

A diffusion approximation theorem for a nonlinear PDE with application to random birefringent optical fibers.

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Abstract

In this article, we propose a generalization of the theory of diffusion approximation for random ODE to a nonlinear system of random Schrödinger equations. This system arises in the study of pulse propagation in randomly birefringent optical fibers. We first show existence and uniqueness of solutions for the random PDE and the limiting equation. We follow the work of Garnier-Marty [16, 20], where a linear electric field is considered, and we get an asymptotic dynamic for the nonlinear electric field.

1 Introduction

The Manakov PMD equation has been introduced by Wai and Menyuk in [30] to study light propagation over long distance in random birefringent optical fibers. Due to the various length scales present in this problem, a small parameter ϵ appears in the rescaled equation. Our aim in this paper is to prove a diffusion limit theorem for this equation for which we will have to generalize the perturbed test function method [4, 18, 22] to the case of infinite dimension. In [16, 20], a limit theorem is proved for the linear part of the Manakov PMD equation using the Fourier transform and the theory of diffusion approximation for random ODE. Obviously the method in [16, 20] does not work for a nonlinear PDE. In [10, 20], a limit theorem is proved for a non linear scalar PDE driven by a one dimensional noise. The proof relies on the fact that the solution processes are continuous functions of the noise. These methods are no longer applicable to the limit equation that we will consider which is driven by a three dimensional noise, because the solution cannot be written as a continuous function of the noise. Indeed in a general setting a strong solution of a stochastic equation is only a measurable function of the initial data and the Brownian Motion driving the equation. However in the case of a one dimensional noise, Doss [12] and Sussman [26] proved that the solution of such an equation can be written as a continuous function of the Brownian motion. This result has been extended by Yamato [31] to multidimensional Brownian Motions when the Lie algebra generated by the vector fields of the equation is nilpotent of step p . He actually proves the equivalence between the nilpotent hypothesis and the fact that the solution can be written as a continuous function of iterated Stratonovich integrals. In our case the vector fields driving the Manakov PMD Equation are functions of the Pauli matrices and the nilpotent hypothesis of Yamato is not satisfied. This motivates the use of the perturbed test function method. Note that the method has been used for a linear PDE in [11] and a PDE with bounded diffusion coefficients in [23].

We are also interested in the mathematical analysis of both the Manakov PMD and the limit equations. Using a unitary transformation, we are able to establish Strichartz estimates for the transformed equation, that are not available for the Manakov PMD equation. This result will then enable us to prove global existence of solutions. The limiting equation is also studied. Since the nilpotent hypothesis is not satisfied, we use a compactness method to study the existence and uniqueness of solutions.

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1.1 Presentation of the model

Optical fibers are thin, transparent and flexible fibers along which the light propagates to transmit information over long distances and so are of huge interest in modern communications. In a perfect fiber, the two transverse components of the electric field are degenerate in the sense that they propagate with the same characteristics : group velocity, chromatic dispersion, refractive indices ($n_1 = n_2$), etc. However during the fabrication process the fiber may present defects like an ellipticity of the core or suffer from mechanical distortions like stress constraints or twisting [1, 2]. These phenomena induce modal birefringence ($n_1 \neq n_2$) characterized by an orientation angle θ and an amplitude b . If $n_1 > n_2$, we then define a slow axe and a rapid axe corresponding respectively to the mode indices n_1 and n_2 . The orientation angle θ describes the rotation of the local polarization axes with respect to the initial axes. The birefringence strength (or degree of modal birefringence) is given by $b = |n_1 - n_2| = \frac{k_1 - k_2}{k_0}$ where k_1, k_2 are the components of the wave vector and k_0 the wavenumber of the incident light in Vacuum. The beat length $L_B = \frac{2\pi}{k_1 - k_2}$ indicates the length required for the polarization to return to its initial states. There exist several types of birefringence that do not have the same effect on the electric field. Usually linearly birefringent fiber is studied (in the absence of Kerr effect, a linearly polarized light remains linearly polarized), although it has been shown that the birefringence could also be elliptic (occurring in case of twisting, see Menyuk [21]). In case of a uniform anisotropy along the fiber, the birefringence parameters (θ, b) are constant. However in realistic configurations, the anisotropy is not uniform along the fiber. We assume, as in [27, 28, 29, 30], that the birefringence is randomly varying, implying Polarization Mode Dispersion (PMD). The difference of velocity of the two modes, due to random change of the birefringence (and so of the refractive indices), induces coupling between the two polarized modes and pulse spreading : PMD is the main limiting effect of high bit rate transmission.

In [30], Wai and Menyuk assumed that there is no polarization-dependent loss and considered that communication fibers are nearly linearly birefringent. We here use one of the models introduced in [30] for which the local axes of birefringence are bended with an angle θ randomly varying along the propagation axe and that b and b' (the frequency derivative of b) are constant along this axe. Let us recall that the Pauli matrices are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and let us consider the coupled nonlinear Schrödinger equation transformed into the frame of the local axes of birefringence ([19, 30])

$$i \frac{\partial \Psi}{\partial t} + \tilde{\Sigma}(t) \Psi + ib' \sigma_3 \frac{\partial \Psi}{\partial x} + \frac{d_0}{2} \frac{\partial^2 \Psi}{\partial x^2} + \frac{5}{6} |\Psi|^2 \Psi + \frac{1}{6} (\Psi^* \sigma_3 \Psi) \sigma_3 \Psi + \frac{1}{3} N(\Psi) = 0, \quad (1.1)$$

where d_0 is the group velocity dispersion parameter, $N(\Psi) = (\overline{\Psi_1 \Psi_2^2}, \overline{\Psi_2 \Psi_1^2})^t$ and

$$\tilde{\Sigma}(t) = \begin{pmatrix} b & -\frac{i}{2} \frac{d\theta(t)}{dt} \\ \frac{i}{2} \frac{d\theta(t)}{dt} & -b \end{pmatrix}.$$

We recall that in the context of fiber optics, x corresponds to the retarded time while t corresponds to the distance along the fiber. We introduce a new vector field $\tilde{\Psi} = \exp(-ibt\sigma_3) \Psi$. The evolution of $\tilde{\Psi}$ is given by the previous equation (1.1) replacing $\tilde{\Sigma}$ and $N(\Psi)$ respectively by

$$\tilde{\tilde{\Sigma}}(t) = \begin{pmatrix} 0 & -\frac{i}{2} \frac{d\theta(t)}{dt} e^{-2ibt} \\ \frac{i}{2} \frac{d\theta(t)}{dt} e^{2ibt} & 0 \end{pmatrix} \quad \text{and} \quad N(\tilde{\Psi}) = \begin{pmatrix} \overline{\tilde{\Psi}_1 \tilde{\Psi}_2^2} e^{-4ibt} \\ \overline{\tilde{\Psi}_2 \tilde{\Psi}_1^2} e^{4ibt} \end{pmatrix}.$$

Following Wai and Menyuk [19, 28, 29, 30] we consider long distance communication fibers and we denote by l the fiber length. We also denote by l_d the dispersion length scale and l_{nl} the nonlinear length scale related to Kerr effect. The fiber autocorrelation length l_c is the length over which two polarization components remain correlated. We consider, as in [30], a typical configuration where

$l \gg l_d \sim l_{nl} \gg l_c \gg L_B$. Under these assumptions, the term $N(\tilde{\Psi})$ is rapidly oscillating and can be neglected ([2, 19, 30]), its effect being averaged out to zero. As in [19, 30], we introduce a unitary matrix

$$T(t) = \begin{pmatrix} u_1(t) & \overline{u_2(t)} \\ -u_2(t) & \overline{u_1(t)} \end{pmatrix}, \quad (1.2)$$

solution of

$$i \frac{\partial T(t)}{\partial t} + \tilde{\Sigma}(t)T(t) = 0. \quad (1.3)$$

We also consider, for $t \in \mathbb{R}_+$, the matrix :

$$\begin{aligned} \sigma(u(t)) &= \begin{pmatrix} |u_1|^2 - |u_2|^2 & 2\overline{u_1 u_2} \\ 2u_1 u_2 & |u_2|^2 - |u_1|^2 \end{pmatrix} = \begin{pmatrix} m_3 & m_1 - im_2 \\ m_1 + im_2 & -m_3 \end{pmatrix} \\ &= \sigma_1 m_1(t) + \sigma_2 m_2(t) + \sigma_3 m_3(t), \end{aligned} \quad (1.4)$$

which characterizes the linear birefringence and where m_1, m_2, m_3 are real valued processes. Then we can remove the rapid variation of the state of polarization in the evolution of $\tilde{\Psi}$ using the change of variable $\tilde{\Psi}(t) = T(t)X(t)$. We obtain

$$i \frac{\partial X}{\partial t} + ib' \sigma(u(t)) \frac{\partial X}{\partial x} + \frac{d_0}{2} \frac{\partial^2 X}{\partial x^2} + \frac{5}{6} |X|^2 X + \frac{1}{6} (X^* \sigma_3 X) \sigma_3 X + \frac{1}{6} N_u(X) = 0, \quad (1.5)$$

where $N_u(X) = (N_{1,u}(X), N_{2,u}(X))^t$ satisfy

$$\begin{aligned} N_{1,u}(X) &= (m_1^2 + m_2^2) (2|X_2|^2 - |X_1|^2) X_1 + (m_1 - im_2) m_3 (2|X_1|^2 - |X_2|^2) X_2 \\ &\quad + (m_1 - im_2)^2 X_2^2 \overline{X_1} + (m_1 + im_2) m_3 X_1^2 \overline{X_2} \end{aligned} \quad (1.6)$$

$$\begin{aligned} N_{2,u}(X) &= (m_1^2 + m_2^2) (2|X_1|^2 - |X_2|^2) X_2 - (m_1 + im_2) m_3 (2|X_2|^2 - |X_1|^2) X_1 \\ &\quad - (m_1 - im_2) m_3 X_2^2 \overline{X_1} + (m_1 + im_2)^2 X_1^2 \overline{X_2}. \end{aligned} \quad (1.7)$$

Assuming, as in [16, 20, 29], that the correlation length of $d\theta/dt$ is much shorter than the birefringence beat length and that $|d\theta/dt| \ll b$, we may replace the process u by ν , with ([16, 20])

$$\begin{aligned} d\nu(t) &= i\sqrt{\gamma_c} (\sigma_1 \nu(t) \circ dW_1(t) + \sigma_2 \nu(t) \circ dW_2(t)) + i\gamma_s \sigma_3 \nu(t) dt \\ &= i\sqrt{\gamma_c} (\sigma_1 \nu(t) dW_1(t) + \sigma_2 \nu(t) dW_2(t)) + i\gamma_s \sigma_3 \nu(t) dt - \gamma_c \nu(t) dt, \end{aligned} \quad (1.8)$$

where $|\nu_1(0)|^2 + |\nu_2(0)|^2 = 1$, $W = (W_1, W_2)$ is a $2d$ real valued Brownian motion and \circ denotes the Stratonovich product. The second equation is the corresponding Ito equation. In addition γ_c, γ_s are two constants determined by θ . Then $\nu(t) \in \mathbb{S}^3$ a.s, the unit sphere in $\mathbb{C}^2 \sim \mathbb{R}^4$. We denote by Λ the unique invariant probability measure of ν (see Section 5) and by $\mathbb{E}_\Lambda(\cdot)$ the expectation with respect to Λ . Thus replacing u by ν in (1.5), we obtain a new equation describing the evolution of the electric field envelope $X = (X_1, X_2)^t$:

$$i \frac{\partial X}{\partial t} + \frac{d_0}{2} \frac{\partial^2 X}{\partial x^2} + \frac{8}{9} |X|^2 X = -ib' \sigma(\nu(t)) \frac{\partial X}{\partial x} - \frac{1}{6} (N_\nu(X) - \mathbb{E}_\Lambda(N_\nu(X))); \quad (1.9)$$

indeed, the process $m = (m_1, m_2, m_3)$ is now defined as a function g of ν , $m = (g_1(\nu), g_2(\nu), g_3(\nu))$ and it can be proved (see Section 5) that

$$\mathbb{E}_\Lambda(N_{1,\nu}(X)) = \frac{2}{3} (2|X_2|^2 - |X_1|^2) X_1, \quad \mathbb{E}_\Lambda(N_{2,\nu}(X)) = \frac{2}{3} (2|X_1|^2 - |X_2|^2) X_2.$$

We set

$$F_{\nu(t)}(X(t)) = \frac{8}{9} |X|^2 X - \frac{1}{6} (N_\nu(X) - \mathbb{E}_\Lambda(N_\nu(X))). \quad (1.10)$$

Equation (1.9) is of great interest for the study of dispersion because the main effects leading to signal distortions (Kerr effect, chromatic dispersion, PMD) can be easily identified : on the left hand side, the first term describes the evolution of the pulse along the fiber. The second

one corresponds to the chromatic dispersion and the last term to the Kerr effect averaged on the Poincaré sphere. On the right hand side of the equation, the first term describes the linear PMD effect and the second term describes nonlinear PMD.

The Manakov PMD equation (1.9) is written in dimensionless form. According to the length scales we consider, we set $X_\epsilon(t, x) = \frac{1}{\epsilon} X\left(\frac{t}{\epsilon^2}, \frac{x}{\epsilon}\right)$ and $\nu_\epsilon(t) = \nu\left(\frac{t}{\epsilon^2}\right)$ where ν is solution of (1.8); then the electric field X_ϵ has the following evolution

$$i \frac{\partial X_\epsilon(t)}{\partial t} + \frac{ib'}{\epsilon} \sigma(\nu_\epsilon(t)) \frac{\partial X_\epsilon(t)}{\partial x} + \frac{d_0}{2} \frac{\partial^2 X_\epsilon(t)}{\partial x^2} + F_{\nu_\epsilon(t)}(X_\epsilon(t)) = 0, \quad (1.11)$$

where the term $F_{\nu_\epsilon(t)}(X_\epsilon(t))$ is given by (1.10).

In various physical situations, the long time behavior of a phenomenon subject to random perturbations requires to take care of the different characteristic length scales of the problem. In this context Papanicolaou-Stroock-Varadhan [22] and Blankenship-Papanicolaou [4] introduced the approximation diffusion theory for random ordinary differential Equations. This method has been used to study wave propagation in random media [15] and in particular in randomly birefringent fibers [16, 20] but only few results exist on limit theorems for random PDEs. In the latter, the authors studied the evolution, in an optical fiber, of the linear field envelope X_ϵ given by

$$i \frac{\partial X_\epsilon(t)}{\partial t} + \frac{ib'}{\epsilon} \sigma(\nu_\epsilon(t)) \frac{\partial X_\epsilon(t)}{\partial x} + \frac{d_0}{2} \frac{\partial^2 X_\epsilon(t)}{\partial x^2} = 0,$$

and proved that the asymptotic dynamics, when ϵ goes to zero, is given by

$$idX(t) + \left(\frac{d_0}{2} \frac{\partial^2 X(t)}{\partial x^2} \right) dt + i\sqrt{\gamma} \sum_{k=1}^3 \sigma_k \frac{\partial X(t)}{\partial x} \circ dW_k(t) = 0,$$

where $W = (W_1, W_2, W_3)$ is a 3d Brownian motion, and $\gamma = (b')^2/6\gamma_c$. Note that the linear PMD effect reduces to one single parameter γ in front of the three Brownian motions. Generalizing the perturbed test function method, we will prove that the asymptotic dynamic of (1.11) is given by the stochastic nonlinear evolution :

$$idX(t) + \left(\frac{d_0}{2} \frac{\partial^2 X(t)}{\partial x^2} + F(X(t)) \right) dt + i\sqrt{\gamma} \sum_{k=1}^3 \sigma_k \frac{\partial X(t)}{\partial x} \circ dW_k(t) = 0, \quad (1.12)$$

where the nonlinear function F reduces to $F(X(t)) = \frac{8}{9} |X(t)|^2 X(t)$. We will also make use of the following equivalent Ito formulation :

$$idX(t) + \left(\left(\frac{d_0}{2} - \frac{3i\gamma}{2} \right) \frac{\partial^2 X(t)}{\partial x^2} + F(X(t)) \right) dt + i\sqrt{\gamma} \sum_{k=1}^3 \sigma_k \frac{\partial X(t)}{\partial x} dW_k(t) = 0. \quad (1.13)$$

This paper is organized as follows : in Section 1.2 we give notations that will be used along the paper and state the main results. Section 2 is devoted to the proof of well-posedness for the Manakov PMD equation. In Section 3 we study the local well-posedness of the limiting Equation (1.12). Finally in Section 4 we prove the convergence in law of X_ϵ to X as ϵ goes to zero. This paper ends with Section 5 where we recall some results obtained in [16, 20] about the driving process ν , and Section 6 where proofs of technical results used in Section 4 are gathered.

1.2 Notations and main results

Before stating the main results of this article, let us give some definitions and notations.

For all $p \geq 1$, we define $\mathbb{L}^p(\mathbb{R}) = (L^p(\mathbb{R}; \mathbb{C}))^2$ the Lebesgue spaces of functions with values in \mathbb{C}^2 . Identifying \mathbb{C} with \mathbb{R}^2 , we define a scalar product on $\mathbb{L}^2(\mathbb{R})$ by

$$(u, v)_{\mathbb{L}^2} = \sum_{i=1}^2 \mathcal{R}e \left\{ \int_{\mathbb{R}} u_i \overline{v_i} dx \right\}.$$

We denote by $\mathbb{W}^{m,p}$, $m \in \mathbb{N}^*$, $p \in \mathbb{N}^*$ the space of functions in \mathbb{L}^p such that their m first derivatives are in \mathbb{L}^p . If $p = 2$, then we denote $\mathbb{H}^m(\mathbb{R}) = \mathbb{W}^{m,2}(\mathbb{R})$, $m \in \mathbb{N}$. We will also use \mathbb{H}^{-m} the topological dual space of \mathbb{H}^m and denote $\langle \cdot, \cdot \rangle$ the pairing between \mathbb{H}^m and \mathbb{H}^{-m} . The Fourier transform of a tempered distribution $v \in \mathcal{S}'(\mathbb{R})$ is either denoted by \widehat{v} or $\mathcal{F}v$. If $s \in \mathbb{R}$ then \mathbb{H}^s is the fractional Sobolev space of tempered distributions $v \in \mathcal{S}'(\mathbb{R})$ such that $(1 + |\xi|^2)^{s/2} \widehat{v} \in \mathbb{L}^2$. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces. We denote by $\mathcal{L}(E, F)$ the space of linear continuous functions from E into F , endowed with its natural norm. If I is an interval of \mathbb{R} and $1 \leq p \leq +\infty$, then $L^p(I; E)$ is the space of strongly Lebesgue measurable functions f from I into E such that $t \mapsto \|f(t)\|_E$ is in $L^p(I)$. The space $L^p(\Omega, E)$ is defined similarly where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. We denote by $L_w^p(I, E)$ the space $L^p(I, E)$ endowed with the weak (or weak star) topology. For a real number $0 < \alpha < 1$ and $p \geq 1$, we denote by $W^{\alpha,p}([0, T], E)$ the fractional Sobolev space of functions u in $L^p(0, T; E)$ satisfying

$$\int_0^T \int_0^T \frac{\|u(t) - u(s)\|_E^p}{|t - s|^{\alpha p + 1}} ds dt < +\infty.$$

The space $C^\beta([0, T]; E)$ is the space of Hölder continuous functions of order $\beta > 0$ with values in E and we denote by $\mathcal{M}(E)$ the set of probability measures on E , endowed with the topology of the weak convergence $\sigma(\mathcal{M}(E), C_b(E))$.

We will use the space

$$\mathcal{K} = (C([0, T], \mathbb{H}_{loc}^1) \cap C_w([0, T], \mathbb{H}^1) \cap L_w^\infty(0, T; \mathbb{H}^2)) \times C([0, T], \mathbb{R}),$$

where $C_w([0, T], \mathbb{H}^m)$, $m \in \mathbb{Z}$ is the space of functions f in $L^\infty(0, T; \mathbb{H}^m)$, weakly continuous from $[0, T]$ into \mathbb{H}^m .

Let $(\mathcal{A}, \mathcal{G}, \mathbb{Q})$ be a probability space endowed with the complete filtration $(\mathcal{G}_t)_{t \geq 0}$ generated by a two dimensional Brownian Motion $W = (W_1, W_2)$ which is driving the diffusion process ν given by (1.8). We first state an existence and uniqueness result for Equation (1.11).

Theorem 1.1. *Let $\epsilon > 0$ and suppose that $X_\epsilon(0) = v \in \mathbb{L}^2(\mathbb{R})$, then there exists a unique global solution X_ϵ to Equation (1.11) such that, \mathbb{Q} -almost surely,*

$$X_\epsilon \in C(\mathbb{R}_+, \mathbb{L}^2) \cap C^1(\mathbb{R}_+, \mathbb{H}^{-2}) \cap \mathbb{L}_{loc}^8(\mathbb{R}_+, \mathbb{L}^4).$$

Moreover Equation (1.11) preserves the \mathbb{L}^2 norm i.e for all $t \in \mathbb{R}_+$:

$$\|X_\epsilon(t)\|_{\mathbb{L}^2} = \|v\|_{\mathbb{L}^2}.$$

If in addition $X_\epsilon(0) = v \in \mathbb{H}^1$ (resp. \mathbb{H}^2 , resp. \mathbb{H}^3) then corresponding solution is in $C(\mathbb{R}_+, \mathbb{H}^1)$ (resp. $C(\mathbb{R}_+, \mathbb{H}^2)$, resp. $C(\mathbb{R}_+, \mathbb{H}^3)$).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a 3-dimensional real valued Brownian motion $W = (W_1, W_2, W_3)$. We denote by $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ the complete filtration generated by W . The next theorem gives existence and uniqueness of local solution for (1.12)

Theorem 1.2. *Let $X_0 = v \in \mathbb{H}^1(\mathbb{R})$ then there exists a maximal stopping time $\tau^*(v, \omega)$ and a unique strong solution X (in the probabilistic sense) to (1.12), such that $X \in C([0, \tau^*), \mathbb{H}^1(\mathbb{R}))$ \mathbb{P} -a.s. Furthermore the \mathbb{L}^2 norm is almost surely preserved, i.e, $\forall t \in [0, \tau^*)$, $\|X(t)\|_{\mathbb{L}^2} = \|v\|_{\mathbb{L}^2}$ and the following alternative holds for the maximal existence time of the solution :*

$$\tau^*(v, \omega) = +\infty \quad \text{or} \quad \limsup_{t \nearrow \tau^*(v, \omega)} \|X(t)\|_{\mathbb{H}^1} = +\infty.$$

Moreover if $v \in \mathbb{H}^2$, then $X \in C([0, \tau^*), \mathbb{H}^2(\mathbb{R}))$ and τ^* satisfies

$$\tau^*(v, \omega) = +\infty \quad \text{or} \quad \lim_{t \nearrow \tau^*(v, \omega)} \|X(t)\|_{\mathbb{H}^1} = +\infty. \quad (1.14)$$

Note that we do not obtain global existence for Equation (1.12), due to the lack of control of the evolution of the \mathbb{H}^1 norm (see Remark 3.1).

Using these existence theorems, we are able to prove a diffusion approximation result for the nonlinear system of PDEs (1.11).

Theorem 1.3. *Let $X_\epsilon(0) = X_0 = v$ be in $\mathbb{H}^3(\mathbb{R})$. For any stopping time τ with $\tau < \tau^*$ a.s., we consider the solution X_ϵ^τ of (1.11) given by Theorem 1.1, and the solution X^τ of (1.12), both stopped at time τ ; then X_ϵ^τ converges in law to X^τ on $C([0, T], \mathbb{H}^1)$ i.e for all functions f in $C_b(C([0, T], \mathbb{H}^1))$,*

$$\lim_{\epsilon \rightarrow 0} \mathcal{L}(X_\epsilon^\tau)(f) = \mathcal{L}(X^\tau)(f).$$

Note that we consider here the Manakov PMD equation (1.11), but the method may be carried out to other nonlinear Schrödinger equations. Let us first emphasize the key points that allow us to prove Theorem 1.3.

The first point is that the noise term is a linear function of the unknown X_ϵ . This particular structure leads to a stochastic partial differential equation for the limiting equation. The second point is the fact that the Pauli matrices are hermitian. This is important to obtain the conservation of the \mathbb{L}^2 norm for both equations. Finally we use that the driving process ν is a homogeneous Markov ergodic process defined on a compact state space such that $\mathbb{E}_\Lambda(\sigma(y)) = 0$. The hypothesis on the driving noise may be weakened as in the case of random ordinary differential equation assuming good mixing properties (for example exponential decay of the covariance function). The boundedness of $\sigma(\nu_\epsilon(t))$ seems to be necessary. It is used to prove uniform bounds in Lemma 4.5 for tightness. On the other hand, the lack of Strichartz estimates for the limiting equation (1.12) is a negative aspect. Thus we use that $F(v)$ is locally lipschitz in $\mathbb{H}^1(\mathbb{R})$ to prove existence and uniqueness of a local solution to Equation (1.12). But if $\sigma(\nu_\epsilon(t))$ were a one dimensional process, larger dimension and larger power in the nonlinear term could be considered.

Other types of nonlinear Schrödinger equations may be considered replacing, for example, $i \frac{\partial X_\epsilon}{\partial x}$ by X_ϵ and assuming that the matrices σ_k are real valued and symmetric. This latter equation is simpler to handle using Strichartz estimates for the fundamental solution and because $\sigma(\nu_\epsilon(t)) X_\epsilon(t)$ can be treated as a perturbation as far as we are concerned with existence of solutions.

2 The Manakov PMD equation : proof of Theorem 1.1

The point here is that no Strichartz estimates are available for (1.11) because of the lack of commutativity of the matrix σ at different time : $\sigma(\nu(t))\sigma(\nu(s)) \neq \sigma(\nu(s))\sigma(\nu(t))$. Consequently only local existence and uniqueness for initial data in \mathbb{H}^1 can be easily proved directly on Equation (1.11). The idea of the proof is then to find a unitary transformation such that Strichartz estimates are available for the transformed equation. This change of unknown is given in the next result :

Lemma 2.1. *Let us denote for $t \in \mathbb{R}_+$*

$$Z_\epsilon(t) = \begin{pmatrix} \nu_{1,\epsilon}(t) & \overline{\nu_{2,\epsilon}(t)} \\ -\nu_{2,\epsilon}(t) & \overline{\nu_{1,\epsilon}(t)} \end{pmatrix},$$

where $\nu_\epsilon = \nu(t/\epsilon^2)$, ν given by (1.8). Assuming that $X_\epsilon \in C([0, T], \mathbb{L}^2)$, we set $\Psi_\epsilon(t) = Z_\epsilon(t)X_\epsilon(t)$; then the evolution of the electric field Ψ_ϵ is given by the stochastic Itô equation

$$\begin{aligned} id\Psi_\epsilon(t) + \left\{ \frac{ib'}{\epsilon} \sigma_3 \frac{\partial \Psi_\epsilon}{\partial x} + \frac{d_0}{2} \frac{\partial^2 \Psi_\epsilon}{\partial x^2} + \frac{5}{6} |\Psi_\epsilon|^2 \Psi_\epsilon + \frac{1}{6} (\Psi_\epsilon^* \sigma_3 \Psi_\epsilon) \sigma_3 \Psi_\epsilon \right\} dt \\ + \frac{\gamma_s}{\epsilon^2} \sigma_3 \Psi_\epsilon dt + \frac{i\gamma_c}{\epsilon^2} \Psi_\epsilon dt - \frac{\sqrt{\gamma_c}}{\epsilon} \left(\sigma_1 \Psi_\epsilon d\widetilde{W}_1(t) + \sigma_2 \Psi_\epsilon d\widetilde{W}_2(t) \right) = 0, \end{aligned} \quad (2.1)$$

where $\widetilde{W}_j(t) = \epsilon W_j(t/\epsilon^2)$, $j = 1, 2$, and with initial conditions

$$\Psi_\epsilon(0) = \begin{pmatrix} \nu_{1,\epsilon}(0)v_1 + \overline{\nu_{2,\epsilon}(0)}v_2 \\ -\nu_{2,\epsilon}(0)v_1 + \overline{\nu_{1,\epsilon}(0)}v_2 \end{pmatrix} = \psi_0.$$

Proof. Using the equation satisfied by ν_ϵ and because $|\nu_{1,\epsilon}(t)|^2 + |\nu_{2,\epsilon}(t)|^2 = 1$ for any $t \geq 0$, we obtain :

$$idZ_\epsilon(t)Z_\epsilon^{-1}\Psi_\epsilon(t) = -\frac{\gamma_s}{\epsilon^2}\sigma_3\Psi_\epsilon dt - \frac{i\gamma_c}{\epsilon^2}\Psi_\epsilon dt + \frac{\sqrt{\gamma_c}}{\epsilon}\sigma_1\Psi_\epsilon d\widetilde{W}_1(t) + \frac{\sqrt{\gamma_c}}{\epsilon}\sigma_2\Psi_\epsilon d\widetilde{W}_2(t).$$

The nonlinear part of Equation (2.1) is obtained as in the derivation of Equation (1.5). \square

We first investigate the behavior of the linear equation :

$$i\frac{\partial\Psi_\epsilon}{\partial t} + \frac{1}{\epsilon}ib'\sigma_3\frac{\partial\Psi_\epsilon}{\partial x} + \frac{d_0}{2}\frac{\partial^2\Psi_\epsilon}{\partial x^2} = 0, \quad (2.2)$$

with initial condition $\Psi_\epsilon(0) = \psi_0 \in \mathbb{L}^2$.

Proposition 2.1. *The unbounded matrix operator $H_\epsilon = \frac{id_0}{2}I_2\frac{\partial^2}{\partial x^2} - \frac{b'}{\epsilon}\sigma_3\frac{\partial}{\partial x}$ defined on $\mathcal{D}(H_\epsilon) = \mathbb{H}^2$ is the infinitesimal generator of a unique strongly continuous unitary group $U_\epsilon(t)$ on \mathbb{L}^2 . Moreover $U_\epsilon(t)$ may be expressed as a convolution kernel i.e for $\psi_0 \in \mathcal{S}(\mathbb{R})$*

$$U_\epsilon(t)\psi_0 = A_\epsilon(t) \star \psi_0 = \frac{1}{\sqrt{2\pi id_0 t}} \begin{pmatrix} \exp\left\{\frac{i}{2}\frac{(x-b't/\epsilon)^2}{d_0 t}\right\} & 0 \\ 0 & \exp\left\{\frac{i}{2}\frac{(x+b't/\epsilon)^2}{d_0 t}\right\} \end{pmatrix} \star \psi_0.$$

Proof of Proposition 2.1. Assuming $\psi_0 \in \mathcal{S}(\mathbb{R})$ and taking the Fourier transform in space of Equation (2.2) we obtain readily

$$\frac{\partial\widehat{\Psi}_\epsilon}{\partial t} = -\frac{1}{\epsilon}ib'\sigma_3\xi\widehat{\Psi}_\epsilon - i\frac{d_0\xi^2}{2}\widehat{\Psi}_\epsilon.$$

Since σ_3 does not depend on time, we obtain

$$\widehat{\Psi}_\epsilon(t) = R_\epsilon(t)\widehat{\psi}_0 = \begin{pmatrix} \exp\left\{-\frac{id_0}{2}\xi^2 t - ib'\xi t\right\} & 0 \\ 0 & \exp\left\{-\frac{id_0}{2}\xi^2 t + ib'\xi t\right\} \end{pmatrix} \widehat{\psi}_0.$$

The statement of Proposition 2.1 follows then in a classical way, setting $A_\epsilon(t) = \mathcal{F}^{-1}(R_\epsilon(t))$. \square

The explicit formulation of the kernel given in Proposition 2.1 allows immediately to get the following dispersive estimates : if $p \geq 2$, $t \neq 0$, then $U_\epsilon \in \mathcal{L}(\mathbb{L}^{p'}, \mathbb{L}^p)$ where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$ and for all $\psi_0 \in \mathbb{L}^{p'}$,

$$\|U_\epsilon(t)\psi_0\|_{\mathbb{L}^p} \leq (2\pi|d_0||t|)^{-1/2+1/p} \|\psi_0\|_{\mathbb{L}^{p'}}. \quad (2.3)$$

Using then classical arguments (see [6, 17]) one may prove Strichartz inequalities for $U_\epsilon(t)$.

Proposition 2.2. *The following properties hold :*

1. *For every $\psi_0 \in \mathbb{L}^2(\mathbb{R})$, $U_\epsilon(\cdot)\psi_0 \in L^8(\mathbb{R}; \mathbb{L}^4) \cap C(\mathbb{R}; \mathbb{L}^2)$. Furthermore, there exists a constant C such that*

$$\|U_\epsilon(\cdot)\psi_0\|_{L^8(\mathbb{R}; \mathbb{L}^4)} \leq C \|\psi_0\|_{\mathbb{L}^2} \quad \text{for every } \psi_0 \in \mathbb{L}^2.$$

2. *Let I be an interval of \mathbb{R} and $t_0 \in I$. Let $f \in L^{8/7}(I, \mathbb{L}^{4/3})$ then the function*

$$t \mapsto \int_{t_0}^t U_\epsilon(t-s)f(s)ds,$$

belongs to $L^8(I, \mathbb{L}^4) \cap C(I, \mathbb{L}^2)$. Furthermore, there exists a constant C independent of I such that for every $f \in L^{8/7}(I, \mathbb{L}^{4/3})$

$$\left\| \int_{t_0}^\cdot U_\epsilon(\cdot - s)f(s)ds \right\|_{L^8(I, \mathbb{L}^4) \cap L^\infty(I, \mathbb{L}^2)} \leq C \|f\|_{L^{8/7}(I, \mathbb{L}^{4/3})}.$$

We now turn to the study of the nonlinear problem. We will use, as is classical, a cut-off argument on the nonlinear term which is not lipschitz. The cut off we consider here is of the same form as the one considered in [7]. We first prove an existence and uniqueness result for this truncated equation, then deduce from this result the existence of a unique solution for Equation (2.1). We denote :

$$f(\Psi_\epsilon) = \frac{5}{6} |\Psi_\epsilon|^2 \Psi_\epsilon + \frac{1}{6} (\Psi_\epsilon^* \sigma_3 \Psi_\epsilon) \sigma_3 \Psi_\epsilon.$$

Let $\Theta \in C_c^\infty(\mathbb{R})$ with $\text{supp}\Theta \subset [-2; 2]$ such that $\Theta(x) = 1$ for $|x| \leq 1$ and $0 \leq \Theta(x) \leq 1$ for $x \in \mathbb{R}$. Let $R > 0$ and $\Theta_R(x) = \Theta(x/R)$. We then consider the following equation

$$\begin{aligned} \Psi_\epsilon^R(t) &= U_\epsilon(t)\psi_0 + \frac{i\gamma_s}{\epsilon^2} \int_0^t U_\epsilon(t-s)\sigma_3\Psi_\epsilon^R(s)ds - \frac{\gamma_c}{\epsilon^2} \int_0^t U_\epsilon(t-s)\Psi_\epsilon^R(s)ds \\ &\quad + i \int_0^t U_\epsilon(t-s)\Theta_R\left(\|\Psi_\epsilon^R\|_{L^8(0,s;\mathbb{L}^4)}\right) f(\Psi_\epsilon^R(s)) ds \\ &\quad - \frac{i\sqrt{\gamma_c}}{\epsilon} \int_0^t U_\epsilon(t-s)\sigma_1\Psi_\epsilon^R(s)d\widetilde{W}_1(s) - \frac{i\sqrt{\gamma_c}}{\epsilon} \int_0^t U_\epsilon(t-s)\sigma_2\Psi_\epsilon^R(s)d\widetilde{W}_2(s), \end{aligned} \quad (2.4)$$

which is the mild form of the Ito equation :

$$\begin{aligned} id\Psi_\epsilon^R(t) + \left\{ \frac{ib'}{\epsilon} \sigma_3 \frac{\partial \Psi_\epsilon^R(t)}{\partial x} + \frac{d_0}{2} \frac{\partial^2 \Psi_\epsilon^R(t)}{\partial x^2} + \frac{\gamma_s}{\epsilon^2} \sigma_3 \Psi_\epsilon^R(t) + \frac{i}{\epsilon^2} \gamma_c \Psi_\epsilon^R(t) \right\} dt \\ - \frac{\sqrt{\gamma_c}}{\epsilon} \sigma_1 \Psi_\epsilon^R d\widetilde{W}_1(t) - \frac{\sqrt{\gamma_c}}{\epsilon} \sigma_2 \Psi_\epsilon^R d\widetilde{W}_2(t) + \Theta_R\left(\|\Psi_\epsilon^R\|_{L^8(0,t;\mathbb{L}^4)}\right) f(\Psi_\epsilon^R(t)) dt = 0, \end{aligned} \quad (2.5)$$

with initial condition $\Psi_\epsilon^R(0) = \psi_0$.

Proposition 2.3. *Let $\Psi_\epsilon^R(0) = \psi_0 \in \mathbb{L}^2(\mathbb{R})$. Let $T > 0$ and $\mathcal{U}_c^T = C([0, T]; \mathbb{L}^2) \cap L^8(0, T; \mathbb{L}^4)$; then Equation (2.4) has a unique strong adapted solution $\Psi_\epsilon^R \in L^8(\mathcal{A}; \mathcal{U}_c^T)$, for any $T > 0$.*

Proof of Proposition 2.3. We use a fixed point argument in the Banach space $L^8(\mathcal{A}; \mathcal{U}_c^T)$ for sufficiently small time T depending on R . We first need to establish estimates on the stochastic integrals

$$J_{j,\epsilon}\Psi_\epsilon(t) = \int_0^t U_\epsilon(t-s)\sigma_j\Psi_\epsilon(s)d\widetilde{W}_j(s), \quad j = 1, 2.$$

Lemma 2.2. *Let $T > 0$; then for each adapted process $\Psi_\epsilon \in L^8(\mathcal{A}; \mathcal{U}_c^T)$ and for $j = 1, 2$ the stochastic integral $J_{j,\epsilon}\Psi_\epsilon$ belongs to $L^8(\mathcal{A}; \mathcal{U}_c^T)$. Moreover for any $T > 0$ and t in $[0, T]$ we have the estimates*

$$\mathbb{E}\left(\|J_{j,\epsilon}\Psi_\epsilon\|_{L^8(0,T;\mathbb{L}^4) \cap L^\infty(0,T;\mathbb{L}^2)}^8\right) \leq CT^4 \mathbb{E}\left(\|\Psi_\epsilon\|_{L^\infty(0,T;\mathbb{L}^2)}^8\right).$$

Proof of Lemma 2.2. Since $\Psi_\epsilon \in L^8(\mathcal{A}; \mathcal{U}_c^T)$ and is adapted, we may apply the Burkholder Davis Gundy inequality in the Banach space $\mathbb{L}^4(\mathbb{R})$ (which is UMD space [5]) :

$$\begin{aligned} \mathbb{E}\left(\|J_{j,\epsilon}\Psi_\epsilon\|_{L^8(0,T;\mathbb{L}^4)}^8\right) &= \mathbb{E}\left(\int_0^T \left\| \int_0^t U_\epsilon(t-s)\sigma_j\Psi_\epsilon(s)d\widetilde{W}_j(s) \right\|_{\mathbb{L}^4}^8 dt\right) \\ &\leq \int_0^T \mathbb{E}\left(\sup_{0 \leq u \leq t} \left\| \int_0^u U_\epsilon(t-s)\sigma_j\Psi_\epsilon(s)d\widetilde{W}_j(s) \right\|_{\mathbb{L}^4}^8\right) dt \\ &\leq \mathbb{E}\left(\int_0^T \left(\int_0^t \|U_\epsilon(t-s)\sigma_j\Psi_\epsilon(s)\|_{\mathbb{L}^4}^2 ds\right)^4 dt\right). \end{aligned}$$

Using Hölder inequality in time, Fubini and a change of variable :

$$\mathbb{E}\left(\int_0^T \left(\int_0^t \|U_\epsilon(t-s)\sigma_j\Psi_\epsilon(s)\|_{\mathbb{L}^4}^2 ds\right)^4 dt\right) \leq T^3 \mathbb{E}\left(\int_0^T \|U_\epsilon(\cdot)\sigma_j\Psi_\epsilon(s)\|_{L^8(0,T;\mathbb{L}^4)}^8 ds\right).$$

On the other hand, by Proposition 2.2,

$$\begin{aligned} \mathbb{E} \left(\int_0^T \|U_\epsilon(\cdot)\sigma_j\Psi_\epsilon(s)\|_{L^8(0,T;\mathbb{L}^4)}^8 ds \right) &\leq C\mathbb{E} \left(\int_0^T \|\Psi_\epsilon(s)\|_{\mathbb{L}^2}^8 ds \right) \\ &\leq CTE \left(\|\Psi_\epsilon\|_{L^\infty(0,T;\mathbb{L}^2)}^8 \right). \end{aligned}$$

Combining these inequalities leads to the estimate in $L^8(0,T;\mathbb{L}^4)$. The other estimate is proved using Burkholder inequality in Hilbert space and the unitary property of the group U_ϵ . Finally $U_\epsilon(t)$ being a unitary semigroup in \mathbb{L}^2 , Theorem 6.10 in [24] tells us that, provided $\Psi_\epsilon \in L^8(\mathcal{A}, L^2(0,T;\mathbb{L}^2))$, then $J_{j,\epsilon}\Psi_\epsilon(\cdot)$ has continuous modification with values in $\mathbb{L}^2(\mathbb{R})$. \square

Given $\Psi_\epsilon^R \in L^8(\mathcal{A};\mathcal{U}_\epsilon^T)$, we denote by $\mathcal{T}\Psi_\epsilon^R(t)$ the right hand side of (2.4). Since the group $U_\epsilon(\cdot)$ maps $\mathbb{L}^2(\mathbb{R})$ into $C(\mathbb{R},\mathbb{L}^2(\mathbb{R}))$, Proposition 2.2 and Lemma 2.2 easily imply that the mapping \mathcal{T} maps $L^8(\mathcal{A};\mathcal{U}_\epsilon^T)$ into itself. Let now Ψ_ϵ^R and Φ_ϵ^R being adapted processes with values in $L^8(\mathcal{A};\mathcal{U}_\epsilon^T)$, then using Proposition 2.2, the same arguments as in [7] for the cut-off and Lemma 2.2 applied to $J_{j,\epsilon}(\Phi_\epsilon^R(t) - \Psi_\epsilon^R(t))$, we get

$$\mathbb{E} \left(\|\mathcal{T}\Psi_\epsilon^R - \mathcal{T}\Phi_\epsilon^R\|_{\mathcal{U}_\epsilon^T}^8 \right)^{1/8} \leq \left(\frac{CT}{\epsilon^2} + \frac{CT^{1/2}}{\epsilon} + C(R)T^{1/2} \right) \mathbb{E} \left(\|\Psi_\epsilon^R - \Phi_\epsilon^R\|_{\mathcal{U}_\epsilon^T}^8 \right)^{1/8}.$$

We conclude that \mathcal{T} is a contraction mapping if T is chosen such that $CT/\epsilon^2 + CT^{1/2}/\epsilon + C(R)T^{1/2} < 1$. As usual, iterating the procedure, we deduce the existence of a unique solution of Equation (2.4) in $L^8(\mathcal{A};\mathcal{U}_\epsilon^T)$ for all $T > 0$. \square

Our aim is now to get global existence for the process Ψ_ϵ , solution of Equation (2.1) which may be constructed from the above results. Let us set

$$\kappa_\epsilon^R(\psi_0, \omega) = \inf \left\{ t \geq 0, \|\Psi_\epsilon^R\|_{L^8(0,t;\mathbb{L}^4)} \geq R \right\},$$

which is a $\mathcal{G}_\epsilon(t)$ stopping time. It can be proved using Strichartz estimates and the integral formulation (2.4) (see [7, 8]) that κ_ϵ^R is nondecreasing with R and that $\Psi_\epsilon^R = \Psi_\epsilon^{R'}$ on $[0, \kappa_\epsilon^R]$ for $R < R'$. Thus we are able to define a local solution Ψ_ϵ to Equation (2.1) on the random interval $[0, \kappa_\epsilon^*(\psi_0))$, where $\kappa_\epsilon^*(\psi_0) = \lim_{R \rightarrow +\infty} \kappa_\epsilon^R$, by setting $\Psi_\epsilon(t) = \Psi_\epsilon^R(t)$ on $[0, \kappa_\epsilon^R]$. It remains to prove that $\kappa_\epsilon^* = +\infty$ almost surely. From the construction of the stopping time κ_ϵ^* it is clear that a.s,

$$\text{if } \kappa_\epsilon^*(\psi_0) < +\infty \text{ then } \lim_{t \nearrow \kappa_\epsilon^*(\psi_0)} \|\Psi_\epsilon^R\|_{L^8(0,t;\mathbb{L}^4)} = +\infty. \quad (2.6)$$

The arguments are adapted from [7]. We first prove the following lemma :

Lemma 2.3. *Let $\Psi_\epsilon(0) = \psi_0$ be as in Proposition 2.3 and Ψ_ϵ^R be the corresponding solution of (2.5); then for any $t < T$*

$$\|\Psi_\epsilon^R(t)\|_{\mathbb{L}^2} = \|\psi_0\|_{\mathbb{L}^2} \text{ a.s,}$$

and there is a constant $M_\epsilon > 0$, depending on T and $\|\psi_0\|_{\mathbb{L}^2}$, but independent of R , such that

$$\mathbb{E} \left(\|\Psi_\epsilon^R\|_{L^8(0,T;\mathbb{L}^4)} \right) \leq M_\epsilon(T). \quad (2.7)$$

Proof. To prove that the \mathbb{L}^2 norm of the solution Ψ_ϵ^R of (2.5) is constant in time, we apply formally the Ito formula to $\frac{1}{2} \|\Psi_\epsilon^R(t)\|_{\mathbb{L}^2}^2$ and notice that by integration by parts

$$\left(b' \sigma_3 \frac{\partial \Psi_\epsilon^R}{\partial x}, \Psi_\epsilon^R \right)_{\mathbb{L}^2} = - \left(\Psi_\epsilon^R, b' \sigma_3 \frac{\partial \Psi_\epsilon^R}{\partial x} \right)_{\mathbb{L}^2} = 0.$$

Since $\sigma_j^* = \sigma_j$, $j = 1, 2, 3$ we get $(\Psi_\epsilon^R(t), i\sigma_j \Psi_\epsilon^R(t))_{\mathbb{L}^2} = 0$, for $j = 1, 2, 3$. Moreover because the Itô corrections cancel with the damping term $-\frac{\gamma_\epsilon}{\epsilon} \Psi_\epsilon^R$ of Equation (2.5), we get $\|\Psi_\epsilon^R(t)\|_{\mathbb{L}^2} = \|\psi_0\|_{\mathbb{L}^2}$, $\forall t \leq T$. The computations can be made rigorous by a regularization procedure.

In order to prove (2.7), we follow the procedure in [7, 8]. Using the integral formulation (2.4), the conservation of the \mathbb{L}^2 -norm and Proposition 2.2, we obtain for a.e $\omega \in \Omega$ and for all time T_1 such that $T \geq T_1 > 0$:

$$\|\Psi_\epsilon^R\|_{L^8(0, T_1; \mathbb{L}^4)} \leq K_\epsilon(\omega) + CT_1^{1/2} \|\Psi_\epsilon^R\|_{L^8(0, T_1; \mathbb{L}^4)}^3, \quad (2.8)$$

where

$$K_\epsilon(\omega) = C \left(1 + \frac{T}{\epsilon^2} \right) \|\psi_0\|_{\mathbb{L}^2} + \frac{1}{\epsilon} \sum_{j=1}^2 \|J_{j,\epsilon} \Psi_\epsilon^R\|_{L^8(0, T; \mathbb{L}^4)}.$$

From inequality (2.8) it follows that $\|\Psi_\epsilon\|_{L^8(0, T_1; \mathbb{L}^4)} \leq 2K_\epsilon(\omega)$ if T_1 is chosen for example such that $T_1(\omega) = \inf \left(T, 2^{-6} (C^{1/2} K_\epsilon)^{-4} \right)$. If $T_1 < T$ we can reiterate the process on small time intervals $[lT_1, (l+1)T_1] \subset [0, T]$ (keeping R fixed and varying l) to get $\|\Psi_\epsilon\|_{L^8(lT_1, (l+1)T_1; \mathbb{L}^4)} \leq 2K_\epsilon(\omega)$. Summing these estimates, using $T_1 = 2^{-6} C^{-2} (K_\epsilon)^{-4}$ and Young inequality, we obtain

$$\|\Psi_\epsilon^R\|_{L^8(0, T; \mathbb{L}^4)} \leq C(T) (K_\epsilon(\omega))^5.$$

Taking the expectation in the above inequality, using Holder inequality and Lemma 2.2 we get the following estimate

$$\mathbb{E} \left(\|\Psi_\epsilon^R\|_{L^8(0, T; \mathbb{L}^4)} \right) \leq C(T) \left(\left(1 + \frac{T}{\epsilon^2} \right)^5 \|\psi_0\|_{\mathbb{L}^2}^5 + \frac{CT^{5/2}}{\epsilon^5} \|\psi_0\|_{\mathbb{L}^2}^5 \right), \quad (2.9)$$

from which (2.7) follows. \square

We easily deduce from Lemma 2.3 and (2.6) that $\kappa_\epsilon^* = +\infty$ a.s. and as in [7] the existence and uniqueness of a solution Ψ_ϵ of (2.1), a.s. in \mathcal{U}_c^T for any $T > 0$.

To end the proof of Theorem 1.1 we have to extend those results to the process X_ϵ . For a.e ω in \mathcal{A} and for each $t \geq 0$ we set $X_\epsilon(t) = Z_\epsilon^{-1}(t) \Psi_\epsilon(t)$. By definition of the process $Z_\epsilon^{-1}(t)$ (which in particular is measurable with respect to $\mathcal{G}_\epsilon(t)$) and properties of Ψ_ϵ , we easily deduce that $X_\epsilon(t)$ is adapted and continuous with values in \mathbb{L}^2 , and satisfy (1.11), hence is C^1 with values in \mathbb{H}^{-2} . By unitarity of Z_ϵ we also deduce that for all $t \geq 0$

$$\|\Psi_\epsilon(t)\|_{\mathbb{L}^2}^2 = (X_\epsilon(t), Z_\epsilon^{-1}(t) Z_\epsilon(t) X_\epsilon(t))_{\mathbb{L}^2} = \|X_\epsilon(t)\|_{\mathbb{L}^2}^2,$$

and since the coefficients of $Z_\epsilon^{-1}(t)$ are a.s uniformly bounded, $X_\epsilon \in L_{loc}^8(\mathbb{R}_+, \mathbb{L}^4)$ a.s; Theorem 1.1 is proved. \square

We now extend the previous global existence results to more regular initial data. T being fixed, we denote

$$\mathcal{V}^T = L^\infty(0, T; \mathbb{H}^1) \cap L^8(0, T; \mathbb{W}^{1,4}) \quad \text{and} \quad \mathcal{V}_c^T = C(0, T; \mathbb{H}^1) \cap L^8(0, T; \mathbb{W}^{1,4}).$$

Proposition 2.4. *Let $\Psi_\epsilon(0) = \psi_0 \in \mathbb{H}^1$ and let $T > 0$; then Equation (2.1) has a unique strong solution Ψ_ϵ with trajectories in $C(0, T; \mathbb{H}^1)$.*

Proof of Proposition 2.4. Let ψ_0 be in \mathbb{H}^1 . Given $\Psi_\epsilon^R \in L^8(\mathcal{A}; \mathcal{V}^T)$ we denote by $\mathcal{T} \Psi_\epsilon^R(t)$ the right hand side of (2.4) and $\mathcal{U}^T = L^\infty(0, T; \mathbb{L}^2) \cap L^8(0, T; \mathbb{L}^4)$. By Proposition 2.2, Lemma 2.2 applied to $\partial_x \Psi_\epsilon^R$ and Hölder inequality we deduce that

$$\mathbb{E} \left(\|\mathcal{T} \partial_x \Psi_\epsilon^R\|_{\mathcal{U}^T}^8 \right)^{1/8} \leq C \|\partial_x \psi_0\|_{\mathbb{L}^2} + \left(\frac{CT}{\epsilon^2} + \frac{CT^{1/2}}{\epsilon} + CT^{1/2} 4R^2 \right) \mathbb{E} \left(\|\partial_x \Psi_\epsilon^R\|_{\mathcal{U}^T}^8 \right)^{1/8}.$$

Therefore we conclude that choosing $R_0 = 2C \|\Psi_0\|_{\mathbb{H}^1}$, \mathcal{T} maps the closed ball of $L^8(\mathcal{A}; \mathcal{V}^T)$ with radius R_0 into itself, provided T is small enough depending only on R and ϵ , but not on R_0 . Combining with the fact that \mathcal{T} is a contraction in $L^8(\mathcal{A}; \mathcal{U}^T)$ and that the balls of $L^8(\mathcal{A}; \mathcal{V}^T)$ are closed for the norm in $L^8(\mathcal{A}; \mathcal{U}^T)$, we conclude to the existence of a unique fixed point $\Psi_\epsilon^R \in$

$L^8(\mathcal{A}; \mathcal{V}^T)$. Using Proposition 2.2 and Lemma 2.2, we get continuity of the solution in \mathbb{H}^1 . Since the cut-off only depends on the $L^8(0, T; \mathbb{L}^4(\mathbb{R}))$ norm, we deduce that there is a unique global solution Ψ_ϵ to (2.1) with paths in $C([0, T]; \mathbb{H}^1)$. Since the transformation Z_ϵ does not depend on x , we conclude that these results still hold true for X_ϵ . \square

Proposition 2.5. *Let $\Psi_\epsilon(0) = \psi_0 \in \mathbb{H}^m$, $m = 2, 3$. Let $T > 0$; then Equation (2.1) has a unique strong solution Ψ_ϵ with paths in $C([0, T]; \mathbb{H}^m)$, $m = 2, 3$.*

Proof. We consider Equation (2.5) but with $\Theta_R \left(\|\Psi_\epsilon^R\|_{L^8(0, t; \mathbb{L}^4)} \right)$ replaced by $\Theta_R \left(\|\Psi_\epsilon^R(t)\|_{\mathbb{H}^1}^2 \right)$. Given Ψ_ϵ^R in $L^8(\mathcal{A}; L^\infty(0, T; \mathbb{H}^2(\mathbb{R})))$ we denote by $\mathcal{T}\Psi_\epsilon^R(t)$ the right hand side of the integral formulation of this equation. We easily prove that \mathcal{T} maps the closed ball of $L^8(\mathcal{A}; L^\infty(0, T; \mathbb{H}^2(\mathbb{R})))$ with radius R_0 into itself, for $R_0 = 2C \|\Psi_0\|_{\mathbb{H}^2}$, provided that T is small enough, depending only on R and ϵ , but not on R_0 . Using that this ball is closed for the norm in $L^8(\mathcal{A}; L^\infty(0, T; \mathbb{H}^1(\mathbb{R})))$ and that \mathcal{T} is a contraction for the norm in $L^8(\mathcal{A}; L^\infty(0, T; \mathbb{H}^1(\mathbb{R})))$, we deduce that there exists a unique solution Ψ_ϵ with paths in $C(0, T; \mathbb{H}^2(\mathbb{R}))$ a.s, which is global since the solution is global in \mathbb{H}^1 . Existence and uniqueness in \mathbb{H}^3 can be proved by the same arguments. Again those results are easily extended to X_ϵ and this concludes the proof of Theorem 1.1. \square

3 The limiting equation : proof of Theorem 1.2

In order to prove a local existence and uniqueness result for the system (1.12) we use a compactness approach (see for example [14]) motivated by the fact that we do not know if Strichartz estimates are available for (1.12). We first prove existence of a unique solution in \mathbb{H}^1 for the linear part of the equation, defining then a random propagator, and then consider the nonlinear part as a perturbation. We will strongly use the fact that the nonlinearity is locally lipschitz in \mathbb{H}^1 . The regularity in \mathbb{H}^2 will follow with the same arguments as for Equation (2.1). Let us consider the linear part of Equation (1.12)

$$\begin{aligned} dX(t) &= \left(i \frac{d_0}{2} \frac{\partial^2 X}{\partial x^2} \right) dt - \sqrt{\gamma} \sum_{k=1}^3 \sigma_k \frac{\partial X(t)}{\partial x} \circ dW_k(t) \\ &= \left(i \frac{d_0}{2} + \frac{3\gamma}{2} \right) \frac{\partial^2 X}{\partial x^2} dt - \sqrt{\gamma} \sum_{k=1}^3 \sigma_k \frac{\partial X(t)}{\partial x} dW_k(t), \end{aligned} \quad (3.1)$$

with initial data $X(0) = v \in \mathbb{H}^2$. We introduce, for $\eta > 0$, the mollifier $J_\eta = \left(I - \eta \frac{\partial^2}{\partial x^2} \right)^{-1}$. We denote by X_η the solution of the regularized Ito equation

$$dX_\eta(t) = \left(i \frac{d_0}{2} + \frac{3\gamma}{2} \right) \frac{\partial^2 J_\eta X_\eta}{\partial x^2} dt - \sqrt{\gamma} \sum_{k=1}^3 \sigma_k \frac{\partial J_\eta X_\eta(t)}{\partial x} dW_k(t), \quad (3.2)$$

and $X_\eta(0) = v \in \mathbb{H}^2$. Since the operators $\partial_x^2 J_\eta$ and $\partial_x J_\eta$ are bounded from \mathbb{H}^1 into \mathbb{H}^1 (with constants depending on η), we easily get, thanks to the Doob inequality, Fubini theorem, the Ito isometry and the independence of $(W_k)_{k=1,2,3}$, the existence and uniqueness of a solution X_η to (3.2) with paths in $C([0, T], \mathbb{H}^2)$ for any $T > 0$. Moreover it is easy to see that the \mathbb{H}^2 norm of X_η is conserved since the Pauli matrices are hermitian. Consequently the process

$$M_\eta(t) = -X_\eta(t) + X_\eta(0) + \int_0^t \left(i \frac{d_0}{2} + \frac{3\gamma}{2} \right) \frac{\partial^2 J_\eta X_\eta}{\partial x^2} ds,$$

is a \mathcal{F}_t martingale with paths in $C([0, T], \mathbb{L}^2)$. Let us compute the quadratic variation. Let $a = (a_1, a_2)^t$ and $b = (b_1, b_2)^t$ be in \mathbb{L}^2 and $T \geq t \geq s \geq 0$; then

$$\begin{aligned} &\mathbb{E} \left((a, M_\eta(t))_{\mathbb{L}^2} (b, M_\eta(t))_{\mathbb{L}^2} - (a, M_\eta(s))_{\mathbb{L}^2} (b, M_\eta(s))_{\mathbb{L}^2} \middle| \mathcal{F}_s \right) \\ &= \gamma \sum_{k=1}^3 \mathbb{E} \left(\int_s^t \left(a, \sigma_k \frac{\partial J_\eta X_\eta}{\partial x} \right)_{\mathbb{L}^2} \left(b, \sigma_k \frac{\partial J_\eta X_\eta}{\partial x} \right)_{\mathbb{L}^2} du \middle| \mathcal{F}_s \right). \end{aligned}$$

We deduce that the quadratic variation of $M_\eta(t)$ is given by :

$$(b, \ll M_\eta(t) \gg a)_{\mathbb{L}^2} = \gamma \sum_{k=1}^3 \int_0^t \left(a, \sigma_k \frac{\partial J_\eta X_\eta}{\partial x} \right)_{\mathbb{L}^2} \left(b, \sigma_k \frac{\partial J_\eta X_\eta}{\partial x} \right)_{\mathbb{L}^2} du. \quad (3.3)$$

Using the conservation of the \mathbb{H}^2 norm and Equation (3.2) we get for all $0 \leq \alpha < \frac{1}{2}$

$$\mathbb{E} \left(\|X_\eta\|_{C^\alpha([0,T];\mathbb{L}^2)} \right) \leq C_\alpha(T), \quad (3.4)$$

where $C_\alpha(T)$ is a constant independent of η . Using Ascoli-Arzelà and Banach Alaoglu theorems, Markov inequality and inequality (3.4), we get that the sequence $(\mathcal{L}(X_\eta))_{\eta>0}$ is tight on $C_w([0,T], \mathbb{H}^1(\mathbb{R})) \cap L_w^\infty(0,T, \mathbb{H}^2)$. The Skorokhod theorem ([3],[13]) implies that on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$, there exist a sequence of stochastic processes $(\tilde{X}_\eta)_{\eta>0}$, and a process \tilde{X} , such that :

$$\mathcal{L}(\tilde{X}_\eta) = \mathcal{L}(X_\eta), \quad \mathcal{L}(\tilde{X}) = \mathcal{L}(X),$$

and $\lim_{\eta \rightarrow 0} \tilde{X}_\eta = \tilde{X}$, $\tilde{\mathbb{P}} - a.s$ in $C_w([0,T], \mathbb{H}^1) \cap L_w^\infty(0,T, \mathbb{H}^2)$. For all $\eta > 0$ and $t \in [0,T]$ we define the process

$$\widetilde{M}_\eta(t) = -\tilde{X}_\eta(t) + \tilde{X}_\eta(0) + \int_0^t \left(\frac{id_0}{2} + \frac{3\gamma}{2} \right) \frac{\partial^2 J_\eta \tilde{X}_\eta}{\partial x^2}(s) ds.$$

We deduce from the above laws equality that $\widetilde{M}_\eta(t)$ is a square integrable continuous martingale with values in \mathbb{L}^2 with respect to the filtration $\tilde{\mathcal{F}}_t$ and that the quadratic variation $\ll \widetilde{M}_\eta(t) \gg$ is given by formula (3.3) replacing X_η by \tilde{X}_η . Let $a \in \mathbb{H}^1$, then by the above martingale property we get for all $s \leq t$:

$$\mathbb{E} \left(\left(a, \widetilde{M}_\eta(t) - \widetilde{M}_\eta(s) \right)_{\mathbb{L}^2} \middle| \tilde{\mathcal{F}}_s \right) = 0.$$

Using the almost sure convergence in $C_w([0,T], \mathbb{H}^1(\mathbb{R}))$ of X_η , the boundedness in \mathbb{H}^{-1} of the operator J_η and the conservation of the \mathbb{H}^1 norm, we get the almost sure convergence in $C_w([0,T], \mathbb{H}^{-1}(\mathbb{R}))$ of \widetilde{M}_η to \widetilde{M} , where

$$\widetilde{M}(t) = \tilde{X}(t) - \tilde{X}(0) - \int_0^t \left(\frac{id_0}{2} + \frac{3\gamma}{2} \right) \frac{\partial^2 \tilde{X}}{\partial x^2}(s) ds.$$

Hence \widetilde{M} is a weakly continuous martingale with values in \mathbb{H}^{-1} . Moreover using the a.s convergence in $C_w([0,T], \mathbb{H}^1(\mathbb{R}))$ and dominated convergence theorem, we get for all $t, s \in [0,T], t \geq s$ and for any $a, b \in \mathbb{H}^1$,

$$\lim_{\eta \rightarrow 0} \mathbb{E} \left(\left\langle b, \ll \widetilde{M}_\eta(t) \gg a \right\rangle \middle| \tilde{\mathcal{F}}_s \right) = \gamma \sum_{k=1}^3 \mathbb{E} \left(\int_0^t \left\langle a, \sigma_k \frac{\partial \tilde{X}}{\partial x}(u) \right\rangle \left\langle b, \sigma_k \frac{\partial \tilde{X}}{\partial x}(u) \right\rangle du \middle| \tilde{\mathcal{F}}_s \right).$$

Thus the quadratic variation $\langle b, \ll \widetilde{M}(t) \gg a \rangle$ is given, for all $t \in [0,T]$, by

$$\langle b, \ll \widetilde{M}(t) \gg a \rangle = \gamma \sum_{k=1}^3 \int_0^t \left\langle a, \sigma_k \frac{\partial \tilde{X}}{\partial x}(u) \right\rangle \left\langle b, \sigma_k \frac{\partial \tilde{X}}{\partial x}(u) \right\rangle du. \quad (3.5)$$

Noticing that $\widetilde{M}(0) = 0$ and using the representation theorem for continuous square integrable martingales we obtain that, on a possibly enlarged space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$, one can find a Brownian motion $\widetilde{W} = (\widetilde{W}_1, \widetilde{W}_2, \widetilde{W}_3)$ such that

$$\langle a, \widetilde{M}(t) \rangle = \sqrt{\gamma} \int_0^t \sum_{k=1}^3 \left\langle a, \sigma_k \frac{\partial \tilde{X}}{\partial x}(s) \right\rangle d\widetilde{W}_k(s).$$

Thus we deduce that (\tilde{X}, \tilde{W}) is a weak solution of Equation (3.1) on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ with values in $C_w([0, T], \mathbb{H}^1(\mathbb{R})) \cap L^\infty(0, T, \mathbb{H}^2)$. To conclude the proof we have to prove pathwise uniqueness of the solution and strong continuity in \mathbb{H}^1 . Since $\tilde{X} \in L^\infty(0, T, \mathbb{H}^2)$ is solution of (3.1), we easily deduce that $\tilde{X} \in C^\alpha([0, T], \mathbb{L}^2)$ for any $\alpha \in [0, 1/2[$. By interpolation we obtain that $\tilde{X} \in C([0, T], \mathbb{H}^1)$. It follows, using Itô formula, that pathwise uniqueness holds for Equation (3.1) in $C([0, T], \mathbb{H}^1)$. This implies, by the Yamada-Watanabe theorem, that the solution exists in the strong sense. Thus we can define a random unitary propagator $U(t, s)$ which is strongly continuous from \mathbb{H}^2 into \mathbb{H}^1 . This random propagator can be extended to a random propagator from \mathbb{H}^1 into \mathbb{H}^1 using the continuity of X in \mathbb{H}^1 , the density of \mathbb{H}^2 into \mathbb{H}^1 and the isometry property of $U(t, s)$ in \mathbb{H}^1 .

The local existence of the non linear problem (1.12) in \mathbb{H}^1 follows from the construction of the random propagator U : we consider a cut-off function $\Theta \in C_c^\infty(\mathbb{R})$, $\Theta \geq 0$ satisfying

$$\Theta_R \left(\|X(t)\|_{\mathbb{H}^1}^2 \right) = \begin{cases} 1 & \text{if } \|X(t)\|_{\mathbb{H}^1}^2 \leq R \\ 0 & \text{if } \|X(t)\|_{\mathbb{H}^1}^2 \geq 2R. \end{cases}$$

and first construct a solution X^R of the cut-off equation :

$$idX^R(t) + \left(\frac{d_0}{2} \frac{\partial^2 X^R}{\partial x^2} + \Theta_R \left(\|X^R(t)\|_{\mathbb{H}^1}^2 \right) F(X^R(t)) \right) dt + i\sqrt{\gamma} \sum_{k=1}^3 \sigma_k \frac{\partial X^R(t)}{\partial x} \circ dW_k(t) = 0, \quad (3.6)$$

with initial data $X^R(0) = v \in \mathbb{H}^1$ and whose integral formulation is given a.e by

$$X^R(t) = U(t, 0)v + i \int_0^t \Theta_R \left(\|X^R(s)\|_{\mathbb{H}^1}^2 \right) U(t, s) F(X^R(s)) ds. \quad (3.7)$$

The existence and uniqueness of $X^R \in L^\rho(\Omega; C(0, T; \mathbb{H}^1))$, solution of (3.7), is easily obtained by a fixed point argument since the nonlinear term is globally lipschitz. Introducing the nondecreasing stopping time

$$\tau^R = \inf \left\{ t \geq 0, \|X^R(t)\|_{\mathbb{H}^1}^2 \geq R \right\},$$

we may then define a local solution X to Equation (1.12) on a random interval $[0, \tau^*(v))$, where $\tau^*(v) = \lim_{R \rightarrow +\infty} \tau^R$ almost surely, by setting $X(t) = X^R(t)$ on $[0, \tau^R]$. Then for any stopping time $\tau < \tau^*$ we have constructed a unique local solution with paths a.s in $C([0, \tau], \mathbb{H}^1)$. It follows from the construction of the stopping time τ^* that if $\tau^* < +\infty$ then $\limsup_{t \rightarrow \tau^*} \|X(t)\|_{\mathbb{H}^1} = +\infty$. Let us now prove that if $v \in \mathbb{H}^2$ then the maximal stopping time satisfies the following alternative

$$\tau^* < +\infty \text{ or } \lim_{t \rightarrow \tau^*} \|X(t)\|_{\mathbb{H}^1} = +\infty. \quad (3.8)$$

We note that the random propagator commutes with derivation. Hence if $v \in \mathbb{H}^2$, then $U(\cdot, 0)v \in C([0, T], \mathbb{H}^2)$. We easily deduce, using Equation (3.1) and interpolating \mathbb{H}^1 between \mathbb{H}^2 and \mathbb{L}^2 , that $U(\cdot, 0)v \in C^\beta([0, T], \mathbb{H}^1)$ for $\beta \in [0, 1/4[$. By a fixed point argument in \mathbb{H}^2 and Equation (3.7), we conclude that $X \in C^\beta([0, \tau], \mathbb{H}^1)$ for any stopping time $\tau < \tau^*$ and for the same maximal time existence τ^* . Hence using the condition on τ^* and uniform continuity of X in \mathbb{H}^1 , we get that (3.8) holds.

Remark 3.1. *We were not able to prove the global wellposedness for Equation (1.12). Due to the lack of Strichartz estimates, we cannot control the evolution of the \mathbb{H}^1 norm. Moreover the evolution of the energy associated to the deterministic part of Equation (1.12) which is given in the next Lemma does not seem to provide such a control.*

Lemma 3.1. *Let the functional H be defined for $u \in \mathbb{H}^1(\mathbb{R})$ by*

$$H(u) = \frac{1}{4} \int_{\mathbb{R}} \left| \frac{\partial u}{\partial x} \right|^2 dx - \frac{2}{9} \int_{\mathbb{R}} |u|^4 dx.$$

Then for any stopping time τ such that $\tau < \tau^*$, we have

$$\begin{aligned}
H(X(\tau)) &= H(X_0) + \sqrt{\gamma} \frac{8}{9} \sum_{k=1}^3 \int_0^\tau \left\langle |X|^2 X, \sigma_k \frac{\partial X}{\partial x} \right\rangle \circ dW_k(s) \\
&= H(X_0) + \sqrt{\gamma} \frac{8}{9} \sum_{k=1}^3 \int_0^\tau \left\langle |X|^2 X, \sigma_k \frac{\partial X}{\partial x} \right\rangle dW_k(s) \\
&\quad + \frac{2\gamma}{9} \int_0^\tau \int_{\mathbb{R}} \left(\partial_x |X_1|^2 + \partial_x |X_2|^2 \right)^2 dx ds - \frac{4}{9} \gamma \int_0^\tau \int_{\mathbb{R}} \left| X_1 \frac{\partial X_2}{\partial x} - \frac{\partial X_1}{\partial x} X_2 \right|^2 dx ds \\
&\quad + \frac{12}{9} \gamma \int_0^\tau \int_{\mathbb{R}} \partial_x |X_1|^2 \partial_x |X_2|^2 dx ds.
\end{aligned}$$

Proof. The first equality follows by Stratonovich differential calculus applied to the functional H and because the process X is solution of (1.12). The calculation can be made rigorous by localization (H is C^2 but not bounded) and regularization through convolution. The second equality is obtained writing the evolution of H in its Ito formulation, that is

$$\begin{aligned}
H(X(\tau)) &= H(X_0) + \sqrt{\gamma} \frac{8}{9} \sum_{k=1}^3 \int_0^\tau \left\langle |X|^2 X, \sigma_k \frac{\partial X}{\partial x} \right\rangle dW_k(s) \\
&\quad + \frac{24}{9} \gamma \int_0^\tau \langle X, \partial_x X \mathcal{R}e(X \cdot \partial_x \bar{X}) \rangle ds - \frac{8}{9} \gamma \sum_{k=1}^3 \int_0^\tau \langle X, \sigma_k \partial_x X \mathcal{R}e(X \cdot \bar{\sigma}_k \partial_x \bar{X}) \rangle ds,
\end{aligned}$$

where we used the unitary of the Pauli matrices and $\sigma_k = \sigma_k^*$, for $k = 1, 2, 3$. Easy calculations lead to the expression given above. \square

4 Diffusion limit of the Manakov-PMD equation : Proof of Theorem 1.3

The aim of this part is the proof of the convergence result given in Theorem 1.3. For this purpose we have to cut-off Equation (1.11) in order to get uniform bounds, with respect to ϵ , of high order moments of the \mathbb{H}^2 norm of the solution. Let us denote by X_ϵ^R the solution of the cut-off equation

$$\begin{cases} i \frac{\partial X_\epsilon^R(t)}{\partial t} + \frac{ib'}{\epsilon} \boldsymbol{\sigma}(\nu_\epsilon(t)) \frac{\partial X_\epsilon^R}{\partial x} + \frac{d_0}{2} \frac{\partial^2 X_\epsilon^R}{\partial x^2} + \Theta_R \left(\|X_\epsilon^R(t)\|_{\mathbb{H}^1}^2 \right) F_{\nu_\epsilon(t)}(X_\epsilon^R) = 0 \\ X_0 = v \in \mathbb{H}^3(\mathbb{R}). \end{cases} \quad (4.1)$$

The proof will consist of the following steps :

- 5.1 We prove uniform bounds on the solution X_ϵ^R of (4.1). These bounds will enable us to prove tightness on \mathcal{K} .
- 5.2 We use the perturbed test function method to get convergence of the generators in some sense [15, 18, 22]. This method formally gives a candidate for the limit process.
- 5.3 Setting $Z_\epsilon^R = \left(X_\epsilon^R, \|X_\epsilon^R(\cdot)\|_{\mathbb{H}^1}^2 \right)$, we then prove that the family of laws $\mathcal{L}(Z_\epsilon^R) = \mathbb{P} \circ (Z_\epsilon^R)^{-1}$ is tight on \mathcal{K} and we deduce that the process Z_ϵ^R converges in law, up to a subsequence.
- 5.4 Combining the previous steps and using the martingale problem formulation, we identify the limit and conclude to the weak convergence of the whole sequence. Finally we get rid of the cut off and we conclude that for any stopping time $\tau < \tau^*$, the sequence $(X_\epsilon^\tau)_{\epsilon > 0}$ converges in law to X^τ in $C([0, T], \mathbb{H}^1)$ using the Skorokhod Theorem.

4.1 Uniform bounds on X_ϵ^R

Recall that a unique solution $\Psi_\epsilon^R \in C(\mathbb{R}_+, \mathbb{H}^3)$ of the following equation exists (see Section 2).

$$\begin{aligned} id\Psi_\epsilon^R(t) + \left\{ \frac{ib'}{\epsilon} \sigma_3 \frac{\partial \Psi_\epsilon^R(t)}{\partial x} + \frac{d_0}{2} \frac{\partial^2 \Psi_\epsilon^R(t)}{\partial x^2} + \frac{\gamma_s}{\epsilon^2} \sigma_3 \Psi_\epsilon^R(t) + \frac{i}{\epsilon^2} \gamma_c \Psi_\epsilon^R(t) \right\} dt \\ - \frac{\sqrt{\gamma_c}}{\epsilon} \sigma_1 \Psi_\epsilon^R d\widetilde{W}_1(t) - \frac{\sqrt{\gamma_c}}{\epsilon} \sigma_2 \Psi_\epsilon^R d\widetilde{W}_2(t) + \Theta_R \left(\|\Psi_\epsilon^R(t)\|_{\mathbb{H}^1}^2 \right) f(\Psi_\epsilon^R(t)) dt = 0. \end{aligned} \quad (4.2)$$

A solution X_ϵ^R to (4.1) is then easily deduced from $X_\epsilon^R(t) = Z_\epsilon^{-1}(t) \Psi_\epsilon^R(t)$.

Lemma 4.1. *Let $\psi_0 \in \mathbb{H}^3$ and Ψ_ϵ^R be the solution of (4.2); then for all $T > 0$ there exists a positive constant $C(R, T)$ independent of ϵ , such that, a.s for every t in $[0, T]$,*

$$\|\Psi_\epsilon^R(t)\|_{\mathbb{H}^3} \leq C(R, T).$$

Similar bounds hold for $X_\epsilon^R(t) = Z_\epsilon^{-1}(t) \Psi_\epsilon^R(t)$ for any $t \in [0, T]$ since Z_ϵ^{-1} is almost surely bounded.

Proof of Lemma 4.1. The bounds on the \mathbb{H}^3 norm are obtained using an energy method. Using a regularization procedure, Ito formula applied to $\|\partial_x \Psi_\epsilon^R(t)\|_{\mathbb{L}^2}^2$ and Equation (4.2), we obtain for all $t \in [0, T]$

$$\begin{aligned} \|\partial_x \Psi_\epsilon^R(t)\|_{\mathbb{L}^2}^2 &\leq \|\partial_x \psi_0\|_{\mathbb{L}^2}^2 + 2 \int_0^t \Theta_R \left(\|\Psi_\epsilon^R(s)\|_{\mathbb{H}^1}^2 \right) \|\partial_x f(\Psi_\epsilon^R(s))\|_{\mathbb{L}^2} \|\partial_x \Psi_\epsilon^R(s)\|_{\mathbb{L}^2} ds \\ &\leq \|\partial_x \psi_0\|_{\mathbb{L}^2}^2 + C(R) \int_0^t \|\partial_x \Psi_\epsilon^R(s)\|_{\mathbb{L}^2}^2 ds. \end{aligned}$$

By Gronwall Lemma we deduce that

$$\|\partial_x \Psi_\epsilon^R(t)\|_{\mathbb{L}^2}^2 \leq \|\partial_x \psi_0\|_{\mathbb{L}^2}^2 \exp(C(R)T).$$

Using the same procedure for $\|\partial_x^2 X_\epsilon^R\|_{\mathbb{L}^2}^2$, Gagliardo-Nirenberg and Young inequalities,

$$\begin{aligned} \|\partial_x^2 \Psi_\epsilon^R(t)\|_{\mathbb{L}^2}^2 - \|\partial_x^2 \psi_0\|_{\mathbb{L}^2}^2 \\ \leq C \int_0^t \Theta_R \left(\|\Psi_\epsilon^R(s)\|_{\mathbb{H}^1}^2 \right) \left(\left(\|\Psi_\epsilon^R(s)\|_{\mathbb{L}^\infty}^2 + 1 \right) \|\partial_x^2 \Psi_\epsilon^R(s)\|_{\mathbb{L}^2}^2 + \|\Psi_\epsilon^R(s)\|_{\mathbb{L}^\infty}^4 \|\partial_x \Psi_\epsilon^R(s)\|_{\mathbb{L}^2}^6 \right) ds. \end{aligned}$$

By Sobolev embeddings, properties of the cut off function and again Gronwall Lemma, we conclude

$$\|\partial_x^2 \Psi_\epsilon^R(t)\|_{\mathbb{L}^2}^2 \leq \|\partial_x^2 \psi_0\|_{\mathbb{L}^2}^2 C(R, T).$$

A bound on $\|\partial_x^3 X_\epsilon^R\|_{\mathbb{L}^2}^2$ may be obtained similarly using the previous estimates and Gronwall Lemma. \square

Remark 4.1. *To prove the convergence result we need an initial data in $\mathbb{H}^3(\mathbb{R})$. We will explain later where exactly we need this extra regularity but this is mainly due to the fact that we prove tightness in $C([0, T], \mathbb{H}^1)$.*

Remark 4.2. *Note that we first prove convergence in law for the couple of random variables $(X_\epsilon^R, \|X_\epsilon^R(\cdot)\|_{\mathbb{H}^1}^2)$. This is due to the fact that the cut off is not continuous for the weak topology in \mathbb{H}^1 neither for the strong topology in \mathbb{H}_{loc}^1 . These arguments have already been used in [9].*

4.2 The perturbed test function method

Note that the process X_ϵ^R is not Markov due to the presence of ν_ϵ . However $(X_\epsilon^R, \nu_\epsilon)$ is Markov, by construction of ν . We denote by \mathcal{L}_ϵ^R its infinitesimal generator. Let us compute $\mathcal{L}_\epsilon^R f$ for f sufficiently smooth such that f maps $\mathbb{H}^{-1} \times \mathbb{S}^3$ into \mathbb{R} and is of class C_b^2 . Let $\langle \cdot, \cdot \rangle$ be the duality

product between \mathbb{H}^1 and \mathbb{H}^{-1} . Then, for $\epsilon > 0$ and for X_ϵ^R solution of the Manakov-PMD Equation (4.1),

$$\begin{aligned} f(X_\epsilon^R(t), \nu_\epsilon(t)) - f(v, y) &= f(X_\epsilon^R(t), \nu_\epsilon(t)) - f(v, \nu_\epsilon(t)) + f(v, \nu_\epsilon(t)) - f(v, y) \\ &= \langle D_v f(v, \nu_\epsilon(t)), X_\epsilon^R(t) - v \rangle + R(X_\epsilon^R(t), v) + f(v, \nu_\epsilon(t)) - f(v, y), \end{aligned}$$

where

$$R(X_\epsilon^R(t), v) = \int_0^1 (1 - \theta) \langle D_v^2 f(v + \theta(X_\epsilon^R(t) - v))(X_\epsilon^R(t) - v), X_\epsilon^R(t) - v \rangle d\theta,$$

and $D_v^2 f(v) \in \mathcal{L}(\mathbb{H}^{-1}, \mathbb{H}^1)$. Thus

$$\begin{aligned} &\frac{1}{t} \mathbb{E} \left(f(X_\epsilon^R(t), \nu_\epsilon(t)) - f(v, y) \mid (X(0), \nu(0)) = (v, y) \right) \\ &= \mathbb{E} \left(\left\langle D_v f(v, \nu_\epsilon(t)), \frac{X_\epsilon^R(t) - v}{t} \right\rangle \mid (X(0), \nu(0)) = (v, y) \right) \\ &\quad + \mathbb{E} \left(\frac{R(X_\epsilon^R(t), v)}{t} \mid X(0) = v \right) + \mathbb{E} \left(\frac{f(v, \nu_\epsilon(t)) - f(v, y)}{t} \mid \nu(0) = y \right). \end{aligned}$$

We know by Theorem 1.1 that if $v \in \mathbb{H}^3$ then $X_\epsilon^R \in C^1([0, T], \mathbb{H}^1)$. Thus by the mean value Theorem, Equation (4.1), the almost sure boundedness of ν , Lemma 4.1 and the conservation of the \mathbb{L}^2 norm :

$$\begin{aligned} \frac{1}{t} \|X_\epsilon^R(t) - v\|_{\mathbb{L}^2} &\leq \sup_{s \in [0, t]} \|\partial_s X_\epsilon^R(s)\|_{\mathbb{L}^2} \\ &\leq \sup_{s \in [0, t]} \left(\left\| \frac{b'}{\epsilon} \sigma(\nu_\epsilon(s)) \partial_x X_\epsilon^R(s) \right\|_{\mathbb{L}^2} + \left\| \frac{d_0}{2} \partial_x^2 X_\epsilon^R(s) \right\|_{\mathbb{L}^2} + \left\| \Theta_R \left(\|X_\epsilon^R(s)\|_{\mathbb{H}^1}^2 \right) F_{\nu_\epsilon(s)}(X_\epsilon^R(s)) \right\|_{\mathbb{L}^2} \right) \\ &\leq \left(\frac{b'}{\epsilon} + \frac{d_0}{2} \right) C(R, T) + 2RC \|v\|_{\mathbb{L}^2}. \end{aligned}$$

Thus by the boundedness of $D_v^2 f$, the continuity of $t \mapsto X_\epsilon^R(t)$ in \mathbb{L}^2 and the previous bounds, we conclude that

$$\frac{R(X_\epsilon^R(t), v)}{t} \leq C(R, T, \epsilon) \sup_{w \in \mathbb{H}^1} \|D_v^2 f(w)\|_{\mathcal{L}(\mathbb{H}^{-1}, \mathbb{H}^1)} (1 + \|v\|_{\mathbb{L}^2}) \|X_\epsilon(t) - v\|_{\mathbb{L}^2} \xrightarrow[t \rightarrow 0]{} 0.$$

Now, we perform the change of variables $t' = t/\epsilon^2$, to get

$$\frac{1}{t} \mathbb{E} (f(v, \nu_\epsilon(t)) - f(v, y) \mid \nu(0) = y) = \frac{1}{\epsilon^2 t'} \mathbb{E} (f(v, \nu(t')) - f(v, y) \mid \nu(0) = y).$$

Thus, using the Markov property of the process ν , and using Equation (4.1) again, we get an expression of the infinitesimal generator \mathcal{L}_ϵ^R of the Markov process $(X_\epsilon^R, \nu_\epsilon)$:

$$\begin{aligned} \mathcal{L}_\epsilon^R f(v, y) &= \lim_{t \rightarrow 0} \frac{1}{t} \left(\mathbb{E} \left(f(X_\epsilon^R(t), \nu_\epsilon(t)) - f(v, y) \mid (X(0), \nu(0)) = (v, y) \right) \right) \\ &= \langle D_v f(v, y), \partial_t X_\epsilon^R(t) |_{t=0} \rangle + \frac{1}{\epsilon^2} \mathcal{L}_\nu f(v, y) \\ &= \left\langle D_v f(v, y), \frac{id_0}{2} \frac{\partial^2 v}{\partial x^2} + i \Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v) \right\rangle \\ &\quad - \frac{1}{\epsilon} \left\langle D_v f(v, y), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle + \frac{1}{\epsilon^2} \mathcal{L}_\nu f(v, y), \end{aligned} \tag{4.3}$$

where \mathcal{L}_ν is the infinitesimal generator of ν and \mathcal{D}_ν its domain. The perturbed test function method gives (by identifying its infinitesimal generator) an idea of the limit law of the sequence $(X_\epsilon^R)_{\epsilon > 0}$. It provides in addition convergences that are useful to prove the weak convergence of the sequence of measures $(\mathcal{L}(X_\epsilon^R))_{\epsilon > 0}$.

Proposition 4.1 (Perturbed test function method). *There exists a limiting infinitesimal generator $(\mathcal{L}^R, \mathcal{D}^R)$ such that for all sufficiently smooth and real valued functions $f \in \mathcal{D}^R$ and for all positive ϵ , there exists a test function f_ϵ and positive constants $C_1(K)$ and $C_2(K)$ satisfying*

$$\sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} |f_\epsilon(v, y) - f(v)| \leq \epsilon C_1(K) \quad (4.4)$$

$$\sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} |\mathcal{L}_\epsilon^R f_\epsilon(v, y) - \mathcal{L}^R f(v)| \leq \epsilon C_2(K), \quad (4.5)$$

where $\mathcal{B}(K)$ denotes the closed ball of $\mathbb{H}^3(\mathbb{R})$ with radius K .

Proof. The idea is to prove that for all suitable test function f , one can find a function f_ϵ of the form

$$f_\epsilon(v, y) = f(v) + \epsilon f^1(v, y) + \epsilon^2 f^2(v, y), \quad (4.6)$$

such that Proposition 4.1 holds. We plug this expression of f_ϵ into (4.3) and formally compute the expression of $\mathcal{L}_\epsilon^R f_\epsilon$:

$$\begin{aligned} \mathcal{L}_\epsilon^R f_\epsilon(v, y) &= \left\langle D_v f(v), \frac{id_0}{2} \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v) \right\rangle - \left\langle D_v f^1(v, y), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle + \mathcal{L}_\nu f^2(v, y) \\ &\quad + \frac{1}{\epsilon} \mathcal{L}_\nu f^1(v, y) - \frac{1}{\epsilon} \left\langle D_v f(v), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle \\ &\quad + \epsilon \left\langle D_v f^1(v, y), \frac{id_0}{2} \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v) \right\rangle - \epsilon \left\langle D_v f^2(v, y), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle \\ &\quad + \epsilon^2 \left\langle D_v f^2(v, y), \frac{id_0}{2} \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v) \right\rangle, \end{aligned} \quad (4.7)$$

and we notice that $\mathcal{L}_\nu f(v)$ is identically zero because f does not depend on $\nu = (\nu_1, \nu_2)$. The aim is to wisely choose the functions f^1 and f^2 and the regularity of f so that $\mathcal{L}_\epsilon^R f_\epsilon$ is well defined and that f_ϵ and $\mathcal{L}_\epsilon^R f_\epsilon$ converge in the sense of Proposition 4.1. In particular, we need to cancel the terms with a factor $1/\epsilon$ and we need the terms with factors ϵ or ϵ^2 to be $\mathcal{O}(\epsilon)$ on bounded sets. In order to cancel the $1/\epsilon$ terms, we look for a function f^1 solution of the Poisson equation :

$$\mathcal{L}_\nu f^1(v, y) = \left\langle D_v f(v), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle. \quad (4.8)$$

By Corollary 5.1, we know that

$$\mathbb{E}_\Lambda(g_j(\nu)) = 0 \quad \forall j = 1, 2, 3.$$

We deduce that $\left\langle D_v f(v), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle$, which is a linear combination of $m_j = g_j(y)$ (see (1.4)), is of null mass with respect to the invariant measure Λ . Hence $\left\langle D_v f(v), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle$ is a function of $y \in \mathbb{S}^3$, which satisfies the assumptions of Proposition 5.1, provided that f is sufficiently smooth i.e $f \in C^1(\mathbb{H}^{-1})$ and $v \in \mathbb{L}^2$. It follows that the solution f^1 of the Poisson Equation (4.8) can be written as :

$$\begin{aligned} f^1(v, y) &= \mathcal{L}_\nu^{-1} \left(\left\langle D_v f(v), b' \sigma(\cdot) \frac{\partial v}{\partial x} \right\rangle \right) (y) \\ &= - \left\langle D_v f(v), b' \tilde{\sigma}(y) \frac{\partial v}{\partial x} \right\rangle, \end{aligned} \quad (4.9)$$

where

$$\tilde{\sigma}(y) = \int_0^{+\infty} \mathbb{E}(\sigma(\nu(t)) | \nu(0) = y) dt. \quad (4.10)$$

By Proposition 5.1, there is a positive constant M such that

$$\|\tilde{\sigma}(y)\|_\infty \leq M, \quad \forall y \in \mathbb{S}^3, \quad (4.11)$$

and $f^1(v, y)$ is a continuous bounded function of y for $v \in \mathbb{L}^2$. We now have to choose the function f^2 , but we cannot choose $\mathcal{L}_\nu f^2$ cancelling the terms

$$\left\langle D_v f(v), \frac{id_0}{2} \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v) \right\rangle - \left\langle D_v f^1(v, y), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle,$$

because they do not satisfy the null mass condition with respect to Λ . Hence we look for a solution f^2 of the Poisson equation :

$$\begin{aligned} \mathcal{L}_\nu f^2(v, y) &= - \left\langle D_v f(v), i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v) \right\rangle + \left\langle D_v f(v), i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F(v) \right\rangle \\ &\quad + \left\langle D_v f^1(v, y), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle - \mathbb{E}_\Lambda \left(\left\langle D_v f^1(v, y), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle \right), \end{aligned} \quad (4.12)$$

where, due to (4.9),

$$\begin{aligned} &\left\langle D_v f^1(v, y), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle \\ &= -(b')^2 \left\langle D_v^2 f(v) \tilde{\sigma}(y) \frac{\partial v}{\partial x}, \sigma(y) \frac{\partial v}{\partial x} \right\rangle - (b')^2 \left\langle D_v f(v), \tilde{\sigma}(y) \sigma(y) \frac{\partial^2 v}{\partial x^2} \right\rangle. \end{aligned} \quad (4.13)$$

Moreover thanks to expression (4.13), Fubini Theorem and Corollary 5.1

$$\begin{aligned} &-\mathbb{E}_\Lambda \left(\left\langle D_v f^1(v, y), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle \right) \\ &= (b')^2 \sum_{j,k=1}^3 \left\langle D_v^2 f(v) \sigma_k \frac{\partial v}{\partial x}, \sigma_j \frac{\partial v}{\partial x} \right\rangle \int_0^{+\infty} \mathbb{E}_\Lambda (g_k(\nu(t)) g_j(\nu(0))) dt \\ &\quad + (b')^2 \sum_{j,k=1}^3 \left\langle D_v f(v), \sigma_k \sigma_j \frac{\partial^2 v}{\partial x^2} \right\rangle \int_0^{+\infty} \mathbb{E}_\Lambda (g_k(\nu(t)) g_j(\nu(0))) dt \\ &= \frac{\gamma}{2} \sum_{k=1}^3 \left\langle D_v^2 f(v) \sigma_k \frac{\partial v}{\partial x}, \sigma_k \frac{\partial v}{\partial x} \right\rangle + \frac{3\gamma}{2} \left\langle D_v f(v), \frac{\partial^2 v}{\partial x^2} \right\rangle, \end{aligned} \quad (4.14)$$

where $\gamma = (b')^2 / 6\gamma_c$. Provided that f is of class $C^2(\mathbb{H}^{-1})$ and $v \in \mathbb{H}^1$ and because $f^1(v, \cdot)$ is of class $C_b^2(\mathbb{S}^3)$ for any $v \in \mathbb{H}^1$, we can now define, by Proposition 5.1, a unique solution, up to a constant, to the Poisson Equation (4.12). This solution f^2 is expressed as :

$$\begin{aligned} f^2(v, y) &= \mathcal{L}_\nu^{-1} \left(\left\langle D_v f(v), i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) (F_y(v) - F(v)) \right\rangle \right) \\ &\quad - \mathcal{L}_\nu^{-1} \left(\left\langle D_v f^1(v, y), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle - \mathbb{E}_\Lambda \left(\left\langle D_v f^1(v, y), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle \right) \right) \\ &= \left\langle D_v f(v), i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) \tilde{F}(v, y) \right\rangle \\ &\quad - (b')^2 \sum_{k,l=1}^3 \left\langle D_v^2 f(v) \sigma_k \frac{\partial v}{\partial x}, \sigma_l \frac{\partial v}{\partial x} \right\rangle \tilde{g}_{k,l}(y) - \left\langle D_v f(v), (b')^2 \tilde{\sigma}(y) \frac{\partial^2 v}{\partial x^2} \right\rangle, \end{aligned} \quad (4.15)$$

where

$$\tilde{F}(v, y) = \int_0^{+\infty} \mathbb{E} (F_{\nu(t)}(v) - F(v) | \nu(0) = y) dt,$$

and

$$\tilde{g}_{k,l}(y) = \int_0^{+\infty} \left(\int_t^{+\infty} \mathbb{E} (g_k(\nu(s)) g_l(\nu(t)) | \nu(0) = y) ds - \frac{\gamma}{2(b')^2} \delta_{kl} \right) dt,$$

and

$$\tilde{\sigma}(y) = \int_0^{+\infty} \left(\int_t^{+\infty} \mathbb{E} (\sigma(\nu(s)) \sigma(\nu(t)) | \nu(0) = y) ds - \frac{3\gamma}{2(b')^2} \right) dt.$$

Replacing $\mathcal{L}_v f^1$ and $\mathcal{L}_v f^2$ in Equation (4.7), respectively by the right hand side of (4.8) and (4.12) and using expression (4.14) we get :

$$\begin{aligned}
\mathcal{L}_\epsilon^R f_\epsilon(v, y) &= \left\langle D_v f(v), \left(\frac{id_0}{2} + \frac{3\gamma}{2} \right) \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F(v) \right\rangle \\
&\quad + \frac{\gamma}{2} \sum_{k=1}^3 \left\langle D_v^2 f(v) \sigma_k \frac{\partial v}{\partial x}, \sigma_k \frac{\partial v}{\partial x} \right\rangle \\
&\quad + \epsilon \left\langle D_v f^1(v, y), \frac{id_0}{2} \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v) \right\rangle \\
&\quad - \epsilon \left\langle D_v f^2(v, y), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle \\
&\quad + \epsilon^2 \left\langle D_v f^2(v, y), \frac{id_0}{2} \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v) \right\rangle,
\end{aligned} \tag{4.16}$$

and we define the limiting operator by :

$$\begin{aligned}
\mathcal{L}^R f(v) &= \left\langle D_v f(v), \left(\frac{id_0}{2} + \frac{3\gamma}{2} \right) \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F(v) \right\rangle \\
&\quad + \frac{\gamma}{2} \sum_{k=1}^3 \left\langle D_v^2 f(v) \sigma_k \frac{\partial v}{\partial x}, \sigma_k \frac{\partial v}{\partial x} \right\rangle.
\end{aligned} \tag{4.17}$$

Hence if we define \mathcal{D}^R as the space of functions which are the restriction to \mathbb{H}^3 of functions f from \mathbb{H}^{-1} into \mathbb{R} of class $C^3(\mathbb{H}^{-1})$ and such that f and its first three derivatives are bounded on bounded sets of \mathbb{H}^{-1} , then the functions f^1 and f^2 are well defined for $f \in \mathcal{D}^R$. Moreover if $f \in \mathcal{D}^R$ then $\mathcal{L}_\epsilon^R f_\epsilon$ is well defined for $v \in \mathbb{H}^3$.

We now write that

$$\sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} |f_\epsilon(v, y) - f(v)| \leq \epsilon \sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} |f^1(v, y)| + \epsilon^2 \sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} |f^2(v, y)|,$$

and use the following result, which is proved in Section 6.

Lemma 4.2. *Let $f \in \mathcal{D}^R$ and f^1 and f^2 be respectively solution of Equation (4.8) and (4.12). Then*

$$\sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} |f^1(v, y)| \leq C_1(K) \quad \text{and} \quad \sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} |f^2(v, y)| \leq C_2(K).$$

This proves the first convergence of Proposition 4.1. With $\mathcal{L}^R f(v)$ given by (4.17), the second convergence (4.5) in Proposition 4.1 follows from (4.16) and the next Lemma, which is proved in Section 6. \square

Lemma 4.3. *Let $f \in \mathcal{D}^R$ and f^1, f^2 be respectively solutions of Equation (4.8) and (4.12). Then*

$$\begin{aligned}
\sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} \left| \left\langle D_v f^1(v, y), \frac{id_0}{2} \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v) \right\rangle \right| &\leq C_1(K), \\
\sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} \left| \left\langle D_v f^2(v, y), b' \sigma(y) \frac{\partial v}{\partial x} \right\rangle \right| &\leq C_2(K), \\
\sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} \left| \left\langle D_v f^2(v, y), \frac{id_0}{2} \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v) \right\rangle \right| &\leq C_3(K).
\end{aligned}$$

4.3 Tightness of the family of probability measures $(\mathcal{L}(Z_\epsilon^R))_{\epsilon>0}$

To prove tightness on \mathcal{K} of the sequence of probability measure $\mathcal{L}(Z_\epsilon^R) = \mathbb{P} \circ (Z_\epsilon^R)^{-1}$, we need to obtain uniform bounds in ϵ on Z_ϵ^R in the space

$$(C([0, T], \mathbb{H}^2) \cap C^\alpha([0, T], \mathbb{H}^{-1})) \times C^\delta([0, T], \mathbb{R}),$$

for suitable $\alpha, \delta > 0$. Note that uniform bound of X_ϵ^R in $C([0, T], \mathbb{H}^2)$ are given by Lemma 4.1. The perturbed test function method will enable us to get uniform bound in $C^\alpha([0, T], \mathbb{H}^{-1})$. Such bounds can not be directly obtained using Equation (4.1) because of the $1/\epsilon$ term. In order to obtain such bounds we use again the perturbed test function method for convenient test functions. Let $(\widehat{e}_j)_{j \in \mathbb{N}^*}$ be a complete orthonormal system in \mathbb{L}^2 . Recall that $\langle \cdot, \cdot \rangle$ is the duality product between \mathbb{H}^1 - \mathbb{H}^{-1} and $(\cdot, \cdot)_{\mathbb{L}^2}$ the inner product in \mathbb{L}^2 . By definition of $\mathbb{H}^s, s \in \mathbb{R}$ we can define a complete orthonormal system $(e_j)_{j \in \mathbb{N}^*}$ on \mathbb{H}^1 from $(\widehat{e}_j)_{j \in \mathbb{N}^*}$

$$\|v\|_{\mathbb{H}^{-1}}^2 = \left\| (1 + \xi^2)^{-1/2} \widehat{v} \right\|_{\mathbb{L}^2}^2 = \sum_{j=1}^{+\infty} \left((1 + \xi^2)^{-1/2} \widehat{v}, \widehat{e}_j \right)_{\mathbb{L}^2}^2 = \sum_{j=1}^{+\infty} \langle e_j, v \rangle^2,$$

where $e_j = \mathcal{F}^{-1} \left((1 + \xi^2)^{-1/2} \widehat{e}_j \right)$ for any $j \in \mathbb{N}^*$. We denote by $(f_j)_{j \in \mathbb{N}^*}$ the family of test functions in \mathcal{D}^R defined by

$$\begin{aligned} f_j : \mathbb{H}^{-1} &\rightarrow \mathbb{R} \\ v &\mapsto f_j(v) = \langle e_j, v \rangle. \end{aligned}$$

For $v \in \mathbb{H}^3$, we also consider particular perturbed test functions $f_{j,\epsilon}$ of the form

$$f_{j,\epsilon}(v, y) = f_j(v) + \epsilon f_j^1(v, y), \quad (4.18)$$

where, for all j in \mathbb{N}^* , $f_j^1(v, y) = \langle e_j, \varphi^1(v, y) \rangle$ for a given function φ^1 with values in \mathbb{H}^2 . We now choose φ^1 as a solution of the Poisson equation in y :

$$\mathcal{L}_\nu \varphi^1(v, y) - b' \sigma(y) \frac{\partial v}{\partial x} = 0, \quad (4.19)$$

whose explicit formulation is given by (see Proposition 5.1):

$$\varphi^1(v, y) = -b' \widetilde{\sigma}(y) \frac{\partial v}{\partial x}, \quad (4.20)$$

where $\widetilde{\sigma}(y)$ is given by (4.10). We point out that φ^1 behaves in its first variable like $\frac{\partial}{\partial x}$ and is linear in v . Consequently for all j in \mathbb{N}^* :

$$\begin{aligned} &\mathcal{L}_\epsilon^R f_{j,\epsilon}(X_\epsilon^R(t), \nu_\epsilon(t)) \\ &= \left\langle e_j, \frac{id_0}{2} \frac{\partial^2 X_\epsilon^R(t)}{\partial x^2} + i\Theta_R \left(\|X_\epsilon^R(t)\|_{\mathbb{H}^1}^2 \right) F_{\nu_\epsilon(t)}(X_\epsilon^R(t)) \right\rangle \\ &\quad + \left\langle e_j, (b')^2 \widetilde{\sigma}(\nu_\epsilon(t)) \sigma(\nu_\epsilon(t)) \frac{\partial^2 X_\epsilon^R(t)}{\partial x^2} \right\rangle \\ &\quad - \epsilon \left\langle e_j, b' \widetilde{\sigma}(\nu_\epsilon(t)) \frac{\partial}{\partial x} \left(\frac{id_0}{2} \frac{\partial^2 X_\epsilon^R(t)}{\partial x^2} + i\Theta_R \left(\|X_\epsilon^R(t)\|_{\mathbb{H}^1}^2 \right) F_{\nu_\epsilon(t)}(X_\epsilon^R(t)) \right) \right\rangle. \end{aligned} \quad (4.21)$$

For all $t \in [0, T]$, we define the process M_ϵ^R with values in \mathbb{H}^{-1} given for any j in \mathbb{N}^* by:

$$\begin{aligned} \langle e_j, M_\epsilon^R(t) \rangle &= f_{j,\epsilon}(X_\epsilon^R, \nu_\epsilon)(t) - f_{j,\epsilon}(v, y) - \int_0^t \mathcal{L}_\epsilon^R f_{j,\epsilon}(X_\epsilon^R(s), \nu_\epsilon(s)) ds \\ &= \langle e_j, X_\epsilon^R - v \rangle + \epsilon \langle e_j, \varphi^1(X_\epsilon^R, \nu_\epsilon) - \varphi^1(v, y) \rangle - \int_0^t \mathcal{L}_\epsilon^R f_{j,\epsilon}(X_\epsilon^R(s), \nu_\epsilon(s)) ds. \end{aligned}$$

Given the fact that \mathcal{L}_ϵ^R is the infinitesimal generator of the continuous Markov process $(X_\epsilon^R, \nu_\epsilon)$ and $\mathcal{L}_\epsilon^R f_{j,\epsilon}$ is well defined because $f_j \in \mathcal{D}^R$, then $\langle e_j, M_\epsilon^R(t) \rangle$ is a real valued continuous martingale. Moreover it is a square integrable martingale, as follows from the bounds on the \mathbb{H}^3 norm of X_ϵ^R obtained in Lemma 4.1. To prove tightness of the family of probability measures $\mathcal{L}(Z_\epsilon^R)$ on \mathcal{K} we need estimates of moments on the processes X_ϵ^R and $\|X_\epsilon^R(\cdot)\|_{\mathbb{H}^1}^2$. Before proving these estimates we introduce a process Y_ϵ^R close in probability to X_ϵ^R for which it will be easier to get those estimates, using in particular the Kolmogorov criterion. The idea is to use Lemma 4.4 below to get tightness of the family $\mathcal{L}(Z_\epsilon^R)$ from convergence in law of a subsequence of Y_ϵ^R .

Lemma 4.4. *Let us define the process Y_ϵ^R as*

$$X_\epsilon^R(t) - Y_\epsilon^R(t) = \epsilon (\varphi^1(v, y) - \varphi^1(X_\epsilon^R(t), \nu_\epsilon(t))), \quad \forall t \in [0, T]; \quad (4.22)$$

then for all $\delta > 0$:

$$\mathbb{P} \left(\|X_\epsilon^R - Y_\epsilon^R\|_{C([0, T], \mathbb{H}^1)} > \delta \right) \leq \frac{\epsilon}{\delta} C_1(T, R).$$

Proof of Lemma 4.4. Using Markov inequality and Lemma 4.1 we get for all $\delta > 0$,

$$\begin{aligned} & \mathbb{P} \left(\sup_{t \in [0, T]} \|X_\epsilon^R(t) - Y_\epsilon^R(t)\|_{\mathbb{H}^1} > \delta \right) \\ & \leq \frac{\epsilon}{\delta} \mathbb{E} \left(\sup_{t \in [0, T]} \|\varphi^1(X_\epsilon^R(t), \nu_\epsilon(t)) - \varphi^1(v, y)\|_{\mathbb{H}^1} \right) \\ & \leq \frac{\epsilon}{\delta} \mathbb{E} \left(\sup_{t \in [0, T]} \left\| b' \tilde{\sigma}(\nu_\epsilon(t)) \frac{\partial X_\epsilon^R(t)}{\partial x} - b' \tilde{\sigma}(y) \frac{\partial v}{\partial x} \right\|_{\mathbb{H}^1} \right) \\ & \leq \epsilon \frac{2M}{\delta} C(T, R), \end{aligned}$$

where M is given by (4.11). □

Note that the process Y_ϵ^R is also defined by the identity, for all j in \mathbb{N}^* :

$$\begin{aligned} \langle e_j, Y_\epsilon^R(t) \rangle &= \langle e_j, X_\epsilon^R(t) \rangle - \epsilon \langle e_j, \varphi^1(v, y) - \varphi^1(X_\epsilon^R(t), \nu_\epsilon(t)) \rangle \\ &= \langle e_j, M_\epsilon^R(t) \rangle + \langle e_j, v \rangle + \int_0^t \mathcal{L}_\epsilon^R f_{j,\epsilon}(X_\epsilon^R(s), \nu_\epsilon(s)) ds, \quad \forall t \in [0, T]. \end{aligned} \quad (4.23)$$

Lemma 4.5. *For all $1 \geq \epsilon > 0$, there exist three positive constants $C_1(T, R)$, $C_2(T, R)$ and $C_3(T, R)$ depending on final time T and on the cut-off radius R , but independent of ϵ , such that*

$$\mathbb{E} \left(\|Y_\epsilon^R\|_{C([0, T], \mathbb{H}^2)}^4 \right) \leq C_1(T, R), \quad (4.24)$$

$$\mathbb{E} \left(\|Y_\epsilon^R\|_{C^\alpha([0, T], \mathbb{H}^{-1})} \right) \leq C_2(T, R), \quad (4.25)$$

$$\mathbb{E} \left(\left\| \|Y_\epsilon^R\|_{\mathbb{H}^1}^2 \right\|_{C^\delta([0, T], \mathbb{R})} \right) \leq C_3(T, R), \quad (4.26)$$

where $0 < \alpha < \frac{1}{2}$ and $\delta = \alpha/3 > 0$.

Proof of Lemma 4.5. Thanks to Lemma 4.1 we know that the solution X_ϵ^R of Equation (4.1) is uniformly bounded, for all ϵ , in \mathbb{H}^3 by a constant C depending on R and T . We conclude, using the explicit formulation of φ_1 given by (4.20) and Equation (4.22), that (4.24) holds.

To prove inequality (4.25), we first need an intermediate estimate that will be proved in Section 6:

Lemma 4.6. *There exists a positive constant $C(R, T)$ such that for all $t, s \in [0, T]$*

$$\mathbb{E} \left(\|Y_\epsilon^R(t) - Y_\epsilon^R(s)\|_{\mathbb{H}^{-1}}^4 \right) \leq C(R, T)(t - s)^2.$$

Then we deduce from Lemma 4.6 :

$$\mathbb{E} \left(\|Y_\epsilon^R\|_{W^{\gamma,4}([0,T],\mathbb{H}^{-1})}^4 \right) \leq C(R,T),$$

for any $\gamma < 1/2$. We use the Sobolev embedding $W^{\gamma,4}([0,T],\mathbb{H}^{-1}) \hookrightarrow C^\alpha([0,T],\mathbb{H}^{-1})$ for $\gamma - \alpha > 1/4$ and $\gamma < 1/2$, which implies $\alpha < 1/4$. Thus we deduce the second inequality (4.25).

It remains to prove the last bound (4.26). Note that for $t, s \in [0, T]$

$$\begin{aligned} \left| \|Y_\epsilon^R(t)\|_{\mathbb{H}^1}^2 - \|Y_\epsilon^R(s)\|_{\mathbb{H}^1}^2 \right| &\leq C \sup_{r \in [0, T]} \|Y_\epsilon^R(r)\|_{\mathbb{H}^1} \|Y_\epsilon^R(t) - Y_\epsilon^R(s)\|_{\mathbb{H}^1} \\ &\leq C \sup_{r \in [0, T]} \|Y_\epsilon^R(r)\|_{\mathbb{H}^1} \sup_{r \in [0, T]} \|Y_\epsilon^R(r)\|_{\mathbb{H}^2}^{2/3} \|Y_\epsilon^R(t) - Y_\epsilon^R(s)\|_{\mathbb{H}^{-1}}^{1/3}. \end{aligned}$$

It follows that if $\delta = \alpha/3$,

$$\left\| \|Y_\epsilon^R(\cdot)\|_{\mathbb{H}^1}^2 \right\|_{C^\delta([0,T],\mathbb{R})} \leq C \sup_{r \in [0, T]} \|Y_\epsilon^R(r)\|_{\mathbb{H}^2}^{5/3} \|Y_\epsilon^R\|_{C^\alpha([0,T],\mathbb{H}^{-1})}^{1/3}.$$

Inequality (4.26) is then implied by Hölder inequality, (4.24) and (4.25). \square

Remark 4.3. *The extra \mathbb{H}^3 regularity is needed precisely in the first step of the above proof in order to estimate the \mathbb{H}^2 norm of Y_ϵ^R , which involves the gradient of X_ϵ^R .*

Proposition 4.2. *The family of laws $(\mathcal{L}(Z_\epsilon^R))_{\epsilon > 0}$ is tight on \mathcal{K} .*

Proof of Proposition 4.2. We set $\tilde{Z}_\epsilon^R = \left(Y_\epsilon^R, \|Y_\epsilon^R(\cdot)\|_{\mathbb{H}^1}^2 \right)$. Denoting by $\mathcal{B}(K)$ the closed ball of $(C([0, T]; \mathbb{H}^2(\mathbb{R})) \cap C^\alpha([0, T]; \mathbb{H}^{-1}(\mathbb{R})) \times C^\delta([0, T]; \mathbb{R})) \times C^\delta([0, T]; \mathbb{R})$ with radius K , for α and δ as in Lemma 4.5, we deduce using Ascoli-Arzelà and Banach-Alaoglu theorems that $\mathcal{B}(K)$ is compact in \mathcal{K} . Using Markov inequality and Lemma 4.5, we get

$$\begin{aligned} \mathbb{P} \left(\tilde{Z}_\epsilon^R \notin \mathcal{B}(K) \right) &\leq \frac{1}{K} \mathbb{E} \left(\max \left\{ \|Y_\epsilon^R\|_{C([0,T],\mathbb{H}^2(\mathbb{R}))}, \|Y_\epsilon^R\|_{C^\alpha([0,T],\mathbb{H}^{-1}(\mathbb{R}))}, \left\| \|Y_\epsilon^R\|_{\mathbb{H}^1}^2 \right\|_{C^\delta([0,T],\mathbb{R})} \right\} \right) \\ &\leq \frac{1}{K} \max \left(C_1^{1/4}(T, R), C_2(T, R), C_3(T, R) \right). \end{aligned}$$

We conclude that the family of laws $(\mathcal{L}(\tilde{Z}_\epsilon^R))_{\epsilon > 0}$ is tight on \mathcal{K} and by the Prokhorov theorem we obtain the relative compactness of the sequence of laws $(\mathcal{L}(\tilde{Z}_\epsilon^R))_{\epsilon > 0}$ i.e, up to a subsequence, the sequence $\mathcal{L}(\tilde{Z}_\epsilon^R)$ weakly converges to a probability measure $\mathcal{L}(\hat{Z}^R)$ where $\hat{Z}^R = (\hat{X}^R, \gamma^R)$. We may now use Lemma 4.4 to prove that the family of laws $\mathcal{L}(Z_\epsilon^R)$ is tight. Indeed it easily follows from Lemma 4.4 and the above convergence in law that for all $g \in C_b(\mathcal{K})$

$$\lim_{\epsilon \rightarrow 0} \mathbb{E} (g(Z_\epsilon^R)) = \mathbb{E} (g(\hat{Z}^R)).$$

\square

4.4 Convergence in law of the process X_ϵ^T

In order to get the convergence in law of the whole sequence $(X_\epsilon^R)_{\epsilon > 0}$, it remains to characterize the limit, i.e to prove that $\hat{X}^R = X^R$, the solution of Equation (3.6), and that $\gamma^R(t) = \|X^R(t)\|_{\mathbb{H}^1}^2$ for any $t \in [0, T]$. The tool here will be the use of the martingale problem formulation introduced by Stroock-Varadhan in [25].

Proposition 4.3. *The whole sequence X_ϵ^R converges in law to X^R in $C([0, T], \mathbb{H}^1)$.*

Proof of the Proposition 4.3. In order to prove that any subsequence of X_ϵ^R converges to the same limit X^R , solution of Equation (3.6), we will prove the convergence of the martingale problem for suitable test functions $f \in \mathcal{D}^R$. To this purpose let us define, for $a \in \mathbb{H}^1$ with compact support, the particular test function $f_a(\cdot) = \langle a, \cdot \rangle$, so that $f_a \in \mathcal{D}^R$. From this particular choice, we construct a perturbed test function $f_{a,\epsilon}$

$$f_{a,\epsilon}(v, y) = f_a(v) + \epsilon f_a^1(v, y) + \epsilon^2 f_a^2(v, y),$$

obtained thanks to Proposition 4.1. The correctors f_a^1 and f_a^2 are chosen to be solution of the Poisson equations (4.8) and (4.12) for f_a . Let us denote by Z_ϵ^R a subsequence converging to \widehat{Z}^R and define the \mathbb{H}^{-1} valued process $\mathbf{N}_\epsilon^R(Z_\epsilon^R(t))$, associated to Equation (4.1)

$$\langle a, \mathbf{N}_\epsilon^R(Z_\epsilon^R(t)) \rangle = f_{a,\epsilon}(X_\epsilon^R(t), \nu_\epsilon(t)) - f_{a,\epsilon}(v, y) - \int_0^t \mathcal{L}_\epsilon^R f_{a,\epsilon}(X_\epsilon^R(s), \nu_\epsilon(s)) ds,$$

where \mathcal{L}_ϵ^R is given by (4.7). We also define the process $\mathbf{N}^R(Z_\epsilon^R(t))$

$$\langle a, \mathbf{N}^R(Z_\epsilon^R(t)) \rangle = f_a(X_\epsilon^R(t)) - f_a(v) - \int_0^t \mathcal{L}^R f_a(X_\epsilon^R(s)) ds,$$

where \mathcal{L}^R is given by expression (4.17). Moreover we denote by $\mathcal{L}_{\gamma^R}^R$ the operator whose expression is given by (4.17) replacing $\|\widehat{X}^R(t)\|_{\mathbb{H}^1}$ by $\gamma^R(t)$ in the cut-off function. Let us now define $\langle a, \mathbf{N}^R(\widehat{Z}^R(t)) \rangle$ by

$$\langle a, \mathbf{N}^R(\widehat{Z}^R(t)) \rangle = f_a(\widehat{X}^R(t)) - f_a(v) - \int_0^t \mathcal{L}_{\gamma^R}^R f_a(\widehat{X}^R(s)) ds. \quad (4.27)$$

The process $\langle a, \mathbf{N}_\epsilon^R(Z_\epsilon^R(t)) \rangle$ is a real continuous martingale because $(X_\epsilon^R, \nu_\epsilon)$ is a Markov process and because $\mathcal{L}_\epsilon^R f_{a,\epsilon}$ is well defined since $X_\epsilon^R(t) \in \mathbb{H}^3$. Moreover it is a square integrable martingale, as follows from the bounds on the \mathbb{H}^3 norm of X_ϵ^R obtained in Lemma 4.1. The above martingale property implies that for all $t, s \in [0, T], t \geq s$,

$$\mathbb{E} [\langle a, \mathbf{N}_\epsilon^R(Z_\epsilon^R(t)) - \mathbf{N}_\epsilon^R(Z_\epsilon^R(s)) \rangle | \sigma(Z_\epsilon^R(u), \nu_\epsilon(u)), u \leq s] = 0.$$

It follows in particular that for all test functions $h_1, \dots, h_m \in C_b(\mathbb{H}_{loc}^1 \times \mathbb{R})$ and $0 \leq t_1 \dots < t_m \leq s \leq t$

$$\mathbb{E} \left[\langle a, \mathbf{N}_\epsilon^R(Z_\epsilon^R(t)) - \mathbf{N}_\epsilon^R(Z_\epsilon^R(s)) \rangle \prod_{j=1}^m h_j(Z_\epsilon^R(t_j)) \right] = 0.$$

Using Proposition 4.1, Lemma 4.1 and the boundedness of the functions h_j , we get

$$\mathbb{E} \left(\langle a, \mathbf{N}_\epsilon^R(Z_\epsilon^R(t)) - \mathbf{N}^R(Z_\epsilon^R(t)) - \mathbf{N}_\epsilon^R(Z_\epsilon^R(s)) + \mathbf{N}^R(Z_\epsilon^R(s)) \rangle \prod_{j=1}^m h_j(Z_\epsilon^R(t_j)) \right) \leq \epsilon C(R, T).$$

Let us consider a cut off function $\chi_{R_0} \in C_c^\infty(\mathcal{K})$ satisfying

$$\chi_{R_0}(u) = \begin{cases} 1 & \text{if } u \in \mathcal{B}_\mathcal{K}(R_0) \\ 0 & \text{if } u \notin \mathcal{B}_\mathcal{K}(2R_0) \end{cases}$$

where $\mathcal{B}_\mathcal{K}(R_0)$ denotes the closed ball of radius R_0 of the space \mathcal{K} and R_0 is chosen such that $X_\epsilon^R \in \mathcal{B}_\mathcal{K}(R_0)$ a.s (see Lemma 4.1). Note that, by continuity of the functions χ_{R_0} and $\{h_j\}_{j \in \{1, \dots, m\}}$ respectively in \mathcal{K} and $\mathbb{H}_{loc}^1 \times \mathbb{R}$, by continuity of $f_a(\cdot)$ for the weak topology in \mathbb{H}^1 , by continuity and boundedness of Θ_R in $C([0, T]; \mathbb{R})$, by continuity of F from \mathbb{H}^1 to \mathbb{H}^{-1} and the bounds on $F(X_\epsilon^R(t))$ obtained thanks to Lemma 4.1, the function

$$\langle a, \mathbf{N}^R(Z_\epsilon^R(t)) \rangle \chi_{R_0}(Z_\epsilon^R) \prod_{j=1}^m h_j(Z_\epsilon^R(t_j))$$

is a bounded and continuous function of Z_ϵ^R from \mathcal{K} into \mathbb{R} . We deduce by convergence in law of Z_ϵ^R to \widehat{Z}^R in \mathcal{K} , since the test function a is compactly supported, that for all $t, s \in [0, T], t \geq s$

$$\mathbb{E} \left(\left\langle a, \mathbf{N}^R \left(\widehat{Z}^R(t) \right) - \mathbf{N}^R \left(\widehat{Z}^R(s) \right) \right\rangle \chi_{R_0} \left(\widehat{Z}^R \right) \prod_{j=1}^m h_j \left(\widehat{Z}^R(t_j) \right) \right) = 0. \quad (4.28)$$

Since, almost surely, X_ϵ^R belongs to the closed ball $\mathcal{B}_{\mathcal{K}}(R_0)$, we deduce that almost surely $\widehat{X}^R \in \mathcal{B}_{\mathcal{K}}(R_0)$. Thus we conclude from (4.28) that $\left\langle a, \mathbf{N}^R \left(\widehat{Z}^R(\cdot) \right) \right\rangle$ is a continuous square integrable martingale with respect to the filtration $\mathcal{G}_t = \sigma \left(\widehat{Z}^R(s), s \leq t \right)$ and this holds for any $a \in \mathbb{H}^1$ with compact support.

In order to identify the equation satisfied by \widehat{X}^R , we consider, for $a, b \in \mathbb{H}^1$ with compact support, the function $g_{a,b}(v) = f_a(v)f_b(v) \in \mathcal{D}^R$ and the perturbed test function $g_{a,b,\epsilon}$

$$g_{a,b,\epsilon}(v, y) = g_{a,b}(v) + \epsilon g_{a,b}^1(v, y) + \epsilon^2 g_{a,b}^2(v, y),$$

obtained thanks to Proposition 4.1. Thus functions $g_{a,b}^1(v, y)$ and $g_{a,b}^2(v, y)$ are chosen to be solution of the Poisson Equations (4.8) and (4.12) for $g_{a,b}$. Let us now define the real valued continuous martingale

$$\mathbf{H}_{a,b,\epsilon}^R(Z_\epsilon^R(t)) = g_{a,b,\epsilon}(X_\epsilon^R(t), \nu_\epsilon(t)) - g_{a,b,\epsilon}(v, y) - \int_0^t \mathcal{L}_\epsilon^R g_{a,b,\epsilon}(X_\epsilon^R(s), \nu_\epsilon(s)) ds.$$

Using the same arguments as before, we may prove that

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \mathbb{E} \left(\left(\mathbf{H}_{a,b,\epsilon}^R(Z_\epsilon^R(t)) - \mathbf{H}_{a,b,\epsilon}^R(Z_\epsilon^R(s)) \right) \chi_{R_0}(Z_\epsilon^R) \prod_{j=1}^m h_j(Z_\epsilon^R(t_j)) \right) \\ &= \mathbb{E} \left(\left(\mathbf{H}_{a,b}^R(\widehat{Z}^R(t)) - \mathbf{H}_{a,b}^R(\widehat{Z}^R(s)) \right) \chi_{R_0}(\widehat{Z}^R) \prod_{j=1}^m h_j(\widehat{Z}^R(t_j)) \right), \end{aligned}$$

where

$$\mathbf{H}_{a,b}^R(\widehat{Z}(t)) = g_{a,b}(\widehat{X}^R(t)) - g_{a,b}(v) - \int_0^t \mathcal{L}_{\gamma^R}^R g_{a,b}(\widehat{X}^R(s)) ds.$$

From the above convergence and the martingale property of $\mathbf{H}_{a,b,\epsilon}^R(Z_\epsilon^R(t))$, we deduce that $\mathbf{H}_{a,b}^R(\widehat{Z}^R(\cdot))$ is a continuous real valued martingale. A classical computation then shows that the quadratic variation of the martingale $\mathbf{N}^R(\widehat{Z}^R(t))$ defined in (4.27) is given by

$$\begin{aligned} \left\langle b, \ll \mathbf{N}^R(\widehat{Z}^R(t)) \gg a \right\rangle &= \int_0^t \mathcal{L}_{\gamma^R}^R \left(f_a(\widehat{Z}^R(s)) f_b(\widehat{Z}^R(s)) \right) - f_a(\widehat{Z}^R(s)) \mathcal{L}_{\gamma^R}^R f_b(\widehat{Z}^R(s)) \\ &\quad - f_b(\widehat{Z}^R(s)) \mathcal{L}_{\gamma^R}^R f_a(\widehat{Z}^R(s)) ds. \end{aligned}$$

Applying the operator $\mathcal{L}_{\gamma^R}^R$ respectively to the test functions f_a and $g_{a,b}$, we obtain that

$$\mathcal{L}_{\gamma^R}^R f_a(\widehat{Z}^R(t)) = \left\langle a, \left(\frac{id_0}{2} + \frac{3\gamma}{2} \right) \frac{\partial^2 \widehat{X}^R}{\partial x^2} + i\Theta_R \left(\|\gamma^R(t)\|_{\mathbb{H}^1}^2 \right) F(\widehat{X}^R) \right\rangle,$$

and

$$\begin{aligned} \mathcal{L}_{\gamma^R}^R g_{a,b}(\widehat{Z}^R(t)) &= f_b(\widehat{X}^R(t)) \left\langle a, \left(\frac{id_0}{2} + \frac{3\gamma}{2} \right) \frac{\partial^2 \widehat{X}^R(t)}{\partial x^2} + i\Theta_R(\gamma^R(t)) F(\widehat{X}^R(t)) \right\rangle \\ &\quad + f_a(\widehat{X}^R(t)) \left\langle b, \left(\frac{id_0}{2} + \frac{3\gamma}{2} \right) \frac{\partial^2 \widehat{X}^R(t)}{\partial x^2} + i\Theta_R(\gamma^R(t)) F(\widehat{X}^R(t)) \right\rangle \\ &\quad + \gamma \sum_{k=1}^3 \left\langle a, \sigma_k \frac{\partial \widehat{X}^R(t)}{\partial x} \right\rangle \left\langle b, \sigma_k \frac{\partial \widehat{X}^R(t)}{\partial x} \right\rangle. \end{aligned}$$

We deduce that the quadratic variation is given by formula (3.5) with \tilde{X} replaced by \hat{X}^R . Thus, using the martingale representation theorem, we can write the \mathcal{G}_t -martingale $\mathbf{N}^R(\hat{Z}^R(t))$ as the stochastic integral

$$\left\langle a, \mathbf{N}^R(\hat{Z}^R(t)) \right\rangle = \sqrt{\gamma} \int_0^t \sum_{k=1}^3 \left\langle a, \sigma_k \frac{\partial \hat{X}^R(s)}{\partial x} \right\rangle dW_k(s),$$

where $W = (W_1, W_2, W_3)$ is a real valued Brownian motion on a possibly enlarged space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{P})$. We deduce that (\hat{X}^R, W) is a weak solution in $C([0, T]; \mathbb{H}_{loc}^1) \cap C_w([0, T]; \mathbb{H}^1) \cap L_w^\infty(0, T; \mathbb{H}^2)$ of the equation

$$\begin{cases} id\hat{X}^R(t) + \left(\frac{d_0}{2} \frac{\partial^2 \hat{X}^R(t)}{\partial x^2} + \Theta_R(\gamma^R(t)) F(\hat{X}^R(t)) \right) dt + i\sqrt{\gamma} \sum_{k=1}^3 \sigma_k \frac{\partial \hat{X}^R(t)}{\partial x} \circ dW_k(t) = 0 \\ X_0 = v \in \mathbb{H}^3. \end{cases} \quad (4.29)$$

The next step consists in proving that almost surely $\gamma^R(t) = \|\hat{X}^R(t)\|_{\mathbb{H}^1}^2$. Using the Skorokhod representation theorem, we can construct new random variables (that we still denote Z_ϵ^R, \hat{Z}^R) on a new common probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with respectively $\mathcal{L}(Z_\epsilon^R)$ and $\mathcal{L}(\hat{Z}^R)$ as probability measure and with values in \mathcal{K} such that

$$\lim_{\epsilon \rightarrow 0} Z_\epsilon^R = \hat{Z}^R \quad \mathbb{P} \text{ a.s in } \mathcal{K}.$$

Since $\hat{X}^R \in L^\infty(0, T; \mathbb{H}^2)$, we deduce using Equation (4.29) that $\hat{X}^R \in C([0, T]; \mathbb{L}^2)$. Hence applying Itô formula, it is easy to see, since Θ_R is a real valued function, that almost surely

$$\|\hat{X}^R(t)\|_{\mathbb{L}^2} = \|v\|_{\mathbb{L}^2} = \|X_\epsilon^R(t)\|_{\mathbb{L}^2}, \quad \forall t \in [0, T], \forall \epsilon > 0.$$

Thus we deduce the strong convergence of $X_\epsilon^R(t)$ to $\hat{X}^R(t)$ in \mathbb{L}^2 , a.s for each $t \in [0, T]$. Since X_ϵ^R converges to \hat{X}^R in $L_w^\infty(0, T; \mathbb{H}^2)$, we get using Lemma 4.1 that

$$\|\hat{X}^R\|_{L^\infty(0, T; \mathbb{H}^2)} \leq \liminf_{\epsilon \rightarrow 0} \|X_\epsilon^R\|_{L^\infty(0, T; \mathbb{H}^2)} \leq C(R, T) \quad \mathbb{P} - a.s.$$

Interpolating \mathbb{H}^1 between \mathbb{L}^2 and \mathbb{H}^2 , we conclude that

$$\lim_{\epsilon \rightarrow 0} \|X_\epsilon^R(t) - \hat{X}^R(t)\|_{\mathbb{H}^1} = 0, \quad \forall t \in [0, T], \quad \mathbb{P} - a.s, \quad (4.30)$$

and $\hat{X}^R \in C([0, T]; \mathbb{H}^1)$; it follows that, almost surely for all t in $[0, T]$, $\gamma^R(t) = \|\hat{X}^R(t)\|_{\mathbb{H}^1}^2$ and \hat{X}^R is a solution of (3.6). Thus the limit in law of X_ϵ^R is unique and is given by the solution X^R of Equation (3.6).

The final step consists in recovering the convergence in law in $C([0, T], \mathbb{H}^1)$. Since Y_ϵ^R is uniformly bounded in ϵ in $C^\alpha([0, T], \mathbb{H}^{-1}) \cap C([0, T]; \mathbb{H}^2)$ with $0 \leq \alpha < 1/2$, we deduce that it is a.s uniformly bounded in ϵ in $C^\beta([0, T], \mathbb{H}^1)$ with $\beta = \alpha/3$. Moreover using pointwise convergence (4.30), expression (4.22) and uniform bounds (4.1), we get pointwise convergence in \mathbb{H}^1 of Y_ϵ^R to X^R . We conclude that Y_ϵ^R converges in law to X^R in $C([0, T], \mathbb{H}^1(\mathbb{R}))$ and by Lemma 4.4, the convergence in law of X_ϵ^R to X^R in $C([0, T], \mathbb{H}^1(\mathbb{R}))$ follows. \square

Remark 4.4. Using Arzela-Ascoli and Banach-Alaoglu theorems, Lemma 4.5 and Tychonov Theorem, we deduce that $(\mathcal{L}(X_\epsilon^R))_{R \in \mathbb{N}}$ is tight on $\mathcal{K}^{\mathbb{N}}$. Thus the same arguments as above lead to the convergence in law of $(X_\epsilon^R)_{R \in \mathbb{N}}$ to $(X^R)_{R \in \mathbb{N}}$ (see [9]).

Using the Skorokhod Theorem, we can construct new random variables $\tilde{X}_\epsilon^R, \tilde{X}^R$ on a common probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ and with values in $C([0, T], \mathbb{H}^1)$ such that for any $R > 0$,

$$\begin{cases} \tilde{\mu}_\epsilon^R = \mu_\epsilon^R \\ \tilde{\mu}^R = \mu^R \end{cases} \quad \text{and} \quad \tilde{X}_\epsilon^R \xrightarrow{\epsilon \rightarrow 0} \tilde{X}^R \quad \tilde{\mathbb{P}} \text{ a.s. in } C([0, T], \mathbb{H}^1).$$

We define the escape times $\tilde{\tau}^R$ and $\tilde{\tau}_\epsilon^R$ associated to the cut-off :

$$\tilde{\tau}^R = \inf \left\{ t \in [0, T], \left\| \tilde{X}^R(t) \right\|_{\mathbb{H}^1} > R \right\} \quad \text{and} \quad \tilde{\tau}_\epsilon^R = \inf \left\{ t \in [0, T], \left\| \tilde{X}_\epsilon^R(t) \right\|_{\mathbb{H}^1} > R \right\}.$$

Now, if $\tau < \tau^*$ a.s. is a stopping time, it is then obvious that the process defined by $\tilde{X}_\epsilon(t) = \tilde{X}_\epsilon^R(t)$ for $t < \tau \wedge \tilde{\tau}_\epsilon^R$ converges a.s. to \tilde{X} defined by $\tilde{X}(t) = \tilde{X}^R(t)$ for $t < \tau \wedge \tilde{\tau}^R$, and this concludes the proof of Theorem 1.3. \square

5 Study of the driving process ν

We recall in this appendix some results obtained in [16, 20] about the driving process ν .

Proposition 5.1. *The process $\nu = (\nu_1, \nu_2)^t$ is a Feller process that evolves on the unit sphere \mathbb{S}^3 of $\mathbb{C}^2 \sim \mathbb{R}^4$. Furthermore it admits a unique invariant measure Λ , which is the uniform measure on \mathbb{S}^3 , under which it is ergodic. For all $f \in C_b^2(\mathbb{S}^3)$ satisfying the Fredholm alternative (or null mass condition) $\mathbb{E}_\Lambda(f(\nu)) = \int_{\mathbb{S}^3} f(y) \Lambda(dy) = 0$, the Poisson equation $\mathcal{L}_\nu u(y) + f(y) = 0$ admits a unique solution of class $C_b^2(\mathbb{S}^3)$, up to a constant, which can be written as $u(y) = \int_0^{+\infty} \mathbb{E}[f(\nu(t)) | \nu_0 = y] dt$.*

Let us recall that $\sigma(\nu(t)) = \sigma_1 m_1 + \sigma_2 m_2 + \sigma_3 m_3$ where $m_j(t) = g_j(\nu(t))$. We now state a result related to the effect of the random PMD on the pulse evolution.

Corollary 5.1. *1. The process $m = (m_1, m_2, m_3) \in \mathbb{S}^3$ is a Feller process with a unique invariant measure $\Lambda \circ g^{-1}$ under which it is ergodic.*

2. For $j = 1, 2, 3$: $\mathbb{E}_\Lambda(g_j(\nu)) = \mathbb{E}_{\Lambda \circ g_j^{-1}}(m) = 0$ and $\mathbb{E}_\Lambda(g_j(\nu(t))g_k(\nu(t))) = \delta_{jk}/3$. As a consequence,

$$\mathbb{E}_\Lambda(N_{1,\nu}(X)) = \frac{2}{3} \left(2|X_2|^2 - |X_1|^2 \right) X_1, \quad \mathbb{E}_\Lambda(N_{2,\nu}(X)) = \frac{2}{3} \left(2|X_1|^2 - |X_2|^2 \right) X_2.$$

3. For $j, k = 1, 2, 3$:

$$\int_0^{+\infty} \mathbb{E}_\Lambda[g_j(\nu(0))g_k(\nu(t))] dt = \begin{cases} \frac{1}{12\gamma_c} & \text{if } j = k \\ 0 & \text{if } j \neq k, \end{cases}$$

where γ_c is the constant appearing in (1.8).

6 Proof of technical Lemmas

Proof of Lemma 4.2. Let v be in \mathbb{H}^3 . Using the explicit representation (4.9) of f^1 , we obtain, since $D_v f(v) \in \mathbb{H}^1(\mathbb{R})$, that

$$\begin{aligned} |f^1(v, y)| &= \left| \left\langle D_v f(v), b' \tilde{\sigma}(y) \frac{\partial v}{\partial x} \right\rangle \right| \\ &\leq b' \|D_v f(v)\|_{\mathbb{H}^1} \left\| \frac{\partial v}{\partial x} \right\|_{\mathbb{H}^{-1}} \sum_{j=1}^3 \left| \int_0^{+\infty} \mathbb{E}(g_j(\nu(t)) | \nu(0) = y) dt \right|. \end{aligned}$$

Moreover by Proposition 5.1 the integral $\int_0^{+\infty} \mathbb{E}(g_j(\nu(t)) | \nu(0) = y) dt$ converges because g_j is a bounded function of $\nu \in \mathbb{S}^3$. Since $v \mapsto D_v f(v)$ is a continuous function which is bounded on bounded sets of \mathbb{H}^{-1} , we deduce that :

$$\sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} |f_\epsilon^1(v, y)| \leq b' C(K).$$

The function f^2 given by (4.15) may be bounded using the same arguments. Indeed

$$\left\langle D_v f(v), i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) \tilde{F}(v, y) \right\rangle \leq \|D_v f(v)\|_{\mathbb{H}^1} \left\| \tilde{F}(v, y) \right\|_{\mathbb{H}^{-1}}.$$

Since for all $v \in \mathbb{H}^3$, $y \mapsto F_y(v) - F(v)$ is a function of class C_b^2 on \mathbb{S}^3 , with values in \mathbb{H}^{-1} , satisfying the null mass condition of Proposition 5.1, the term $\tilde{F}(v, y)$ is bounded. Moreover $v \mapsto F_y(v) - F(v)$ is bounded in \mathbb{H}^{-1} on bounded sets of \mathbb{H}^1 by the continuous embeddings $\mathbb{H}^1(\mathbb{R}) \hookrightarrow \mathbb{L}^4(\mathbb{R})$ and $\mathbb{L}^{4/3}(\mathbb{R}) \hookrightarrow \mathbb{H}^{-1}(\mathbb{R})$. In addition

$$\begin{aligned} & \left| (b')^2 \sum_{k,l=1}^3 \left\langle D_v^2 f(v) \sigma_k \frac{\partial v}{\partial x}, \sigma_l \frac{\partial v}{\partial x} \right\rangle \tilde{g}_{k,l}(y) + \left\langle D_v f(v), (b')^2 \tilde{\sigma}(y) \frac{\partial^2 v}{\partial x^2} \right\rangle \right| \\ & \leq C \sum_{k,l=1}^3 \left(\left| \left\langle D_v^2 f(v) \sigma_k \frac{\partial v}{\partial x}, \sigma_l \frac{\partial v}{\partial x} \right\rangle \right| + \left| \left\langle D_v f(v), \sigma_k \sigma_l \frac{\partial^2 v}{\partial x^2} \right\rangle \right| \right) \\ & \leq C \left(\|D_v^2 f(v)\|_{\mathcal{L}(\mathbb{H}^{-1}, \mathbb{H}^1)} \left\| \frac{\partial v}{\partial x} \right\|_{\mathbb{H}^{-1}}^2 + \|D_v f(v)\|_{\mathbb{H}^1} \left\| \frac{\partial^2 v}{\partial x^2} \right\|_{\mathbb{H}^{-1}} \right). \end{aligned}$$

Since $v \mapsto D_v f(v)$ and $v \mapsto D_v^2 f(v)$ are bounded on bounded sets of $\mathbb{H}^{-1}(\mathbb{R})$, we conclude the proof of the lemma. \square

Proof of Lemma 4.3. Replacing f^1 by its expression (4.9) we get

$$\begin{aligned} & \left\langle D_v f^1(v, y), \frac{id_0}{2} \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v) \right\rangle \\ & = - \left\langle D_v^2 f(v) b' \tilde{\sigma}(y) \frac{\partial v}{\partial x}, \frac{id_0}{2} \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v) \right\rangle \\ & \quad - \left\langle D_v f(v), b' \tilde{\sigma}(y) \frac{id_0}{2} \frac{\partial^3 v}{\partial x^3} + i b' \tilde{\sigma}(y) \Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) \partial_x F_y(v) \right\rangle. \end{aligned}$$

By the assumptions on f , $v \mapsto D_v f(v)$ and $v \mapsto D_v^2 f(v)$ are continuous bounded functions on bounded sets of $\mathbb{H}^{-1}(\mathbb{R})$. Moreover $D_v^2 f(v) \in \mathcal{L}(\mathbb{H}^{-1}, \mathbb{H}^1)$, $D_v f(v) \in \mathbb{H}^1$ and $\frac{\partial^3 v}{\partial x^3} \in \mathbb{L}^2$. Using the bound (4.11), we deduce that

$$\sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} \left| \left\langle D_v f^1(v, y), \frac{id_0}{2} \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v) \right\rangle \right| \leq C(K).$$

Let us now compute the first derivative of f^2 using expression (4.15); for all h in \mathbb{H}^1 and v in \mathbb{H}^3 :

$$\begin{aligned} & \langle D_v f^2(v, y), h \rangle \\ & = \left\langle D_v^2 f(v) h, i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) \tilde{F}(v, y) \right\rangle + \left\langle D_v f(v), 2i\Theta'_R \left(\|v\|_{\mathbb{H}^1}^2 \right) (v, h)_{\mathbb{H}^1} \tilde{F}(v, y) \right\rangle \\ & + \left\langle D_v f(v), i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) D_v \tilde{F}(v, y) \cdot h \right\rangle - (b')^2 \sum_{k,l=1}^3 D_v^3 f(v) \cdot \left(\sigma_k \frac{\partial v}{\partial x}, \sigma_l \frac{\partial v}{\partial x}, h \right) \tilde{g}_{k,l}(y) \\ & - 2(b')^2 \sum_{k,l=1}^3 \left\langle D_v^2 f(v) \sigma_k \frac{\partial h}{\partial x}, \sigma_l \frac{\partial v}{\partial x} \right\rangle \tilde{g}_{k,l}(y) \\ & - \left\langle D_v^2 f(v) h, (b')^2 \tilde{\sigma}(y) \frac{\partial^2 v}{\partial x^2} \right\rangle - \left\langle D_v f(v), (b')^2 \tilde{\sigma}(y) \frac{\partial^2 h}{\partial x^2} \right\rangle. \end{aligned}$$

Taking respectively $h = \frac{id_0}{2} \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v)$ and $h = b'\sigma(y) \frac{\partial v}{\partial x}$, we conclude

$$\sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} \left| \left\langle D_v f^2(v, y), \frac{id_0}{2} \frac{\partial^2 v}{\partial x^2} + i\Theta_R \left(\|v\|_{\mathbb{H}^1}^2 \right) F_y(v) \right\rangle \right| \leq C(K),$$

and

$$\sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} \left| \left\langle D_v f^2(v, y), b'\sigma(y) \frac{\partial v}{\partial x} \right\rangle \right| \leq C(K),$$

since $v \mapsto D_v^3 f(v)$ is bounded on bounded set of $\mathbb{H}^{-1}(\mathbb{R})$ with values in $\mathcal{L}_3(\mathbb{H}^{-1}, \mathbb{R})$ and $\frac{\partial^4 v}{\partial x^4} \in \mathbb{H}^{-1}$. \square

Proof of lemma 4.6. Let us recall that the family $\{e_i\}_{i \in \mathbb{N}^*}$ denotes a complete orthonormal system of \mathbb{H}^1 constructed from a complete orthonormal system $\{\tilde{e}_i\}_{i \in \mathbb{N}^*}$ in \mathbb{L}^2 and $\langle \cdot, \cdot \rangle$ is the duality product between $\mathbb{H}^1 - \mathbb{H}^{-1}$. Then

$$\|Y_\epsilon^R(t) - Y_\epsilon^R(s)\|_{\mathbb{H}^{-1}}^4 = \left\{ \sum_{i=1}^{+\infty} \langle e_i, Y_\epsilon^R(t) - Y_\epsilon^R(s) \rangle^2 \right\}^2.$$

Using twice the Young inequality and the expression of Y_ϵ^R given by (4.23) and (4.21), we obtain :

$$\begin{aligned} & \|Y_\epsilon^R(t) - Y_\epsilon^R(s)\|_{\mathbb{H}^{-1}}^4 \\ & \leq C \left\| \frac{d_0}{2} \int_s^t \frac{\partial^2 X_\epsilon^R(t')}{\partial x^2} dt' \right\|_{\mathbb{H}^{-1}}^4 + C \left\| \int_s^t \Theta_R \left(\|X_\epsilon^R(t')\|_{\mathbb{H}^1}^2 \right) F_{\nu_\epsilon(t')}(X_\epsilon^R(t')) dt' \right\|_{\mathbb{H}^{-1}}^4 \\ & + C \left\| \int_s^t (b')^2 \tilde{\sigma}(\nu_\epsilon(t')) \sigma(\nu_\epsilon(t')) \frac{\partial^2 X_\epsilon^R(t')}{\partial x^2} dt' \right\|_{\mathbb{H}^{-1}}^4 + C \left(\sum_{i=1}^{+\infty} \langle e_i, M_\epsilon^R(t) - M_\epsilon^R(s) \rangle^2 \right)^2 \\ & + C\epsilon^4 \left\| \int_s^t b' \tilde{\sigma}(\nu_\epsilon(t')) \frac{\partial}{\partial x} \left(\frac{d_0}{2} \frac{\partial^2 X_\epsilon^R(t')}{\partial x^2} + \Theta_R \left(\|X_\epsilon^R(t')\|_{\mathbb{H}^1}^2 \right) F_{\nu_\epsilon(t')}(X_\epsilon^R(t')) \right) dt' \right\|_{\mathbb{H}^{-1}}^4. \end{aligned}$$

We bound each terms separately. Using Lemma 4.1,

$$\left\| \int_s^t \frac{d_0}{2} \frac{\partial^2 X_\epsilon^R(t')}{\partial x^2} dt' \right\|_{\mathbb{H}^{-1}}^4 \leq C(R, T)(t-s)^4.$$

Using that F is cubic and Lemma 4.1,

$$\left\| \int_s^t \Theta_R \left(\|X_\epsilon^R(t')\|_{\mathbb{H}^1}^2 \right) F_{\nu_\epsilon(t')}(X_\epsilon^R(t')) dt' \right\|_{\mathbb{H}^{-1}}^4 \leq C(R, T)(t-s)^4,$$

and using, Lemma 4.1 and the bound (4.11)

$$\left\| \int_s^t (b')^2 \tilde{\sigma}(\nu_\epsilon(t')) \sigma(\nu_\epsilon(t')) \frac{\partial^2 X_\epsilon^R(t')}{\partial x^2} dt' \right\|_{\mathbb{H}^{-1}}^4 \leq C(R, T)(t-s)^4.$$

Finally we bound the ϵ^4 term that is well defined because X_ϵ^R has values in \mathbb{H}^3 . Using the Cauchy Schwarz inequality, Lemma 4.1 and (4.11), we get for all $\epsilon < 1$:

$$\begin{aligned} & \epsilon^4 \left\| \int_s^t b' \tilde{\sigma}(\nu_\epsilon(t')) \frac{\partial}{\partial x} \cdot \left(\frac{d_0}{2} \frac{\partial^2 X_\epsilon^R(t')}{\partial x^2} + \Theta_R \left(\|X_\epsilon^R(t')\|_{\mathbb{H}^1}^2 \right) F_{\nu_\epsilon(t')}(X_\epsilon^R(t')) \right) dt' \right\|_{\mathbb{H}^{-1}}^4 \\ & \leq \epsilon^4 (b')^4 M^4 \left\| \int_s^t \frac{d_0}{2} \frac{\partial^3 X_\epsilon^R(t')}{\partial x^3} + \frac{\partial}{\partial x} \left(\Theta_R \left(\|X_\epsilon^R(t')\|_{\mathbb{H}^1}^2 \right) F_{\nu_\epsilon(t')}(X_\epsilon^R(t')) \right) dt' \right\|_{\mathbb{H}^{-1}}^4 \\ & \leq C(R, T)(t-s)^4. \end{aligned}$$

Taking the expectation and adding the previous estimates, we deduce that :

$$\mathbb{E} \left(\|Y_\epsilon^R(t) - Y_\epsilon^R(s)\|_{\mathbb{H}^{-1}}^4 \right) \leq C(R, T) (t-s)^4 + C\mathbb{E} \left(\left(\sum_{j \in \mathbb{N}^*} \langle e_j, M_\epsilon^R(t) - M_\epsilon^R(s) \rangle \right)^2 \right).$$

In order to prove a uniform bound, with respect to ϵ , of the second term, we will use the Burkholder-Davis-Gundy inequality and consequently we have to compute the quadratic variation $\ll M_\epsilon^R(t) \gg$ of $M_\epsilon^R(t)$ defined, for all $j \in \mathbb{N}^*$, by

$$\langle e_j, M_\epsilon^R(t) \rangle = f_{j,\epsilon}(X_\epsilon^R(t), \nu_\epsilon(t)) - f_{j,\epsilon}(v, y) - \int_0^t \mathcal{L}_\epsilon^R f_{j,\epsilon}(X_\epsilon^R(s), \nu_\epsilon(s)) ds,$$

where $\mathcal{L}_\epsilon^R f_{j,\epsilon}(X_\epsilon^R(s), \nu_\epsilon(s))$ is given by (4.21). The next Lemma states that the process $\ll M_\epsilon^R(t) \gg$ can be expressed only in terms of the infinitesimal generator \mathcal{L}_ν of the Markov process ν .

Lemma 6.1. *For all j in \mathbb{N}^**

$$\begin{aligned} & \langle e_j, \ll M_\epsilon^R(t) \gg e_j \rangle \\ &= (b')^2 \int_0^t \mathcal{L}_\nu \left(\left\langle e_j, \tilde{\sigma}(\nu_\epsilon(s)) \frac{\partial X_\epsilon^R}{\partial x}(s) \right\rangle^2 \right) - 2 \left\langle e_j, \tilde{\sigma}(\nu_\epsilon(s)) \frac{\partial X_\epsilon^R}{\partial x}(s) \right\rangle \left\langle e_j, \mathcal{L}_\nu \tilde{\sigma}(\nu_\epsilon(s)) \frac{\partial X_\epsilon^R}{\partial x}(s) \right\rangle ds. \end{aligned}$$

Thus using the Burkholder-Davis-Gundy inequality

$$\mathbb{E} \left(\left(\sum_{j=1}^{+\infty} \langle e_j, M_\epsilon^R(t) - M_\epsilon^R(s) \rangle \right)^2 \right) \leq C(R, T) |t-s|^2,$$

thanks to Lemma 4.1 and Proposition 5.1. Adding the previous estimates,

$$\mathbb{E} \left(\|Y_\epsilon^R(t) - Y_\epsilon^R(s)\|_{\mathbb{H}^{-1}}^4 \right) \leq C(R, T) |t-s|^2,$$

and Lemma 4.6 is proved. □

Proof of Lemma 6.1. A classical computation shows that for all $j \in \mathbb{N}^*$:

$$\langle e_j, \ll M_\epsilon^R(t) \gg e_j \rangle = \int_0^t \mathcal{L}_\epsilon^R \left(f_{j,\epsilon}(X_\epsilon^R(s), \nu_\epsilon(s))^2 \right) - 2f_{j,\epsilon}(X_\epsilon^R(s), \nu_\epsilon(s)) \mathcal{L}_\epsilon^R f_{j,\epsilon}(X_\epsilon^R(s), \nu_\epsilon(s)) ds.$$

Now, for all j in \mathbb{N}^* ,

$$\begin{aligned} (f_{j,\epsilon}(X_\epsilon^R(s), \nu_\epsilon(s)))^2 &= \langle e_j, X_\epsilon^R(s) \rangle^2 - 2b'\epsilon \langle e_j, X_\epsilon^R(s) \rangle \left\langle e_j, \tilde{\sigma}(\nu_\epsilon(s)) \frac{\partial X_\epsilon^R}{\partial x}(s) \right\rangle \\ &\quad + (b')^2 \epsilon^2 \left\langle e_j, \tilde{\sigma}(\nu_\epsilon(s)) \frac{\partial X_\epsilon^R}{\partial x}(s) \right\rangle^2. \end{aligned}$$

Thus we get

$$\begin{aligned}
& \mathcal{L}_\epsilon^R (f_{j,\epsilon} (X_\epsilon^R(s), \nu_\epsilon(s)))^2 \\
&= 2 \langle e_j, X_\epsilon^R(s) \rangle \left\langle e_j, \frac{id_0}{2} \frac{\partial^2 X_\epsilon^R(s)}{\partial x^2} + i\Theta_R \left(\|X_\epsilon^R(s)\|_{\mathbb{H}^1}^2 \right) F_{\nu_\epsilon(s)} (X_\epsilon^R(s)) \right\rangle \\
&\quad - 2b'\epsilon \left\langle e_j, \frac{id_0}{2} \frac{\partial^2 X_\epsilon^R(s)}{\partial x^2} + i\Theta_R \left(\|X_\epsilon^R(s)\|_{\mathbb{H}^1}^2 \right) F_{\nu_\epsilon(s)} (X_\epsilon^R(s)) \right\rangle \left\langle e_j, \tilde{\sigma}(\nu_\epsilon(s)) \frac{\partial X_\epsilon^R}{\partial x}(s) \right\rangle \\
&\quad - 2b'\epsilon \left\langle e_j, X_\epsilon^R(s) \right\rangle \left\langle e_j, \tilde{\sigma}(\nu_\epsilon(s)) \frac{\partial}{\partial x} \cdot \left(\frac{id_0}{2} \frac{\partial^2 X_\epsilon^R(s)}{\partial x^2} + i\Theta_R \left(\|X_\epsilon^R(s)\|_{\mathbb{H}^1}^2 \right) F_{\nu_\epsilon(s)} (X_\epsilon^R(s)) \right) \right\rangle \\
&\quad - 2(b')^2 \left\langle e_j, \mathcal{L}_\nu \tilde{\sigma}(\nu_\epsilon(s)) \frac{\partial X_\epsilon^R}{\partial x}(s) \right\rangle \left\langle e_j, \tilde{\sigma}(\nu_\epsilon(s)) \frac{\partial X_\epsilon^R}{\partial x}(s) \right\rangle + (b')^2 \mathcal{L}_\nu \left(\left\langle e_j, \tilde{\sigma}(\nu_\epsilon(s)) \frac{\partial X_\epsilon^R}{\partial x}(s) \right\rangle^2 \right) \\
&\quad + 2b' \left\langle e_j, \tilde{\sigma}(\nu_\epsilon(s)) \frac{\partial}{\partial x} \cdot \left(b'\sigma(\nu_\epsilon(s)) \frac{\partial X_\epsilon^R(s)}{\partial x} \right) \right\rangle \langle e_j, X_\epsilon^R(s) \rangle \\
&\quad + 2(b')^2 \epsilon^2 \left\langle e_j, \tilde{\sigma}(\nu_\epsilon(s)) \frac{\partial X_\epsilon^R}{\partial x}(s) \right\rangle \left\langle e_j, \tilde{\sigma}(\nu_\epsilon(s)) \frac{id_0}{2} \frac{\partial^3 X_\epsilon^R(s)}{\partial x^3} \right\rangle \\
&\quad + 2(b')^2 \epsilon^2 \left\langle e_j, \tilde{\sigma}(\nu_\epsilon(s)) \frac{\partial X_\epsilon^R}{\partial x}(s) \right\rangle \left\langle e_j, i\tilde{\sigma}(\nu_\epsilon(s)) \Theta_R \left(\|X_\epsilon^R(s)\|_{\mathbb{H}^1}^2 \right) \frac{\partial}{\partial x} F_{\nu_\epsilon(s)} (X_\epsilon^R(s)) \right\rangle \\
&\quad - 2(b')^2 \epsilon \left\langle e_j, \tilde{\sigma}(\nu_\epsilon(s)) \frac{\partial X_\epsilon^R}{\partial x}(s) \right\rangle \left\langle e_j, b'\tilde{\sigma}(\nu_\epsilon(s)) \sigma(\nu_\epsilon(s)) \frac{\partial^2 X_\epsilon^R(s)}{\partial x^2} \right\rangle.
\end{aligned}$$

The same kind of computations for the term $2f_{j,\epsilon} \mathcal{L}_\epsilon^R f_{j,\epsilon}$ lead to the result. \square

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