}^{p} \frac{1}{\eta} \int_{J_{\varepsilon,k}} \rho\left(\frac{|x-y|}{\eta}\right) [\Phi_k(m^+(y)) - \Phi_k(m^-(y))] \cdot \nu(y)\, d\mathcal{H}^1(y) \\
&\overset{(21)}{=} \sum_{k=1}^{p} \frac{1}{\eta} \int_{J_\varepsilon} \rho\left(\frac{|x-y|}{\eta}\right) \tilde{c}_S(m^-(y), m^+(y), \nu(y)) \mathbf{1}_{s_k}(m^-(y), m^+(y), \nu(y))\, d\mathcal{H}^1(y) \\
&= \frac{1}{\eta} \int_{J_\varepsilon} \rho\left(\frac{|x-y|}{\eta}\right) \tilde{c}_S(m^-(y), m^+(y), \nu(y))\, d\mathcal{H}^1(y) \overset{\eta \to 0}{\to} \tilde{c}_S(m^-(x), m^+(x), \nu(x))
\end{aligned}$$

(we use that x is a Lebesgue point of $\tilde{c}_S(m^-, m^+, \nu)$ since $\tilde{c}_S$ is continuous),

$$\begin{aligned}
\left| II^{\pm} \right| &= \left| \sum_{k=1}^{p} \frac{1}{\eta} \int_{J_{\varepsilon,k}} \rho\left(\frac{|x-y|}{\eta}\right) [\Phi_k(m^{\pm}(x)) - \Phi_k(m^{\pm}(y))] \cdot \nu(x)\, d\mathcal{H}^1(y) \right| \\
&\le \frac{C}{\eta} \int_{J_\varepsilon} \rho\left(\frac{|x-y|}{\eta}\right) |m^{\pm}(x) - m^{\pm}(y)|\, d\mathcal{H}^1(y) \overset{\eta \to 0}{\to} 0
\end{aligned}$$

(here, we used the fact that Φ_k are Lipschitz maps),

$$\begin{aligned}
\left| III^{\pm} \right| &= \left| \sum_{k=1}^{p} \frac{1}{\eta} \int_{J_{\varepsilon,k}} \rho\left(\frac{|x-y|}{\eta}\right) \Phi_k(m^{\pm}(y)) \cdot (\nu(x) - \nu(y))\, d\mathcal{H}^1(y) \right| \\
&\le \frac{C}{\eta} \int_{J_\varepsilon} \rho\left(\frac{|x-y|}{\eta}\right) |\nu(x) - \nu(y)|\, d\mathcal{H}^1(y) \overset{\eta \to 0}{\to} 0.
\end{aligned}$$

By (23), we know that $\alpha_{k,\eta}(x) = 0$ for $x \in \Omega \setminus J_\varepsilon$ and η small enough, therefore we deduce that

$$\lim_{\eta \downarrow 0} \sum_{k=1}^{p} \alpha_{k,\eta}(x)[\Phi_k(m^+(x)) - \Phi_k(m^-(x))] \cdot \nu(x) = 0.$$

Step 5. End of proof of (19). By Step 3, we have

$$\mathcal{E}_S(m) \;\geq\; \limsup_{\eta\downarrow 0} \sum_{k=1}^{p} \langle\, \mu_{\Phi_k}(m), \frac{\alpha_{k,\eta}}{\|\sum_{j=1}^{p}\alpha_{j,\eta}\|_\infty}\,\rangle \geq \limsup_{\eta\downarrow 0} \sum_{k=1}^{p} \langle \mu_{\Phi_k}(m), \alpha_{k,\eta}\rangle. \tag{24}$$

Then, by (9), the dominated convergence theorem leads via Step 4 to:

$$\begin{aligned}\lim_{\eta\downarrow 0}\sum_{k=1}^{p}\langle\mu_{\Phi_k}(m),\alpha_{k,\eta}\rangle &= \lim_{\eta\downarrow 0}\sum_{k=1}^{p}(\langle\mu_{\Phi_k}(m)\llcorner J_\varepsilon,\alpha_{k,\eta}\rangle + \langle\mu_{\Phi_k}(m)\llcorner(J(m)\setminus J_\varepsilon),\alpha_{k,\eta}\rangle)\\ &= \int_{J_\varepsilon}\tilde{c}_S(m^-,m^+,\nu)\,d\mathcal{H}^1 \overset{(20)}{\geq} \int_{J(m)}\tilde{c}_S(m^-,m^+,\nu)\,d\mathcal{H}^1 - \varepsilon.\end{aligned}$$

By (24),

$$\mathcal{E}_S(m) \;\geq\; \int_{J(m)} \tilde{c}_S(m^-,m^+,\nu) - \varepsilon$$

and therefore, we obtain (19) in the case of a finite set S of entropies by passing to the limit $\varepsilon \to 0$.
Step 6. $\mathcal{E}_S(m) = \mathcal{I}_{c_S}(m)$ in the general case of an arbitrary symmetric and equivariant set $S \subset ENT$. In fact, there exists a countable symmetric set $\tilde{S} = \{\pm\Phi_j\}_{j\geq 1} \subset S$ which is dense in S for the L^∞-norm. We set $S_p := \{\pm\Phi_1, \cdots, \pm\Phi_p\}$. For every $(z^-, z^+, \nu) \in \mathcal{T}$, we have

$$c_S(|z^+ - z^-|) \;=\; \lim_{p\uparrow\infty} \tilde{c}_{S_p}(z^-, z^+, \nu).$$

Integrating this equality on $J(m)$, we conclude by the monotone convergence theorem and Step 5 applied to the finite set S_p (for any $p \in \mathbb{N}$) that $\mathcal{E}_S(m) \;\geq\; \mathcal{I}_{c_S}(m)$. Therefore, from Step 1, we have $\mathcal{E}_S = \mathcal{I}_{c_S}$ on $\mathcal{S}(\Omega)$.
Step 7. It remains to prove that $\mathcal{I}_{c_S} = \overline{\mathcal{I}_{c_S}}$ on $\mathcal{S}(\Omega)$. Obviously, $\mathcal{I}_{c_S} \geq \overline{\mathcal{I}_{c_S}}$ and since $\mathcal{E}_S$ is l.s.c. in L^1 and satisfies $\mathcal{E}_S \leq \mathcal{I}_{c_S}$ on $\mathcal{S}(\Omega)$, we also have $\overline{\mathcal{I}_{c_S}} \geq \mathcal{E}_S$ and we conclude by Step 6 that $\mathcal{I}_{c_S}$ is l.s.c. □

3 Compactness and maximal set of entropies

Energies of type $\mathcal{E}_S$ are lower semicontinuous for the strong convergence in L^1. For proving existence of minimizers of $\mathcal{E}_S$, it is sufficient to establish the compactness of the sublevel sets $\{\mathcal{E}_S(m) < \eta\}$ for the strong L^1-topology.

Proposition 8 *Let $S \subset ENT$ be equivariant and assume that there exists a uniformly bounded sequence $\{\Phi_k\} \subset \mathbf{R}S$ such that for every $z \in S^1$,*

$$\Phi_k(z) \;\overset{k\uparrow\infty}{\longrightarrow}\; \Phi_0(z) := \left\{ \begin{array}{ll} e_1 & \textit{if } z\cdot e_1 > 0 \\ 0 & \textit{if } z\cdot e_1 \leq 0 \end{array} \right\}. \tag{25}$$

Then for every $\eta > 0$, the sublevel set $\{m \in L^1(\Omega, \mathbf{R}^2),\ \mathcal{E}_S(m) < \eta\}$ is relatively compact in $L^1(\Omega)$.

Proof. We closely follow the ideas in [12] based on a combination of arguments dealing with compensated compactness theorem and Young measure theory. First of all, it is easy to derive a weak convergence result. Indeed, let $\eta > 0$ and $\{m_n\}$ be a sequence in the sublevel set $\{m \in$

$L^1(\Omega, \mathbf{R}^2)$, $\mathcal{E}_S(m) < \eta\}$. By definition of $\mathcal{E}_S$, we have that $m_n \in \mathcal{L}(\Omega)$. Therefore, up to a subsequence, there exists $m_0 \in L^\infty(\Omega, \mathbf{R}^2)$ such that

$$m_n \overset{*}{\rightharpoonup} m_0 \quad \text{in } L^\infty(\Omega). \tag{26}$$

As a consequence, one deduce that $\nabla \cdot m_0 = 0$ and $|m_0| \leq 1$ a.e. in Ω. In order to obtain that $m_n \to m_0$ in $L^1(\Omega)$, it is sufficient to prove that $|m_0| = 1$ a.e. in Ω. We will denote by $\{\beta_x\}_{x\in\Omega}$, the Young measure associated to the sequence $\{m_n\}$. The second step consists in showing that $\{\nabla \cdot [\Phi(m_n)]\}$ is compact in $H^{-1}(\Omega)$, for every $\Phi \in S$. For that, we apply Lemma 3.1 in [12] for $\{\Phi(m_n)\}$ since $\{|\Phi(m_n)|^2\}$ is uniformly integrable in Ω and $\{\nabla \cdot [\Phi(m_n)]\}$ is bounded in $\mathcal{M}(\Omega)$ (because $\mathcal{E}_S(m_n) < \eta$). The third step consists in applying the div-curl lemma of Murat-Tartar to $\{\nabla \cdot [\Phi(m_n)], \nabla \times [\tilde{\Phi}^\perp(m_n)] = \nabla \cdot [\tilde{\Phi}(m_n)]\}$ where $\Phi, \tilde{\Phi} \in S$. We obtain:

$$\int_{S^1} \Phi(y) \cdot \tilde{\Phi}^\perp(y)\, d\beta_x(y) = \left(\int_{S^1} \Phi(y)\, d\beta_x(y)\right) \cdot \left(\int_{S^1} \tilde{\Phi}^\perp(y)\, d\beta_x(y)\right), \quad \text{a.e. } x \in \Omega.$$

By hypothesis (25) and the equivariance of S, this identity is also valid for the elementary entropies $R^{-1}\Phi_0 R$ where R is an arbitrary rotation. Finally, Lemma 2.6. in [12] shows that β_x is a Dirac measure for a.e. $x \in \Omega$ and one concludes that the weak convergence (26) turns into a strong L^1-convergence. $\square$

One needs a criteria on f so that the maximal set of entropies S_f satisfies the hypothesis (25) of Proposition 8. This criteria is described in the next Proposition, whose proof is postponed to the end of the Section.

Proposition 9 *Let $f : [0, 2] \to \mathbf{R}_+$ be a cost function (f is lower semicontinuous and $f(0) = 0$).*

(i) If $\liminf_{t\downarrow 0} \frac{f(t)}{t^3} = 0$, then $S_f = \mathbf{R}Id \oplus \mathbf{R}^2$ (where $Id : S^1 \to S^1$ is the identity function).

(ii) If there exists $t_0 \in (0, 2]$ such that $f(t_0) = 0$, then there exists an infinite subset $E \subset 2\mathbb{Z}$ such that the Fourier coefficients $\gamma_p(\lambda)$ vanish for every $p \in E$ and $\Phi \in S_f$ (with λ is given via Φ by (14) and written as $\lambda(\theta) = \sum_{k\in\mathbb{Z}} \gamma_k(\lambda) e^{ik\theta}$, $\theta \in (-\pi, \pi)$).

(iii) If $\inf\limits_{t\in(0,2]} \frac{f(t)}{t^3} > 0$, then $\mathbf{R}S_f = ENT$.

Remark 8 This Proposition implies that if f is a lower-semicontinuous cost function which is positive on $(0, 2]$, then, either S_f contains only the trivial entropies (the space generated by the identity and constant vectors in $\mathbf{R}^2$), or S_f generates the entire space of entropies. For that, the threshold behavior of f at the origin is the cubic power. Indeed, as in the setting of scalar conservation laws, the entropy production across small shocks is cubic in non-degenerate cases (see e.g. Proposition 5 in [17]).

In case *(iii)* where the maximal set of entropies S_f contains all the entropies (up to multiplicative constants), if the cost function associated to S_f achieves f, i.e., $c_{S_f} = f$ on $[0, 2]$ then $\mathcal{I}_f$ satisfies the desired properties: lower semicontinuity and existence of minimizers of $\overline{\mathcal{I}_f}$ over $\mathcal{L}_0(\Omega)$ as stated in Theorem 4.

Proof of Theorem 4. By Proposition 9, we know that $\mathbf{R}S_f = ENT$. By approximation of elementary entropies by smooth entropies (see Lemma 2.5 in [12]), we deduce that (25) holds. Since

$\mathcal{L}_0(\Omega)$ is a closed set in L^1, Proposition 8 shows that $\{\mathcal{E}_{S_f} < \lambda\} \cap \mathcal{L}_0(\Omega)$ (resp. $\{\overline{\mathcal{I}_f} < \lambda\} \cap \mathcal{L}_0(\Omega)$) is relatively compact in $L^1(\Omega)$. Since $\overline{\mathcal{I}_f}$ (resp. $\mathcal{E}_{S_f}$) is lower semicontinuous in L^1, we conclude with the existence of minimizers of $\overline{\mathcal{I}_f}$ (resp. $\mathcal{E}_{S_f}$) over $\mathcal{L}_0(\Omega)$. □

Remark 9 The elementary (non smooth) entropy Φ_0 defined in (25) satisfies property (14) of Proposition 4 in the sense of distributions: $ie^{i\theta}\lambda_0(\theta) = \frac{d}{d\theta}[\Phi_0(e^{i\theta})]$ where λ_0 is the π-periodic distribution given by

$$\lambda_0 \ = \ \delta_{\pi/2} \ = \ 2\sum_{p\geq 0}(-1)^p\cos(2p\,\cdot) \quad \text{ in } \quad \mathcal{D}'([0,\pi)).$$

So, the even Fourier coefficients of λ_0 do not vanish. We conclude that for a cost function f satisfying situation *(i)* or *(ii)* of Lemma 9 (i.e. $\inf_{t\in(0,2]} f(t)/t^3 = 0$), the set $S = S_f$ does not satisfy the hypothesis of Proposition 8. Consequently the existence of minimizers of $\overline{\mathcal{I}_f}$ over $\mathcal{L}_0(\Omega)$ in these cases is still open.

Proof of Proposition 9. For an entropy $\Phi \in S_f$, we will use the notations introduced in Proposition 4. Let $z^\pm = e^{i(x\pm\beta)}$ and $\nu = e^{ix}$ for some arbitrary $x \in \mathbf{R}$, $\beta \in [0,\pi]$. By (15), we have

$$[\Phi(z^+) - \Phi(z^-)]\cdot\nu \ = \int_{-\beta}^{\beta}\lambda(x-\theta)\sin(\theta)\,d\theta,$$

where $\lambda(\theta) = \partial_\theta\Phi(z)\cdot z^\perp$ for every $z = e^{i\theta} \in S^1$. By Taylor's expansion, one writes $\lambda(x-\theta) = \lambda(x) - \theta\frac{d\lambda}{d\theta}(x) + O(\theta^2)$. Therefore,

$$[\Phi(z^+) - \Phi(z^-)]\cdot\nu = -\frac{d\lambda}{d\theta}(x)\int_{-\beta}^{\beta}\theta\sin(\theta)\,d\theta + O(\beta^4) = -\frac{2\beta^3}{3}\frac{d\lambda}{d\theta}(x) + O(\beta^4).$$

In order to conclude, we distinguish the three cases:

(i): $\liminf_{t\downarrow 0}\frac{f(t)}{t^3} = 0$. If $\Phi \in S_f$, for every $x \in \mathbf{R}$ and $\beta \in [0,\pi/2]$, we have

$$f(2|\sin\beta|) \geq \ \left|[\Phi(e^{i(x+\beta)}) - \Phi(e^{i(x-\beta)})]\cdot e^{ix}\right| = \frac{2\beta^3}{3}\left|\frac{d\lambda}{d\theta}\right|(x) + O(\beta^4).$$

Dividing by β^3 and passing to the liminf as $\beta \downarrow 0$, one gets $\frac{d\lambda}{d\theta} \equiv 0$ in $\mathbf{R}$, i.e., $\frac{d\varphi}{d\theta} + \frac{d^3\varphi}{d\theta^3} \equiv 0$ where we use Proposition (4). We conclude that $\varphi \in \mathbf{R} + \mathbf{R}\sin + \mathbf{R}\cos$ which leads to $\Phi \in \mathbf{R}Id + \mathbf{R}^2$, i.e., $S_f \subset \mathbf{R}Id + \mathbf{R}^2$. The converse inclusion is obvious.

(ii): There exists $t_0 \in (0,2]$ *such that* $f(t_0) = 0$. Let $\beta_0 = \arcsin(t_0/2)$ and let $\Phi \in S_f$. Using the notations of Proposition 4 and formula (15), we have by hypothesis $\lambda \star \sin_{\beta_0} \equiv 0$. In terms of Fourier coefficients, this is equivalently with:

$$\forall k \in \mathbb{Z}, \quad \gamma_k(\sin_{\beta_0}) \neq 0 \Longrightarrow \gamma_k(\lambda) = 0.$$

In order to conclude, we have to check that $\gamma_{2p}(\sin_{\beta_0}) \neq 0$ for infinitely many even numbers $k = 2p$. We compute

$$\gamma_{2p}(\sin_{\beta_0}) := \int_{-\pi}^{\pi}\sin_{\beta_0}(x)e^{-2ipx}\,dx \ = \ \frac{4ip}{4p^2-1}\left[\sin\beta_0\cos(2p\beta_0) - \frac{1}{2p}\cos\beta_0\sin(2p\beta_0)\right].$$

Now, assume by contradiction that $\gamma_{2p}(\sin_{\beta_0}) = 0$ for $|p|$ large enough then in particular, $\cos(2p\beta_0)$ tends to 0 as $|p|$ goes to infinity which is wrong (recall that the fractional part of the sequence $\{p\beta_0/\pi\}_p$ is either periodic or ergodic in $[0,1]$). So there exists an infinite subset $E \subset 2\mathbb{Z}$ such that for every $\Phi \in S_f$ and $k \in E$, we have $\gamma_k(\lambda) = 0$.

(iii): There exists $C > 0$ such that $f(t) \geq Ct^3$ for every $0 \leq t \leq 2$. Let $\Phi \in ENT$. For $(z^-, z^+, \nu) \in \mathcal{T}$, there exists $x \in \mathbf{R}$ and $\beta \in [-\pi, \pi]$ such that $(z^-, z^+, \nu) = (e^{i(x-\beta)}, e^{i(x+\beta}, e^{ix})$. First assume that $\beta \in [-\pi/2, \pi/2]$, we have

$$\left|[\Phi(z^+) - \Phi(z^-)] \cdot \nu\right| = \frac{2|\beta|^3}{3}\left|\frac{d\lambda}{d\theta}\right|(x) + O(\beta^4) \;=\; O(|\beta|^3) \;\leq cf(2|\sin\beta|),$$

for some constant $c = c(\Phi) > 0$. Finally, by (17) in Remark 7, we see that this inequality holds for $\beta \in [-\pi, \pi]$. Thus $c^{-1}\Phi \in S_f$ and $\Phi \in \mathbf{R}S_f$. $\square$

4 Some examples of appropriate cost functions

In this Section we consider symmetric and equivariant sets of entropies generated by a single entropy $\Phi \in ENT$:

$$\langle\Phi\rangle \;=\; \{\pm R^{-1}\Phi R \;:\; R \in \mathrm{SO}(2)\}.$$

The corresponding cost function $c_{\langle\Phi\rangle}$ has been introduced in Definition 3. Using formula (15) in Proposition 4 and symmetries (16),(17) in Remark 7, we have

$$c_{\langle\Phi\rangle}(2\sin\beta) \;=\; \sup_{x\in[0,2\pi]} |\lambda \star \sin_\beta|(x), \quad \beta \in [0, \pi/2]. \tag{27}$$

Remark 10 In the case of odd $\pi-$periodic functions λ, (27) is equivalent by considering only the supremum (27) over $x \in [0, \pi/2]$.

In some cases, the supremum (27) is easy to compute. Such a situation is explain in the following:

Proposition 10 *Let $\Phi \in ENT$ and let λ be defined as in Proposition 4. We assume that λ is odd and π-periodic and that its restriction to $(0, \pi/2)$ is convex.*
a) Then $\lambda \star \sin_\beta$ is an even π-periodic function that is nonincreasing on $[0, \pi/2]$, for every $\beta \in [0, \pi/2]$. Consequently,

$$c_{\langle\Phi\rangle}(2\sin\beta) = \max\left\{\lambda \star \sin_\beta(0), -\lambda \star \sin_\beta(\pi/2)\right\}, \quad \beta \in [0, \pi/2]. \tag{28}$$

b) The cost function $c_{\langle\Phi\rangle}$ is continuous, nondecreasing and convex on $[0, 2]$.
c) If moreover, $\lambda(\cdot - \pi/4)$ is even, then $(\lambda \star \sin_\beta)(\cdot - \pi/4)$ is odd and

$$c_{\langle\Phi\rangle}(2\sin\beta) \;=\; \lambda \star \sin_\beta(0). \tag{29}$$

Proof. *a)* It is easy to check that $\lambda\star\sin_\beta$ is an even π-periodic function. Note that $\lambda\star\sin_\beta(\cdot-\pi/2)$ is also even. Let us check that $\lambda \star \sin_\beta$ is nonincreasing on $[0, \pi/2]$. For $x \in [0, \pi/2]$, we have

$$(\lambda \star \sin_\beta)(x) \;=\; \int_0^\beta [\lambda(x - \theta) - \lambda(x + \theta)] \sin\theta \, d\theta, \quad \beta \in [0, \pi/2]. \tag{30}$$

Denoting

$$\pi_1(x-\theta) \ := \ |x-\theta| \in [0,\pi/2], \qquad \pi_2(x+\theta) \ := \ \min(\theta+x, \pi-(\theta+x)) \in [0,\pi/2],$$

one deduces for $x \in [0,\pi/2]$:

$$\frac{d}{dx}(\lambda \star \sin_\beta)(x) \ = \ \int_0^\beta [\lambda'(x-\theta) - \lambda'(x+\theta)] \sin\theta \, d\theta \ = \ \int_0^\beta [\lambda'(\pi_1(x-\theta)) - \lambda'(\pi_2(x+\theta))] \sin\theta \, d\theta,$$

where we have used the symmetries of λ and the fact that $\beta \in [0,\pi/2]$. Then we easily check that $0 \le \pi_1(x-\theta) \le \pi_2(x+\theta) \le \pi/2$ for $\theta \in (0,\beta)$ and $x,\beta \in [0,\pi,2]$, so that the convexity of λ yields $\lambda \star \sin_\beta$ is nonincreasing on $[0,\pi/2]$. Therefore,

$$\sup_{x\in[0,\pi]} |\lambda \star \sin_\beta|(x) \le \max\left\{|\lambda \star \sin_\beta(0)|, |\lambda \star \sin_\beta(\pi/2)|\right\}.$$

We deduce (28) by noticing that $\lambda(\theta) \le 0$, for $\theta \in [0,\pi/2]$, so $\lambda \star \sin_\beta(0) \ge 0 \ge \lambda \star \sin_\beta(\pi/2)$. Indeed, since λ is a odd, $\pi-$periodic and smooth, we have $\lambda(0) = \lambda(\pi/2) = 0$. Then the convexity of λ on $[0,\pi/2]$ yields $\lambda \le 0$ on $[0,\pi/2]$.

b) In view of (28), the cost function $c_{\langle\Phi\rangle}$ is continuous. It is also non-negative and satisfies $c_{\langle\Phi\rangle}(0) = 0$, so we only have to check that it is convex. For this, writing

$$\beta = \arcsin(t/2), \ \text{i.e.} \ t = 2\sin\beta,$$

it is enough to prove that $h_1 \ : \ t \mapsto \lambda \star \sin_\beta(0)$ and $h_2 \ : \ t \mapsto -\lambda \star \sin_\beta(\pi/2)$ are convex; since $\lambda(\pi/2-\cdot)$ satisfies the same hypotheses as λ, we deduce from the identity $h_2(t) = \lambda(\pi/2-\cdot)\star\sin_\beta(0)$ that it is enough to establish the convexity of h_1. For this, we compute $h_1'(t) = -\lambda(\beta)\tan\beta$ and we conclude by proving that h_1' is nondecreasing, that is, equivalently,

$$j(\beta) \ := \ \lambda(\beta) + \frac{\sin(2\beta)}{2}\lambda'(\beta) \ \le \ 0, \qquad \text{for } \beta \in [0,\pi/2].$$

This inequality is obvious if $\lambda'(\beta) \le 0$. On the other hand, if $\lambda'(\beta) \ge 0$, we deduce from the inequality $\sin\theta \le \pi - \theta$ on $[0,\pi]$ and from the convexity of λ that we have $j(\beta) \le \lambda(\beta) + (\pi/2 - \beta)\lambda'(\beta) \le \lambda(\pi/2) = 0$.

c) If moreover $\lambda(\cdot - \pi/4)$ is even, we deduce that $\lambda \star \sin_\beta(\cdot - \pi/4)$ is odd. In particular, $(\lambda \star \sin_\beta)(\pi/2) = -(\lambda \star \sin_\beta)(0) \le 0$ and $c_{\langle\Phi\rangle}(t) \ = \ \lambda \star \sin_\beta(0)$. □

Discussion on Example 1 for Aviles-Giga cost function: The following entropy $\Phi_1 \in ENT$ was introduced in [20]:

$$\Phi_1(z) = 4(z_2^3, z_1^3), \qquad \text{for } z = (z_1,z_2) \in S^1.$$

Using notations in Proposition 4, we compute $\varphi_1(\theta) = \Phi_1(z)\cdot z = 2\sin 2\theta$ for $z = e^{i\theta}$ and $\lambda_1(\theta) = \varphi_1 + \frac{d^2}{d\theta^2}\varphi_1 = -6\sin 2\theta$. So λ_1 satisfies the hypotheses of Proposition 10 *c*) and we compute

$$c_{\langle\Phi_1\rangle}(t) \ = \ \int_{-\arcsin\frac{t}{2}}^{\arcsin\frac{t}{2}} 6\sin 2\theta \sin\theta \, d\theta = t^3 \quad \text{for} \ \ 0 \le t \le 2.$$

We conclude by Theorem 3 that $\mathcal{I}_{t\mapsto t^3} = \mathcal{E}_{\langle\Phi_1\rangle}$ on $\mathcal{S}(\Omega)$ and $\mathcal{I}_{t\mapsto t^3}$ is l.s.c. □

The smoothness assumption in our definition of entropies yields some restrictions on the cost function $c_{\langle\Phi\rangle}$. In particular, they have a cubic power behavior at the origin:

Proposition 11 *Assume that $S_0 \subset ENT$ is a finite set. Then $c_{\langle S_0 \rangle}(t) \overset{t\downarrow 0}{=} O(t^3)$.*

Proof. Since

$$c_{\langle S_0 \rangle} \;=\; \max_{\Phi \in S_0} c_{\langle \Phi \rangle},$$

it is sufficient to prove the result when S_0 is a singleton. Let $S_0 = \{\Phi\}$. For $t \in [0,2]$ we write $\beta = \arcsin(t/2)$. With the notations of Proposition 4, we have $c_{\langle \Phi \rangle}(t) \;=\; \|\lambda \star \sin_\beta\|_\infty$ and we compute

$$|\lambda \star \sin_\beta|(x) \;=\; \left| \int_0^\beta (\lambda(x-\theta) - \lambda(x+\theta)) \sin(\theta)\, d\theta \right| \;\leq \frac{2}{3} \|\lambda'\|_\infty \beta^3 \;=\; O(t^3). \qquad \square$$

As a consequence, we deduce that cost functions with noncubic power behavior at origin (for example, the quadratic power cost function) must be generated by an infinite number of smooth entropies. A way to overcome this difficulty consists in using cost functions generated by *nonsmooth* entropies defined below:

Definition 6 *We will say that $\Phi \in C^0(S^1, \mathbf{R}^2)$ is a nonsmooth entropy if (8) holds in $\mathcal{D}'(S^1)$. We extend Definition 3 of the cost function $c_{\langle \Phi \rangle}$ to these entropies.*

If Φ is absolutely continuous, then $c_{\langle \Phi \rangle}$ is l.s.c. on $[0,2]$ and Propositions 4, 5 and 10 are still valid:

Proposition 12 *Let $\Phi \in W^{1,1}(S^1, \mathbf{R}^2)$.*
a) Then Φ is a nonsmooth entropy if and only if there exists a $2\pi-$periodic $\varphi \in W^{2,1}_{loc}(\mathbf{R})$ such that

$$\Phi(z) = \varphi(\theta) z + \frac{d\varphi}{d\theta}(\theta) z^\perp \qquad \text{in } \mathcal{D}'(S^1). \tag{31}$$

In this case,

$$\frac{d}{d\theta}\Phi(z) = \lambda(\theta) z^\perp$$

where $\lambda \in L^1_{loc}(\mathbf{R})$ is the $2\pi-$periodic function defined by $\lambda = \varphi + \dfrac{d^2}{d\theta^2}\varphi$ in $\mathbf{R}$. With this notation, for $\beta \in [0, \pi/2]$, $t = 2\sin\beta$, we have

$$c_{\langle \Phi \rangle}(t) \;=\; \sup_{x \in \mathbf{R}} |\lambda \star \sin_\beta|(x).$$

b) If Φ is a nonsmooth entropy with λ an odd π-periodic $L^1_{loc}(\mathbf{R})$-function such that the restriction of λ to $(0, \pi/2)$ is nonpositive and convex, then the cost function $c_{\langle \Phi \rangle} \in W^{2,\infty}(0,2)$ is a nondecreasing convex function satisfying

$$c_{\langle \Phi \rangle}(2\sin\beta) = \max\left\{\lambda \star \sin_\beta(0), -\lambda \star \sin_\beta(\pi/2)\right\}, \quad \beta \in [0, \pi/2].$$

If moreover, $\lambda(\cdot - \pi/4)$ is an even function then

$$c_{\langle \Phi \rangle}(2\sin\beta) \;=\; \lambda \star \sin_\beta(0) \geq 0, \quad \beta \in [0, \pi/2].$$

Proof. *a)* The arguments presented in the proof of Propositions 4, 5 are to be repeated also for the absolutely continuous entropy Φ.
b) Since λ is convex on $(0, \pi/2)$, it implies that λ is locally Lipschitz on $(0, \pi/2)$ and we denote $\lambda' \in L^\infty_{loc}(0, \pi/2)$ to be its derivative. Observe that the limits $\lambda(0\pm)$ and $\lambda(\frac{\pi}{2}\pm)$ exist in $\mathbf{R}$ (so, λ may jump at the origin). However, equality (30) holds for a.e. $x \in \mathbf{R}$ and yields $\lambda\star\sin_\beta \in W^{1,1}_{loc}(\mathbf{R})$ is an even π-periodic function. In order to show that $\dfrac{d}{dx}(\lambda \star \sin_\beta) \le 0$ a.e. in $(0, \frac{\pi}{2})$, the argument presented in the proof of Proposition 10 *a)* is to be interpreted in the sense of distributions. More precisely, for every test function $\zeta \in C^\infty_c(0, \frac{\pi}{2})$, $\zeta \ge 0$, one has that for a.e. $\theta \in (0, \frac{\pi}{2})$:

$$
\begin{aligned}
-\int_0^{\pi/2} [\lambda(x-\theta) - \lambda(x+\theta)]\zeta'(x)\,dx = \int_0^{\pi/2} &[\lambda'(\pi_1(x-\theta)) - \lambda'(\pi_2(x+\theta))]\zeta(x)\,dx \\
&+ 2\zeta(\theta)\lambda(0+) + 2\zeta(\frac{\pi}{2} - \theta)\lambda(\frac{\pi}{2}-) \le 0,
\end{aligned}
$$

where π_1 and π_2 are defined in the proof of Proposition 10 *a)* and we used the symmetries, nonpositivity and convexity of λ on $(0, \frac{\pi}{2})$; thus, $\int_0^{\pi/2} \dfrac{d}{dx}(\lambda \star \sin_\beta)\zeta\,dx \le 0$ for every $\beta \in [0, \frac{\pi}{2}]$. The rest of the proof follows as in Proposition 10. $\square$

For an entropy $\Phi \in W^{1,1}(S^1, \mathbf{R}^2)$, the cost function $c_{\langle\Phi\rangle}$ may be obtained as the cost function generated by an infinite subset of smooth entropies through the standard regularization process below:

Definition 7 *For a given $\rho \in C^\infty_c(\mathbf{R}, \mathbf{R}_+)$ satisfying $\int_{\mathbf{R}} \rho(s)\,ds = 1$, we consider the family of mollifiers $\{\rho_\varepsilon := \frac{1}{\varepsilon}\rho(\frac{\cdot}{\varepsilon})\}_{0<\varepsilon\le 1}$. If Φ is an absolutely continuous entropy satisfying (31) for $\varphi \in W^{1,1}(\mathbf{R})$, we define the family $\{\Phi_\varepsilon\}_{0<\varepsilon\le 1} \subset ENT$ of smooth approximations of Φ by*

$$
\Phi_\varepsilon(z) := \varphi_\varepsilon(\theta)z + \frac{d}{d\theta}\varphi_\varepsilon(\theta)z^\perp \qquad \textit{with} \quad \varphi_\varepsilon := \varphi \star \rho_\varepsilon \textit{ and } z = e^{i\theta} \in S^1.
$$

We will denote

$$
\langle\Phi\rangle_{app} \;:=\; \langle\{\Phi_\varepsilon,\, 0 < \varepsilon \le 1\}\rangle
$$

the set of entropies generated by the smooth approximations of Φ.

Remark that a-priori the set $\langle\Phi\rangle_{app}$ depends on the choice of the mollifying generator ρ. However, the cost function associated to $\langle\Phi\rangle_{app}$ is independent of the choice of ρ and coincides with the cost function associated to the nonsmooth set of entropies $\langle\Phi\rangle$:

Proposition 13 *If $\Phi \in W^{1,1}(S^1, \mathbf{R}^2)$, we have that*

$$
c_{\langle\Phi\rangle_{app}} \;=\; c_{\langle\Phi\rangle}.
$$

Proof. With the notations in Proposition 12, we set $\lambda(\theta) = \partial_\theta\Phi(z)\cdot z^\perp \in L^1_{loc}(\mathbf{R})$, $z = e^{i\theta} \in S^1$ and $\lambda_\varepsilon = \lambda \star \rho_\varepsilon$. Let $t \in [0, 2]$, $\beta = \arcsin\frac{t}{2}$ and an arbitrary triplet $(z^-, z^+, \nu) \in \mathcal{T}$ such that $|z^+ - z^-| = t$. Using rotation invariance of $\langle\Phi\rangle_{app}$ and $\langle\Phi\rangle$ and standard properties of mollifiers, Proposition 12 *a)* yields:

$$
c_{\langle\Phi\rangle_{app}}(t) = \sup_{\tilde\Phi\in\langle\Phi\rangle_{app}} \left|[\tilde\Phi(z^+) - \tilde\Phi(z^-)]\cdot\nu\right| \;=\; \sup_{\varepsilon\in(0,1]} \|\lambda_\varepsilon \star \sin_\beta\|_\infty = \|\lambda \star \sin_\beta\|_\infty \;=\; c_{\langle\Phi\rangle}(t).
$$

Indeed, since $\lambda \star \sin_\beta$ is a $2\pi-$periodic continuous function, then there exists $x = x(\beta) \in [0, 2\pi]$ such that $\|\lambda \star \sin_\beta \|_\infty = \lambda \star \sin_\beta(x) = \lim_{\varepsilon\to 0} \lambda_\varepsilon \star \sin_\beta(x) \leq \sup_{\varepsilon\in(0,1]} \|\lambda_\varepsilon \star \sin_\beta \|_\infty$; conversely, for every $y \in \mathbf{R}$ and $\varepsilon \in (0, 1]$, one has $\lambda_\varepsilon \star \sin_\beta(y) = \rho_\varepsilon \star (\lambda \star \sin_\beta)(y) \leq \|\lambda \star \sin_\beta \|_\infty$. $\square$

Discussion on Example 2 for the "cross-tie wall" cost function: It is a first example of nonsmooth entropy introduced in [1] as an entropy adapted to the energetic cost of cross-tie walls. Namely, let $\Phi_2 \in C^{1,1}(S^1, \mathbf{R}^2)$ such that $\frac{d}{d\theta}\Phi_2(z) = \lambda_2(\theta)z^\perp$ for all $\theta \in [0, 2\pi)$, $z = e^{i\theta}$ where $\lambda_2 : \mathbf{R} \to \mathbf{R}$ is the odd $\pi-$periodic Lipschitz function defined by

$$\lambda_2(\theta) = |\theta - \frac{\pi}{4}| - \frac{\pi}{4}, \quad \forall\theta \in (-\frac{\pi}{4}, \frac{3\pi}{4}).$$

This Lipschitz function λ_2 satisfies the hypotheses of Proposition 12 *b)* (since $\lambda_2(\cdot - \frac{\pi}{4})$ is an even function); thus,

$$c_{\langle\Phi_2\rangle}(2\sin\beta) = \int_{-\beta}^{\beta} \lambda_2(-\theta)\sin\theta\, d\theta = \begin{cases} \sin\beta - \beta\cos\beta & \text{if } 0 \leq \beta \leq \pi/4, \\ \sqrt{2} - \left(\frac{\pi}{2} - \beta\right)\cos\beta - \sin\beta & \text{if } \pi/4 < \beta \leq \pi/2. \end{cases}$$

So we conclude by Theorem 3 and Proposition 13 that $\mathcal{I}_{c_{\langle\Phi_2\rangle}} = \mathcal{I}_{c_{\langle\Phi_2\rangle_{app}}} = \mathcal{E}_{\langle\Phi_2\rangle_{app}}$ on $\mathcal{S}(\Omega)$ and $\mathcal{I}_{c_{\langle\Phi_2\rangle}}$ is l.s.c. Moreover, by Theorem 4, the lower semicontinous energy $\mathcal{E}_{\langle\Phi_2\rangle_{app}}$ admits at least a minimizer in $\mathcal{L}_0(\Omega)$. $\square$

In Examples 1 and 2, the constructed cost functions have a cubic power behavior at the origin. One can also construct nonsmooth entropies Φ such that the cost function satisfies $\limsup_{t\to 0} \frac{c_{\langle\Phi\rangle}(t)}{t^3} = +\infty$. An interesting example of cost function with a quadratic power behavior at the origin is given by the following Lipschitz entropy:

Example 3 There exists a Lipschitz entropy Φ_3 such that $c_{\langle\Phi_3\rangle}(t) = \frac{t^2}{2+\sqrt{4-t^2}}$ for $t \in [0, 2]$. Indeed, let $\lambda_3 \in L^\infty(\mathbf{R})$ be the $\pi-$periodic odd function defined by

$$\lambda_3(\theta) = \begin{cases} -1 & \text{if} \quad \theta \in (0, \frac{\pi}{2}), \\ 1 & \text{if} \quad \theta \in (-\frac{\pi}{2}, 0). \end{cases}$$

Using the notations of Proposition 12, we construct a Lipschitz entropy $\Phi_3 : S^1 \to \mathbf{R}^2$ such that $\frac{\partial}{\partial\theta}\Phi_3(\theta) = \lambda_3(\theta)(e^{i\theta})^\perp$ for all $\theta \in [0, 2\pi)$:

$$\Phi_3(e^{i\theta}) = (-1)^k \left(e^{i\theta} - \sqrt{2}e^{i(2k+1)\pi/4}\right) \quad \text{if } \theta \in (k\pi/2, (k+1)\pi/2), k = 0, \ldots, 3.$$

Since λ_3 satisfies the hypotheses of Proposition 12 *b)*, we compute:

$$c_{\langle\Phi_2\rangle}(2\sin\beta) = c_{\langle\Phi_2\rangle_{app}}(2\sin\beta) = \int_{-\beta}^{\beta} \lambda_3(-\theta)\sin\theta\, d\theta = 2(1 - \cos\beta), \quad \beta \in [0, \pi/2]$$

and the conclusion follows by taking $t := 2\sin\beta \in [0, 2]$. $\square$

Example 4 (A nonmonotone and nonconvex cost function) In general the cost function $c_{\langle\Phi\rangle}$ associated to an entropy is not convex and even not increasing. Indeed, using the notations of Proposition 12, we consider a Lipschitz entropy $\Phi_4 \in W^{1,\infty}(S^1, \mathbf{R}^2)$ such that $\lambda_4 \in L^\infty(\mathbf{R})$ is the odd $\pi/2$-periodic function defined by $\lambda_4 = -1$ on $(0, \pi/4)$. We have of course $c_{\langle\Phi_4\rangle}(0) = 0$. Next, using (27), we have

$$c_{\langle\Phi_4\rangle}(\pi/4) \ \geq \lambda_4 \star \sin_{\pi/4}(0) = 2 - \sqrt{2} \approx 0.58579\ldots.$$

On the other hand, using $\lambda_4(\cdot + \pi/4) = -\lambda_4(\cdot)$ and $|\lambda_4| \leq 1$, we have

$$\begin{aligned} c_{\langle\Phi_4\rangle}(3\pi/8) &= \sup_x \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} -\lambda_4(x-\theta) \left[\sin(\theta - \frac{\pi}{4}) - \sin\theta + \sin(\theta + \frac{\pi}{4})\right] d\theta \\ &\leq \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \left|\sin(\theta - \frac{\pi}{4}) - \sin\theta + \sin(\theta + \frac{\pi}{4})\right| d\theta = \left(2 - \sqrt{2+\sqrt{2}}\right)(\sqrt{2}-1) \approx 0.06306\ldots. \end{aligned}$$

So we have $c_{\langle\Phi_4\rangle}(0) < c_{\langle\Phi_4\rangle}(3\pi/8) < c_{\langle\Phi_4\rangle}(\pi/4)$ and $c_{\langle\Phi_4\rangle}$ is non-monotone and not convex.

5 On Conjecture 1: first part, Proof of Theorem 2

Let us know discuss Conjecture 1. We first concentrate on the cost function $f(t) = t^2/2$. We will show the following result:

Proposition 14 *There exists $S \subset ENT$ nonempty, symmetric and equivariant such that $c_S = f$.*

In other words, we are looking for an equivariant and symmetric set $S \subset ENT$ such that for every $(z^-, z^+, \nu) \in \mathcal{T}$, $z^\pm = e^{i(\pm\gamma + x)}$, $\nu = e^{ix}$, we have

$$\sup_{\Phi\in S} \left[\Phi(z^+) - \Phi(z^-)\right]\cdot\nu \ = \ 2\sin^2\gamma \quad (= f(|z^+ - z^-|)).$$

The proof of Proposition 14 relies on the following proposition:

Proposition 15 *There exists a family $\{\lambda_\gamma\}_{\gamma\in[0,\pi/2]} \subset L^\infty(\mathbf{R})$ of π-periodic odd functions such that*

$$F_{\gamma,\gamma}(0) \ = \ 2\sin^2\gamma \qquad \text{for every } \gamma \in [0, \pi/2], \tag{32}$$

$$|F_{\gamma,\beta}(x)| \ \leq \ 2\sin^2\beta \qquad \text{for every } \gamma, \beta, x \in [0, \pi/2]. \tag{33}$$

where we have introduced the notation

$$F_{\gamma,\beta} \ := \ \lambda_\gamma \star \sin_\beta \qquad \text{with} \qquad \sin_\beta(\theta) \ := \ \begin{cases} \sin(\theta) & \text{if } \theta \in [-\beta, \beta] \\ 0 & \text{in the other cases.} \end{cases}$$

As a consequence of Theorem 4 and Proposition 14, we obtain the lower semicontinuity of $\mathcal{I}_f$ as stated in Theorem 2 and the existence of minimizers of $\overline{\mathcal{I}_f}$ over $\mathcal{L}_0(\Omega)$. Let us now prove how Proposition 14 follows from Proposition 15:

Proof of Proposition 14. Assume that Proposition 15 holds. For $\gamma \in [0, \pi/2]$, we define the nonsmooth entropy $\Phi_\gamma \in W^{1,\infty}(S^1, S^1)$ by

$$\Phi_\gamma(z) \;=\; \varphi_\gamma(\theta) z + \frac{d}{d\theta}\varphi_\gamma(\theta) z^\perp \quad \text{for every } z = e^{i\theta} \in S^1,$$

where $\varphi_\gamma \in W^{2,\infty}_{loc}(\mathbf{R})$ is the odd and π-periodic function solving

$$\frac{d^2}{d\theta^2}\varphi_\gamma + \varphi_\gamma \;=\; \lambda_\gamma \quad \text{ in } \mathcal{D}'(\mathbf{R}) \tag{34}$$

(one uses Remark 6 for the existence and uniqueness of such a solution φ_γ). By construction, Φ_γ is a nonsmooth entropy in the sense of Definition 6. By Proposition 12 *a)*, we have for $\beta \in [0, \pi/2]$ and $t = 2\sin\beta$,

$$c_{\langle \Phi_\gamma \rangle}(t) \;=\; \sup_{x\in[0,\pi]} |\lambda_\gamma \star \sin_\beta|(x) = \sup_{x\in[0,\pi/2]} |F_{\gamma,\beta}(x)|$$

(here we used that λ_γ is an odd $\pi-$periodic function). Thus if we set $S_0 := \{\Phi_\gamma \;:\; \gamma \in [0, \pi/2]\}$, we obtain by Proposition 15:

$$c_{\langle S_0 \rangle}(t) \;=\; \sup_\gamma c_{\langle \Phi_\gamma \rangle}(t) \;=\; 2\sin^2\beta \quad \text{ for every } \beta \in [0, \pi].$$

Then, starting from S_0 and using the mollifying process of Definition 7 and Proposition 13, we obtain a family of smooth entropies $S \subset ENT$ with the desired properties. □

Proof of Proposition 15. We begin by constructing the family $\{\lambda_\gamma\}$. For $\gamma \in [0, \pi/2]$, we define for $\theta \in [0, \pi/2]$:

$$\lambda_\gamma(\theta) \;=\; \begin{cases} 0 & \text{if } \; 0 \le \theta \le \psi_\gamma, \\ -c_\gamma & \text{if } \; \psi_\gamma < \theta < \gamma, \\ 0 & \text{if } \; \gamma \le \theta \le \pi/2. \end{cases} \tag{35}$$

where the constants c_γ and $\psi_\gamma \in [0, \gamma]$ will be set below. The function λ_γ is then uniquely extended on $\mathbf{R}$ as an odd, π-periodic function. How do we choose the constants c_γ and ψ_γ?

i) First, we notice that (32) implies

$$\sin^2\gamma = \frac{1}{2} F_{\gamma,\gamma}(0) = c_\gamma(\cos\psi_\gamma - \cos\gamma). \tag{36}$$

Next, assume $c_\gamma = 2$. In this case, (36) is equivalent to

$$\cos\psi_\gamma - \cos\gamma = \frac{1}{2}\sin^2\gamma. \tag{37}$$

It is easy to check that the function $\gamma \mapsto \left(\cos\gamma + \frac{1}{2}\sin^2\gamma\right) \in C^\infty([0, \pi/2])$ decreases from 1 to 1/2 on $[0, \pi/2]$. So we can set

$$\psi_\gamma \;=\; \Psi_1(\gamma) \;:=\; \arccos\left(\cos\gamma + \frac{1}{2}\sin^2\gamma\right). \tag{38}$$

Then $\Psi_1 : [0, \pi/2] \to [0, \pi/3]$ is a continuous increasing bijective function with $\Psi_1(\gamma) \le \gamma$ for $\gamma \in [0, \pi/2]$ (these properties are due to the choice of the constant $c_\gamma = 2$). Now let us observe that this choice of λ_γ is not appropriate for γ large where (33) fails. Indeed, let $g(\gamma) := \gamma + \Psi_1(\gamma)$. The function g is continuous and increasing on $[0, \pi/2]$; since $g(0) = 0 < \pi/2 < 5\pi/6 = g(\pi/2)$, there exists a unique $\gamma_0 \in (0, \pi/2)$ such that

$$\gamma_0 + \Psi_1(\gamma_0) = \pi/2 \tag{39}$$

and $\gamma + \Psi_1(\gamma) \lesseqgtr \pi/2$ for $\gamma \lesseqgtr \gamma_0$, $0 \le \gamma \le \pi/2$. Moreover, $\gamma_0 > \pi/4$ since $\Psi_1(\gamma_0) < \gamma_0$. One can check that if ψ_γ were defined by (38) for $\gamma > \gamma_0$ then we would have for $\varepsilon > 0$ small enough

$$-F_{\gamma,\beta}(\pi/2) > 2\sin^2\beta \qquad \text{for } \gamma \in (\theta_0, \gamma_0 + \varepsilon) \text{ and } \beta = \pi/2 - \psi_\gamma < \gamma.$$

(Moreover, $-F_{\pi/2,\beta}(\pi/2) > 2\sin^2\beta$ for $\beta \in (0, \varepsilon)$.) Both these inequalities violate (33) at $x = \pi/2$. In order to avoid this situation, we do the following choice:

ii) For $\gamma > \gamma_0$, we define ψ_γ to be the symmetric of γ with respect to $\pi/4$ (γ_0 satisfies already this property): in other words, we set

$$\psi_\gamma \;=\; \Psi_2(\gamma) \;:=\; \pi/2 - \gamma, \quad \gamma > \gamma_0. \tag{40}$$

In this case, (32) yields

$$c_\gamma \;=\; C(\gamma) \;:=\; \frac{\sin^2\gamma}{\sin\gamma - \cos\gamma} \in [1, 2], \quad \gamma > \gamma_0. \tag{41}$$

To summarize, we have chosen the family of π-periodic odd function $\{\lambda_\gamma\}_\gamma$ by formula (35) with

$$\begin{cases} c_\gamma \;=\; 2, \quad \psi_\gamma \;=\; \Psi_1(\gamma) & \text{if } 0 \le \gamma < \gamma_0, \\ c_\gamma \;=\; C(\gamma), \quad \psi_\gamma \;=\; \Psi_2(\gamma) & \text{if } \gamma_0 \le \gamma \le \pi/2, \end{cases}$$

where $\gamma_0 \in (0, \pi/2)$ is the unique solution of (39).

With this choice (32) is satisfied. We now have to prove that inequalities (33) also hold. We start by studying the variations of the functions $F_{\gamma,\beta}$ on $[0, \pi/2]$, where

$$F_{\gamma,\beta}(x) \;=\; -\int_{\mathbf{R}} \sin_\beta(y - x)\lambda_\gamma(y)\, dy.$$

These functions are continuous and piecewise smooth. More precisely, λ_γ is constant on each component of its support $\operatorname{supp}\lambda_\gamma = \pm[\psi_\gamma, \gamma] + \pi\mathbb{Z}$ and we have that $F_{\gamma,\beta}$ on $[0, \pi/2$ is C^1 away from the finite set

$$\mathcal{C}_{\gamma,\beta} \;:=\; \{x \in [0, \pi/2] \;:\; \{x \pm \beta\} \cap (\{\pm\psi_\gamma, \pm\gamma\} + \pi\mathbb{Z}) \;\;\neq\;\; \emptyset\}\,, \quad \gamma, \beta \in [0, \pi/2].$$

The complement $(0, \pi/2) \setminus \mathcal{C}_{\gamma,\beta}$ is then a finite union of open intervals. The following Lemma describes the variations of $F_{\gamma,\beta}$:

Lemma 1 *Let $\gamma, \beta \in (0, \pi/2)$ and $I \subset [0, \pi/2]$ be an interval such that $I \cap \mathcal{C}_{\gamma,\beta} = \emptyset$. For $x \in I$, we denote by $n(x)$ the number of elements of the set $\{x \pm \beta\} \cap \operatorname{supp}\lambda_\gamma$, let $J(x) := \operatorname{supp}\sin_\beta(\cdot - x) \cap \operatorname{supp}\lambda_\gamma$ and $p(x)$ the number of connected components of $J(x)$. Then the functions n and p are constant on I. Moreover, if $n = p = 1$ then*

$$|F_{\gamma,\beta}(x)| \;\le\; 2\sin^2\beta \text{ for every } x \in I;$$

otherwise, $F_{\gamma,\beta}$ is nonincreasing on I.

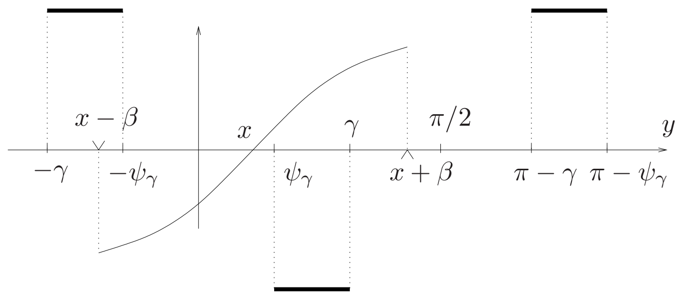

Figure 4: Example of functions $y \mapsto \sin_\beta(y-x)$ and $y \mapsto \lambda_\gamma(y)$ (thick lines).

Proof. We easily see that the functions n and p are constant on I since $I \cap \mathcal{C}_{\gamma,\beta} = \emptyset$. Moreover, $0 \le n \le 2$ and $0 \le p \le 3$. Observe that if $p=3$ then $n=2$. We enumerate and study below all these possibilities.

Case 1: $n=0$. If $p=0$, then $F_{\gamma,\beta}$ vanishes. If $p=1$, then $J(x)=[\psi_\gamma,\gamma]$ and

$$F_{\gamma,\beta}(x) \;=\; c_\gamma \int_{\psi_\gamma}^{\gamma} \sin(y-x)\,dy, \text{ for } x \in I,$$

and $F_{\gamma,\beta}$ is decreasing on I. Finally, if $p=2$ then $J(x)=[-\gamma,-\psi_\gamma]\cup[\psi_\gamma,\gamma]$ or $J(x)=[\psi_\gamma,\gamma]\cup[\pi-\gamma,\pi-\psi_\gamma]$ and (using usual trigonometric relations)

$$F_{\gamma,\beta}(x) = 2c_\gamma \cos x \int_{\psi_\gamma}^{\gamma} \sin y\,dy \quad \text{or} \quad F_{\gamma,\beta}(x) = -2c_\gamma \sin x \int_{\psi_\gamma}^{\gamma} \cos y\,dy \quad \text{for } x \in I,$$

and in both cases $F_{\gamma,\beta}$ is decreasing on I.

Case 2: $n=1$. If $p=1$, then $J(x)$ is an interval and

$$|F_{\gamma,\beta}(x)| \le c_\gamma \int_0^\beta \sin\theta\,d\theta = 2c_\gamma \sin^2(\beta/2) \overset{1\le c_\gamma\le 2}{\le} 4\sin^2(\beta/2).$$

On the other hand, for $\beta \in [0,\pi/2]$ we have $\cos(\beta/2) \ge \sqrt{2}/2$. Thus

$$4\sin^2(\beta/2) \;\le\; 8\sin^2(\beta/2)\cos^2(\beta/2) \;=\; 2\sin^2\beta$$

and we conclude that $|F_{\gamma,\beta}(x)| \;\le\; 2\sin^2\beta$ for every $x \in I$. If $p=2$ we have $J(x)=[x-\beta,-\psi_\gamma]\cup[\psi_\gamma,\gamma]$ or $J(x)=[\psi_\gamma,\gamma]\cup[\pi-\gamma,x+\beta]$. So $F_{\gamma,\beta}(x)$ is the sum of two integrals on the two intervals of $J(x)$ and an easy computation shows that each of these two integrals is a decreasing function of x. Thus, $F_{\gamma,\beta}$ is decreasing on I.

Case 3: $n=2$. If $p=2$, then $J(x)=[x-\beta,-\psi_\gamma]\cup[\psi_\gamma,x+\beta]$ or $J(x)=[x-\beta,\gamma]\cup[\pi-\gamma,x+\beta]$. In the first case we have

$$F_{\gamma,\beta}(x) \;=\; c_\gamma \left(-\int_{x-\beta}^{-\psi_\gamma} \sin(\theta-x)\,d\theta + \int_{\psi_\gamma}^{x+\beta} \sin(\theta-x)\,d\theta \right) \qquad \text{for } x \in I,$$

and we compute

$$F'_{\gamma,\beta}(x) \;=\; c_\gamma(-\sin(\psi_\gamma + x) + \sin(\psi_\gamma - x)) \;=\; -2c_\gamma \cos\psi_\gamma \sin x \;\leq 0 \quad \text{for} \;\; x \in I \subset [0, \pi/2].$$

The same computation yields the same result in the second case. So in both cases, $F_{\gamma,\beta}$ is decreasing on I. If $p = 3$, then $J(x) = [x - \beta, -\psi_\gamma] \cup [\psi_\gamma, \gamma] \cup [\pi - \gamma, x + \beta]$ and $F_{\gamma,\beta}(x)$ is the sum of three integrals which are decreasing functions of x (by the same computations as in the situation $n = 2, p = 2$). Thus, $F_{\gamma,\beta}$ is decreasing on I. $\square$

By Lemma 1, we easily see that it is sufficient to prove (33) for $x = 0$ and $x = \pi/2$. This is the purpose of Lemmas 2 and 3 below which end the proof of Proposition 15. $\square$

Lemma 2 *For $\beta, \gamma \in [0, \pi/2]$ we have $0 \leq F_{\gamma,\beta}(0) \;\leq\; 2\sin^2\beta$.*

Proof. First, if $\beta \leq \psi_\gamma$, we have $F_{\gamma,\beta}(0) = 0$ and the result is obvious. Second, assume $\beta \geq \gamma$. Since $\sin_\beta \equiv \sin_\gamma$ on the support of λ_γ, we have

$$F_{\gamma,\beta}(0) = F_{\gamma,\gamma}(0) \overset{(32)}{=} 2\sin^2\gamma \leq 2\sin^2\beta.$$

Finally, in the case $\psi_\gamma \leq \beta \leq \gamma$, we have to prove that $2c_\gamma(\cos\psi_\gamma - \cos\beta) \leq 2\sin^2\beta$ which is equivalent to $h(\beta) \geq 0$ with $h(\beta) := \sin^2\beta + c_\gamma\cos\beta - c_\gamma\cos\psi_\gamma$. Since $h'(\beta) = \sin\beta(2\cos\beta - c_\gamma)$, we know that either h is decreasing on $[\psi_\gamma, \gamma]$ or h is increasing and then decreasing on this interval; therefore, $\min\limits_{[\psi_\gamma,\gamma]} h = \min(h(\psi_\gamma), h(\gamma)) = h(\gamma) = 0$. (We used relation (36) for this last equality). $\square$

Lemma 3 *For $\beta, \gamma \in [0, \pi/2]$ we have $0 \geq F_{\gamma,\beta}(\pi/2) \;\geq -2\sin^2\beta$.*

Proof. If $\gamma \geq \gamma_0$, we have that $\lambda_\gamma(\cdot - \frac{\pi}{4})$ is even and therefore, $F_{\gamma,\beta}(\pi/2) = -F_{\gamma,\beta}(0)$; the conclusion follows via Lemma 2. We now assume $\gamma < \gamma_0$ (so, $c_\gamma = 2$). We distinguish three cases:
Case 1: $\beta \leq \pi/2 - \gamma$. One computes $F_{\gamma,\beta}(\pi/2) = 0$, so the conclusion is obvious.
Case 2: $\beta \geq \pi/2 - \psi_\gamma$. First, let us show the statement for $\beta_0(\gamma) = \pi/2 - \psi_\gamma$; one has that $F_{\gamma,\beta_0(\gamma)}(\pi/2) \;\geq -2\sin^2\beta_0(\gamma)$ is equivalent to proving that for $\gamma \in [0, \gamma_0]$,

$$\cos\left(\frac{\pi}{2} - \gamma\right) - \cos\beta_0(\gamma) \;\leq\; \frac{\sin^2\beta_0(\gamma)}{2}, \quad \text{i.e.,} \quad \ell(\gamma) := \Psi_1\left(\frac{\pi}{2} - \Psi_1(\gamma)\right) + \gamma \leq \frac{\pi}{2}, \quad \gamma < \gamma_0. \tag{42}$$

To show (42), we compute

$$\Psi_1'(\gamma) = 2\sin\frac{\gamma}{2}\left(1 + \left[\sin\frac{\gamma}{2}\right]^2\right)^{-1/2} \tag{43}$$

that is an increasing positive function on $(0, \pi/2)$ and we check that $\ell(\gamma_0) = \pi/2$. We then deduce (42) from the following statement:

Claim: *For $\gamma \in [0, \pi/2]$, we have $\ell'(\gamma) = 1 - \Psi_1'(\beta_0(\gamma))\Psi_1'(\gamma) \;\geq\; 0$.*

Proof of Claim. By (43), the Claim is equivalent to

$$15\left(\sin\frac{\beta_0(\gamma)}{2}\right)^2\left(\sin\frac{\gamma}{2}\right)^2-\left(\sin\frac{\beta_0(\gamma)}{2}\right)^2-\left(\sin\frac{\gamma}{2}\right)^2\le 1\quad\text{for}\quad\gamma\in[0,\pi/2].\tag{44}$$

One computes $\sin\beta_0(\gamma)=\cos\psi_\gamma=1-2\left(\sin\frac{\gamma}{2}\right)^4$ which implies

$$\left(\sin\frac{\beta_0(\gamma)}{2}\right)^2=\frac{1}{2}-\left(\sin\frac{\gamma}{2}\right)^2\left(1-\left[\sin\frac{\gamma}{2}\right]^4\right)^{1/2}.$$

Plugging this identity in (44) and writing $s=(\sin(\gamma/2))^2$, we are led to prove that

$$k(s)\ :=\ \left(1-2s\sqrt{1-s^2}\right)(15s-1)-2-2s\ \le\ 0\qquad\text{for } 0\le s\le 1/2.$$

This inequality is clear for $0\le s\le 1/15$. If $1/15\le s\le 1/2$, we have $\sqrt{1-s^2}\ge\sqrt{3}/2$ and then,

$$k(s)\ \le\ \left(1-\sqrt{3}s\right)(15s-1)-2-2s\ =\ -3+(13+\sqrt{3})s-15\sqrt{3}s^2\le 0$$

since the above polynomial of second degree has conjugated complex roots. $\square$

Returning to the general case $\pi/2\ge\beta\ge\beta_0(\gamma)$, we conclude

$$F_{\gamma,\beta}(\pi/2)=F_{\gamma,\beta_0(\gamma)}(\pi/2)\ \overset{(42)}{\ge}\ -2\sin^2\beta_0(\gamma)\ge-2\sin^2\beta.$$

Case 3: $\pi/2-\gamma\le\beta\le\pi/2-\psi_\gamma$ We have to prove $h(\beta):=\sin\gamma-\cos\beta-\frac{\sin^2\beta}{2}\ \le\ 0$. By (42), the result holds for $\beta=\pi/2-\psi_\gamma$ and we conclude by noticing that h is increasing on $[0,\pi/2]$. $\square$

6 On Conjecture 1: second part, Proof of Theorem 1

We have seen above that the asymptotic $c_S(t)\overset{t\downarrow 0}{\sim}t^p$ is possible for $p=3$ (in Examples 1 and 2) and for $p=2$ (in Example 3 and Theorem 2). In fact, in these two cases we even have the equality $c_S(t)=t^p$ for $t\in[0,2]$. For proving Theorem 1, we show below that for any $1\le p\le 3$, it is possible to find $S\subset ENT$ symmetric and equivariant such that $c_S(t)=t^p$ for $t\in[0,\sqrt{2}]$. The case $2\le p\le 3$ is treated in Proposition 16 and the case $1\le p<2$ is established in Corollary 1 of Proposition 17 below.

Proposition 16 *For $2\le p\le 3$, there exists $\Phi\in ENT$ such that $c_{\langle\Phi\rangle}(t)=t^p$ for $t\in[0,\sqrt{2}]$.*

Proof. Let $p\in[2,3]$ and let us consider the $\pi-$periodic odd function $\lambda:\mathbf{R}\to\mathbf{R}$ defined by

$$\lambda(\theta)\ =\begin{cases}-2^{p-1}p(\sin\theta)^{p-2}\cos\theta & \text{for }\theta\in(0,\pi/4),\\ -2^{p-1}p\left(\sqrt{2}/2\right)^{p-1}\left(2-\frac{4}{\pi}\theta\right) & \text{for }\theta\in[\pi/4,\pi/2).\end{cases}$$

We easily see that λ is continuous on $\mathbf{R}$ and affine on the interval $(\pi/4,\pi/2)$. We then check that λ is convex in $(0,\pi/2)$. Indeed, for $\theta\in(0,\pi/4)$, we compute

$$\lambda''(\theta)\ =\ 2^{p-1}p(\sin\theta)^{p-4}\cos\theta\left[(3-p)(p-2)(\cos\theta)^2+(3p-5)(\sin\theta)^2\right]\ \ge\ 0;$$

moreover, at $\theta = \pi/4$ (using the inequality $3 - p \le 1 \le 4/\pi$), we have

$$\lim_{\theta\uparrow\pi/4} \lambda'(\theta) \;=\; 2^{p-1}p(3-p)\left(\sqrt{2}/2\right)^{p-1} \;\le\; 2^{p-1}p(4/\pi)\left(\sqrt{2}/2\right)^{p-1} \;=\; \lim_{\theta\downarrow\pi/4} \lambda'(\theta).$$

So the convexity holds in $(0, \pi/2)$. We can apply Proposition 12 b) (first part) to the entropy Φ associated to λ (note that $\lambda(\pi/4-\cdot)$ is not even!). Next, we claim that $\lambda\star\sin_\beta(0) \ge -\lambda\star\sin_\beta(\pi/2)$ for $\beta \in (0, \pi/4)$. Indeed, since λ is convex on $[0, \pi/4]$, we have $\lambda(\theta) \le \frac{4}{\pi}\lambda(\pi/4)\,\theta = \lambda(\pi/2-\theta) \le 0$ for $\theta \in (0, \pi/4)$ which implies our claim. Finally, we get from (28),

$$c_{\langle\Phi\rangle}(2\sin\beta) = \lambda \star \sin_\beta(0) = 2\int_0^\beta \lambda(-\theta)\sin\theta\, d\theta = (2\sin\beta)^p, \qquad \beta \in (0, \pi/4).$$

Therefore, one gets the conclusion for the cost function $c_{\langle\Phi\rangle}(t)$. □

Remark 11 The preceding construction is based on a cost function generated by only one entropy. This is feasible because we only prescribe the value of the cost function on the subinterval $[0, \sqrt{2}]$. In the proof of Theorem 2 or Proposition 17 below, a larger set of entropies is required.

Remark 12 The value $p = 3$ is critical for our method based on entropies. Indeed, if $S \subset ENT$ is such that $\liminf_{t\downarrow 0} c_S(t)/t^3 = 0$ then by Proposition 9 *(i)*, we have $S \subset \mathbf{R}Id \oplus \mathbf{R}^2$ leading to $c_S \equiv 0$.

Proposition 17 *Let $f \;:\; [0,2] \;\to \mathbf{R}_+$ be an increasing cost function such that*

$$\frac{\sin^2\frac{\gamma}{2}}{\sin^2\frac{\beta}{2}} \le \frac{f(2\sin\gamma)}{f(2\sin\beta)} \;\le\; \frac{2\sin(\gamma/2)}{\sin(\beta-\gamma/2)} \quad \textit{for} \quad 0 < \gamma \le \beta \le \pi/4. \tag{45}$$

Then there exists a symmetric and equivariant set $S \subset ENT$ such that $c_S \equiv f$ on $[0, \sqrt{2}]$.

Corollary 1 *For $1 \le p < 2$, there exists $S \subset ENT$ symmetric and equivariant such that $c_S(t) = t^p$ for $t \in [0, \sqrt{2}]$.*

Proof of Corollary 1. We will check that for $1 \le p < 2$, the increasing function $f(t) = t^p$ satisfies the hypotheses of Proposition 17. We define the function $k : (0, \frac{\pi}{2}] \to \mathbf{R}$ by

$$k(\beta) \;:=\; \frac{f(2\sin\beta)}{4\sin^2(\beta/2)}, \quad \beta \in (0, \frac{\pi}{2}]. \tag{46}$$

First, we compute

$$k'(\beta) \;=\; -2^{p-2}\frac{(\sin\beta)^{p-1}}{\sin^2\frac{\beta}{2}}\left(p + 2(1-p)\cos^2\frac{\beta}{2}\right) \;\le\; 0, \quad 0 < \beta \le \pi/2,\; 1 \le p < 2.$$

Thus, k is non-increasing and therefore the first inequality in (45) holds. Let us now prove the second inequality in (45). For this, one observes that $f(2\sin\gamma)/f(2\sin\beta) \le (\sin\gamma)/(\sin\beta)$ for $0 < \gamma \le \beta \le \pi/4$, so it is enough to prove the second inequality in (45) only for $p = 1$. This is equivalent to

$$h_\beta(\gamma) \;:=\; \sin\beta - \sin(\beta-\gamma/2)\cos\frac{\gamma}{2} \;\ge\; 0 \quad \text{for } 0 \le \gamma \le \beta \le \pi/4.$$

We easily check that $h_\beta(0) = 0$ and $h'_\beta(\gamma) = \frac{1}{2}\cos(\beta-\gamma) \geq 0$ if $0 \leq \gamma \leq \beta \leq \pi/4$; thus, h_β is non-decreasing so that the above inequality holds and the conclusion follows. $\square$

Proof of Proposition 17. We proceed as in the proof of Proposition 14. Considering the function k as in (46) non-increasing on $(0, \pi/4]$, let us define the family of odd π-periodic functions $\{\lambda_\gamma\}_{\gamma\in[0,\pi/4]}$ defined on $(0, \pi/2)$ by

$$\lambda_\gamma(\theta) \;=\; \begin{cases} -k(\gamma) & \text{for } \; 0 < \theta \leq \gamma, \\ 0 & \text{for } \; \gamma < \theta < \pi/2. \end{cases}$$

We set $F_{\gamma,\beta} \;:=\; \lambda_\gamma \star \sin_\beta$ for $0 \leq \gamma, \beta \leq \pi/4$. In order to conclude, it is enough to establish that

$$F_{\gamma,\gamma}(0) \;=\; f(2\sin\gamma) \qquad \text{for every } \gamma \in [0, \pi/4], \tag{47}$$

$$|F_{\gamma,\beta}(x)| \;\leq\; f(2\sin\beta) \qquad \text{for every } \gamma, \beta \in [0, \pi/4],\ x \in [0, \pi/2]. \tag{48}$$

Then we will conclude by the same regularization process as in Proposition 14.
First, the identity (47) is obvious. Indeed, by definition of k, we have, for $\gamma \in [0, \pi/4]$,

$$F_{\gamma,\gamma}(0) \;=\; k(\gamma)\int_{-\gamma}^{\gamma} |\sin(\theta)|\, d\theta \;=\; f(2\sin\gamma).$$

Next, let us study the quantities $|F_{\gamma,\beta}(x)|$. We first assume that $0 \leq \beta < \gamma \leq \pi/4$. In this case, using the previous computation, we have

$$|F_{\gamma,\beta}(x)| \;\leq\; k(\gamma)\int_{-\beta}^{\beta} |\sin(\theta)|\, d\theta \;=\; \frac{k(\gamma)}{k(\beta)} f(2\sin\beta), \quad \text{for every } x \in \mathbf{R};$$

since k is non-increasing, it implies that (48) holds in this case. Next we consider the other case $0 \leq \gamma < \beta \leq \pi/4$. Here, $F_{\gamma,\beta}$ is piecewise smooth, we compute

$$F_{\gamma,\beta}(x) \;=\; \begin{cases} 4k(\gamma)\cos x\,(\sin(\gamma/2))^2 & \text{for } 0 \leq x \leq \beta-\gamma, \\ k(\gamma)(2\cos x - \cos(\gamma - x) - \cos\beta) & \text{for } \beta-\gamma \leq x \leq \beta, \\ k(\gamma)(\cos\beta - \cos(x-\gamma)) & \text{for } \beta \leq x \leq \beta+\gamma, \\ 0 & \text{for } \beta+\gamma \leq x \leq \pi/2. \end{cases}$$

In the last interval $\beta + \gamma \leq x \leq \pi/2$, (48) obviously holds. In the first interval $0 \leq x \leq \beta - \gamma$, we have $|F_{\gamma,\beta}(x)| \leq 4k(\gamma)\,(\sin(\gamma/2))^2 = f(2\sin\gamma) \leq f(2\sin\beta)$, so (48)holds. In the third interval $\beta \leq x \leq \beta + \gamma$, we have

$$0 \geq \cos(\beta) - \cos(x-\gamma) \;\geq\; \cos(\beta) - \cos(\beta - \gamma) \;=\; -2\sin(\beta - \gamma/2)\sin(\gamma/2).$$

So, $|F_{\gamma,\beta}(x)| = 2\sin(\beta-\gamma/2)\sin(\gamma/2)k(\gamma)$ and using the second inequality in (45), we get (48). Finally, we consider the second interval $\beta - \gamma \leq x \leq \beta$. For this, we compute

$$F'_{\gamma,\beta}(x) \;=\; -k(\gamma)\Big((2-\cos\gamma)\sin x + \sin\gamma\cos x\Big) \;\leq\; 0,$$

and by continuity of $F_{\gamma,\beta}$, we deduce that (48) holds in this interval from the results on the first and third intervals. This ends the proof of the Proposition. $\square$

7 Is the viscosity solution a minimizer?

Let us start this section by proving that the viscosity solution is a minimizer for the stadium-shaped domain and any appropriate cost function c_S.

Proof of Theorem 5. The jump set of m_0 is the line $J(m_0) = (-L, L)\times\{0\}$ oriented by $\nu_0 = (0, 1)$ and the traces of m_0 on $J(m_0)$ are the constant functions $m_0^+ = (-1, 0)$ and $m_0^- = (1, 0)$. Let $m \in \mathcal{S}_\perp(\Omega)$ be an arbitrary vector field in $\mathcal{S}_\perp(\Omega)$ with $J(m)$ its jump set oriented by ν and $m^\pm$ be the traces of m on $J(m)$. Recall that n is the normal unit outer vector at $\partial\Omega$. Then, on one hand,

$$\begin{aligned}\mathcal{I}_{c_S}(m_0) = \mathcal{H}^1(J(m_0))c_S(|m_0^+ - m_0^-|) = {} & \mathcal{H}^1(J(m_0)) \sup_{\Phi\in S}[(\Phi(m_0^+) - \Phi(m_0^-)) \cdot \nu_0] \\ = \sup_{\Phi\in S}\int_\Omega \nabla\cdot[\Phi(m_0)] \ = {} & \sup_{\Phi\in S}\int_{\partial\Omega} \Phi(n^\perp)\cdot n\, d\mathcal{H}^1.\end{aligned}$$

On the orther hand, for every $\Phi \in S$ and every $m \in \mathcal{S}_\perp(\Omega)$,

$$\begin{aligned}\int_{\partial\Omega} \Phi(n^\perp)\cdot n\, d\mathcal{H}^1 \ = \ \int_\Omega \nabla\cdot[\Phi(m)] \ = & \int_{J(m)} (\Phi(m^+) - \Phi(m^-))\cdot \nu\, d\mathcal{H}^1 \\ \leq \int_{J(m)} c_S(|m^+ - m^-|) d\mathcal{H}^1 \ = & \ \mathcal{I}_{c_S}(m).\end{aligned}$$

Thus $\mathcal{I}_{c_S}(m_0) \leq \mathcal{I}_{c_S}(m)$ and m_0 minimizes $\mathcal{I}_{c_S}$ over $\mathcal{S}_\perp(\Omega)$. □

We now present alternative domains Ω (including nonconvex domains) for which the viscosity solution is still a minimizer of $\mathcal{I}_{c_S}$ in $\mathcal{S}_\perp(\Omega)$ (but with more restrictions on the cost function; in this sense, the next results are somehow weaker than the example of Theorem 5). For that, we first prove the following statement:

Proposition 18 *Let Ω be a Lipschitz bounded domain, and let m_0 be the associated viscosity solution. We note J_0 the jump set of m_0 oriented by ν_0 and $m_0^\pm$ the traces of m_0 on J_0. If $\Phi \in W^{1,\infty}(S^1, \mathbf{R}^2)$ is a nonsmooth entropy such that*

$$[\Phi(m_0^+(x)) - \Phi(m_0^-(x))]\cdot \nu_0(x) = c_{\langle\Phi\rangle}(|m_0^+(x) - m_0^-(x)|) \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in J_0, \tag{49}$$

then the viscosity solution m_0 minimizes $\mathcal{I}_{c_{\langle\Phi\rangle}}$.

Proof of Proposition 18. Let $m \in \mathcal{S}_\perp(\Omega)$ with $J(m)$ its jump set oriented by ν and $m^\pm$ be the traces of m on $J(m)$. As in the proof of Theorem 5, one has:

$$\begin{aligned}\mathcal{I}_{c_{\langle\Phi\rangle}}(m) &= \int_{J(m)} c_{\langle\Phi\rangle}(|m^+ - m^-|)\, d\mathcal{H}^1 \geq \int_{J(m)} (\Phi(m^+) - \Phi(m^-))\cdot\nu\, d\mathcal{H}^1 = \int_\Omega \nabla\cdot[\Phi(m)] \\ &= \int_{\partial\Omega} \Phi(n^\perp)\cdot n = \int_\Omega \nabla\cdot[\Phi(m_0)] = \int_{J_0} [\Phi(m_0^+(x)) - \Phi(m_0^-(x))]\cdot\nu_0\, d\mathcal{H}^1 \overset{(49)}{=} \mathcal{I}_{c_{\langle\Phi\rangle}}(m_0).\end{aligned}$$

Notice that we used Proposition 2 in a general case of a Lipschitz entropy Φ. In fact, (9) still holds since the proof of Proposition 2 only uses the chain rule for $\Phi(e^{i(\cdot)}) \circ \Theta(\cdot)$ that is valid for $\Phi \in W^{1,\infty}(S^1, \mathbf{R}^2)$ and $\Theta \in BV(\Omega, \mathbf{R})$ a lifting of m (see Theorem 3.99 in [3]). □

Remark 13 Proposition 18 covers the cases of domains Ω and cost functions f where the following two conditions are simultaneously satisfied:

- J_0 is a union of vertical lines oriented by $\nu_0(x) \equiv e_1$ with $m_0^{\pm}(x) = e^{\pm i\theta_0(x)}$, $0 \leq \theta_0(x) \leq \pi$;
- $f = c_{\langle\Phi\rangle}$ is generated by one (nonsmooth) entropy Φ satisfying the hypotheses of Proposition 12 b). (Notice that the conditions on the corresponding λ in Proposition 12 b) imply that $\lambda \in L^\infty(\mathbf{R})$ so that $\Phi \in W^{1,\infty}(S^1, \mathbf{R}^2)$.) In particular, the Aviles-Giga cost function of Example 1, the "cross-tie wall" cost function of Example 2 or the cost function in Example 3.

Indeed, by Proposition 12 b), one has for $\mathcal{H}^1$-a.e. $x \in J_0$:

$$c_{\langle\Phi\rangle}(|m_0^+(x) - m_0^-(x)|) = c_{\langle\Phi\rangle}(2\sin\theta_0(x)) = \lambda \star \sin_{\theta_0(x)}(0) = [\Phi(m_0^+(x)) - \Phi(m_0^-(x))] \cdot \nu_0,$$

i.e., (49) holds.

Corollary 2 *Let $f = c_{\langle\Phi\rangle} : [0,2] \to \mathbf{R}_+$ be a cost function generated by one (nonsmooth) entropy Φ satisfying the hypotheses of Proposition 12 b) (i.e., $\lambda : \mathbf{R} \to \mathbf{R}$ defined as $\lambda(\theta) = \partial_\theta \Phi_j(z) \cdot z^\perp$ for every $z = e^{i\theta} \in S^1$ is odd $\pi-$periodic and its restriction to $(0, \frac{\pi}{2})$ is convex, nonpositive and symmetric with respect to $\pi/4$). Then the viscosity solution m_0 is a minimizer of $\mathcal{I}_{c_{\langle\Phi\rangle}}$ if*

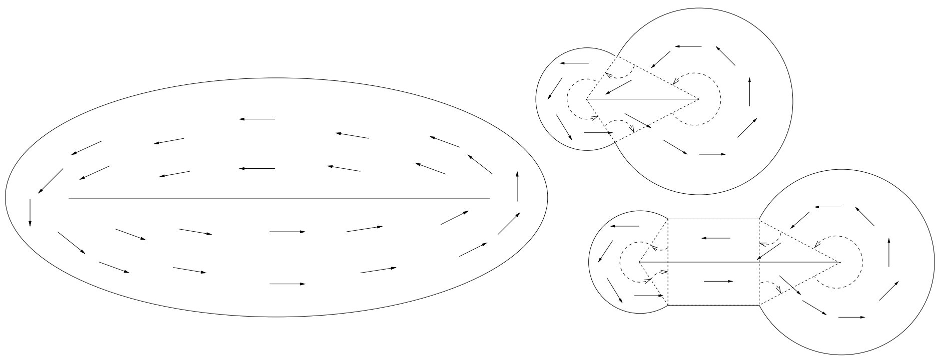

Figure 5: Viscosity solutions for an ellipse, the union of two disks, and a bone-shaped domain (dotted angles indicate vortices structures).

1. *Ω is an ellipse;*
2. *Ω is the (non-convex) union of two discs $B((-\ell, 0), 1) \cup B((L, 0), R)$ with $0 < \ell < 1$ and $0 < L < R$;*
3. *Ω is the (non-convex) bone-shaped domain*

$$B((-s-\ell, 0), r) \cup (-s, s) \times (-1, 1) \cup B((s+L, 0), R), \quad \text{with } 0 < \ell < r,\ 0 < L < R,\ R, r > 1.$$

Proof. In the above cases, the jump set of the viscosity solution m_0 lies on one line oriented by Re_1 for some rotation $R \in SO(2)$. The conclusion is then a direct consequence of Proposition 18 and Remark 13 with the entropy $R^{-1}\Phi R$ (inducing the same cost function f). $\square$

Remark 14 The domain obtained as the union of two identical discs has been discussed in [20]. In that paper (see Theorem 7.1), it is stated that the viscosity solution m_0 of such a domain is not a minimizer of $\mathcal{I}_{t\mapsto t^p}$ for any $p > 0$, $p \neq 3$. This does not contradict our result since for $p \neq 3$, the cost function $t \mapsto t^p$ is not generated by one entropy satisfying (49) on the jump line of m_0.

We now turn to the proof of Theorems 6, 7. Our construction is based upon this remark: the line-energy does not penalize vortices (point singularities carrying a topological degree). So we hope that some divergence-free configurations formed from interior vortices may have a lower energy than the viscosity solution.

Proof of Theorem 6. Let $P_k = \sqrt{2}e^{i(\frac{\pi}{4}+k\frac{\pi}{2})} \in \{(\pm 1, \pm 1) \in \mathbf{R}^2\}$, $k = 0, \ldots, 3$. Let $\Re$ be the full square of size 2 having the vertices $\{P_k\}_{1\le k\le 4}$. We define the nonconvex domain of piecewise Lipschitz boundary

$$\Omega = \cup_{0\le k\le 3} B(P_k, 1) \cup \Re,$$

where $B(P_k, 1)$ stands for the unit disk of center P_k. We consider the function $\tilde{m} : \Omega \to S^1$ given by

$$\tilde{m}(x) = \frac{(x - P_k)^\perp}{|x - P_k|} \qquad \text{if } x \in \Omega \text{ and } \operatorname{Arg} x \in \left(\frac{k\pi}{2}, \frac{(k+1)\pi}{2}\right),\ k = 0, \ldots, 3.$$

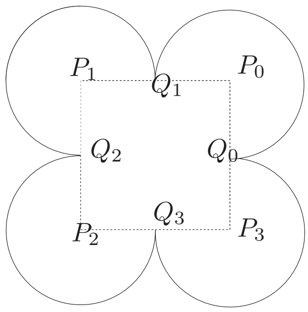

Figure 6: The Lipschitz domain Ω

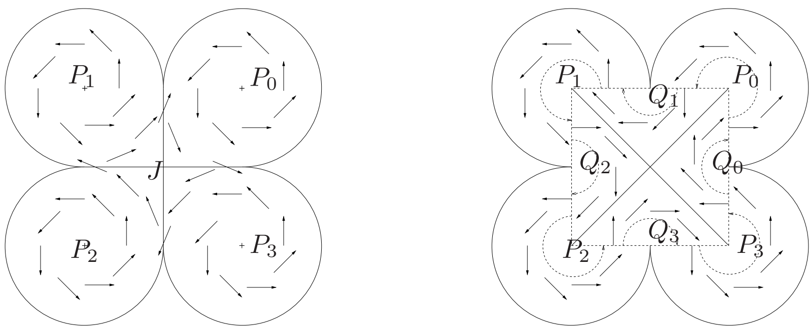

Figure 7: The configurations $\tilde{m}$ (left) and m_0 (right)

Theorem 6 follows by proving the next Lemma. $\square$

Lemma 4 *We have $m_0, \tilde{m} \in \mathcal{S}_\perp(\Omega)$ and $\mathcal{I}_f(\tilde{m}) < \mathcal{I}_f(m_0)$ for every lower semicontinuous function $f : [0,2] \to \mathbf{R}_+$ nonidentically-zero on the segment $(\sqrt{2}, 2)$.*

Proof. Let $Q_k = e^{ik\frac{\pi}{2}}$, $k = 0, \ldots, 3$ and $O = (0,0) \in \mathbf{R}^2$ be the origin. Then an easy computation shows that $m_0, \tilde{m} \in \mathcal{S}_\perp(\Omega)$ and their jump sets are supported by four segments:

$$J(\tilde{m}) = \cup_{0 \le k \le 3}(OQ_k) \quad \text{and} \quad J(m_0) = \cup_{0 \le k \le 3}(OP_k). \tag{50}$$

Now we compute the line-energy of $\tilde{m}$. By symmetry, it is enough to compute the energy cost of the jump line (OQ_0). If $M(t) \in (OQ_0)$ is a moving point with $t = |M(t)Q_0| \in (0,1)$, then the angle $\tilde{\theta}(M(t))$ of the jump of $\tilde{m}$ in $M(t)$ is given by

$$\tilde{\theta}(M(t)) = \frac{\angle P_0 M(t) P_3}{2} = \arctan\frac{1}{t}.$$

Therefore, we obtain

$$\mathcal{I}_f(\tilde{m}) = 4\int_0^1 f\left(2\sin(\arctan\frac{1}{t})\right) dt = 4\int_0^1 f\left(\frac{2}{\sqrt{1+t^2}}\right) dt.$$

Next we compute the line-energy of m_0. By symmetry, it is enough to compute the energy cost of the jump line (OP_0). If $P_0' = \frac{1}{2}P_0$ is the middle point of (OP_0), we consider $M(t) \in (OP_0')$ a moving point with $t = |M(t)P_0'| \in (0, \frac{1}{\sqrt{2}})$. Then the angle $\theta_0(M(t))$ of the jump of m_0 in $M(t)$ is given by

$$\theta_0(M(t)) = \frac{\angle Q_0 M(t) Q_1}{2} = \arctan\frac{1}{t\sqrt{2}}.$$

Observe that if $M(t) \in (P_0'P_0)$, then the angle of the jump of m_0 in $M(t)$ equals with the one in the point $2P_0' - M(t) \in (OP_0')$. Therefore, by symmetry, we deduce

$$\mathcal{I}_f(m_0) = 8\int_0^{1/\sqrt{2}} f\left(2\sin(\arctan\frac{1}{t\sqrt{2}})\right) dt = 4\sqrt{2}\int_0^1 f\left(\frac{2}{\sqrt{1+s^2}}\right) ds.$$

(Here, we used the change of variable $s = \sqrt{2}t$.) Since f is lower semicontinuous and nonidentically-zero on the segment $(\sqrt{2}, 2)$, we deduce $\mathcal{I}_f(m_0) = \sqrt{2}\mathcal{I}_f(\tilde{m}) > \mathcal{I}_f(\tilde{m})$. $\square$

Let us now discuss the case of nonconvex smooth domains:

Proof of Theorem 7. Let $f : [0,2] \to \mathbf{R}_+$ be a bounded lower semicontinuous function that is nonidentically-zero on the segment $(\sqrt{2}, 2)$. Let $\varepsilon = \varepsilon(f) \in (0, \frac{1}{2})$ be a small positive number to be determined in function of f later (see Lemma 5). We consider the nonconvex $C^{1,1}$ domain

$$\Omega_\varepsilon = \{x \in \Omega \,:\, \mathrm{dist}\,(x, \partial\Omega) > \varepsilon\}.$$

Note that the distance function $\psi_0 = \mathrm{dist}\,(x, \partial\Omega_\varepsilon)$ represents the restriction to Ω_ε of the distance function $x \mapsto \mathrm{dist}\,(x, \partial\Omega)$. Since ε is small, the corresponding configuration $m_0 = \nabla^\perp\psi_0$ has the same jump set $J(m_0) \subset \Omega_\varepsilon$ (defined in (50)) and therefore, the same energy:

$$\mathcal{I}_f(m_0) = 4\sqrt{2}\int_0^1 f\left(\frac{2}{\sqrt{1+s^2}}\right) ds.$$

We now construct a configuration $\tilde{m}_\varepsilon$ that has less energy than m_0. In fact, $\tilde{m}_\varepsilon$ will coincide with $\tilde{m}$ away from four small sets $\mathcal{N}_k^\varepsilon$, $k = 0, \ldots, 3$ (see Figure 8). These sets are defined as follows: First, we define the smooth function $\gamma_\varepsilon : [0, t_\varepsilon] \to [0, 2\varepsilon]$ by

$$\gamma_\varepsilon(0) = 2\varepsilon, \quad \gamma_\varepsilon(t_\varepsilon) = 0, \quad \sqrt{t^2 + (1 - \gamma_\varepsilon(t))^2} - \sqrt{t^2 + \gamma_\varepsilon^2(t)} = 1 - 4\varepsilon, \ t \in [0, t_\varepsilon]. \tag{51}$$

(Here, t_ε is defined as the first zero of the decreasing function $\gamma_\varepsilon(t)$ in t.) Then $\mathcal{N}_0^\varepsilon$ corresponds to the interior set surrounded by

$$\partial \mathcal{N}_0^\varepsilon = \{(x_1, \pm\gamma_\varepsilon(1 - x_1)) \,:\, x_1 \in [1 - t_\varepsilon, 1]\} \cup \{(1, x_2) \,:\, x_2 \in [-2\varepsilon, 2\varepsilon]\}$$

and

$$\mathcal{N}_k^\varepsilon = e^{\frac{i\pi k}{2}} \mathcal{N}_0^\varepsilon, \, k = 1, 2, 3,$$

i.e., these regions are equivalent up to a rotation. Using notation introduced in Section 7, we define $\tilde{m}_\varepsilon : \Omega_\varepsilon \to S^1$ as follows:

$$\tilde{m}_\varepsilon(x) = \begin{cases} -\frac{(x-Q_0)^\perp}{|x-Q_0|} & x \in \mathcal{N}_0^\varepsilon \cap B(Q_0, 2\varepsilon) \cap \Omega_\varepsilon, \\ \frac{(x-Q_0)^\perp}{|x-Q_0|} & x \in \mathcal{N}_0^\varepsilon \setminus B(Q_0, 2\varepsilon), \\ e^{\frac{i\pi k}{2}} \tilde{m}_\varepsilon(e^{-\frac{i\pi k}{2}} x) & x \in \mathcal{N}_k^\varepsilon, \, k = 1, 2, 3, \\ \tilde{m}(x) & x \in \Omega_\varepsilon \setminus \cup_{0 \le k \le 3} \mathcal{N}_k^\varepsilon. \end{cases}$$

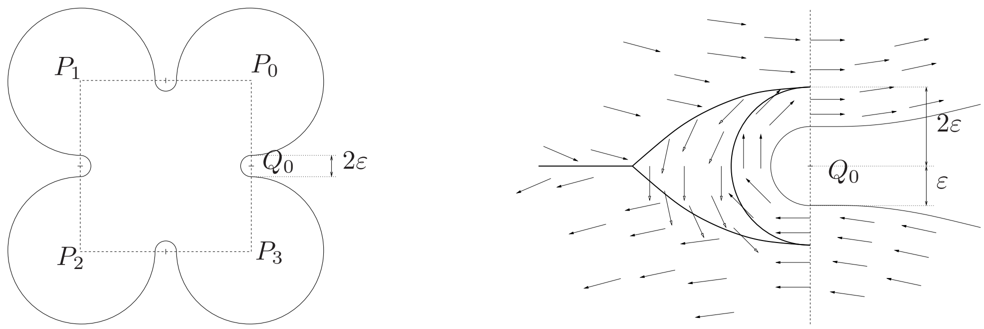

Figure 8: The domain Ω_ε (left) and the configuration $\tilde{m}_0$ in the neighborhood of Q_0 (right).

Theorem 7 is then the consequence of the following result. $\square$

Lemma 5 *We have $m_0, \tilde{m} \in \mathcal{S}_\perp(\Omega_\varepsilon)$ and $\mathcal{I}_f(\tilde{m}_\varepsilon) < \mathcal{I}_f(m_0)$ if $\varepsilon < \varepsilon(f)$ (one can choose $\varepsilon(f) := \min\left\{\frac{1}{20}, \frac{(\sqrt{2}-1)\int_0^1 f\left(\frac{2}{\sqrt{1+s^2}}\right) ds}{25\|f\|_{L^\infty([0,2])}}\right\}$).*

Proof. We divide the proof in several steps.
Step 1. $t_\varepsilon < 5\varepsilon$. By (51), we have that $t^2 + (1 - \gamma_\varepsilon(t))^2 = (\sqrt{t^2 + \gamma_\varepsilon(t)^2} + 1 - 4\varepsilon)^2$ which leads to

$$\sqrt{t^2 + \gamma_\varepsilon(t)^2} = \frac{4\varepsilon - \gamma_\varepsilon(t) - 8\varepsilon^2}{1 - 4\varepsilon} \le 5\varepsilon, \quad t \in [0, t_\varepsilon],$$

if $\varepsilon \le 1/20$. Therefore, $t_\varepsilon \le 5\varepsilon$.

Step 2. $\tilde{m}_\varepsilon \in \mathcal{S}_\perp(\Omega_\varepsilon)$. One can easily check that $\tilde{m}_\varepsilon \in BV(\Omega_\varepsilon, S^1)$, $m = n^\perp$ on $\partial\Omega_\varepsilon$ and

$$J(\tilde{m}_\varepsilon) = \left(J(\tilde{m}) \setminus \bigcup_{0\le k\le 3} \mathcal{N}_k^\varepsilon \right) \cup \left(\bigcup_{0\le k\le 3} (\partial B(Q_k, 2\varepsilon) \cap \mathcal{N}_k^\varepsilon) \right) \cup \left(\bigcup_{0\le k\le 3} (\partial \mathcal{N}_k^\varepsilon \setminus B(Q_k, 2\varepsilon)) \right). \quad (52)$$

By definition, it follows that $\tilde{m}_\varepsilon$ is divergence free in $\Omega_\varepsilon \setminus J(\tilde{m}_\varepsilon)$. It remains to show that the normal component $\tilde{m}_\varepsilon \cdot \nu$ of $\tilde{m}_\varepsilon$ is continuous across the jump set $J(\tilde{m}_\varepsilon)$. Since the continuity of $\tilde{m}_\varepsilon \cdot \nu$ is easy to check for the first two components of $J(\tilde{m}_\varepsilon)$ in (52), it is sufficient by symmetry to prove that $\tilde{m}_\varepsilon^+ \cdot \nu = \tilde{m}_\varepsilon^- \cdot \nu$ for every point $M(t) \in \{(1-t, \gamma_\varepsilon(t)) \,:\, t \in [0, t_\varepsilon]\}$. Indeed, if we denote

$$\nu(M(t)) = \frac{(\gamma_\varepsilon'(t), 1)}{\sqrt{1+\gamma_\varepsilon'(t)^2}}, \quad \overrightarrow{M(t)P_0} = (t, 1-\gamma_\varepsilon(t)), \quad \overrightarrow{M(t)Q_0} = (t, -\gamma_\varepsilon(t)),$$

then

$$\tilde{m}_\varepsilon^+(M(t)) = \frac{(1-\gamma_\varepsilon(t), -t)}{\sqrt{t^2 + (1-\gamma_\varepsilon(t))^2}} \quad \text{and} \quad \tilde{m}_\varepsilon^-(M(t)) = \frac{(-\gamma_\varepsilon(t), -t)}{\sqrt{t^2 + \gamma_\varepsilon(t)^2}}$$

and the divergence free condition at $M(t)$ holds true by differentiating (51).

Step 3. Computation of $\mathcal{I}_f(\tilde{m})$. First, we estimate the length of $J(\tilde{m}_\varepsilon) \setminus J(\tilde{m})$. For that, we have that the length of $\partial\mathcal{N}_0^\varepsilon \setminus B(Q_0, 2\varepsilon)$ is bounded by

$$\mathcal{H}^1(\partial\mathcal{N}_0^\varepsilon \setminus B(Q_0, 2\varepsilon)) = 2\int_0^{t_\varepsilon} \sqrt{1+\gamma_\varepsilon'(t)^2}\, dt \le 2(t_\varepsilon + 2\varepsilon) \le 14\varepsilon.$$

Since the length of $\partial B(Q_0, 2\varepsilon) \cap \mathcal{N}_0^\varepsilon$ is equal to $2\pi\varepsilon$, we deduce by symmetry that

$$\mathcal{H}^1(J(\tilde{m}_\varepsilon) \setminus J(\tilde{m})) \le 4(14\varepsilon + 2\pi\varepsilon) \le 100\varepsilon.$$

Therefore,

$$\mathcal{I}_f(\tilde{m}_\varepsilon) \le \mathcal{I}_f(\tilde{m}) + \|f\|_{L^\infty([0,2])}\mathcal{H}^1(J(\tilde{m}_\varepsilon)\setminus J(\tilde{m})) \le \mathcal{I}_f(\tilde{m}) + 100\varepsilon\|f\|_{L^\infty([0,2])} \le \mathcal{I}_f(m_0)$$

provided that $\varepsilon \le \varepsilon(f)$ given in the statement. □

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