ECOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES APPLIQUÉES

UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 46 00. Fax: 01 69 33 46 46

http://www.cmap.polytechnique.fr/

Homogenization and concentration for a diffusion equation with large convection in a bounded domain

Grégoire Allaire, Irina Pankratova,

Andrey Piatnitski

R.I. 713 *May 2011*

Homogenization and concentration for a diffusion equation with large convection in a bounded domain

G. Allaire * I. Pankratova † A. Piatnitski ‡

May 24, 2011

Abstract

We consider the homogenization of a non-stationary convection-diffusion equation posed in a bounded domain with periodically oscillating coefficients and homogeneous Dirichlet boundary conditions. Assuming that the convection term is large, we give the asymptotic profile of the solution and determine its rate of decay. In particular, it allows us to characterize the "hot spot", i.e., the precise asymptotic location of the solution maximum which lies close to the domain boundary and is also the point of concentration. Due to the competition between convection and diffusion the position of the "hot spot" is not always intuitive as exemplified in some numerical tests.

Keywords: Homogenization, convection-diffusion, localization.

1 Introduction

The goal of the paper is to study the homogenization of a convection-diffusion equation with rapidly periodically oscillating coefficients defined in a bounded domain. Namely, we consider the following initial boundary problem:

$$\begin{cases} \partial_t u^\varepsilon(t,x) + A^\varepsilon\, u^\varepsilon(t,x) = 0, & \text{in } (0,T)\times\Omega, \\ u^\varepsilon(t,x) = 0, & \text{on } (0,T)\times\partial\Omega, \\ u^\varepsilon(0,x) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

*Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex France (allaire@cmap.polytechnique.fr)

†Narvik University College, Postbox 385, 8505 Narvik, Norway; Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex France (pankratova@cmap.polytechnique.fr)

‡Narvik University College, Postbox 385, 8505 Narvik, Norway; Lebedev Physical Institute RAS, Leninski ave., 53, 119991 Moscow, Russia (andrey@sci.lebedev.ru)

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, u_0 belongs to $L^2(\Omega)$ and A^ε is an operator defined by

$$A^\varepsilon u^\varepsilon = -\frac{\partial}{\partial x_i}\Big(a_{ij}\Big(\frac{x}{\varepsilon}\Big)\frac{\partial u^\varepsilon}{\partial x_j}\Big) + \frac{1}{\varepsilon}\, b_j\Big(\frac{x}{\varepsilon}\Big)\frac{\partial u^\varepsilon}{\partial x_j},$$

where we employ the convention of summation over repeated Latin indices. As usual ε, which denotes the period of the coefficients, is a small positive parameter intended to tend to zero. Note the large scaling in front of the convective term which corresponds to the convective and diffusive terms having both the same order of magnitude at the small scale ε (this is a classical assumption in homogenization [5], [12], [13], [21]). We make the following assumptions on the coefficients of the operator A^ε.

(H1) The coefficients $a_{ij}(y), b_j(y)$ are measurable bounded functions defined on the unit cell $Y = (0,1]^d$, that is $a_{ij}, b_j \in L^\infty(Y)$. Moreover, $a_{ij}(y), b_j(y)$ are Y-periodic.

(H2) The $d \times d$ matrix $a(y)$ is uniformly elliptic, that is there exists $\Lambda > 0$ such that, for all $\xi \in \mathbb{R}^d$ and for almost all $y \in \Omega$,

$$a_{ij}(y)\xi_i\xi_j \geq \Lambda|\xi|^2.$$

For the large convection term we do not suppose that the effective drift (the weighted average of b defined below by (2.4)) is zero, nor that the vector field $b(y)$ is divergence-free. Some additional assumptions on the smoothness and compact support of the initial data u_0 will be made in Section 2 after introducing auxiliary spectral cell problems. In view of **(H1)** and **(H2)**, for any $\varepsilon > 0$, problem (1.1) has a unique weak solution $u^\varepsilon \in L^\infty[0,T; L^2(\Omega)] \cap L^2[0,T; H^1(\Omega)]$ (see [6]).

Our main goal is to describe the asymptotic behavior of the solution $u^\varepsilon(t,x)$ of problem (1.1) as ε goes to zero. There are of course many motivations to study such a problem (one of them being the transport of solutes in porous media [17]). However, if (1.1) is interpreted as the heat equation in a fluid domain (the fluid velocity being given by $\varepsilon^{-1}b(x/\varepsilon)$), we can paraphrase the famous "hot spot" conjecture of J. Rauch [23], [7], [10], and ask a simple question in plain words. If the initial temperature u_0 has its maximum inside the domain Ω, where shall this maximum or "hot spot" go as time evolves ? More precisely, we want to answer this question asymptotically as ε goes to zero. Theorem 2.1 (and the discussion following it) gives a complete answer to this question. The "hot spot" is a concentration point x_c, located asymptotically close to the boundary $\partial\Omega$ (see Figure 1), which maximizes the linear function $\Theta \cdot x$ on Ω where the vector parameter Θ is determined as an optimal parameter in an auxiliary cell problem (see Lemma 2.1). Surprisingly Θ is not some average of the velocity field but is the result of an intricate interaction between convection and diffusion in the periodicity cell (even in the case of constant coefficients ; see the numerical examples of Section 7). Furthermore, Theorem 2.1 gives the asymptotic profile of the solution, which is localized in the vicinity of the "hot spot" x_c, in terms of a homogenized equation with an initial condition that depends on the geometry of the support of the initial data u_0.

Before we explain our results in greater details, we briefly review previous results in the literature. In the case when the vector-field $b(y)$ is solenoidal and has zero mean-value, problem (1.1) has been studied by the classical homogenization methods (see, e.g, [8], [25]). In particular, the sequence of solutions is bounded in $L^\infty[0,T;L^2(\Omega)]\cap L^2[0,T;H^1(\Omega)]$ and converges, as $\varepsilon\to 0$, to the solution of an effective or homogenized problem in which there is no convective term. For general vector-fields $b(y)$, and if the domain Ω is the whole space $\mathbb{R}^d$, the convection might dominate the diffusion and we cannot expect a usual convergence of the sequence of solutions $u^\varepsilon(t,x)$ in the fixed spatial reference frame. Rather, introducing a frame of moving coordinates $(t, x-\bar b t/\varepsilon)$, where the constant vector $\bar b$ is the so-called effective drift (or effective convection) which is defined by (2.4) as a weighted average of b, it is known that the translated sequence $u^\varepsilon(t, x-\bar b t/\varepsilon)$ converges to the solution of an homogenized parabolic equation [5], [13]. Note that the notion of effective drift was first introduced in [21]. Of course, the convergence in moving coordinates cannot work in a bounded domain. The purpose of the present work is to study the asymptotic behavior of (1.1) in the case of a bounded domain Ω.

Bearing these previous results in mind, intuitively, it is clear that in a bounded domain the initial profile should move rapidly in the direction of the effective drift $\bar b$ until it reaches the boundary, and then dissipate due to the homogeneous Dirichlet boundary condition, as t grows. Since the convection term is large, the dissipation increases, as $\varepsilon\to 0$, so that the solution asymptotically converges to zero at finite time. Indeed, introducing a rescaled (short) time $\tau=\varepsilon^{-1}t$, we rewrite problem (1.1) in the form

$$\begin{cases} \partial_\tau u^\varepsilon - \varepsilon\,\mathrm{div}\big(a^\varepsilon\,\nabla u^\varepsilon\big) + b^\varepsilon\cdot\nabla u^\varepsilon = 0, & \text{in } (0,\varepsilon^{-1}\,T)\times\Omega,\\ u^\varepsilon(\tau,x)=0, & \text{on } (0,\varepsilon^{-1}\,T)\times\partial\Omega,\\ u^\varepsilon(0,x)=u_0(x), & x\in\Omega. \end{cases} \tag{1.2}$$

Applying the classical two-scale asymptotic expansion method [8], one can show that, for any $\tau\geq 0$

$$\int_\Omega |u^\varepsilon(\tau,x)-u^0(\tau,x)|^2\,dx\to 0,\quad \varepsilon\to 0,$$

where the leading term of the asymptotics u^0 satisfies the following first-order equation

$$\begin{cases} \partial_\tau u^0(\tau,x) + \bar b\cdot\nabla u^0(\tau,x) = 0, & \text{in } (0,+\infty)\times\Omega,\\ u^0(\tau,x)=0, & \text{on } (0,+\infty)\times\partial\Omega_{\bar b},\\ u^0(0,x)=u_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

with $\bar{b}$ being the vector of effective convection defined by (2.4). Here $\partial\Omega_{\bar{b}}$ is the subset of $\partial\Omega$ such that $\bar{b} \cdot n < 0$ where n stands for the exterior unit normal on $\partial\Omega$. One can construct higher order terms in the asymptotic expansion for u^ε. This expansion will contain a boundary layer corrector in the vicinity of $\partial\Omega \setminus \partial\Omega_{\bar{b}}$. A similar problem in a more general setting has been studied in [9].

The solution of problem (1.3) can be found explicitly,

$$u^0(\tau, x) = \begin{cases} u_0(x - \bar{b}\tau), & \text{for } (\tau, x) \text{ such that } x,\ (x - \bar{b}\tau) \in \Omega, \\ 0, & \text{otherwise,} \end{cases}$$

which shows that u^0 vanishes after a finite time $\tau_0 = O(1)$. In the original coordinates (t, x) we have

$$\int_\Omega |u^\varepsilon(t, x) - u_0(x - \varepsilon^{-1}\bar{b}\, t)|^2\, dx \to 0, \quad \varepsilon \to 0.$$

Thus, for $t = O(\varepsilon)$ the initial profile of u^ε moves with the velocity $\varepsilon^{-1}\bar{b}$ until it reaches the boundary of Ω and then dissipates. Furthermore, any finite number of terms in the two-scale asymptotic expansion of $u^\varepsilon(\tau, x)$ vanish for $\tau \geq \tau_0 = O(1)$ and thus for $t \geq t_0$ with an arbitrary small $t_0 > 0$. On the other hand, if u_0 is positive, then by the maximum principle, $u^\varepsilon > 0$ for all t. Thus, the method of two-scale asymptotic expansion in this short-time scaling is unable to capture the limit behaviour of $u^\varepsilon(t, x)$ for positive time. The goal of the present paper is therefore to perform a more delicate analysis and to determine the rate of vanishing of u^ε, as $\varepsilon \to 0$.

The homogenization of the spectral problem corresponding to (1.1) in a bounded domain for a general velocity $b(y)$ was performed in [11], [12]. Interestingly enough the effective drift does not play any role in such a case but rather the key parameter is another constant vector $\Theta \in \mathbb{R}^d$ which is defined as an optimal exponential parameter in a spectral cell problem (see Lemma 2.1). More precisely, it is proved in [11], [12] that the first eigenfunction concentrates as a boundary layer on $\partial\Omega$ in the direction of Θ. We shall prove that the same vector parameter Θ is also crucial in the asymptotic analysis of (1.1).

Notice that for large time and after a proper rescaling the solution of (1.1) should behave like the first eigenfunction of the corresponding elliptic operator, and thus concentrates in a small neighbourhood of $\partial\Omega$ in the direction of Θ. We prove that this guess is correct, not only for large time but also for any time $t = O(1)$, namely that $u^\varepsilon(t, x)$ concentrates in the neighbourhood of the "hot spot" or concentration point $x_c \in \partial\Omega$ which depends on Θ. The value of Θ can be determined in terms of some optimality property of the first eigenvalue of an auxiliary periodic spectral problem (see Section 2). It should be stressed that, in general, Θ does not coincide with $\bar{b}$. As a consequence, it may happen that the concentration point x_c does not even belong to the subset of $\partial\Omega$ consisting of points which are attained by translation of the initial data support along $\bar{b}$. This phenomenon is illustrated by numerical examples in Section 7.

The paper is organized as follows. In Section 2 we introduce auxiliary spectral problems in the unit cell Y and impose additional conditions on the geometry of the compact support of u_0. We then state our main result (see Theorem 2.1) and give its geometric interpretation. In Section 3, in order to simplify the original problem (1.1), we use a factorization principle, as in [24], [18], [26], [11], based on the first eigenfunctions of the auxiliary spectral problems. As a result, we obtain a reduced problem, where the new convection is divergence-free and has zero mean-value. Studying the asymptotic behaviour of the Green function of the reduced problem, performed in Section 4, is an important part of the proof. It is based on the result obtained in [1] for a fundamental solution of a parabolic operator with lower order terms. Asymptotics of u^ε is derived in Section 5. In Section 6 we study the case when the boundary of the support of u_0 has a flat part. To illustrate the main result of the paper, in Section 7 we present direct computations of u^ε using the software FreeFEM++ [15]. A number of basic facts from the theory of almost periodic functions is given in Section 8.

2 Auxiliary spectral problems and main result

We define an operator A and its adjoint A^* by

$$Au = -\mathrm{div}(a\nabla u) + b\cdot\nabla u, \quad A^* v = -\mathrm{div}(a^T\nabla v) - \mathrm{div}(b\, v),$$

where a^T is the transposed matrix of a. Following [8], for $\theta\in\mathbb{R}^d$, we introduce two parameterized families of spectral problems (direct and adjoint) in the periodicity cell $Y = [0,1)^d$.

$$\begin{cases} e^{-\theta\cdot y}\, A\, e^{\theta\cdot y}\, p_\theta(y) = \lambda(\theta)\, p_\theta(y), \quad Y, \\ y\to p_\theta(y) \quad \text{Y-periodic.} \end{cases} \tag{2.1}$$

$$\begin{cases} e^{\theta\cdot y}\, A^*\, e^{-\theta\cdot y}\, p^*_\theta(y) = \lambda(\theta)\, p^*_\theta(y), \quad Y, \\ y\to p^*_\theta(y) \quad \text{Y-periodic.} \end{cases} \tag{2.2}$$

The next result, based on the Krein-Rutman theorem, was proved in [11], [12].

Lemma 2.1. *For each $\theta\in\mathbb{R}^d$, the first eigenvalue $\lambda_1(\theta)$ of problem (2.1) is real, simple, and the corresponding eigenfunctions p_θ and p^*_θ can be chosen positive. Moreover, $\theta\to\lambda_1(\theta)$ is twice differentiable, strictly concave and admits a maximum which is obtained for a unique $\theta=\Theta$.*

The eigenfunctions p_θ and p^*_θ defined by Lemma 2.1, can be normalized by

$$\int_Y |p_\theta(y)|^2 dy = 1 \quad \text{and} \quad \int_Y p_\theta(y)\, p^*_\theta(y)\, dy = 1.$$

Differentiating equation (2.1) with respect to θ_i, integrating against p^*_θ and writing down the compatibility condition for the obtained equation yield

$$\frac{\partial\lambda_1}{\partial\theta_i} = \int_Y \big(b_i\, p_\theta\, p^*_\theta + a_{ij}(p_\theta\,\partial_{y_j}p^*_\theta - p^*_\theta\,\partial_{y_j}p_\theta) - 2\,\theta_j\, a_{ij}\, p_\theta\, p^*_\theta\big)\, dy. \tag{2.3}$$

Obviously, $p_{\theta=0} = 1$, and, thus,

$$\frac{\partial \lambda_1}{\partial \theta_i}(\theta = 0) = \int\limits_Y \big(b_i\, p^*_{\theta=0} + a_{ij}\, \partial_{y_j} p^*_{\theta=0}\big)\, dy := \bar{b}_i, \tag{2.4}$$

which defines the components $\bar{b}_i$ of the so-called effective drift. In the present paper we assume that $\bar{b} \neq 0$ (or, equivalently, $\Theta \neq 0$). The case $\bar{b} = 0$ can be studied by classical methods (see, for example, [25]). The equivalence of $\bar{b} = 0$ and $\Theta = 0$ is obvious since $\lambda_1(\theta)$ is strictly concave with a unique maximum.

We need to make some assumptions on the geometry of the support ω (a closed set as usual) of the initial data u_0 with respect to the direction of Θ. One possible set of conditions is the following.

(H3) The initial data $u_0(x)$ is a continuous function in Ω, has a compact support $\omega \Subset \Omega$ and belongs to $C^2(\omega)$. Moreover, ω is a C^2-class domain.

(H4) The "source" point $\bar{x} \in \partial\omega$, at which the minimum in $\min_{x\in\omega} \Theta\cdot x$ is achieved, is unique (see Figure 1(a)). In other words

$$\Theta \cdot (x - \bar{x}) > 0, \quad x \in \omega \setminus \{\bar{x}\}. \tag{2.5}$$

(H5) The point $\bar{x}$ is elliptic and $\partial\omega$ is locally convex at $\bar{x}$, that is the principal curvatures at $\bar{x}$ have the same sign. More precisely, in local coordinates the boundary of ω in some neighborhood $U_\delta(\bar{x})$ of the point $\bar{x}$ can be defined by

$$z_d = (Sz', z') + o(|z'|^2)$$

for some positive definite $(d-1)\times(d-1)$ matrix S. Here $z' = (z_1, \cdots z_{d-1})$ are the orthonormal coordinates in the tangential hyperplane at $\bar{x}$, and z_d is the coordinate in the normal direction.

(H6) $\nabla u_0(\bar{x}) \cdot \Theta \neq 0$.

Remark 2.1. *In assumption* (**H3**) *it is essential that the support ω is a strict subset of Ω, i.e., does not touch the boundary $\partial\Omega$ (see Remark 5.3 for further comments on this issue). However, the continuity assumption on the initial function u_0 is not necessary. It will be relaxed in Theorem 5.2 where $u_0(x)$ still belongs to $C^2(\overline{\omega})$ but is discontinuous through $\partial\omega$. Of course, assuming continuity or not will change the order of convergence and the multiplicative constant in front of the asymptotic solution.*

Note that assumption (**H4**) *implies that $\Theta \neq 0$ is a normal vector to $\partial\omega$ at $\bar{x}$.*

Eventually, assumption (**H6**) *is required because, u_0 being continuous in Ω, we have $u_0(\bar{x}) = 0$.*

To avoid excessive technicalities for the moment, we state our main result in a loose way (see Theorem 5.1 for a precise statement).

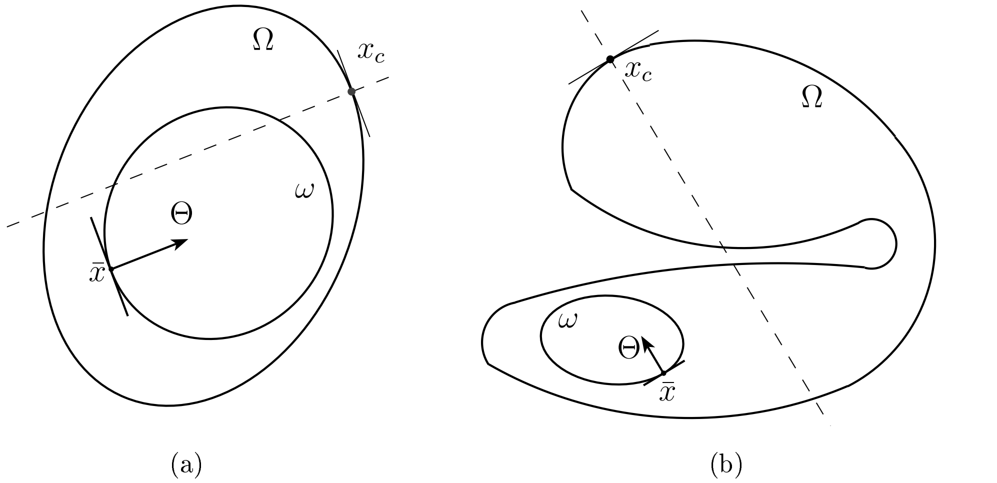


Figure 1: Definition of the source point $\bar{x}$ and of the concentration point x_c.

Theorem 2.1. *Suppose conditions* (**H1**) – (**H6**) *are satisfied and* $\Theta \neq 0$. *If* u^ε *is a solution of problem* (1.1), *then, for any* $t_0 > 0$ *and* $t \geq t_0$

$$u^\varepsilon(t,x) \approx \varepsilon^2 \, \varepsilon^{\frac{d-1}{2}} \, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}} \, e^{\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}} \, M_\varepsilon \, p_\Theta\big(\frac{x}{\varepsilon}\big) \, u(t,x), \quad \varepsilon \to 0,$$

where $(\lambda_1(\Theta), p_\Theta)$ *is the first eigenpair defined by Lemma 2.1 and* $u(t,x)$ *solves the homogenized problem*

$$\begin{cases} \partial_t u = \operatorname{div}(a^{\text{eff}} \, \nabla u), & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = \nabla u_0(\bar{x}) \cdot \dfrac{\Theta}{|\Theta|} \, \delta(x - \bar{x}), & x \in \Omega. \end{cases} \tag{2.6}$$

Here a^{eff} *is a positive definite matrix, defined by (4.7),* M_ε *is a constant, defined in Theorem 5.1, depending on* p_Θ*, on the geometry of* $\partial\omega$ *at* $\bar{x}$ *and on the relative position of* $\bar{x}$ *in* εY *(see Remark 5.1 and Figure 2), and* $\delta(x - \bar{x})$ *is the Dirac delta-function at the point* $\bar{x}$.

The interpretation of Theorem 2.1 in terms of concentration or finding the "hot spot" is the following. Up to a multiplicative constant $\varepsilon^2 \, \varepsilon^{\frac{d-1}{2}} M_\varepsilon$, the solution u^ε is asymptotically equal to the product of two exponential terms, a periodically oscillating function $p_\Theta\big(\frac{x}{\varepsilon}\big)$ (which is uniformly positive and bounded) and the homogenized function $u(t,x)$ (which is independent of ε). The first exponential term $e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}$ indicates a fast decay in time, uniform in space. The second exponential term $e^{\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}$ is the root of a localization phenomenon. Indeed, it is maximum at those points on the boundary, $x_c \in \partial\Omega$, which have a maximal coordinate $\Theta \cdot x$, independently of the position of $\bar{x}$ (see Figure 1(b)). These (possibly multiple) points x_c are the "hot spots". Everywhere else in Ω the solution is exponentially smaller, for any positive time. This behaviour can clearly be checked on the numerical examples of Section 7. It is of course similar to the behavior of the corresponding first eigenfunction as studied in [12].

The proof of Theorem 2.1 consists of several steps. First, using a factorization principle (see, for example, [24], [18], [26], [11]) in Section 3 we make a change of unknown function in such a way that the resulting equation is amenable to homogenization. After that, the new unknown function $v^\varepsilon(t,x)$ is represented in terms of the corresponding Green function $K^\varepsilon(t,x,\xi)$. Studying the asymptotic behaviour of K^ε is performed in Section 4. Finally, we turn back to the original problem and write down the asymptotics for u^ε in Section 5 which finishes the proof of Theorem 2.1.

Remark 2.2. *Theorem 2.1 holds true even if we add a singular zero-order term of the type $\varepsilon^{-2}c(\frac{x}{\varepsilon})u^\varepsilon$ in the equation (1.1). This zero-order term will be removed by the factorization principle and the rest of the proof is identical. With some additional work Theorem 2.1 can be generalized to the case of so-called cooperative systems for which a maximum principle holds. Such systems of diffusion equations arise in nuclear reactor physics and their homogenization (for the spectral problem) was studied in [12].*

3 Factorization

We represent a solution u^ε of the original problem (1.1) in the form

$$u^\varepsilon(t,x) = e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, v^\varepsilon(t,x), \tag{3.1}$$

where Θ and p_Θ are defined in Lemma 2.1. Notice that the change of unknowns is well-defined since p_Θ is positive and continuous. Substituting (3.1) into (1.1), multiplying the resulting equation by $p^*_\Theta\big(\frac{x}{\varepsilon}\big)$ and using (2.2), one obtains the following problem for v^ε:

$$\begin{cases} \varrho_\Theta\big(\dfrac{x}{\varepsilon}\big)\,\partial_t v^\varepsilon + A^\varepsilon_\Theta\, v^\varepsilon = 0, & (t,x)\in(0,T)\times\Omega, \\ v^\varepsilon(t,x) = 0, & (t,x)\in(0,T)\times\partial\Omega, \\ v^\varepsilon(0,x) = \dfrac{u_0(x)}{p_\Theta\big(\frac{x}{\varepsilon}\big)}\, e^{-\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}, & x\in\Omega, \end{cases} \tag{3.2}$$

where $\varrho_\Theta(y) = p_\Theta(y)\,p^*_\Theta(y)$ and

$$A^\varepsilon_\Theta v = -\frac{\partial}{\partial x_i}\big(a^\Theta_{ij}\big(\frac{x}{\varepsilon}\big)\,\frac{\partial v}{\partial x_j}\big) + \frac{1}{\varepsilon}\, b^\Theta_i\big(\frac{x}{\varepsilon}\big)\,\frac{\partial v}{\partial x_i},$$

and the coefficients of the operator A^ε_Θ are given by

$$\begin{aligned} &a^\Theta_{ij}(y) = \varrho_\Theta(y)\,a_{ij}(y); \\ &b^\Theta_i(y) = \varrho_\Theta(y)\,b_j(y) - 2\,\varrho_\Theta(y)\,a_{ij}(y)\,\Theta_j \\ &+a_{ij}(y)\,\big[p_\Theta(y)\,\partial_{y_j}p^*_\Theta(y) - p^*_\Theta(y)\,\partial_{y_j}p_\Theta(y)\big]. \end{aligned} \tag{3.3}$$

Obviously, the matrix a^Θ is positive definite since both p_Θ and p_Θ^* are positive functions. Moreover, it has been shown in [11] that, for any $\theta \in \mathbb{R}^d$, the vector-field b^θ is divergence-free and that, for $\theta = \Theta$, it has zero mean-value

$$\int\limits_Y b^\Theta(y)\, dy = 0; \quad \operatorname{div} b^\theta = 0, \ \forall\, \theta. \tag{3.4}$$

Remark 3.1. *This computation leading to the simple problem* (3.2) *for v^ε does not work if the coefficients are merely locally periodic, namely of the type $a(x, x/\varepsilon)$, $b(x, x/\varepsilon)$. Indeed there would be additional terms in* (3.2) *due to the partial derivatives with respect to the slow variable x because $\lambda_1(\Theta)$ and p_Θ would depend on x.*

Although problem (3.2) is not self-adjoint, the classical approach of homogenization (based on energy estimates in Sobolev spaces) would apply, thanks to (3.4), if the initial condition were not singular (the limit of $e^{-\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}$ is 0 or $+\infty$ almost everywhere). This singular behavior of the initial data (which formally has a limit merely in the sense of distributions) requires a different methodology for homogenizing (3.2). In order to overcome this difficulty, we use the representation of v^ε in terms of the corresponding Green function

$$v^\varepsilon(t, x) = \int\limits_\Omega K_\varepsilon(t, x, \xi)\, \frac{u_0(\xi)}{p_\Theta\big(\frac{\xi}{\varepsilon}\big)}\, e^{-\frac{\Theta\cdot(\xi-\bar{x})}{\varepsilon}}\, d\xi, \tag{3.5}$$

where, for any given ξ, K_ε, as a function of (t, x), solves the problem

$$\begin{cases} \varrho_\Theta\big(\dfrac{x}{\varepsilon}\big)\, \partial_t K_\varepsilon(t, x, \xi) + A_\Theta^\varepsilon\, K_\varepsilon(t, x, \xi) = 0, & (t, x) \in (0, T) \times \Omega, \\ K_\varepsilon(t, x, \xi) = 0, & (t, x) \in (0, T) \times \partial\Omega, \\ K_\varepsilon(0, x, \xi) = \delta(x - \xi), & x \in \Omega, \end{cases} \tag{3.6}$$

The strategy is now to replace the Green function K_ε by an ansatz in (3.5) and to study the limit, as $\varepsilon \to 0$, of the resulting singular integral. The next section is devoted to the study of the asymptotic behavior of K_ε.

4 Asymptotics of the Green function K_ε

The main goal of this section is to prove the following statement.

Lemma 4.1. *Assume that conditions* (**H1**) – (**H2**) *are satisfied. Let K_ε be the Green function of problem* (3.2)*. Then, for any $t_0 > 0$ and any compact subset $B \Subset \Omega$, there exists a constant C such that, for all $t \geq t_0 > 0$, $\xi \in B$,*

$$\int\limits_\Omega |K_\varepsilon(t, x, \xi) - K_0(t, x, \xi)|^2\, dx \leq C\, \varepsilon^2,$$
$$|K_\varepsilon(t, x, \xi) - K_0(t, x, \xi)| \leq C\, \varepsilon^\gamma, \quad x \in \Omega,$$

where the constant C depends on $t_0, \mathrm{dist}(B, \partial\Omega), \Omega, \Lambda, d$ and is independent of ε, $\gamma = \gamma(\Omega, \Lambda, d) > 0$, and K_0 is the Green function of the homogenized problem (2.6), i.e., as a function of (t, x), it solves

$$\begin{cases} \partial_t K_0(t,x,\xi) = \mathrm{div}(a^{\mathrm{eff}} \nabla K_0(t,x,\xi)), & (t,x) \in (0,T) \times \Omega, \\ K_0(t,x,\xi) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ K_0(0,x,\xi) = \delta(x-\xi), & x \in \Omega, \end{cases} \tag{4.1}$$

with the constant positive definite matrix a^{eff} defined by (4.7).

Proof. The main difficulty in studying the asymptotics of the Green function K_ε, defined as a solution of (3.6), is the presence of the delta function in the initial condition. To overcome this difficulty, we consider the difference

$$V_\varepsilon(t,x,\xi) = \Phi_\varepsilon(t,x,\xi) - K_\varepsilon(t,x,\xi),$$

where Φ_ε is the Green function of the same parabolic equation in the whole space, that is, for $\xi \in \mathbb{R}^d$, Φ_ε, as a function of (t,x), is a solution of the problem

$$\begin{cases} \varrho_\Theta\big(\frac{x}{\varepsilon}\big)\, \partial_t \Phi_\varepsilon(t,x,\xi) + A^\varepsilon_\Theta \Phi_\varepsilon(t,x,\xi) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ \Phi_\varepsilon(0,x,\xi) = \delta(x-\xi), & x \in \mathbb{R}^d. \end{cases} \tag{4.2}$$

In this way, for all $\xi \in \Omega$, V_ε, as a function of (t,x), solves the problem

$$\begin{cases} \varrho_\Theta\big(\frac{x}{\varepsilon}\big)\, \partial_t V_\varepsilon(t,x,\xi) + A^\varepsilon_\Theta\, V_\varepsilon(t,x,\xi) = 0, & (t,x) \in (0,T) \times \Omega, \\ V_\varepsilon(t,x,\xi) = \Phi_\varepsilon(t,x,\xi), & (t,x) \in (0,T) \times \partial\Omega, \\ V_\varepsilon(0,x,\xi) = 0, & x \in \Omega. \end{cases} \tag{4.3}$$

We emphasize that V_ε, in contrast with K_ε, is Hölder continuous for all $t \geq 0$ provided that $\xi \notin \partial\Omega$.

Notice that, by a proper rescaling in time and space, Φ_ε can be identified with the fundamental solution of an operator which is independent of ε. Indeed,

$$\Phi_\varepsilon(t,x,\xi) = \varepsilon^{-d}\, \Phi\big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\big), \tag{4.4}$$

where $\Phi(\tau, y, \eta)$ is defined, for $\eta \in \mathbb{R}^d$, as the solution in (τ, y) of

$$\begin{cases} \varrho_\Theta(y)\ \partial_\tau \Phi(\tau,y,\eta) + A_\Theta \Phi(\tau,y,\eta) = 0, & \tau > 0,\ y \in \mathbb{R}^d, \\ \Phi(0,y,\eta) = \delta(y-\eta), & y \in \mathbb{R}^d. \end{cases} \tag{4.5}$$

Here, for brevity, we denote by A_Θ the rescaled version of A^ε_Θ

$$A_\Theta \Phi(\tau,y,\eta) = -\mathrm{div}_y(a^\Theta(y) \nabla_y \Phi(\tau,y,\eta)) + b^\Theta(y) \cdot \nabla_y \Phi(\tau,y,\eta).$$

We also introduce the fundamental solution $\Phi_0(t,x,\xi)$ for the effective operator

$$\begin{cases} \partial_t \Phi_0 = \mathrm{div}_x(a^{\mathrm{eff}} \nabla_x \Phi_0), & (t,x) \in (0,T) \times \mathbb{R}^d, \\ \Phi_0(0,x,\xi) = \delta(x-\xi), & x \in \mathbb{R}^d. \end{cases} \tag{4.6}$$

The homogenized matrix a^{eff} is classically [8], [25] given by

$$
\begin{aligned}
a_{ij}^{\text{eff}} &= \int\limits_Y \big(a_{ij}^{\Theta}(y) + a_{ik}^{\Theta}(y)\partial_{y_k} N_j(y) - b_i^{\Theta}(y)\, N_j(y)\big)\; dy \\
&= \int\limits_Y \big(a_{ji}^{\Theta}(\eta) + a_{ki}^{\Theta}(\eta)\partial_{y_k} N_j^*(\eta) + b_i^{\Theta}(\eta)\, N_j^*(\eta)\big)\; d\eta,
\end{aligned}
\tag{4.7}
$$

where the vector-valued functions $N = (N_i)_{1\le i\le d}$ and $N^* = (N_i^*)_{1\le i\le d}$ solve the direct and adjoint cell problems, respectively,

$$
\begin{cases}
-\operatorname{div}(a^{\Theta}\nabla N_i) + b^{\Theta}\cdot\nabla N_i = \partial_{y_j} a_{ij}^{\Theta}(y) - b_i^{\Theta}(y), \quad Y, \\
y \mapsto N_i \quad Y-\text{periodic};
\end{cases}
\tag{4.8}
$$

$$
\begin{cases}
-\operatorname{div}((a^{\Theta})^T\nabla N_i^*) - b^{\Theta}\cdot\nabla N_i^* = \partial_{y_j} a_{ji}^{\Theta}(y) + b_i^{\Theta}(y), \quad Y, \\
y \mapsto N_i^* \quad Y-\text{periodic}.
\end{cases}
\tag{4.9}
$$

The matrix a^{eff} is positive definite (see, for example, [8], [20], [25]) and is exactly the same homogenized matrix as in the homogenization of the spectral problem [11]. Note that N and N^* are Hölder continuous functions (see [16]). The solution of problem (4.6) can be written explicitly:

$$
\Phi_0(t,x,\xi) = \frac{1}{(4\pi t)^{d/2}}\,\frac{1}{\det a^{\text{eff}}}\,\exp\Big\{-\frac{(x-\xi)^T(a^{\text{eff}})^{-1}(x-\xi)}{4t}\Big\}.
$$

The first-order approximation for the Green function Φ, solution of (4.5), is defined as follows

$$
\Phi_1(\tau,y,\eta) = \Phi_0(\tau,y,\eta) + N(y)\cdot\nabla_x\Phi_0(\tau,y,\eta) + N^*(\eta)\cdot\nabla_\xi\Phi_0(\tau,y,\eta). \tag{4.10}
$$

By means of Bloch wave analysis it has been shown in [1] that, under assumption (3.4), there exists a constant C such that, for any $\tau \ge 1$ and $y,\eta\in\mathbb{R}^d$,

$$
\begin{aligned}
|\Phi(\tau,y,\eta) - \Phi_0(\tau,y,\eta)| &\le \frac{C}{\tau^{(d+1)/2}}, \\
|\Phi(\tau,y,\eta) - \Phi_1(\tau,y,\eta)| &\le \frac{C}{\tau^{(d+2)/2}}.
\end{aligned}
\tag{4.11}
$$

Thus, in view of the rescaling (4.4), there exists a constant $C>0$, which does not depend on ε, such that, for any $t\ge\varepsilon^2$, $x,\xi\in\mathbb{R}^d$,

$$
\begin{aligned}
|\Phi_\varepsilon(t,x,\xi) - \Phi_0(t,x,\xi)| &\le \frac{C\,\varepsilon}{t^{(d+1)/2}}; \\
|\Phi_\varepsilon(t,x,\xi) - \Phi_1^\varepsilon(t,x,\xi)| &\le \frac{C\,\varepsilon^2}{t^{(d+2)/2}}.
\end{aligned}
\tag{4.12}
$$

Here $\Phi_1^\varepsilon(t,x,\xi) = \varepsilon^{-d}\,\Phi_1\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon},\frac{\xi}{\varepsilon}\big)$, namely

$$
\Phi_1^\varepsilon(t,x,\xi) = \Phi_0(t,x,\xi) + \varepsilon\, N\big(\frac{x}{\varepsilon}\big)\cdot\nabla_x\Phi_0(t,x,\xi) + \varepsilon\, N^*\big(\frac{\xi}{\varepsilon}\big)\cdot\nabla_\xi\Phi_0(t,x,\xi). \tag{4.13}
$$

Next, we study the asymptotic behavior of V_ε, solution of (4.3). The (formal) two-scale asymptotic expansion method suggests to approximate V_ε by a first-order ansatz defined by

$$V_1^\varepsilon(t,x,\xi) = V_0(t,x,\xi) + \varepsilon\, N\big(\frac{x}{\varepsilon}\big)\cdot\nabla_x V_0(t,x,\xi) + \varepsilon\, N^*\big(\frac{\xi}{\varepsilon}\big)\cdot\nabla_\xi V_0(t,x,\xi), \quad (4.14)$$

where N and N^* are the solutions of cell problems (4.8) and (4.9), respectively, and, for fixed ξ, V_0, as a function of (t,x), is the solution of the effective problem

$$\begin{cases} \partial_t V_0(t,x,\xi) = \mathrm{div}_x(a^{\mathrm{eff}}\nabla_x V_0(t,x,\xi)), & (t,x)\in(0,T)\times\Omega, \\ V_0(t,x,\xi) = \Phi_0(t,x,\xi), & (t,x)\in(0,T)\times\partial\Omega, \\ V_0(0,x,\xi) = 0, & x\in\Omega. \end{cases} \quad (4.15)$$

Due to the maximum principle and to the explicit formula for Φ_0, there exists a constant C, which depends only on Λ and d, such that, for any compact subset $B\Subset\Omega$, $\xi\in B, (t,x)\in[0,T]\times\Omega$,

$$0\le V_0(t,x,\xi)\le \max_{(t,x)\in[0,T)\times\partial\Omega}\Phi_0(t,x,\xi)\le \frac{C}{\mathrm{dist}(B,\partial\Omega)^d}. \quad (4.16)$$

Moreover, combining (4.16) with the local estimates of the derivatives of V_0 gives

$$\Big|\partial_t^k\partial_{x_j}^l\partial_{\xi_j}^m V_0(t,x,\xi)\Big| \le \frac{C}{\mathrm{dist}(B,\partial\Omega)^{d+2k+l+m}},\ (t,x,\xi)\in[0,T]\times\Omega\times B. \quad (4.17)$$

To finish the proof of Lemma 4.1 we need the following intermediate result.

Lemma 4.2. *Let V_ε and V_0 be solutions of problems* (4.3) *and* (4.15)*, respectively. Then, for any compact subset $B\Subset\Omega$, there exists a positive constant C, only depending on* $\mathrm{dist}(B,\partial\Omega),\Omega,d,\Lambda$*, such that, for any $(t,\xi)\in[0,T]\times B$,*

$$\int_\Omega |V_\varepsilon(t,x,\xi)-V_0(t,x,\xi)|^2\,dx \le C\,\varepsilon^2.$$

Proof. Let V_1^ε be the first-order approximation of V_ε defined by (4.14). Evaluating the remainder after substituting the difference $\widetilde{V}^\varepsilon = V_1^\varepsilon - V_\varepsilon$ into problem (4.3), we get

$$\begin{cases} \varrho_\Theta\big(\frac{x}{\varepsilon}\big)\,\partial_t\widetilde{V}^\varepsilon + A_\Theta^\varepsilon\widetilde{V}^\varepsilon = F\big(t,x,\xi;\frac{x}{\varepsilon},\frac{\xi}{\varepsilon}\big) \\ \qquad +\varepsilon\, f\big(t,x,\xi;\frac{x}{\varepsilon},\frac{\xi}{\varepsilon}\big), & (t,x)\in(0,T)\times\Omega, \\ \widetilde{V}^\varepsilon = G_\varepsilon\big(t,x,\xi;\frac{x}{\varepsilon},\frac{\xi}{\varepsilon}\big), & (t,x)\in(0,T)\times\partial\Omega, \\ \widetilde{V}^\varepsilon(0,x,\xi)=0, & x\in\Omega, \end{cases} \quad (4.18)$$

with F, f and G defined by

$$F(t,x,\xi;y,\eta) = \varrho_\Theta(y)\,\partial_t V_0 - \mathrm{div}_y(a^\Theta(y)\nabla_x(N(y)\,\nabla_x V_0(t,x,\xi)))$$

$$-\mathrm{div}_y(a^\Theta(y)\nabla_x(N^*(\eta)\,\nabla_\xi V_0(t,x,\xi))) - \mathrm{div}_x(a^\Theta(y)\nabla_x V_0(t,x,\xi))$$

$$-\mathrm{div}_x(a^\Theta(y)\nabla_y(N(y)\nabla_x V_0(t,x,\xi))) + b^\Theta(y)\cdot\nabla_x(N(y)\,\nabla_x V_0(t,x,\xi)))$$

$$+b^\Theta(y)\cdot\nabla_x(N^*(\eta)\,\nabla_\xi V_0(t,x,\xi));$$

$$f(t,x,\xi;y,\eta) = N(y)\cdot\partial_t\nabla_x V_0(t,x,\xi) + N^*(\eta)\cdot\partial_t\nabla_\xi V_0(t,x,\xi)$$

$$-\mathrm{div}_x(a^\Theta(y)\nabla_x(N(y)\cdot\nabla_x V_0(t,x,\xi)))$$

$$-\mathrm{div}_x(a^\Theta(y)\nabla_x(N^*(y)\cdot\nabla_\xi V_0(t,x,\xi)));$$

$$G_\varepsilon(t,x,\xi;y,\eta) = \Phi_0(t,x,\xi) - \Phi_\varepsilon(t,x,\xi)$$

$$+\varepsilon\,N(y)\cdot\nabla_x V_0(t,x,\xi) + \varepsilon\,N^*(\eta)\cdot\nabla_\xi V_0(t,x,\xi).$$

By linearity, we represent $\widetilde{V}^\varepsilon$ as a sum $\widetilde{V}^\varepsilon = \widetilde{V}_1^\varepsilon + \widetilde{V}_2^\varepsilon$, where $\widetilde{V}_1^\varepsilon$ and $\widetilde{V}_2^\varepsilon$ are solutions of the follwoing problems

$$\begin{cases} \varrho_\Theta\big(\dfrac{x}{\varepsilon}\big)\,\partial_t\widetilde{V}_1^\varepsilon + A_\Theta^\varepsilon\widetilde{V}_1^\varepsilon = F\big(t,x,\xi;\dfrac{x}{\varepsilon},\dfrac{\xi}{\varepsilon}\big) \\ \qquad +\varepsilon\,f\big(t,x,\xi;\dfrac{x}{\varepsilon},\dfrac{\xi}{\varepsilon}\big), \ (t,x)\in(0,T)\times\Omega, \\ \widetilde{V}_1^\varepsilon = 0, \qquad\qquad (t,x)\in(0,T)\times\partial\Omega, \\ \widetilde{V}_1^\varepsilon(0,x,\xi) = 0, \qquad\qquad x\in\Omega; \end{cases} \tag{4.19}$$

$$\begin{cases} \varrho_\Theta\big(\dfrac{x}{\varepsilon}\big)\,\partial_t\widetilde{V}_2^\varepsilon + A_\Theta^\varepsilon\widetilde{V}_2^\varepsilon = 0, \ (t,x)\in(0,T)\times\Omega, \\ \widetilde{V}_2^\varepsilon = G_\varepsilon\big(t,x,\xi;\dfrac{x}{\varepsilon},\dfrac{\xi}{\varepsilon}\big), \quad (t,x)\in(0,T)\times\partial\Omega, \\ \widetilde{V}_2^\varepsilon(0,x,\xi) = 0, \qquad\qquad x\in\Omega. \end{cases} \tag{4.20}$$

The trick is to estimate $\widetilde{V}_1^\varepsilon$ by standard energy estimates and $\widetilde{V}_2^\varepsilon$ by the maximum principle. First, we estimate $\widetilde{V}_1^\varepsilon$. Taking into account (4.17) and the boundedness of N, N^*, after integration by parts one has, for $\xi\in B\Subset\Omega$,

$$\Big|\int_Y F(t,x,\xi;y,\eta)w(y)\,dy\Big| \le C\|w\|_{H^1_\#(Y)}, \quad \forall w\in H^1_\#(Y),$$

where $H^1_\#(Y)$ stands for the closure of Y-periodic smooth functions with respect to the $H^1(Y)$ norm. Thus, as a function of y, F belongs to the dual space $H^{-1}_\#(Y)$ uniformly in (t,x,ξ,η). As is usual in the method of two-scale asymptotic expansion, equating the Y-average of F to zero yields the homogenized equation (4.15). Therefore, it is no surprise that, in view of (3.4), (4.15) and the periodicity of a^Θ_{ij}, N, N^*, we compute

$$\int_Y F(t,x,\xi;y,\eta)\,dy = 0.$$

Thus, for any t, x, ξ there exists a Y-periodic with respect to y vector function $\chi = \chi(t, x, \xi; y, \eta)$, which belongs to $L^2_\#(Y; \mathbb{R}^d)$, such that

$$
\begin{aligned}
&F(t, x, \xi; y, \eta) = \mathrm{div}_y \chi(t, x, \xi; y, \eta) \\
&\int\limits_Y |\chi(t, x, \xi; y, \eta)|^2 \, dy \le C, \ \ \xi \in B \Subset \Omega.
\end{aligned}
\tag{4.21}
$$

By rescaling we obtain

$$
F(t, x, \xi; y, \xi/\varepsilon) = \varepsilon \, \mathrm{div}_x \Big(\chi(t, x, \xi; x/\varepsilon, \eta) \Big) - \varepsilon \, \Big(\mathrm{div}_x \chi \Big)(t, x, \xi; x/\varepsilon, \eta). \tag{4.22}
$$

Since b^Θ is divergence-free, the a priori estimates are then obtained in the classical way. Multiplying the equation in (4.19) by $\widetilde{V}_1^\varepsilon$, integrating by parts and using (4.21), (4.22) yield

$$
\int\limits_\Omega |\widetilde{V}_1^\varepsilon(t, x, \xi)|^2 \, dx \le C \, \varepsilon^2, \quad (t, x) \in [0, T] \times \Omega, \ \xi \in B \Subset \Omega. \tag{4.23}
$$

Second, we estimate $\widetilde{V}_2^\varepsilon$, solution of (4.20), by using the maximum principle. Our next goal is to prove that

$$
|G_\varepsilon\big(t, x, \xi; \frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\big)| \le C \, \varepsilon, \quad (t, x) \in [0, T] \times \partial\Omega, \ \xi \in B \Subset \Omega. \tag{4.24}
$$

By (4.12), for any $\beta \le 2$ and $t \ge \varepsilon^\beta$,

$$
|\Phi_\varepsilon(t, x, \xi) - \Phi_1^\varepsilon(t, x, \xi)| \le C \, \varepsilon^{2 - (d+2)\beta/2}. \tag{4.25}
$$

In (4.25) we find $2 - (d+2)\beta/2 \ge 1$ if and only if $\beta \le (1 + d/2)^{-1}$ which is always smaller than 2. For $x \in \partial\Omega, \xi \in B \Subset \Omega$, uniformly with respect to $t \ge 0$, we have

$$
|\nabla_x \Phi_0(t, x, \xi)| \le \frac{C \, |x - \xi|}{t^{1+d/2}} \, e^{-\frac{C_0 |x-\xi|^2}{t}} \le C
$$

and a similar bound for $\nabla_\xi \Phi_0$. Thus, from (4.13) we deduce

$$
|\Phi_1^\varepsilon(t, x, \xi) - \Phi_0(t, x, \xi)| \le C \, \varepsilon, \quad t \ge 0, \ x \in \partial\Omega, \ \xi \in B \Subset \Omega. \tag{4.26}
$$

Combining (4.25) and (4.26) yields, for any $0 < \beta \le (1 + d/2)^{-1}$,

$$
|\Phi_\varepsilon(t, x, \xi) - \Phi_0(t, x, \xi)| \le C \, \varepsilon, \quad t \ge \varepsilon^\beta, x \in \partial\Omega, \xi \in B \Subset \Omega. \tag{4.27}
$$

To estimate $\Phi_\varepsilon - \Phi_0$ for small $t \in [0, \varepsilon^\beta)$ we make use of the Aronson estimates [6]. Taking into account (3.4) and (4.4), we see that Φ_ε admits the following bound

$$
0 \le \Phi_\varepsilon(t, x, \xi) = \varepsilon^{-d} \, \Phi\big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\big) \le \frac{C}{t^{d/2}} \exp\Big\{ - \frac{C_0 |x - \xi|^2}{t} \Big\}
$$

with the constants C_0, C independent of ε. Thus, for sufficiently small ε, we obtain

$$
\begin{aligned}
&|\Phi_\varepsilon(t, x, \xi) - \Phi_0(t, x, \xi)| \le |\Phi_\varepsilon(t, x, \xi)| + |\Phi_0(t, x, \xi)| \\
&\le \frac{C}{t^{d/2}} \exp\Big\{ - \frac{C_0 |x - \xi|^2}{t} \Big\} \le \frac{C}{\varepsilon^{d\beta/2}} \exp\Big\{ - \frac{C_0 |x - \xi|^2}{\varepsilon^\beta} \Big\}.
\end{aligned}
\tag{4.28}
$$

Thus, for $t \in [0, \varepsilon^\beta)$, $x \in \partial\Omega$ and $\xi \in B \Subset \Omega$, the difference $|\Phi_\varepsilon(t,x,\xi) - \Phi_0(t,x,\xi)|$ is exponentially small if $\beta > 0$. Combining (4.27) and (4.28) yields

$$|\Phi_\varepsilon(t,x,\xi) - \Phi_0(t,x,\xi)| \le C\,\varepsilon, \quad (t,x) \in [0,T] \times \partial\Omega,\ \xi \in B \Subset \Omega, \tag{4.29}$$

with the constant C depending on $\mathrm{dist}(B,\Omega), \Lambda, d$. The boundedness of N, N^*, estimates (4.17) and (4.29) imply (4.24).

Then, we use the maximum principle in (4.20) to deduce from (4.24) that

$$|\widetilde{V}_2^\varepsilon(t,x,\xi)| \le C\,\varepsilon, \quad (t,x,\xi) \in [0,T) \times \Omega \times B. \tag{4.30}$$

In view of (4.23) and (4.30), we conclude

$$\int_\Omega |V_\varepsilon(t,x,\xi) - V_1^\varepsilon(t,x,\xi)|^2\,dx \le C\,\varepsilon^2, \quad t \in [0,T],\ \xi \in B \Subset \Omega.$$

Recalling the definition of V_1^ε and using estimate (4.17) complete the proof of Lemma 4.2. □

Turning back to the proof of Lemma 4.1, the Green function $K_0(t,x,\xi)$, which is defined as the solution of (4.1), satisfies $K_0 = V_0 - \Phi_0$. Similarly, by definition, $K_\varepsilon = V_\varepsilon - \Phi_\varepsilon$. Taking into account (4.12), Lemma 4.2 implies

$$\int_\Omega |K_\varepsilon(t,x,\xi) - K_0(t,x,\xi)|^2\,dx \le C\,\varepsilon^2, \quad t \ge t_0 > 0,\ \xi \in B \Subset \Omega.$$

We would like to emphasize that the constant C in the last estimate only depends on t_0, $\mathrm{dist}(B, \partial\Omega)$, Λ, d, Ω. Due to the Nash-De Giorgi estimates for the parabolic equations (see, for example, [19]), K_ε is Hölder continuous (of course K_0 is), and, thus, one can deduce a uniform estimate

$$|K_\varepsilon(t,x,\xi) - K_0(t,x,\xi)| \le C\,\varepsilon^\gamma, \quad t \ge t_0 > 0,\ x \in \Omega,\ \xi \in B \Subset \Omega \tag{4.31}$$

for some $\gamma > 0$ depending on Ω, Λ and d. We emphasize that the constants C, γ do not depend on ε. Indeed, due to condition (3.4), problem (3.2) can be rewritten in divergence form, without any convective term and without any ε-factor in front of the coefficients. The proof of Lemma 4.1 is complete. □

Remark 4.1. *Estimate* (4.31) *is enough for our purpose, but we emphasize that it can be improved. Namely, constructing sufficiently many terms in the asymptotic expansion for V_ε, one can show that*

$$|K_\varepsilon(t,x,\xi) - K_0(t,x,\xi)| \le C\,\varepsilon, \quad t \ge t_0 > 0,\ x \in \Omega,\ \xi \in B \Subset \Omega.$$

5 Asymptotics of u^ε or v^ε

The goal of this section is to prove our main result Theorem 2.1 and actually to give a more precise statement of it in Theorem 5.1. By the factorization principle

(3.1) it is equivalent to find a precise asymptotic expansion of v^ε. Recall that v^ε, as a solution of (3.2), can be represented in terms of the corresponding Green function K_ε by using formula (3.5). Bearing in mind Lemma 4.1, we rearrange (3.5) as follows

$$v^\varepsilon(t,x) = I_1^\varepsilon + I_2^\varepsilon \tag{5.1}$$

with

$$I_1^\varepsilon = \int\limits_\Omega K_0(t,x,\xi)\,\frac{u_0(\xi)}{p_\Theta\big(\frac{\xi}{\varepsilon}\big)}\,e^{-\frac{\Theta\cdot(\xi-\bar{x})}{\varepsilon}}\,d\xi,$$
$$I_2^\varepsilon = \int\limits_\Omega \big(K_\varepsilon(t,x,\xi) - K_0(t,x,\xi)\big)\,\frac{u_0(\xi)}{p_\Theta\big(\frac{\xi}{\varepsilon}\big)}\,e^{-\frac{\Theta\cdot(\xi-\bar{x})}{\varepsilon}}\,d\xi.$$

Of course, because of (4.31), the second integral in (5.1) is going to be, at least, ε^γ times smaller that the first one. Recall that, by assumption **(H3)**, u_0 has a compact support $\omega \Subset \Omega$ so we are able to use the previous estimates of Lemma 4.1. Let us compute approximately the first integral I_1^ε. Since $\Theta\cdot(x-\bar{x}) > 0$ for $x \in \omega\setminus\{\bar{x}\}$, it is clear that the main contribution is given by integrating over a neighborhood of the point $\bar{x}$. We consider the case of general position, when condition **(H5)** is fulfilled, that is, in local coordinates in a neighborhood $U_\delta(\bar{x})$ of the point $\bar{x}$, $\partial\omega$ can be defined by

$$z_d = (Sz', z') + o(|z'|^2)$$

for some positive definite $(d-1)\times(d-1)$ matrix S. Here $(z_1,\cdots,z_d)$ is an orthonormal basis such that the coordinates $z' = (z_1,\cdots,z_{d-1})$ are tangential to $\partial\omega$ and the axis z_d is the interior normal at $\bar{x}$. Note that, by assumption **(H4)**, Θ is directed along z_d. The neighborhood of $\bar{x}$ is defined by

$$U_\delta(\bar{x}) = \{z\in\omega:\ |z'|\le\delta,\ 0\le z_d\le\delta^2\|S\|\},$$

where $\|S\| = \max_{|x'|=1}|Sx'|$. Choosing $\delta = \varepsilon^{1/4}$ guaranties that the integral over the complement to $U_\delta(\bar{x})$ is negligible. Indeed,

$$\Big|\int\limits_{\omega\setminus U_\delta(\bar{x})} K_0(t,x,\xi)\,\frac{u_0(\xi)}{p_\Theta\big(\frac{\xi}{\varepsilon}\big)}\,e^{-\frac{\Theta\cdot(\xi-\bar{x})}{\varepsilon}}\,d\xi\Big| = O(e^{-\frac{1}{\sqrt{\varepsilon}}}).$$

Let us now compute the integral over $U_\delta(\bar{x})$, $\delta = \varepsilon^{1/4}$. Expanding K_0 and u_0 (which is of class C^2 in ω) into Taylor series about $\bar{x}$ and taking into account assumption (**H6**), for $t\ge t_0>0$, we obtain

$$I_1^\varepsilon = K_0(t,x,\bar{x})\,\frac{\partial u_0}{\partial\Theta}(\bar{x}) \int\limits_{U_\delta(\bar{x})} \frac{\Theta}{|\Theta|}\cdot(\xi-\bar{x})\,\big(p_\Theta\big(\frac{\xi}{\varepsilon}\big)\big)^{-1}\,e^{-\frac{\Theta\cdot(\xi-\bar{x})}{\varepsilon}}\,d\xi + O(\varepsilon^3\,\varepsilon^{\frac{d-1}{2}})$$
$$= K_0(t,x,\bar{x})\,\frac{\partial u_0}{\partial\Theta}(\bar{x}) \int\limits_{U_\delta(0)} \frac{\Theta}{|\Theta|}\cdot\xi\,\big(p_\Theta\big(\frac{\xi}{\varepsilon}+\frac{\bar{x}}{\varepsilon}\big)\big)^{-1}\,e^{-\frac{\Theta\cdot\xi}{\varepsilon}}\,d\xi + O(\varepsilon^3\,\varepsilon^{\frac{d-1}{2}}).$$

where $\partial u_0/\partial\Theta := \nabla u_0\cdot\Theta/|\Theta|$ is the directional derivative of u_0 along Θ (the tangential derivative of u_0 vanishes at $\bar{x}$ because u_0 is continuous and equal to

0 outside ω). Note that we have anticipated the precise order of the remainder term which will be clear once we compute the leading integral. Let us introduce the rotation matrix $\mathfrak{R}$ which defines the local coordinate system $(z_1, z_2, \cdots, z_d) = (z', z_d)$ previously defined. By definition it satisfies $\xi = \mathfrak{R}^{-1} z$ and $\Theta \cdot \xi = |\Theta| \, z_d$. Applying this change of variables we get

$$p_\Theta\big(\frac{\xi}{\varepsilon} + \{\frac{\bar{x}}{\varepsilon}\}\big) = p_\Theta\big(\mathfrak{R}^{-1}\big(\frac{z}{\varepsilon} + \mathfrak{R}\{\frac{\bar{x}}{\varepsilon}\}\big)\big) \equiv P_\Theta\big(\frac{z}{\varepsilon} + \bar{z}^\varepsilon\big), \tag{5.2}$$

where $\{\bar{x}/\varepsilon\}$ is the fractional part of $\bar{x}/\varepsilon$ and $\bar{z}^\varepsilon = \mathfrak{R}\{\bar{x}/\varepsilon\}$. In the case when $\Theta_1, \Theta_2, \cdots, \Theta_d$ are rationally dependent in pairs, P_Θ remains periodic with another period. Otherwise P_Θ is merely almost periodic. It happens, for example, when all $\Theta_k, k = 1, ..., d$ are rationally independent in pairs.

We turn to the computation of the integral over $U_\delta(0)$. By the above change of variables we get

$$\begin{aligned} I_1^\varepsilon &= K_0(t, x, \bar{x}) \frac{\partial u_0}{\partial \Theta}(\bar{x}) \\ &\times \int\limits_{|z'| \le \delta} dz' \int\limits_{(Sz', z')}^{\delta^2 \|S\|} z_d \, P_\Theta^{-1}\big(\frac{z}{\varepsilon} + \bar{z}^\varepsilon\big) \, e^{-\frac{|\Theta| z_d}{\varepsilon}} \, dz_d + o(\varepsilon^2 \, \varepsilon^{\frac{d-1}{2}}). \end{aligned} \tag{5.3}$$

To blow-up the integral in (5.3) we make a (parabolic) rescaling of the space variables

$$\zeta' = \frac{z'}{\sqrt{\varepsilon}}, \quad \zeta_d = \frac{z_d}{\varepsilon},$$

and recalling that $\delta = \varepsilon^{1/4}$, we arrive at the following integral

$$\begin{aligned} I_1^\varepsilon &= \varepsilon^2 \, \varepsilon^{\frac{(d-1)}{2}} \, K_0(t, x, \bar{x}) \frac{\partial u_0}{\partial \Theta}(\bar{x}) \\ &\times \int\limits_{\mathbb{R}^{d-1}} d\zeta' \int\limits_{(S\zeta', \zeta')}^{+\infty} \zeta_d \, P_\Theta^{-1}\big(\frac{\zeta'}{\sqrt{\varepsilon}} + (\bar{z}^\varepsilon)', \zeta_d + \bar{z}_d^\varepsilon\big) \, e^{-|\Theta| \zeta_d} \, d\zeta_d + o(\varepsilon^2 \, \varepsilon^{\frac{d-1}{2}}), \end{aligned}$$

where the reaminder term takes into account the fact that the domain of integration is now infinite. Changing the order of integration we have

$$\begin{aligned} I_1^\varepsilon &= \varepsilon^2 \, \varepsilon^{\frac{(d-1)}{2}} \, K_0(t, x, \bar{x}) \frac{\partial u_0}{\partial \Theta}(\bar{x}) \\ &\times \int\limits_0^{+\infty} \zeta_d \, e^{-|\Theta| \zeta_d} \, d\zeta_d \int\limits_{(S\zeta', \zeta') \le \zeta_d} P_\Theta^{-1}\big(\frac{\zeta'}{\sqrt{\varepsilon}} + (\bar{z}^\varepsilon)', \zeta_d + \bar{z}_d^\varepsilon\big) \, d\zeta' + o(\varepsilon^2 \, \varepsilon^{\frac{d-1}{2}}). \end{aligned}$$

The function $P_\Theta^{-1}\big(\eta', \tau_d\big)$ is uniformly continuous; moreover, it is almost periodic with respect to the first variable. Thus, for any bounded Borel set $B \subset \mathbb{R}^{d-1}$, the following limit exists

$$\mathcal{M}\{P_\Theta^{-1}(\cdot, \tau_d)\} = \lim_{t \to \infty} \frac{1}{|tB|} \int\limits_{tB} P_\Theta^{-1}\big(\eta' + \tau', \tau_d\big) \, d\eta'. \tag{5.4}$$

We emphasize that the convergence is uniform with respect to τ' and τ_d, and the limit does not depend on τ'. Therefore, by Lemma 8.2, as $\varepsilon \to 0$, we eventually deduce

$$
\begin{aligned}
I_1^\varepsilon &= \varepsilon^2\, \varepsilon^{\frac{d-1}{2}}\, K_0(t,x,\bar{x})\, \frac{\partial u_0}{\partial \Theta}(\bar{x}) \\
&\times \int\limits_{\mathbb{R}^{d-1}} d\zeta' \int\limits_{(S\zeta',\zeta')}^{+\infty} \zeta_d\, e^{-|\Theta|\zeta_d}\, \mathcal{M}\{P_\Theta^{-1}\big(\cdot, \zeta_d + \bar{z}_d^\varepsilon\big)\}\, d\zeta_d + o(\varepsilon^2\, \varepsilon^{\frac{d-1}{2}}),
\end{aligned}
\tag{5.5}
$$

where the remainder term is asymptotically smaller than the leading order term (uniformly in $t \ge 0, x \in \overline{\Omega}$) but we cannot say how much since there is no precise speed of convergence for averages of almost periodic functions in Lemma 8.2.

The case of the second integral I_2^ε is then very similar. Taking into account the positiveness of p_Θ, and Lemma 4.1, for $t \ge t_0 > 0$, we obtain

$$
|I_2^\varepsilon| \le C\, \varepsilon^\gamma \int\limits_\omega |u_0(x)|\, e^{-\frac{\Theta\cdot(\xi-\bar{x})}{\varepsilon}}\, d\xi,
$$

where C does not depend on ε. The same computation as above (but without the necessity of considering almost periodic functions) yields

$$
\begin{aligned}
|I_2^\varepsilon| &\le C\, \varepsilon^\gamma \left|\frac{\partial u_0}{\partial \Theta}(\bar{x})\right| \int\limits_\omega \left|\frac{\Theta}{|\Theta|}\cdot(\xi-\bar{x})\right|\, e^{-\frac{\Theta\cdot(\xi-\bar{x})}{\varepsilon}}\, d\xi \\
&\le C\, \varepsilon^\gamma \left|\frac{\partial u_0}{\partial \Theta}(\bar{x})\right| \int\limits_{\mathbb{R}^{d-1}} dz' \int\limits_{S_0|z'|^2}^{+\infty} z_d\, e^{-\frac{|\Theta| z_d}{\varepsilon}}\, dz_d \\
&\le C\, \varepsilon^{2+\gamma} \left|\frac{\partial u_0}{\partial \Theta}(\bar{x})\right| \int\limits_{\mathbb{R}^{d-1}} (1 + S_0\, |z'|^2\, \varepsilon^{-1})\, e^{-\frac{|\Theta| S_0 |z'|^2}{\varepsilon}} dz' \\
&\le C\, \varepsilon^{2+\gamma}\, \varepsilon^{\frac{d-1}{2}},
\end{aligned}
$$

for some constant $S_0 > 0$ and $C = C(S_0, \Theta)$. Finally, we have derived the following asymptotics of v^ε, as $\varepsilon \to 0$,

$$
\begin{aligned}
v^\varepsilon(t,x) &= \varepsilon^2\, \varepsilon^{\frac{d-1}{2}} \Big(1 + r_\varepsilon(t,x)\Big) K_0(t,x,\bar{x})\, \frac{\partial u_0}{\partial \Theta}(\bar{x}) \\
&\times \int\limits_{\mathbb{R}^{d-1}} d\zeta' \int\limits_{(S\zeta',\zeta')}^{+\infty} \zeta_d\, e^{-|\Theta|\zeta_d}\, \mathcal{M}\{P_\Theta^{-1}\big(\cdot, \zeta_d + \bar{z}_d^\varepsilon\big)\}\, d\zeta_d,
\end{aligned}
$$

where $r_\varepsilon(t,x)$ converges to zero uniformly with respect to $(t,x) \in [t_0, T] \times \overline{\Omega}$ with any $t_0 > 0$.

We summarize the result, just obtained, by formulating a more precise version of Theorem 2.1, describing the asymptotics of $u^\varepsilon(t,x)$.

Theorem 5.1. *Suppose conditions* **(H1)** $-$ **(H6)** *are satisfied and* $\Theta \neq 0$. *Let* u^ε *be the solution of problem* (1.1). *Then, for* $t \ge t_0 > 0$,

$$
u^\varepsilon(t,x) = \varepsilon^2\, \varepsilon^{\frac{d-1}{2}} \Big(1 + r_\varepsilon(t,x)\Big) e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}\, M_\varepsilon\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, u(t,x),
$$

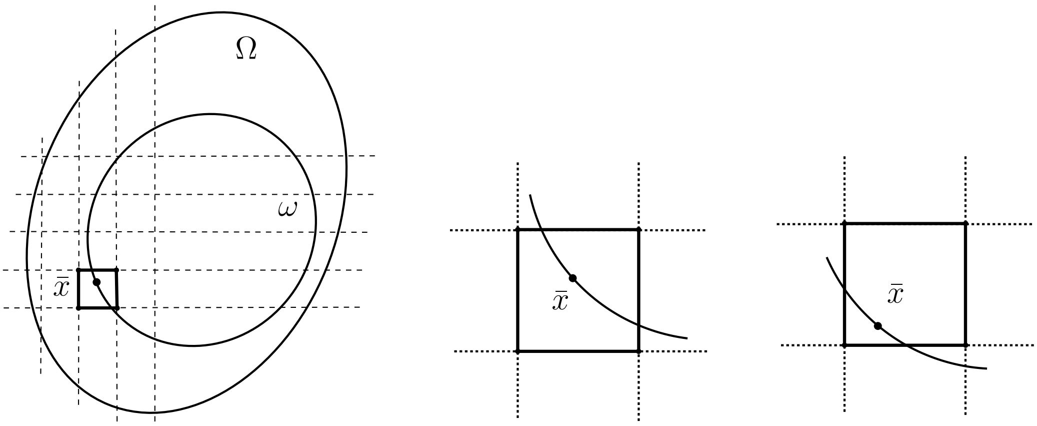
where $(\lambda_1(\Theta), p_\Theta)$ *is the first eigenpair defined by Lemma 2.1 and* $r_\varepsilon(t,x) \to 0$*, as* $\varepsilon \to 0$*, uniformly with respect to* $(t,x) \in [t_0, T] \times \overline{\Omega}$*. The function* $u(t,x)$ *solves the homogenized problem*

$$\begin{cases} \partial_t u = \operatorname{div}(a^{\text{eff}}\,\nabla u), & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = \nabla u_0(\bar{x}) \cdot \dfrac{\Theta}{|\Theta|}\,\delta(x - \bar{x}), & x \in \Omega, \end{cases} \tag{5.6}$$

with a^{eff} *being a positive definite matrix given by* (4.7)*,* $\delta(x-\bar{x})$ *is the Dirac delta-function at the point* $\bar{x}$*. The constant* M_ε *is defined by*

$$M_\varepsilon = \int\limits_{\mathbb{R}^{d-1}} d\zeta' \int\limits_{(S\zeta',\zeta')}^{+\infty} \zeta_d\, e^{-|\Theta|\zeta_d}\, \mathcal{M}\{P_\Theta^{-1}\big(\cdot, \zeta_d + \bar{z}_d^\varepsilon\big)\}\, d\zeta_d, \tag{5.7}$$

where $\mathcal{M}\{P_\Theta^{-1}\big(\cdot,\tau_d\big)\}$ *is the mean-value of the almost periodic function* $\eta' \to P_\Theta^{-1}(\eta', \tau_d)$ *(see* (5.4)*),* P_Θ *is given by* (5.2) *and* $\bar{z}_d^\varepsilon = \mathfrak{R}\{\bar{x}/\varepsilon\} \cdot \frac{\Theta}{|\Theta|}$.



(a) (b) (c)

Figure 2: Position of $\bar{x}$ in εY for different values of ε

Remark 5.1. *The constant* M_ε *defined by* (5.7) *depends on* $\bar{z}_d^\varepsilon = \mathfrak{R}\{\bar{x}/\varepsilon\} \cdot \frac{\Theta}{|\Theta|}$*, that is on the component, parallel to* Θ*, of the fractional part of* $\bar{x}/\varepsilon$*, or, in other words, on the relative position of* $\bar{x}$ *inside the cell* εY *(see Figure 2). Notice that* M_ε *is bounded, thus, up to a subsequence, it converges to some* M^**, as* $\varepsilon \to 0$*. The choice of the converging subsequence is only a matter of the geometric definition of the periodic medium. For example, if* $\bar{x}$ *is known, we may decide to make it the origin and to define the periodic microstructure relative to this origin. Then* $\bar{x} = 0$*,* $\bar{z}^\varepsilon = 0$ *is fixed in the periodicity cell, and* $M_\varepsilon = M$ *is independent of* ε*.*

It might happen that the vector Θ is such that its components Θ_d and Θ_k are rationally independent for all $k \neq d$. In such a case, it turns out that the constant M_ε does not depend on ε and, moreover, can be explicitly computed. This is the topic of the following result.

Corollary 5.1. *Let conditions of Theorem 5.1 be satisfied. And assume that the vector* Θ *is such that* Θ_d *and* Θ_k*, for any* $k = 1, \cdots, (d-1)$*, are rationally independent. Then* M_ε *is independent of* ε *and is given by*

$$M_\varepsilon = \frac{(d-1)}{|\Theta|^2}\Big(\frac{\pi}{|\Theta|}\Big)^{\frac{d-1}{2}} (\det S)^{1/2} \int\limits_Y p_\Theta^{-1}(y)\,dy.$$

In other words, for $t \ge t_0 > 0$,

$$u^\varepsilon(t,x) = \Big(\frac{\varepsilon}{|\Theta|}\Big)^{2+\frac{d-1}{2}} K\, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, u(t,x)\,\Big(1 + r_\varepsilon(t,x)\Big),$$

where $r_\varepsilon(t,x) \to 0$*, as* $\varepsilon \to 0$*, uniformly with respect to* $(t,x) \in [t_0,T]\times\overline{\Omega}$*;* $u(t,x)$ *solves the homogenized problem* (5.6)*. The constant* K *is given by*

$$K = (d-1)\,\pi^{\frac{d-1}{2}}\,(\det S)^{1/2} \int\limits_Y p_\Theta^{-1}(y)\,dy.$$

Proof. It is sufficient to notice that in the case when Θ_d and Θ_k, $k = 1, 2, \cdots (d-1)$, are rationally independent, the mean value of the almost periodic function $P_\Theta^{-1}(\zeta', \tau_d)$ with respect to the first variable ζ', for any τ_d, coincides with its volume average

$$\mathcal{M}\{P_\Theta^{-1}(\cdot, \tau_d)\} = \int\limits_Y p_\Theta^{-1}(y)\,dy.$$

Thus, the constant M_ε given by (5.7) does not depend on ε and has the following form

$$M_\varepsilon = \Big(\int\limits_Y p_\Theta^{-1}(y)\,dy\Big) \int\limits_{\mathbb{R}^{d-1}} d\zeta' \int\limits_{(S\zeta',\zeta')}^{+\infty} \zeta_d\, e^{-|\Theta|\zeta_d}\, d\zeta_d.$$

Evaluating the last integral we obtain

$$M_\varepsilon = \frac{(d-1)}{|\Theta|^2}\Big(\frac{\pi}{|\Theta|}\Big)^{\frac{d-1}{2}} (\det S)^{1/2} \int\limits_Y p_\Theta^{-1}(y)\,dy$$

that implies the desired result. □

Remark 5.2. *Theorem 5.1 does not provide any rate of convergence due to several reasons. First of all, without specifying the remainder in hypothesis* ***(H5)****, one cannot expect any estimate in* (5.3)*. One possible option would be to assume that in local coordinates, in the neighbourhood of the point* $\bar{x}$*,* $\partial\omega$ *is defined by*

$$z_d = (Sz', z') + O(|z|^3).$$

Then in (5.3) *one would obtain the error* $O(\varepsilon^3\,\varepsilon^{(d-1)/2})$.

The second reason for the lack of estimates is concealed in Lemma 8.2. In contrast with the classical mean value theorem for periodic functions, Lemma 8.2 does not provide any rate of convergence. However, if all the components of the vector Θ *are rationally dependent, then* P_Θ *remains periodic (maybe with another* *period), and one can apply the mean value theorem for smooth periodic functions that gives an error $O(\varepsilon)$, and, consequently, $O(\varepsilon^3\,\varepsilon^{(d-1)/2})$ in (5.5).*

Finally, estimate (4.31) guaranties that the second integral in (5.1) is ε^γ smaller than the first one, where $0 < \gamma \le 1$ depends on Λ, Ω, d.

Remark 5.3. *We stress that if condition **(H3)** is violated and the support of u_0 touches the boundary of Ω, then the two integrals in (5.1) are of the same order, and we cannot neglect the second integral any more. In this case it is necessary to construct not only the leading term of the asymptotics for K_ε, but also a corrector term together with a boundary layer corrector. It is possible in some particular cases, for example, when $\bar{x}$ belongs to a flat part of the boundary of Ω, or when the coefficients of the equation are constant. But it is well known that boundary layers in homogenization are very difficult to build in the case of a non flat boundary. Simple cases (flat boundaries, cylindrical domains) will be considered in our forthcoming paper [3].*

Another typical situation arises when we do not assume anymore that the initial data u_0 is continuous on Ω but merely that it has compact support and is C^2 inside its support. In particular, in this new situation we may have $u_0(\bar{x}) \neq 0$. The next theorem, characterizing the asymptotic behaviour of u^ε in this case, can be proved in exactly the same way as Theorem 5.1.

Theorem 5.2. *Suppose conditions* **(H1)**, **(H2)**, **(H4)**, **(H5)** *are satisfied and $\Theta \neq 0$. Assume that u_0 has compact support $\omega \Subset \Omega$, $u_0 \in C^2(\overline{\omega})$ and $u_0(\bar{x}) \neq 0$. If u^ε is a solution of problem (1.1), then, for $t \ge t_0 > 0$*

$$u^\varepsilon(t,x) = \varepsilon\,\varepsilon^{\frac{d-1}{2}}\Big(1 + r_\varepsilon(t,x)\Big)e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\,e^{\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}\,M_\varepsilon\,p_\Theta\big(\frac{x}{\varepsilon}\big)\,u(t,x),$$

where $r_\varepsilon(t,x) \to 0$, as $\varepsilon \to 0$, uniformly with respect to $(t,x) \in [t_0,T] \times \overline{\Omega}$. Here, $u(t,x)$ solves the effective problem

$$\begin{cases} \partial_t u = \operatorname{div}(a^{\text{eff}}\,\nabla u), & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(\bar{x})\,\delta(x-\bar{x}), & x \in \Omega. \end{cases}$$

The constant M_ε is now given by

$$M_\varepsilon = \int\limits_{\mathbb{R}^{d-1}} d\zeta' \int\limits_{(S\zeta',\zeta')}^{+\infty} e^{-|\Theta|\zeta_d}\,\mathcal{M}\{P_\Theta^{-1}\big(\cdot,\zeta_d + \bar{z}_d^\varepsilon\big)\}\,d\zeta_d,$$

with the same definitions of the mean-value $\mathcal{M}$, of the almost periodic function P_Θ and of $\bar{z}_d^\varepsilon$ as in Theorem 5.1.

Remark 5.4. *Yet another possible situation is that $u_0 = \partial u_0/\partial\Theta = 0$ in the neighborhood of $\bar{x}$. If we assume that $u_0 \in C^3(\omega)$ and replace condition* **(H6)** *by*

$$\frac{\partial^2 u_0}{\partial\Theta^2}(\bar{x}) = \frac{\partial}{\partial\Theta}\Big(\frac{\partial u_0}{\partial\Theta}\Big)(\bar{x}) \neq 0,$$

where $\partial u_0/\partial\Theta$ is the directional derivative of u_0 in the direction of Θ, then we can prove in this case that, for $t \ge t_0 > 0$,

$$u^\varepsilon(t,x) = \varepsilon^3\, \varepsilon^{\frac{d-1}{2}} \Big(1 + r_\varepsilon(t,x)\Big) e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{-\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}\, M_\varepsilon\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, u(t,x),$$

where $r_\varepsilon(t,x) \to 0$, as $\varepsilon \to 0$, uniformly with respect to $(t,x) \in [t_0,T] \times \overline{\Omega}$ and $u(t,x)$ is a solution of

$$\begin{cases} \partial_t u = \operatorname{div}(a^{\text{eff}}\, \nabla u), & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = \dfrac{1}{2}\, \dfrac{\partial^2 u_0}{\partial \Theta^2}(\bar{x})\, \delta(x-\bar{x}), & x \in \Omega. \end{cases}$$

The constant M_ε is now given by

$$M_\varepsilon = \int\limits_{\mathbb{R}^{d-1}} d\zeta' \int\limits_{(S\zeta',\zeta')}^{+\infty} \zeta_d^2\, e^{-|\Theta|\zeta_d}\, \mathcal{M}\{P_\Theta^{-1}\big(\cdot, \zeta_d + \bar{z}_d^\varepsilon\big)\}\, d\zeta_d.$$

The case when u_0 vanishes on the boundary of ω together with its derivatives up to order k, can be treated similarly.

It should be noticed that a statement similar to that of Corollary 5.1 remains valid for Theorem 5.2 and Remark 5.4.

6 The case of a flat boundary of ω

In the previous sections we analyzed the case when the quadratic form of the surface $\partial\omega$ is non-degenerate at the point $\bar{x}$. The asymptotics of the solution of problem (1.1) can also be constructed when $\bar{x}$ belongs to a flat part Σ of $\partial\omega$ and the vector Θ is orthogonal to Σ.

More precisely, we replace the previous assumptions **(H4), (H5), (H6)** with the following ones.

(H4') The set of points $\bar{x}$ which provide the minimum in $\min_{x\in\omega} \Theta\cdot x$ is a subset Σ of $\partial\omega$ which is included in an hyperplane of $\mathbb{R}^d$ and Σ has a positive $(d-1)$-measure.

(H5') $u_0(y) = 0$ for all $y \in \Sigma$. There exists $\bar{x} \in \Sigma$ such that $\dfrac{\partial u_0}{\partial\Theta}(\bar{x}) \neq 0$.

Remark 6.1. *Assumption* **(H4')** *implies that*

$$\Theta\cdot(x-\bar{x}) > 0 \textit{ for all } x \in \omega\setminus\Sigma,\ \bar{x}\in\Sigma,$$

and Θ is orthogonal to Σ and directed inside ω (see Figure 3). Furthermore, $\bar{x}_\Theta = \bar{x}\cdot\frac{\Theta}{|\Theta|}$ is the same for all $\bar{x}\in\Sigma$.

In this case we prove the following result.

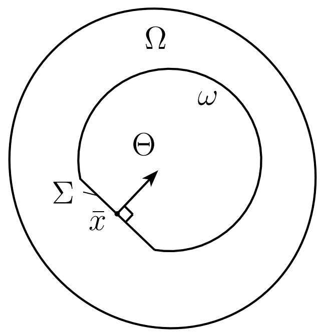


Figure 3: The case of a flat part of the boundary $\partial\omega$

Theorem 6.1. *Assume that conditions **(H1)-(H3)** and **(H4')-(H5')** are fulfilled, and $\Theta \neq 0$. Then, for $t \geq t_0 > 0$, the asymptotic behaviour of the solution u^ε of problem* (1.1) *is described by*

$$u^\varepsilon(t,x) = \varepsilon^2 \, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}} \, e^{\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}} \, (1 + r_\varepsilon(t,x)) \, M_\varepsilon \, p_\Theta\big(\frac{x}{\varepsilon}\big) \, u(t,x),$$

where $r_\varepsilon(t,x) \to 0$, as $\varepsilon \to 0$, uniformly with respect to $(t,x) \in [t_0,T] \times \overline{\Omega}$, $(\lambda_1(\Theta), p_\Theta)$ is the first eigenpair defined by Lemma 2.1, $\bar{x}$ is an arbitrary point on Σ and $u(t,x)$ solves the homogenized problem

$$\begin{cases} \partial_t u = \operatorname{div}(a^{\text{eff}} \, \nabla u), & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = \dfrac{\partial u_0}{\partial \Theta}(x) \, \delta_\Sigma, & x \in \Omega. \end{cases} \tag{6.1}$$

Here a^{eff} is still defined by (4.7)*, δ_Σ is the Dirac delta-function on Σ and the constant M_ε is given by*

$$M_\varepsilon = \int_0^{+\infty} \zeta_d \, e^{-|\Theta|\,\zeta_d} \, \mathcal{M}\{P_\Theta^{-1}(\cdot, \zeta_d + \frac{\bar{x}_\Theta}{\varepsilon})\} d\zeta_d$$

with $\mathcal{M}\{P_\Theta^{-1}(\cdot, \tau_d)\}$ being the mean value of the almost periodic function $P_\Theta^{-1}(\cdot, \tau_d)$ (see (5.4)*), $P_\Theta(z)$ being the rotation of p_Θ in the local coordinates of Σ: $P_\Theta(\zeta) = p_\Theta(\mathcal{R}^{-1}\zeta)$, where $\mathcal{R}$ is the rotation matrix.*

Proof. The proof starts, like that of Theorem 5.1, by using the representation formula (3.5) for the solution v^ε of (3.2) in terms of the Green function K_ε. Writing $K_\varepsilon = K_0 + (K_\varepsilon - K_0)$ we arrive at (5.1), namely

$$v^\varepsilon(t,x) = I_1^\varepsilon + I_2^\varepsilon.$$

By Lemma 4.1, we can estimate I_2^ε, passing to local coordinates, as in the proof of Theorem 5.1,

$$
\begin{aligned}
|I_2^\varepsilon| &\le C\,\varepsilon^\gamma \int\limits_\omega |u_0(\xi)|\, e^{-\frac{\Theta\cdot(\xi-\bar x)}{\varepsilon}}\, d\xi \\
&\le C\,\varepsilon^\gamma \int\limits_\Sigma \big|\frac{\partial u_0}{\partial\Theta}(z',\bar x_\Theta)\big| dz' \int\limits_0^{+\infty} z_d\, e^{-\frac{|\Theta| z_d}{\varepsilon}}\, dz_d
\end{aligned}
$$

for some $\gamma=\gamma(\Lambda,\Omega,d)>0$ defined in (4.31). Making the change of variables $\zeta_d = z_d/\varepsilon$, we see that

$$
|I_2^\varepsilon| \le C\,\varepsilon^{2+\gamma} \int\limits_\Sigma \big|\frac{\partial u_0}{\partial\Theta}(z',\bar{x_\Theta})\big| dz' \int\limits_0^{+\infty} \zeta_d\, e^{-|\Theta|\zeta_d}\, d\zeta_d \le C\,\varepsilon^{2+\gamma}.
$$

In order to compute approximately I_1^ε, we again pass to the local coordinates. Namely, we rotate coordinates $z=\mathfrak{R}\,\xi$ in such a way that Θ is directed along z_d. It is obvious that only the neighborhood of Σ contributes in I_1^ε. Expanding K_0 and u_0 into a Taylor series with respect to z_d and making the change of variables $\zeta_d = z_d/\varepsilon$ leads to

$$
I_1^\varepsilon = \varepsilon^2 \int\limits_0^{+\infty} \zeta_d\, e^{-|\Theta|\,\zeta_d}\, d\zeta_d \int\limits_\Sigma K_0(t,x,z',\bar x_\Theta)\,\frac{\partial u_0}{\partial\Theta}(z',\bar x_\Theta)\, P_\Theta^{-1}\big(\frac{z'}{\varepsilon},\zeta_d+\frac{\bar x_\Theta}{\varepsilon}\big)\, dz' + o(\varepsilon^2).
$$

where $P_\Theta(\zeta)\equiv p_\Theta(\mathcal{R}^{-1}\zeta)$ with $\mathcal{R}$ being the rotation matrix.

Since $P_\Theta^{-1}(\zeta',\tau_d)$ is uniformly continuous, and, moreover, almost periodic with respect to ζ', by Lemma 8.1, we have

$$
I_1^\varepsilon = \varepsilon^2\, M_\varepsilon \int\limits_\Sigma K_0(t,x,z',\bar x_\Theta)\,\frac{\partial u_0}{\partial\Theta}(z',\bar x_\Theta)\ dz' + o(\varepsilon^2),
$$

where

$$
M_\varepsilon = \int\limits_0^{+\infty} \zeta_d\, e^{-|\Theta|\,\zeta_d}\, \mathcal{M}\{P_\Theta^{-1}\big(\cdot,\zeta_d+\frac{\bar x_\Theta}{\varepsilon}\big)\} d\zeta_d.
$$

Here $\mathcal{M}\{P_\Theta^{-1}(\cdot,\tau_d)\}$ is the mean value of the almost periodic function $P_\Theta^{-1}(\cdot,\tau_d)$ (see (5.4)).

Consequently, as $\varepsilon\to 0$,

$$
v^\varepsilon(t,x) = \varepsilon^2\, M_\varepsilon \int\limits_\Sigma K_0(t,x,z',\bar x_\Theta)\,\frac{\partial u_0}{\partial\Theta}(z',\bar x_\Theta)\ dz' + o(\varepsilon^2).
$$

Recalling that K_0 is the Green function of the effective problem (4.1) completes the proof. □

Corollary 6.1. *Let conditions of Theorem 6.1 be fulfilled. Assume that the vector Θ is such that Θ_d and Θ_k, for any $k=1,\cdots,(d-1)$, are rationally independent. Then, for $t\ge t_0>0$,*

$$
u^\varepsilon(t,x) = \Big(\frac{\varepsilon}{|\Theta|}\Big)^2 e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta\cdot(x-\bar x)}{\varepsilon}}\,(1+r_\varepsilon(t,x))\,p_\Theta\big(\frac{x}{\varepsilon}\big)\Big(\int_Y p_\Theta^{-1}\,dy\Big)\,u(t,x),
$$

where $r_\varepsilon(t,x) \to 0$*, as* $\varepsilon \to 0$*, uniformly with respect to* $(t,x) \in [t_0,T] \times \overline{\Omega}$ *and* $u(t,x)$ *solves the homogenized problem* (6.1)*.*

Corollary 6.1 is proved in the same way as Corollary 5.1.

7 Numerical examples

In this section we illustrate the results obtained in the previous sections by direct computations performed with the free software FreeFEM++ ([15]).

When studying convection-diffusion equation, the so-called effective convection (effective drift) defined by (2.4) plays an important role. As was already noticed, condition $\bar{b}_i \neq 0$ yields $\Theta_i \neq 0$. The question arises, if $\bar{b}$ coincide with Θ or not. The answer is negative, and the corresponding example is given below.

Example 1. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Consider the following boundary value problem with constant coefficients:

$$\begin{cases} \partial_t u^\varepsilon - \dfrac{\partial^2 u^\varepsilon}{\partial x_1^2} - 2\,\dfrac{\partial^2 u^\varepsilon}{\partial x_1\,\partial x_2} - 2\,\dfrac{\partial^2 u^\varepsilon}{\partial x_2^2} + \dfrac{1}{\varepsilon}\,b\,\dfrac{\partial u^\varepsilon}{\partial x_2} = 0, & \text{in } (0,T)\times\Omega, \\ u^\varepsilon(t,x) = 0, & \text{on } (0,T)\times\partial\Omega, \\ u^\varepsilon(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{7.1}$$

Here $b > 0$ is a real parameter and it is obvious that the effective drift is $\bar{b} = \{0, b\}$. To find Θ, one should consider the spectral problem (2.1) on the periodicity cell. Since the coefficients of the equation are constant, $\lambda_1(\theta)$ can be found easily:

$$\lambda_1(\theta) = -\theta_1^2 - 2\,\theta_1\,\theta_2 - 2\,\theta_2^2 + b\,\theta_2.$$

The maximum of λ_1 is attained at $\Theta = \{-b/2, b/2\} \neq \bar{b}$.

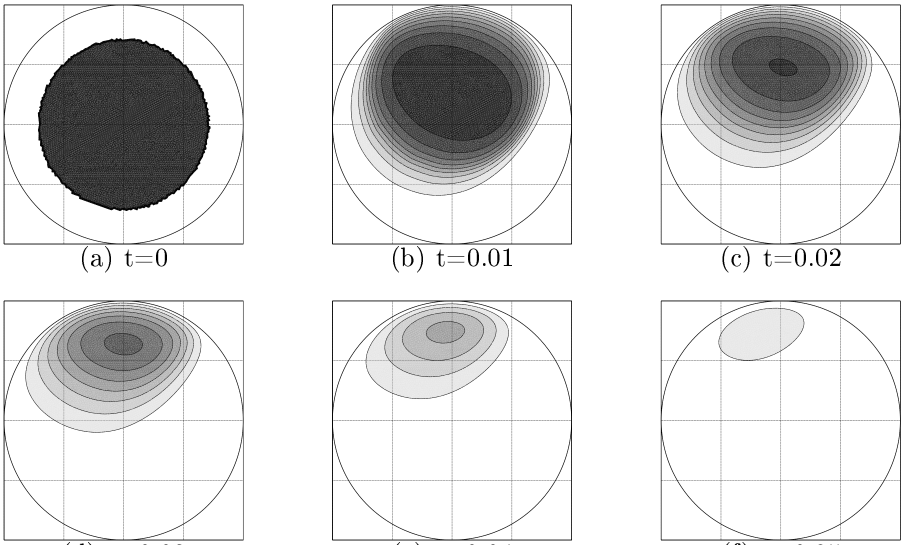
For the numerical computations, we choose Ω to be the unit circle $\Omega = \{x : |x_1-1|^2 + |x_2-1|^2 \leq 1\}$, u_0 being the characteristic function of the smaller circle $\{x : |x_1-1|^2+|x_2-1|^2 \leq 0.5\}$ (see Figure 4(a)), $b = 1$ and $\varepsilon = 0.03$. Theorem 2.1 predicts that the "hot spot" or concentration point of the solution u_ε will be at the point $x_c = (1-\sqrt{2}/2, 1+\sqrt{2}/2)$ where Θ is orthogonal to $\partial\Omega$.

The presence of the large parameter in front of the convection in (1.1) suggests to use Characteristics-Galerkin Method (see [14], [22]). As a finite element space, a space of piecewise linear continuous functions has been chosen. The number of triangles is 21192. The result of the direct computations at different times are presented on Figure 4.

Splitting each triangle of the mesh in 9, we have compared two solutions, u_1 defined on the original mesh and u_2 on the refined one, and computed the relative L^2-error for small t

$$\sup_t \frac{\|u_1 - u_2\|_{L^2(\Omega)}}{\|u_1\|_{L^2(\Omega)}} \approx 0.002.$$

It is small enough so we can conclude that convergence under mesh refinement is attained. It can be seen from Figure 4 that the solution profile, vanishing with



(a) t=0 (b) t=0.01 (c) t=0.02

(d) t=0.03 (e) t=0.04 (f) t=0.05

Figure 4: Isolines of u^ε for small values of t

time, moves first in the vertical direction (along the effective drift) and then to the left. Because of the very fast decay, it is not possible to plot the solution itself at large time. Thus, instead of u^ε we consider $\tilde{u}^\varepsilon = u^\varepsilon / \max_\Omega u^\varepsilon$. On Figure 5 the isolines of $\tilde{u}^\varepsilon$ are presented. One can see that indeed the concentration occurs at the point $(1-\sqrt{2}/2, 1+\sqrt{2}/2)$, not the point $(1,2)$ where $\bar{b}$ is normal to $\partial\Omega$.

We perform another numerical test in a nonconvex domain for the same values of the parameters in (7.1). The isolines of the rescaled solution $\tilde{u}^\varepsilon$ are ploted on Figure 6. It is interesting to see how the initial profile first moves in the direction of the effective drift, then vanishes and reappear afterwards to concentrate at the "hot spot" where $\Theta \cdot x$ attains its maximum, as predicted by Theorem 2.1. Such an example is clearly non-intuitive (at least to the authors).

8 Some results from the theory of almost periodic functions.

Denote by $\mathrm{Trig}(\mathbb{R}^d)$ the set of all trigonometric polynomials

$$\mathrm{Trig}(\mathbb{R}^d) = \big\{\mathcal{P}(x) \big| \ \mathcal{P}(x) = \sum_{\xi\in\mathbb{R}^d} c_\xi \, e^{ix\cdot\xi}\big\},$$

where in the sum only finite number of $c_\xi \neq 0$. We designate by $\mathrm{CAP}(\mathbb{R}^d)$ (set of almost periodic functions) a closure of $\mathrm{Trig}(\mathbb{R}^d)$ with respect to the norm $\sup\limits_{\mathbb{R}^d} |\mathcal{P}(x)|$. For any almost periodic function $g \in \mathrm{CAP}(\mathbb{R}^d)$, there exists a mean value

$$\mathcal{M}\{g\} = \lim_{t\to\infty} \frac{1}{|t\,\mathcal{B}|} \int\limits_{t\,\mathcal{B}} g(x)\, dx, \tag{8.1}$$

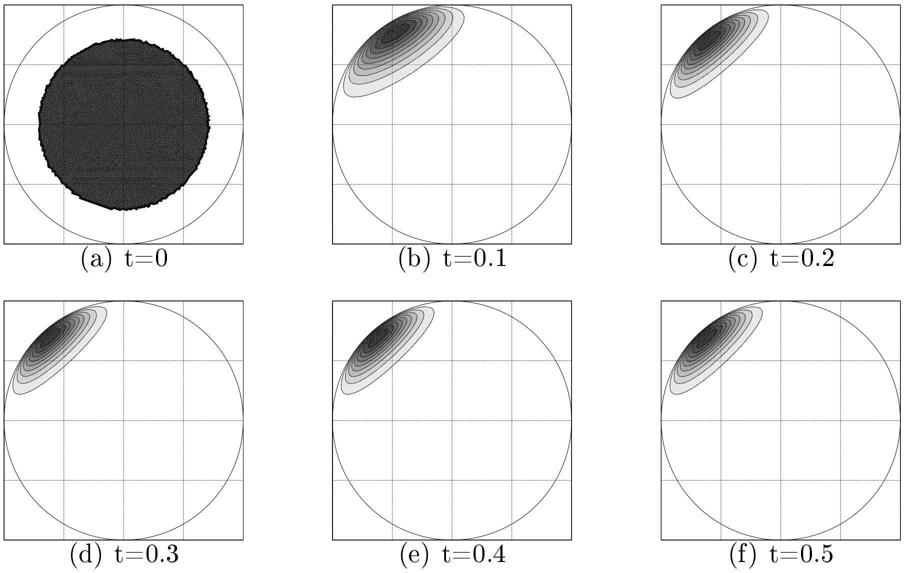


Figure 5: Isolines of rescaled u^ε for different values of t

where $\mathcal{B} \subset \mathbb{R}^d$ is a Borel set, $|\mathcal{B}|$ - its volume. The mean-value theorem takes place for almost periodic functions ([18]).

Lemma 8.1. *Given $g \in \mathrm{CAP}(\mathbb{R}^d)$ and $v \in L^2(Q)$, $Q \subset \mathbb{R}^d$, the following equality holds true:*

$$\lim_{\varepsilon \to 0} \int\limits_Q g\big(\frac{x}{\varepsilon}\big)\, v(x)\, dx = \mathcal{M}\{g\} \int\limits_Q v(x)\, dx,$$

where $\mathcal{M}\{g\}$ is given by formula (8.1).

Lemma 8.1 can be formulated also in more general form.

Lemma 8.2. *Given a function $g(x,y) \in C[\overline{Q}; \mathrm{CAP}(\mathbb{R}^d)]$, $Q \subset \mathbb{R}^d$, the following equality holds:*

$$\lim_{\varepsilon \to 0} \int\limits_Q g\big(x, \frac{x}{\varepsilon}\big)\, dx = \int\limits_Q \mathcal{M}\{g(x,\cdot)\}\, dx,$$

where

$$\mathcal{M}\{g(x,\cdot)\} = \lim_{t \to \infty} \frac{1}{|t\,\mathcal{B}|} \int\limits_{t\,\mathcal{B}} g(x,y)\, dy.$$

The last statement can be proved combining the approximation of $g(x,y)$ by finite sums of the type $\sum f_1(x)\, f_2(y)$ and the result of Lemma 8.1.

Acknowledgments The final version of this paper was written during the stay of the last two authors at CMAP, Ecole Polytechnique, the hospitality of which is kindly acknowledged. This research was partially supported by the GNR MOMAS CNRS-2439 (Modélisation Mathématique et Simulations numériques liées aux problèmes de gestion des déchets nucléaires) (PACEN/CNRS, ANDRA, BRGM, CEA, EDF, IRSN). G. A. is a member of the DEFI project at INRIA Saclay Ile-de-France and is partially supported by the Chair "Mathematical modelling and numerical simulation, F-EADS - Ecole Polytechnique - INRIA".

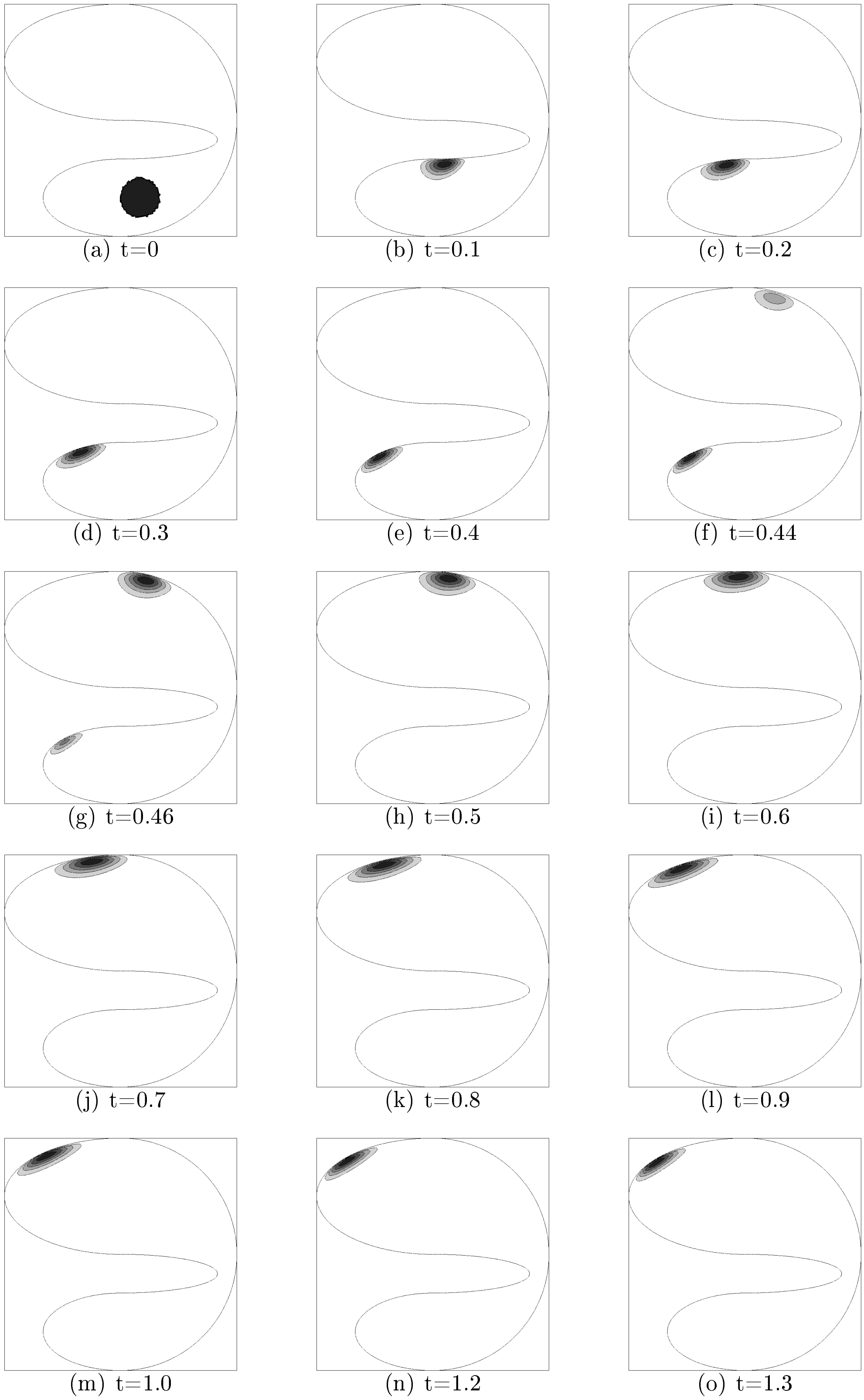


Figure 6: Isolines of rescaled u^ε for different values of t in a non-convex domain

References

[1] (1445458) I. A. Aleksandrova, *The spectral method in asymptotic diffusion problems with drift.* Math. Notes 59 (1996), no. 5-6, 554–556.

[2] (2098106) G. Allaire, Y. Capdeboscq, A. Piatnitski, V. Siess, M. Vanninathan, *Homogenization of periodic systems with large potentials.* Arch. Ration. Mech. Anal. 174 (2004), no. 2, 179–220.

[3] G. Allaire, I. Pankratova, A. Piatnitski, *Homogenization of a nonstationary convection-diffusion equation with large convection in a thin rod and in a layer* preprint.

[4] (1900560) G. Allaire, A. Piatnitski, *Uniform spectral asymptotics for singularly perturbed locally periodic operators* Comm. Partial Differential Equations 27 (2002), no. 3-4, 705–725.

[5] (2324490) G. Allaire, A.-L. Raphael, *Homogenization of a convection-diffusion model with reaction in a porous medium.* C. R. Math. Acad. Sci. Paris 344 (2007), no. 8, 523–528.

[6] (0435594) D.G. Aronson, *Non-negative solutions of linear parabolic equations.* Ann. Scuola Norm. Sup. Pisa (3) 22 (1968), 607–694.

[7] R. Bañuelos, K. Burdzy, *On the "hot spots" conjecture of J. Rauch* J. Funct. Anal. 164 (1999), no. 1, 1-33.

[8] (0503330) A. Bensoussan, J. L. Lions, G. Papanicolaou, Asymptotic Analysis for Periodic Structure, Studies in Mathematics and its Applications, 5. North-Holland Publishing Co., Amsterdam-New York, 1978.

[9] (1948866) A. Bourgeat, M. Jurak, A. Piatnitski *Averaging a transport equation with small diffusion and oscillating velocity.* (English). Mathematical Methods in the Applied Sciences, 26, No.2, 95-117, (2003).

[10] K. Burdzy, W. Werner, *A counterexample to the "hot spots" conjecture* Ann. of Math. (2) 149 (1999), no. 1, 309-317.

[11] (1663726) Y. Capdeboscq, *Homogenization of a diffusion equation with drift.* C. R. Acad. Sci. Paris Ser. I Math. 327 (1998), no. 9, 807–812.

[12] Y. Capdeboscq, *Homogenization of a neutronic critical diffusion problem with drift.* Proc. Roy. Soc. Edinburgh Sect. A 132 (2002), no. 3, 567–594.

[13] (2233176) P. Donato, A. Piatnitski, *Averaging of nonstationary parabolic operators with large lower order terms.* Multi Scale Problems and Asymptotic Analysis, GAKUTO Internat. Ser. Math. Sci. Appl., **24**, 153–165 (2005).

[14] (0672564) J. Douglas, T. Russell, *Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures.* SIAM J. Numer. Anal. 19 (1982), no. 5, 871–885.

[15] F. Hecht, O. Pironneau, A. Le Hyaric, K. Ohtsuka, Freefem++, Version 2.0-0, http://www.freefem.org//ff++, Laboratoire Jacques Louis Lions, Université Pierre et Marie Curie, Paris.

[16] (1814364) D. Gilbarg and N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[17] U. Hornung, Homogenization and Porous Media, Interdiscip. Appl. Math. 6, Springer-Verlag, New York, 1997.

[18] (0737902) S.M. Kozlov, *Reducibility of Quasiperiodic Differential Operators and Averaging.* Transc. Moscow Math. Soc. 1984, 2, 101-126.

[19] (MR0241822) O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and quasilinear equations of parabolic type. Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I. 1967.

[20] (2505654) I. Pankratova, A. Piatnitski, *On the behaviour at infinity of solutions to stationary convection-diffusion equation in a cylinder*, DCDS-B, **11** (4) (2009).

[21] (0699735) A. Piatnitski, *Averaging of a singularly perturbed equation with rapidly oscillating coefficients in a layer* Math. USSR-Sb. **49** (1984), no. 1, 19–40.

[22] (0654100) O. Pironneau, *On the transport-diffusion algorithm and its applications to the Navier-Stokes equations* Numer. Math. **38** (1981/82), no. 3, 309-332.

[23] J. Rauch, *Five problems: an introduction to the qualitative theory of partial differential equations* Partial differential equations and related topics (Program, Tulane Univ., New Orleans, La., 1974), pp. 355-369. Lecture Notes in Math., Vol. 446, Springer, Berlin, 1975

[24] (0635561) M. Vanninathan, *Homogenization of eigenvalue problems in perforated domains*, Proc. Indian Acad. Sci. Math. Sci., 90:239-271 (1981).

[25] (1329546) V.V. Zhikov, S.M. Kozlov, O.A. Oleinik, Homogenization of differential operators and integral functionals, Springer-Verlag, Berlin, 1994.

[26] (0737901) V.V. Zhikov, *Asymptotic behavior and stabilization of solutions of a second-order parabolic equation with lowest terms.* Transc. Moscow Math. Soc., 2(1984).