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Generic properties of the spectrum of the Stokes system with Dirichlet boundary condition in $\mathbb{R}^3$

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Abstract. Let (SD_Ω) be the Stokes operator defined in a bounded domain Ω of $\mathbb{R}^3$ with Dirichlet boundary conditions. We first prove that, generically with respect to the domain Ω, all the eigenvalues of (SD_Ω) are simple. That answers positively a question raised by J. H. Ortega and E. Zuazua in [18, Section 6]. Our second result states that, generically with respect to the domain Ω, the spectrum of (SD_Ω) verifies a non resonant property introduced by C. Foias and J. C. Saut in [10] and used to linearize the Navier-Stokes system in a bounded domain Ω of $\mathbb{R}^3$ with Dirichlet boundary conditions. The proofs of these results follow a standard strategy based on a contradiction argument requiring shape differentiation. However, one needs to shape differentiate twice the initial problem in the direction of carefully chosen domain variations. The main step of the contradiction argument amounts to study the evaluation of Dirichlet-to-Neumann operators associated to these domain variations.

Contents

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1 Introduction and main results

In this paper, we study the eigenvalue problem for the Stokes system with Dirichlet boundary conditions defined in a bounded open subset Ω of $\mathbb{R}^3$ with C^3 boundary,

$$(SD_\Omega) \qquad \begin{cases} -\Delta\phi + \nabla p &= \lambda\phi \quad \text{in } \Omega, \\ \operatorname{div}\phi &= 0 \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \partial\Omega, \\ \int_\Omega p(x)dx &= 0. \end{cases}$$

Here we use $\phi \in \mathbb{R}^3$ and $p \in \mathbb{R}$ to denote respectively the velocity field and the pressure. It is well-known that (SD_Ω) admits an increasing sequence of positive eigenvalues $(\lambda_n)_{n\geq 1}$ tending to infinity as n goes to infinity. Consider the following property, referred as (Simple), regarding the spectrum of (SD_Ω)

(Simple) All the eigenvalues of (SD_Ω) are simple.

We note that (Simple) is not always true, for instance if Ω is a disk (cf. [13, Chapter 4, pages 49-50]). First of all, recall that the set of bounded domains of $\mathbb{R}^3$ with C^3 boundary denoted by $\mathbb{D}_3$ can be endowed with the following topology: the base of open neighborhoods is given by the sets $V(\Omega, \varepsilon)$ defined, for any domain $\Omega \in \mathbb{D}_3$ and $\varepsilon > 0$, as the images of Ω by $\mathrm{Id} + u$, with $u \in W^{4,\infty}(\Omega, \mathbb{R}^3)$ and $\|u\|_{W^{4,\infty}} < \varepsilon$ (cf. [13] and [23]). Then ε is chosen small enough so that $\mathrm{Id} + u : \Omega \to (\mathrm{Id} + u)(\Omega)$ is a diffeomorphism. As shown by A. M. Michelleti in [17] (see also [13, Appendix 2]), $V(\Omega, \varepsilon)$ is metrizable using a Courant-type distance, denoted by

d_4, and each $(V(\Omega, \varepsilon), d_4)$ is complete and separable. For any domain $\Omega \in \mathbb{D}_3$, we use $\mathbb{D}_3(\Omega)$ to denote the Banach manifold obtained as the set of images $(\mathrm{Id}+u)(\Omega)$ by $u \in W^{4,\infty}(\Omega, \mathbb{R}^3)$, which are diffeomorphic to Ω. A property (P) will thus be referred to as "being generic with respect to $\Omega \in \mathbb{D}_3$" if, for every $\Omega \in \mathbb{D}_3$, the set of of domains of $\mathbb{D}_3(\Omega)$ where (P) holds true contains a countable intersection of open and dense subsets of $\mathbb{D}_3(\Omega)$.

One of the goals of the present paper consists in proving the following theorem.

Theorem 1.1. *Generically with respect to $\Omega \in \mathbb{D}_3$,* (Simple) *holds true, i.e., all the eigenvalues of (SD_Ω) are simple.*

Remark 1.1. In [18], several properties for the Stokes system with Dirichlet boundary conditions (in particular (Simple)) were proved to be generic for domains in $\mathbb{R}^2$. Moreover, in the same paper, the three dimensional case was considered in Section 6, pointing out why techniques developed in [18] could only handle the two dimensional case. In this regard, Theorem 1.1 answers positively the open question of Section 6 in [18].

In a second time, we aim at showing another generic property for the spectrum of (SD_Ω). For that purpose, we need the following definition, cf. [10, Definition 1].

Definition 1.1. We call *resonance* in the spectrum of (SD_Ω) a relation of the type

$$\lambda_{k+1} = \sum_{j=1}^{k} m_j \lambda_j, \text{ where } m_j \in \mathbb{N}, \qquad 1 \leq j \leq k. \tag{1}$$

If no resonance occurs in the spectrum of (SD_Ω), then (SD_Ω) will be called nonresonant.

The concept of resonance was introduced by C. Foias and J. C. Saut in [10] in order to linearize the Navier-Stokes system and obtain a normal form for it as well as a useful asymptotic expansion for its solutions in case when the corresponding Stokes operator (SD_Ω) is nonresonant. As noticed in [10], nonresonance does not occur for periodic boundary conditions. However the authors conjectured that nonresonance should be generic for Dirichlet boundary conditions. In this paper, we confirm that conjecture.

Theorem 1.2. *Generically with respect to $\Omega \in \mathbb{D}_3$, the operator (SD_Ω) is nonresonant.*

Remark 1.2. Theorem 1.1 is of course a particular case of Theorem 1.2, but we need to establish first generic simplicity of the spectrum of (SD_Ω) , and then, in a second step, the Foias-Saut conjecture in full generality.

We now describe the main steps of our arguments. As it is standard since [1], the reasoning goes by contradiction and is based on shape differentiation.

We start with a description of the proof of Theorem 1.1. Fix a domain $\Omega_0 \in \mathbb{D}_3$. For every integer l, we define A_l as the (open) subset of $\mathbb{D}_3(\Omega_0)$ whose elements Ω verify that the l first eigenvalues of (SD_Ω) are simple. Clearly, by Baire's lemma, proving Theorem 1.1 amounts to show that A_{l+1} is dense in A_l for every $l \geq 0$. We argue by contradiction and assume that there exists an integer l, a domain Ω with C^3 boundary in A_l and $\varepsilon > 0$, such that, for every $u \in W^{4,\infty}(\Omega, \mathbb{R}^3)$ with $\|u\|_{W^{4,\infty}} < \varepsilon$, the domain $(\mathrm{Id} + u)\Omega$, or simply $\Omega + u$, belongs to A_l but not to A_{l+1}. Let $m \geq 2$ be the multiplicity of λ, the value of the $(l + 1)$-th eigenvalue of (SD_Ω) and ϕ_i, $i = 1, \cdots, m$, orthonormal eigenfunctions associated to λ. Finally, let n be the outward unit normal vector field on $\partial\Omega$. By computing the shape derivative of the $(l + 1)$-th eigenvalue of (SD_Ω), J. H. Ortega and E. Zuazua obtained in [18] that, at every $x \in \partial\Omega$, one has, for $i, j = 1, \cdots, m$, and $i \neq j$

$$\frac{\partial \phi_i}{\partial n} \cdot n = 0, \qquad \frac{\partial \phi_i}{\partial n} \cdot \frac{\partial \phi_j}{\partial n} = 0, \qquad \|\frac{\partial \phi_i}{\partial n}\|^2 = \|\frac{\partial \phi_j}{\partial n}\|^2. \tag{2}$$

If $m > 2$, then there necessarily exists $1 \leq i \leq m$ so that $\dfrac{\partial \phi_i}{\partial n} \equiv 0$ on $\partial\Omega$ and one reaches a contradiction using a unique continuation result due to Osses (cf. [20]). However, in order to obtain generic simplicity ($m = 1$), it was not clear how to pursue the reasoning by contradiction, i.e., showing that relations in (2) do not hold true generically with respect to the domains of $\mathbb{R}^3$ if $m = 2$. Note that, for questions involving scalar PDEs, if one wants to prove generic simplicity of the spectrum of a self-adjoint operator with Dirichlet boundary conditions, then it is standard to follow the lines of the above mentioned contradiction argument and to reach Eq. (2). The second equation there is now a product of real numbers and a contradiction follows readily by unique continuation, cf. [1] and [13]. Therefore, the difficulty for showing the generic simplicity of the spectrum of (SD_Ω) stems, at this stage of the argument, from the vectorial character of ϕ_i, i.e., the fact that we are dealing with a system of PDEs.

In this paper, we push further the contradiction argument by computing the shape derivative of the $(l+1)$-th eigenvalue of $(S)_{\Omega+u}$ at every $u \in W^{4,\infty}(\Omega, \mathbb{R}^2)$ with $\|u\|_{W^{4,\infty}} < \varepsilon$ small enough. The relations obtained in Eq. (2) for Ω are now valid for every domain $\Omega + u$ with u small enough. At this stage, we are not able to derive a contradiction. So we again take the shape derivative of the above relations on $\partial\Omega$ and end up with expressions of the type

$$M'(u)(x) = -(u \cdot n)(x) \frac{\partial M(0)}{\partial n}(x), \quad x \in \partial\Omega, \tag{3}$$

for $\|u\|_{W^{4,\infty}} < \varepsilon$ and where

$$M(\cdot) := \frac{\partial \phi_i}{\partial n} \cdot n, \quad \frac{\partial \phi_i}{\partial n} \cdot \frac{\partial \phi_j}{\partial n}, \quad \text{or} \quad \|\frac{\partial \phi_i}{\partial n}\|^2 - \|\frac{\partial \phi_j}{\partial n}\|^2.$$

Taking into account the expression of M, its shape derivative $M'(u)$ can also be expressed in terms of Neumann data of the shape derivatives of the eigenfunctions whose Dirichlet data have the regularity of $u \cdot n$. By standard elliptic theory, if $u \cdot n$ belongs to the Sobolev space $H^s(\partial\Omega)$, $M'(u)$ *a priori* belongs to $H^{s-1}(\partial\Omega)$. Then, the key observation is that a *gap* of regularity exists between the two sides of Eq. (3). The whole point now comes down to prove that the gap of regularity actually leads to a contradiction. Two angles of attack, essentially equivalent but technically different, are possible. In this paper, we reformulate the issue at hand as follows: how to extract pointwise information (i.e., for $x \in \partial\Omega$) reflecting the aforementioned gap of regularity and thus allow us to pursue the reasoning by contradiction? This rather elementary line of attack, first considered in [7] and also applied in [4], consists in choosing appropriate variations u "localized" at an arbitrary point $x \in \partial\Omega$. We note that problems treated in [7] and [4] concerned planar domains and, therefore, equations of the type (3) were valid on closed C^3 curves of $\mathbb{R}^2$. In that case, the localization procedure is easier to handle. Indeed, the strategy adopted in [7] and [4] consisted in extending $M'(u)$ for variations u defined on $\partial\Omega$ as continuous functions except at some point $x \in \partial\Omega$. More particularly, $u = u_x$ can be taken as a Heaviside function admitting a single jump of discontinuity at x. In order to exploit the gap of regularity, the singular part of $M'(u_x)(\cdot)$ at x (in the distributional sense) had to be computed, to eventually obtain the following expression,

$$M'(u_x)(\sigma) = M_0 \text{ p.v.} \left(\frac{1}{\sigma}\right) + R(\sigma),$$

where σ denotes the arclength (with $\sigma = 0$ corresponding to x) and $R(\cdot)$ belongs to $H^{1/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon > 0$. Plugging back the above expression into Eq. (3), one deduces that $M_0(\cdot) \equiv 0$ on $\partial\Omega$. In [7], the previous relation provided additional information and allowed to conclude the contradiction argument. However, in [4], it turns out that $M_0(\cdot)$ is proportional to $M(0)(\cdot)$ and hence is trivially equal to zero. To determine the first non trivial term in the "singular" expansion of $M'(u_x) + (u_x \cdot n)\frac{\partial M(0)}{\partial n}$ at x, in the distributional sense, a detailed study of Dirichlet-to-Neumann operators associated to several Helmholtz equations was required.

In the present paper, the "localization" procedure, i.e., the choice of appropriate variations u for any arbitrary point $x \in \partial\Omega$, must be performed for functions defined on a surface $\partial\Omega$ and not anymore on a curve, as in [7] and [4]. For that purpose, after fixing an arbitrary point $x \in \partial\Omega$, we will choose sequences of smooth functions $u_{\varepsilon,x_\varepsilon}$ approximating the Dirac distribution at x_ε as ε tends to zero, the point $x_\varepsilon \in \partial\Omega$ being any point at distance ε of x. The gap of regularity between the two sides of Eq. (3) will be now quantified in terms of powers of $\frac{1}{\varepsilon}$, namely, the rigth-hand side of Eq. (3) is a $O(\frac{1}{\varepsilon^2})$ meanwhile we will establish that the left-hand side of of Eq. (3) is equal to $\frac{w_\varepsilon}{\varepsilon^3} + O(\frac{1}{\varepsilon^2})$, where w_ε is bounded independently of ε. Letting ε tend to zero, one deduces that $\lim_{\varepsilon\to 0} w_\varepsilon = 0$ and finally one concludes the contradiction argument. Theorem 1.1 is proved in this manner. Note that the exact characterization of w_ε requires, as in [4], a detailed study of certain Dirichlet-to-Neumann operators, but here, associated to the Stokes system. That study heavily uses many technical results borrowed from [13], not only for handling certain weakly singular operators but also for the material which is necessary to evaluate integrals defined on the surface $\partial\Omega$. It is noteworthy that, to perform the evaluation of the surface integrals, we choose charts based at $x_\varepsilon \in \partial\Omega$ near the fixed point $x \in \partial\Omega$, but not exactly at x. This trick turns out to be crucial for handling the singularities in computations involving layer potentials. More importantly, it also provides two degrees of freedom, namely the distance and the angle between x_ε and x, and functions of these two variables being equal to zero give additional information to yield a contradiction.

Let us now briefly mention how goes the argument for Theorem 1.2. Since the resonance relations of the type (1) are clearly of countable number, we can start a contradiction argument similar to the abovementioned one. Therefore, there exists a resonance relation of the type (1) and denoted here by (RR), a domain Ω with C^3 boundary and $\varepsilon > 0$, such that, for every $u \in W^{4,\infty}(\Omega, \mathbb{R}^3)$ with $\|u\|_{W^{4,\infty}} < \varepsilon$, the domain $\Omega + u$ verifies (RR). Moreover, since Theorem 1.1 holds true, one can assume that the eigenvalues involved in (RR) are all simple for $\Omega + u$ with $\|u\|_{W^{4,\infty}} < \varepsilon$. We then take the shape derivative of (RR) but we are unable to derive any contradiction. Assuming thus that this shape derivative is equal to zero for $\Omega + u$ with $\|u\|_{W^{4,\infty}}$ small enough, we again differentiate the shape derivative of (RR) at $u = 0$. We then consider the variations $u_{\varepsilon,x_\varepsilon}$ introduced previously and embark into the characterization of the main term of the second shape derivative of (RR). After lengthy computations, we get a contradiction and conclude. It is interesting to notice the following difference betwen the proofs of Theorem 1.1 and Theorem 1.2 respectively. Indeed, for the first result, one uses, in the contradiction argument, the parameter defined by the angular part between x and x_ε whereas for the second result, it is the radial part between x and x_ε which plays a crucial role. Both parameters actually result from the vectorial character of our variations and that enables one to adequately address the fact that (SD_Ω) is a system of PDEs. Therefore, one should emphasize the flexibility of the approach proposed in this paper, which can be applied to genericity questions for other systems of PDEs.

Before passing to the plan of the paper, we must mention that [13] provides the best update for genericity questions related to PDEs, where genericity is meant with respect to the domain Ω. Moreover, many new genericity results are proven there and in several situations, the author (essentially) arrives to the same critical issue as the one explained previously, i.e., equations of the type (3) and the gap of regularity they exhibit. D. Henry proposes the second angle of attack to derive a contradiction from the gap remark. He chooses to reformulate such an equation as the fact that a certain pseudo-differential operator has finite rank. Then, to contradict that finiteness assumption, D. Henry applies the operator to rapidly oscillating functions, which is a strategy much more general than ours but which is more complicated to implement when one deals with systems of PDEs, such as in here with the Stokes system (see [22] for a nice application of the strategy advocated in [13] and also [21] for an extension of [13, Chapter 8]).

The paper is organized as follows. In Section 2, we present the necessary material on the Stokes system, shape differentiation and the result displayed in Eq. (2) and first established in [18]. The third section is devoted to the proof of Theorem 1.1 assuming that the expansion of a Dirichlet-to-Neuman operator in terms of inverse powers of ε is available. Then, in Section 4, the argument to achieve such an expansion is provided using technical results on representation formulas for Dirichlet-to-Neuman operators gathered in Section 6. The proof of Theorem 1.2 is given in Section 5. Background materials on layer potentials and integral representation formulas for the Stokes system as well as the proofs of computational lemmas are gathered in Appendices A and B.

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2 Definitions and preliminary results

We start by defining precisely in Section 2.1 the topology for the set of domains in $\mathbb{R}^d$ with C^l boundary. The material is standard and borrowed from [13] and [23]. We then recall in Section 2.2 the definition of the Stokes operator and its spectrum. The presentation adopted in this section is inspired by [9, Chapter II], [25, Chapter 5] and [18]. Results on the regularity of the eigenvalues and eigenfunctions of the Stokes operator with respect to domain variations are derived in Section 2.3 based on [15, Chapter 7] and [2]. Section 2.4 is devoted to the shape differentiation for the Stokes system following the strategy of [23]. We finally recall in Section 2.5 J. H. Ortega and E. Zuazua's result obtained in [18] and provide an alternative proof. This result will be the starting point of our proof for Theorem 1.1.

2.1 Topology on the domains

In this section, we provide the basic definitions needed in the paper. We work in this section in $\mathbb{R}^d$, $d \geq 2$, even though we will only be interested by the case $d = 3$. A domain Ω of $\mathbb{R}^d$, $d \geq 2$, is an open bounded subset of $\mathbb{R}^d$. We provide now the standard topology for domains with a regular boundary as well as basic results relative to shape differentiation. For $l \geq 1$, the set of domains Ω of $\mathbb{R}^d$ with C^l boundary will be denoted by $\mathbb{D}_l$. Following [23], we can define a topology on $\mathbb{D}_l$. Consider the Banach space $W^{l+1,\infty}(\Omega, \mathbb{R}^d)$ equipped with its standard norm defined by

$$\|u\|_{l+1,\infty} := \text{supess}\{\|D^\alpha u(x)\|;\ 0 \leq \alpha \leq l+1,\ x \in \Omega\}.$$

For $\Omega \in \mathbb{D}_l$, $u \in W^{l+1,\infty}(\Omega, \mathbb{R}^d)$, let $\Omega + u := (\text{Id} + u)(\Omega)$ be the subset of points $y \in \mathbb{R}^d$ such that $y = x + u(x)$ for some $x \in \Omega$ and $\partial\Omega + u := (\text{Id} + u)(\partial\Omega)$ its boundary. For $\varepsilon > 0$, let $V(\Omega, \epsilon)$ be the set of all $\Omega + u$ with $u \in W^{l+1,\infty}(\Omega, \mathbb{R}^d)$ and $\|u\|_{W^{l+1,\infty}} \leq \varepsilon$, small enough so that $\text{Id} + u : \Omega \to (\text{Id} + u)(\Omega)$ is a diffeomorphism. The topology of $\mathbb{D}_l$ is defined by taking the sets $V(\Omega, \varepsilon)$ with ε small enough as a base of open neighborhoods of Ω.

A. M. Michelleti in [17] (and also reported in [13, Appendix 2]) considered a Courant-type metric, denoted d_{l+1} in this paper, so that $V(\Omega, \varepsilon)$ is metrizable and each $(V(\Omega, \varepsilon), d_{l+1})$ is complete and separable. For any domain $\Omega \in \mathbb{D}_3$, we use $\mathbb{D}_l(\Omega)$ to denote the the set of images $(\text{Id} + u)(\Omega)$ by $u \in W^{l+1,\infty}(\Omega, \mathbb{R}^d)$, which are diffeomorphic to Ω. Then $\mathbb{D}_l(\Omega)$ is a Banach manifold modeled on $u \in W^{l+1,\infty}(\partial\Omega, \mathbb{R}^d)$ as proved in [13, Theorem A.10]. In the sequel, we will sometimes identify, without further notice, the neighborhoods $V(\Omega, \varepsilon)$ with the corresponding open balls of $W^{l+1,\infty}(\Omega, \mathbb{R}^d)$ centered at 0.

Definition 2.1. We say that a property (P) is generic in $\mathbb{D}_l$ if, for every $\Omega \in \mathbb{D}_l$, the set of domains of $\mathbb{D}_l(\Omega)$ on which Property (P) holds true is residual i.e., contains a countable intersection of open and dense subsets of $\mathbb{D}_l(\Omega)$.

2.2 Spectrum of the Stokes operator with Dirichlet boundary conditions

The presentation here is inspired by [9, Chapter II], [25, Chapter 5] and [18]. Let Ω be a domain of $\mathbb{R}^d$, $d \geq 1$ with C^1 boundary. We use $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ to denote respectively the space of C^∞ functions with compact support in Ω and the space of distributions on Ω. The duality bracket will be denoted by $\langle \cdot, \cdot \rangle_{\mathcal{D}' \times \mathcal{D}}$.

Consider the following fundamental functional spaces for the Stokes system:

$$\begin{aligned} V(\Omega) &:= \{v \in (H_0^1(\Omega))^d \mid \text{div } v = 0\}, \\ H(\Omega) &:= \{v \in (L^2(\Omega))^d \mid \text{div } v = 0 \ \text{ in } \Omega,\ v \cdot n = 0 \ \text{ on } \partial\Omega\}. \end{aligned}$$

The space $V(\Omega)$ is equipped with the scalar product of $(H_0^1(\Omega))^d$ defined by

$$\langle u, v \rangle_V := \int_\Omega \nabla u \cdot \nabla v := \sum_{i,j=1}^d \int_\Omega \frac{\partial u^i}{\partial x_j} \frac{\partial v^i}{\partial x_j} dx, \tag{4}$$

for $u := (u^1, \cdots, u^d)$ and $v := (v^1, \cdots, v^d)$ in $V(\Omega)$. The space $H(\Omega)$ is equipped with the scalar product of $(L^2(\Omega))^d$ which will be denoted by $\langle \cdot, \cdot \rangle_H$. Note that $V(\Omega)$ and $H(\Omega)$ are separable Hilbert spaces as they are closed sub-spaces of respectively $(H_0^1(\Omega))^d$ and $(L^2(\Omega))^d$. We use $L_0^2(\Omega)$ to denote the subspace of $L^2(\Omega)$ made of the functions f with zero mean, i.e. $\displaystyle\int_\Omega f(x)dx = 0$.

Remark 2.1. If we define $\mathcal{V}(\Omega) := \{v \in (\mathcal{D}(\Omega))^d \mid \text{div } v = 0\}$, one can show that $V(\Omega)$ is the closure of $\mathcal{V}$ in $(H^1(\Omega))^d$ (cf. [27, Th.1.6, p.18]), and $H(\Omega)$ is the closure of $\mathcal{V}(\Omega)$ in $(L^2(\Omega))^d$ (cf. [27, Theorem 1.4, page 15] and [11, Theorem 2.8, page 30]).

Let $f \in H$. Since the linear form on $V(\Omega)$ defined by $\ell(v) := \displaystyle\int_\Omega f \cdot v$, for $v \in V(\Omega)$, is continuous, by Lax-Milgram's Theorem, there exists a unique $w \in V(\Omega)$ such that, for every $v \in V(\Omega)$, $\langle w, v\rangle_V = \ell(v)$ and $\|w\|_V \leq C(\Omega)\|f\|_H$, where the constant $C(\Omega)$ only depends on Ω. Therefore, the linear operator L from $H(\Omega)$ to $H(\Omega)$ defined by $Lf = w$ is continuous. As L is also self-adjoint and compact (cf. [6, Theorem IX.16, page 169]), then, by classical spectral theory (cf. [6, Theorem VI.11, page 97]), the operator L admits a non-increasing sequence of positive eigenvalues $(\mu_i)_{i\in\mathbb{N}}$ tending to 0, and the corresponding eigenfunctions $(\phi_i)_{i\in\mathbb{N}}$ can be taken so that they constitute an orthonormal basis of H. In particular, one has

$$\int_\Omega \nabla\phi_i \cdot \nabla v = \lambda_i \int_\Omega \phi_i \cdot v, \qquad \forall\, v \in V, \tag{5}$$

where $\lambda_i := \dfrac{1}{\mu_i}$. Note that $(\lambda_i)_{i\in\mathbb{N}}$ is a non-decreasing sequence tending to infinity. We use $m(\lambda)$ to denote the multiplicity of the eigenvalue λ.

For $v \in \mathcal{V}$, Eq. (5) is equivalent to

$$\langle \Delta\phi_i + \lambda_i\phi_i, v\rangle_{\mathcal{D}'\times\mathcal{D}} = 0. \tag{6}$$

Theorem 2.1 (de Rham-Lions). *Let $q \in (\mathcal{D}'(\Omega))^d$ such that*

$$\langle q, v\rangle_{\mathcal{D}'\times\mathcal{D}} = 0, \qquad \forall\, v \in \mathcal{V}. \tag{7}$$

Then, there exists $p \in \mathcal{D}'(\Omega)$ such that $q = \nabla p$. As a consequence of Theorem 2.1, one deduces from Eq. (5) that there exists $p_i \in \mathcal{D}'(\Omega)$ such that

$$\Delta\phi_i + \lambda_i\phi_i = \nabla p_i. \tag{8}$$

Remark 2.2. Note that p in Theorem 2.1 is unique up to an additive constant.

Remark 2.3. Theorem 2.1 is a consequence of a more general result due to de Rham (cf. [8, Theorem 17', page 95]). This version is due to Lions, also stated in [27, Proposition 1.1, page 14]. A constructive proof can be found in [24].

Remark 2.4. There exists an equivalent presentation of the eigenvalue poblem for the Stokes system based on the Stokes operator T_S, which is defined as the operator defined on $V \cap W^2(\Omega)$ by $T_S u \in H$ being the unique element satisfying

$$\Delta u + T_S u = \nabla p,$$

for some harmonique pressure field p, cf. [9, Chapter II]. Then, one has $T_S = -\mathcal{P}\Delta$ where $\mathcal{P}$ is the Leray projector. One then proceeds by standard functional analysis arguments.

The following regularity result holds for ϕ_i and p_i (cf. [27, Section 2.6, page 38]).

Theorem 2.2 (Regularity). *If the domain Ω is of class C^m, for an integer $m \geq 2$, then, for $i \in \mathbb{N}$, $\phi_i \in H^m(\Omega)$ and $p_i \in H^{m-1}(\Omega)$. If Ω is of class C^∞, then, for $i \in \mathbb{N}$, $\phi_i \in C^\infty(\overline{\Omega})$ and $p_i \in C^\infty(\overline{\Omega})$.*

We now summarize some computational results related to the Stokes system. We start by providing several notions of "normal" derivatives used in this context. If $\phi = (\phi^i)_{1\le i\le 3}$, the Jacobian matrix of ϕ defined as $(\dfrac{\partial \phi^i}{\partial x_j})_{1\le i,j\le 3}$ will be denoted by $\nabla\phi$. We use n to denote the outward unit normal to $\partial\Omega$ and the letter T used below denotes the transpose of a matrix. The corresponding normal derivative is given by

$$\frac{\partial \phi}{\partial n} := \nabla\phi \cdot n, \tag{9}$$

and we also have

$$\frac{\partial \phi}{\partial N} := (\nabla\phi + \nabla\phi^T) \cdot n. \tag{10}$$

Finally, the conormal derivative $\frac{\partial \phi}{\partial \nu}$ on $\partial\Omega$ is defined as follows

$$\frac{\partial \phi}{\partial \nu} := \frac{\partial \phi}{\partial N} - pn. \tag{11}$$

Moreover, we will use n_x or $n(x)$, with $x \in \partial\Omega$, to denote the value of the outward normal vector at the point x.

Definition 2.2. For a and b are C^1 functions defined on an open neighborhood of Ω, we use $\nabla a : \nabla b$ to denote the following function

$$\nabla a : \nabla b = \frac{1}{2}(\nabla a + \nabla^T a) \cdot (\nabla b + \nabla^T b),$$

where $\cdot$ is defined in Eq. (4) as the Hadamard product of two matrices.

We recall the following Green's formula (cf. [16]).

Lemma 2.3. *Assume that $d = 3$. The following Green's formula*

$$\int_{\partial\Omega} a \cdot \frac{\partial b}{\partial \nu} + \int_{\Omega} \nabla(\text{div } b)a + \int_{\Omega} q(\text{div } a) = \int_{\Omega} (\nabla a : \nabla b) + \int_{\Omega} a \cdot (\Delta b - \nabla q), \tag{12}$$

holds true for every a and (b, q), C^1 functions defined on an open neighborhood of Ω, taking values in $\mathbb{R}^3$ and $\mathbb{R}^3 \times \mathbb{R}$ respectively.

A direct consequence of Lemma 2.3 is the following second Green's formula.

Corollary 2.4. *Assume that $d = 3$. The formula*

$$\int_{\partial\Omega} a \cdot \frac{\partial b}{\partial \nu} - \int_{\partial\Omega} b \cdot \frac{\partial a}{\partial \nu} = \int_{\Omega} a \cdot ((\Delta + \eta)b - \nabla q) - \int_{\Omega} b \cdot ((\Delta + \eta)a - \nabla p), \tag{13}$$

holds for every $\eta \in \mathbb{R}$ and for every pairs (a, p) and (b, q) of C^1 functions defined on an open neighborhood of Ω, taking values in $\mathbb{R}^3 \times \mathbb{R}$ and satisfying div $a =$ div $b = 0$.

Remark 2.5. One has noticed that the dot "$\cdot$" has been used for scalar product a well as for the Hadamard product in Eq. (4). We will make that abuse of notation throughout the paper.

We also need the following obvious result.

Lemma 2.5. *Let $d \ge 2$ be an integer, $a \in (C^1(\overline{\Omega}))^d \cap (H^1_0(\Omega))^d$ and $\Omega \subset \mathbb{R}^d$ be an open domain of class C^1. Then,*

$$\nabla a = \frac{\partial a}{\partial n} n^T, \qquad \text{on } \partial\Omega. \tag{14}$$

2.3 Regularity of the eigenvalues and eigenfunctions with respect to the shape perturbation parameter

Let Ω be a domain in $\mathbb{D}_3$. We consider perturbations u in the space $W^{4,\infty}(\mathbb{R}^3,\mathbb{R}^3)$ with its standard norm $\|\cdot\|_{4,\infty}$. To study perturbations of eigenvalues, we adopt the strategy described in [15, Chapter 7, Section 6.5, pages 423-425].

Recall that the eigenvalue problem associated to the Stokes system on Ω with Dirichlet boundary condition is given by

$$(SD_\Omega) \qquad \begin{cases} -\Delta\phi + \nabla p &= \lambda\phi \quad \text{in } \Omega,\\ \mathrm{div}\phi &= 0 \quad \text{in } \Omega,\\ \phi &= 0 \quad \text{on } \partial\Omega,\\ \displaystyle\int_\Omega p(x)dx &= 0. \end{cases}$$

Consider any smooth map $t \to T_t$ defined for t small enough so that $T_0 = \mathrm{Id}$ and T_t is a diffeomorphism from Ω onto its image $\Omega_t := T_t(\Omega)$. Let (ϕ_t, p_t, λ_t) be the solution of

$$(SD_{\Omega_t}) \qquad \begin{cases} -\Delta\phi_t + \nabla p_t &= \lambda_t\phi_t \quad \text{in } \Omega_t,\\ \mathrm{div}\ \phi_t &= 0 \quad \text{in } \Omega_t,\\ \phi_t &= 0 \quad \text{on } \partial\Omega_t,\\ \displaystyle\int_{\Omega_t} p_t(y^t)dy^t &= 0. \end{cases}$$

We next turn to the variational formulation of the above eigenvalue problem.

For every $(w,q) \in (H_0^1(\Omega_t))^3 \times L_0^2(\Omega_t)$, it comes

$$\int_{\Omega_t} \nabla\phi_t : \nabla w\ dy^t - \int_{\Omega_t} p_t\ \mathrm{div}(w)\ dy^t + \int_{\Omega_t} \mathrm{Tr}(\nabla\phi_t) q\ dy^t = \int_{\Omega_t} \lambda_t\ \phi_t w\ dy^t.$$

We set $\phi^t := \phi_t \circ T_t \in H_0^1(\Omega)$, $p^t := p_t \circ T_t$. Define the change of variables $y^t := T_t(y)$ and set $z(y) := w(y^t)$ and $r(y) := q(y^t)$. Then, one shows that (ϕ^t, p^t) satisfies the following identity

$$\int_\Omega A(t)\nabla\phi^t : \nabla z - \int_\Omega p^t \mathrm{Tr}(B(t)\nabla z)\gamma(t) + \int_\Omega \mathrm{Tr}(B(t)\nabla\phi^t) r\gamma(t) = \int_\Omega \lambda_t\ \phi^t z\gamma(t), \tag{15}$$

where $\gamma(t) = \det(DT_t)$, where $A(t) = \gamma(t)(DT_t^{-1})^*(DT_t^{-1})$ and where $B(t) = (DT_t^{-1})^*$. It follows that (ϕ^t, p^t) satisfies

$$\begin{cases} -\mathrm{div}(A(t)\nabla\phi^t) + \mathrm{div}(p^t\gamma(t)B(t)^*) &= \lambda_t\ \phi^t\gamma(t) \quad \text{in } \Omega,\\ \mathrm{Tr}(B(t)\nabla\phi^t) &= 0 \quad \text{in } \Omega,\\ \phi^t &= 0 \quad \text{on } \partial\Omega,\\ \displaystyle\int_\Omega p^t(y)\gamma(t)dy &= 0. \end{cases} \tag{16}$$

Let $L^2_{T_t}$ be the Hilbert space equipped with the scalar product

$$\langle\phi,\psi\rangle_{T_t} = \int_\Omega \phi(x)\psi(x)\gamma(t)\ dx, \tag{17}$$

and define $L^2_{0,T_t} := \left\{ v \in L^2 \ : \ \int_\Omega v(x)\gamma(t)dx = 0 \right\}$. We consider $\mathcal{C}(t)$ and $\mathcal{L}(t)$ the two operators on $H^1_0(\Omega_t)^3$ given by

$$\mathcal{C}(t)v = -\frac{1}{\gamma(t)}\mathrm{div}(A(t)\nabla v), \tag{18}$$

and

$$\mathcal{L}v = -\mathrm{Tr}\left((DT^{-1})^*\nabla v\right). \tag{19}$$

The following result holds true.

Theorem 2.6 (cf. [2])**.**

1. *The operator $\mathcal{C}(t)$ is self-adjoint with respect to the scalar product of $L^2_{T_t}(\Omega)$ and $\mathcal{C}(t)^{-1}$ is coercive, i.e., there exists $C > 0$ such that, for every $g \in H^{-1}(\Omega)$, one has $\langle g, \mathcal{C}(t)^{-1} g\rangle \geq C \parallel g \parallel_{H^{-1}}$.*
2. *The range of $\mathcal{L}(t)$ is closed and the adjoint $\mathcal{L}(t)^*$ of $\mathcal{L}$ is given by*

$$\mathcal{L}^* q(t) = \frac{1}{\gamma(t)}\mathrm{div}(q\gamma(t)). \tag{20}$$

Moreover, the null space of $\mathcal{L}(t)$ is made of constant functions on Ω and its range is equal to $L^2_{0,T_t}(\Omega)$.

Using the operators $\mathcal{C}(t)$ and $\mathcal{L}(t)$, we rewrite System (16) as

$$\left\{ \begin{array}{rcll} \mathcal{C}(t)\phi^t + \mathcal{L}(t)^* p^t &=& \lambda_t \phi^t, & \text{in } \Omega, \\ \mathcal{L}(t)\phi^t &=& 0 & \text{in } \Omega, \\ \phi^t &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{21}$$

Since the operator $\mathcal{C}(t) : (H^1_0(\Omega))^3 \mapsto (H^{-1}(\Omega))^3$ is an isomorphism, we can write

$$\phi^t + \mathcal{C}(t)^{-1}\mathcal{L}^*(t)p^t = \lambda_t \mathcal{C}(t)^{-1}\phi^t, \tag{22}$$

and since $\mathcal{L}(t)\phi^t = 0$, one has

$$\mathcal{L}(t)\mathcal{C}(t)^{-1}\mathcal{L}(t)^* p^t = \lambda_t \mathcal{L}(t)\mathcal{C}(t)^{-1}\phi^t.$$

Thanks to the coercivity of $\mathcal{C}(t)^{-1}$, one concludes that $\mathcal{L}(t)\mathcal{C}(t)^{-1}\mathcal{L}(t)^*$ is continuous and one-to-one in the space orthogonal to the null space of $\mathcal{L}(t)^*$. It follows that

$$p^t = \lambda_t \left(\mathcal{L}(t)\mathcal{C}(t)^{-1}\mathcal{L}(t)^*\right)^{-1} \mathcal{L}(t)\mathcal{C}(t)^{-1}\phi^t.$$

Finally, reporting this expression of p^t into (22), we derive that

$$\mathcal{C}(t)\phi^t + \lambda_t \mathcal{L}(t)^* \left(\mathcal{L}(t)\mathcal{C}(t)^{-1}\mathcal{L}(t)^*\right)^{-1} \mathcal{L}(t)\mathcal{C}(t)^{-1}\phi^t = \lambda_t\phi^t, \tag{23}$$

or equivalently

$$\mathcal{C}(t)\phi^t = \lambda_t \mathcal{A}(t)\phi^t, \tag{24}$$

where we have set

$$\mathcal{A}(t) := \left[\left(I - \mathcal{L}(t)^* \left(\mathcal{L}(t)\mathcal{C}(t)^{-1}\mathcal{L}(t)^*\right)^{-1} \mathcal{L}(t)\right) \mathcal{C}(t)^{-1} \right].$$

We are in the same situation as [15, Eq. (6.42), p. 42]. Indeed, (21) shows that the operators $\mathcal{A}$ and $\mathcal{C}$ are dependent on the parameter t but are defined on a fixed Hilbert space, i.e., independent of t. We also have that

- the operators $\mathcal{A}(t)$ and $\mathcal{C}(t)$ are closed operators.
- Assume that $t \mapsto T_t$ is analytic in a neighborhood of $t = 0$. Then, the operators $t \mapsto \mathcal{C}(t)$ and $t \mapsto \mathcal{C}(t)^{-1}$ are analytic in a neighborhood of $t = 0$, from $W_0^{1,\infty}$ into $\mathcal{L}((H_0^1(\Omega)^3), (H^{-1}(\Omega)^3))$ and $\mathcal{L}((L^2(\Omega)^3), (H^{-1}(\Omega)^3))$ respectively, and so is the inversion of continuous operators. This shows that the mapping $t \mapsto \mathcal{A}(t)$ is analytic in a neighbourhood of $t = 0$. Furthermore, $\mathcal{A}(t)$ is bounded when t is sufficiently small.

From [15, Chapter 7, Sections 6.2 and 6.5], we deduce that (λ_t, ϕ^t, p^t) defined in (21) is analytic in a neighborhood of $t = 0$. Moreover, if $\lambda = \lambda(0)$ is an eigenvalue of multiplicity h, by applying a standard Lyapunov-Schmidt argument (cf. for instance [15, Chapter 7], [12] or [13]), one gets the following result when $T_t = \mathrm{Id} + tu$, with $u \in W^{4,\infty}(\Omega, \mathbb{R}^3)$.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain of class C^3. Assume that λ is an eigenvalue of multiplicity $m(\lambda) = h$ of the Stokes system with Dirichlet boundary condition on the domain Ω. Then, there exist h real-valued continuous functions, $u \mapsto \lambda_i(u)$ defined in a neighborhood V of 0 in $W^{4,\infty}(\Omega, \mathbb{R}^3)$ such that the following properties hold,*

- *$\lambda_i(0) = \lambda$, for $i = 1, ..., h$;*
- *for every open interval $I \subset \mathbb{R}$, such that the intersection of I with the set of eigenvalues of (SD_Ω) contains only λ, there exists a neighborhood $V_I \subset V$ such that, for every $u \in U_I$, there exist exactly h eigenvalues counting with multiplicity, $\lambda_i(u)$, $1 \le i \le h$, of $(SD_{(Id+u)\Omega})$ contained in I;*
- *for every $u \in W^{4,\infty}(\Omega, \mathbb{R}^3)$ and $1 \le i \le h$, consider the map*

$$\Psi_i : \begin{array}{cccccc} J & \to & \mathbb{R} & \times\ H_0^1(\Omega) & \times & L_0^2(\Omega) \\ t & \mapsto & (\ \lambda_i^t(u), & \phi_i^t(u), & & p_i^t(u)\ ) \end{array}$$

with $J \subset \mathbb{R}$ an open interval containing 0, for $1 \le i \le h$, $\phi_i^t(u) := \phi_{t,i}(u) \circ (\mathrm{Id} + tu)$ and $p_i^t(u) := p_{t,i}(u) \circ (\mathrm{Id} + tu)$, where $\phi_{t,i}(u)$ and $p_{t,i}(u)$ are respectively eigenfunction and eigenpressure of $(SD_{\Omega+u})$. Then, for $1 \le i \le h$, Ψ_i is analytic in a neighborhood of $t = 0$. Moreover, the family $(\phi_{t,1}(u), \ldots, \phi_{t,h}(u))$ is orthonormal in $H_0^1(\Omega + u)$.

Remark 2.6. This result is actually the Stokes system's version of [19, Theorem 3]. It is important to insist on the fact that at $t = 0$ the orthonormal family

$$(\phi_{0,1}(u), \ldots, \phi_{0,h}(u)),$$

of eigenfunctions associated to λ does depend in general on u and continuity of the eigenfunctions with respect to the shape parameter u does not hold true. Therefore, only *directional* continuity and derivability with respect to u can be achieved and this is the object of the next paragraph.

2.4 Shape differentiation

The subsequent developments follow a standard strategy (cf. [23, Theorem 2.13] for instance) but seem to be new for the Stokes system with Dirichlet boundary conditions. Fix $u \in W^{4,\infty}(\Omega, \mathbb{R}^3)$ and set $T_t = \mathrm{Id} + tu$ for t small enough. In this section, we define and calculate the differential systems verified by the derivatives at $t = 0$ of the eigenfunctions $(\phi_{i,t}(u), p_{i,t}(u))$ defined in Theorem 2.7. For that purpose, we must first consider the derivatives of the maps $\phi_i^t(u)$ and $p_i^t(u)$. Since we perform such a computation along a fixed analytic branch $(\lambda_i^t(u), \phi_{t,i}(u), p_{t,i}(u))$, the index i is omitted for the rest of the paragraph.

According to Theorem 2.7, $(\phi^t(u), p^t(u))$ is analytic in a neighborhood of $t = 0$ and we set

$$\dot{\phi}(u) := \frac{d\phi^t(u)}{dt}\Big|_{t=0}, \qquad \dot{p}(u) := \frac{dp^t(u)}{dt}\Big|_{t=0}. \tag{25}$$

We next proceed in a similar way as in [23, Theorem 2.13]. For every open set ω whose closure is included in Ω, we consider $(\phi_t(u))|_\omega$ and $(p_t(u))|_\omega$, the restrictions of $\phi_t(u)$ and $p_t(u)$ respectively to ω. As compositions of two analytic maps in a neighborhood of $t = 0$, $(\phi_t(u))|_\omega$ and $(p_t(u))|_\omega$ are also analytic in a neighborhood of $t = 0$ and their derivatives at $t = 0$ are equal to $(\dot{\phi}(u) - \nabla\phi \cdot u)|_\omega$ and $(\dot{p}(u) - \nabla p \cdot u)|_\omega$ respectively. It is then easy to see that these formulas are actually valid over the whole Ω and thus, if we use $\phi'(u)$ and $p'(u)$ to denote the derivatives at $t = 0$ of ϕ_t and p_t respectively, one finally gets that

$$\phi'(u) = \dot{\phi}(u) - \nabla\phi \cdot u, \quad p'(u) = \dot{p}(u) - \nabla p \cdot u, \text{ in } \Omega. \tag{26}$$

We refer to $\phi'(u)$ and $p'(u)$ as the shape derivatives in the direction u of the eigenfunction and eigenpressure (ϕ, p) associated to λ.

According to Theorem 2.2, $\dot{\phi}(u)$ and $\dot{p}(u)$ at least belong to $H_0^3(\Omega)$ and $H^2(\Omega)$ respectively and thus admit traces on $\partial\Omega$ in $H^{5/2}(\partial\Omega)$ and $H^{3/2}(\partial\Omega)$ respectively. From Eqs. (26) and (16), we deduce at once, by using Eq. (14) that $p'(u) + \mathrm{div}(up) \in L_0^2(\Omega)$ and

$$\phi'(u) + (u \cdot n)\frac{\partial\phi}{\partial n} = 0 \text{ on } \partial\Omega.$$

It remains to determine the relations satisfied by the derivatives $\phi'(u)$ and $p'(u)$ inside the domain Ω. For that end, we take the derivative with respect to time evaluated at $t = 0$ of Eq. (15). For arbitrary test functions $(z, r) \in \mathcal{D}(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R})$, we obtain

$$\int_\Omega \Big(A'(0)\nabla\phi + \nabla\dot{\phi}(u)\Big) : \nabla z - \int_\Omega \Big(\dot{p}(u)\mathrm{div}(z) + p\mathrm{Tr}(B'(0)\nabla z) + p\mathrm{div}(z)\gamma'(0)\Big)$$
$$+ \int_\Omega \Big(\mathrm{Tr}(B'(0)\nabla\phi) + \mathrm{div}(\dot{\phi}(u)) + \mathrm{div}(\phi)\gamma'(0)\Big) r = \int_\Omega \Big(\lambda'(u)\phi + \lambda\dot{\phi}(u) + \lambda\phi\gamma'(0)\Big) z. \tag{27}$$

To simplify the previous equation, we use the following relations between time derivatives and shape derivatives,

$$A'(0) = \mathrm{div}(u)\mathrm{Id} - (\nabla u + \nabla^T u) \text{ and } B'(0) = -\nabla^T u.$$

We first use the boundary conditions for ϕ and notice that the term multiplied by $\gamma'(0) = \mathrm{div}(u)$ in the integrand of Eq. (27) is the PDE satisfied by ϕ. Eq. (27) then reduces to

$$\int_\Omega \Big(\nabla\phi'(u) + \nabla(\nabla\phi \cdot u) - (\nabla u + \nabla^T u)\nabla\phi\Big) : \nabla z - \int_\Omega \Big((p'(u) + \nabla p \cdot u)\mathrm{div}(z) + p\mathrm{Tr}(\nabla^T u \nabla z)\Big)$$
$$= \int_\Omega \Big(\lambda'\phi + \lambda\phi'(u) + \lambda\nabla\phi \cdot u\Big) z,$$

and

$$\int_\Omega \Big(- \mathrm{Tr}(\nabla^T u \nabla \phi) + \mathrm{div}(\phi'(u)) + \mathrm{div}(\nabla \phi \cdot u) \Big) r = 0.$$

After some integrations by parts and using the boundary conditions, one deduces the two identities

$$\int_\Omega \nabla \phi'(u) : \nabla z + \int_\Omega \nabla p'(u) \cdot z = \int_\Omega \Big(\lambda'(u)\phi + \lambda \phi'(u) \Big) z,$$

and

$$\int_\Omega \mathrm{div}(\phi'(u)) r = 0.$$

These identities hold for every $(z, r) \in \mathcal{D}(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R})$, and they yield to the equations which are valid in Ω

$$-(\Delta + \lambda)\phi'(u) + \nabla p'(u) = -\lambda'(u)\phi, \qquad \mathrm{div}(\phi'(u)) = 0.$$

In summary, the shape derivatives $\phi'(u)$ and $p'(u)$ satisfy the following inhomogeneous Stokes system of PDEs

$$\left\{ \begin{array}{ccccc} -(\Delta + \lambda)\phi'(u) + \nabla p'(u) & = & -\lambda'(u)\phi & \text{in } \Omega, \\ \mathrm{div}\ \phi'(u) & = & 0 & \text{in } \Omega, \\ \phi'(u) + (u \cdot n)\dfrac{\partial \phi}{\partial n} & = & 0 & \text{on } \partial\Omega, \\ p'(u) + \mathrm{div}(up) & \in & L^2_0(\Omega). \end{array} \right. \tag{28}$$

2.5 Ortega-Zuazua's result

Our argument for establishing Theorem 1.1 requires to perform shape differentiation of the eigenvalue problem (SD_Ω). The first step of the contradiction argument (i.e., assuming that Property (Simple) is not generic) was already conducted by J. H. Ortega and E. Zuazua in [18]. We next recall precisely the main result they obtained and, for that purpose, we introduce the following definition.

Definition 2.3. Let $d \geq 2$. A domain $\Omega \in \mathbb{D}_3$ verifies Property $(P_{OZ})_d$ if, for every λ eigenvalue of the Stokes operator with Dirichlet boundary conditions (SD_Ω), one has $m(\lambda) \leq d-1$ and if $m(\lambda) = d-1$, for $1 \leq i, j \leq d-1$ and $i \neq j$, the following three conditions must hold on $\partial\Omega$,

$$\frac{\partial \phi_i}{\partial n} \cdot n = 0, \tag{29}$$

$$\frac{\partial \phi_i}{\partial n} \cdot \frac{\partial \phi_j}{\partial n} = 0, \tag{30}$$

$$\left\| \frac{\partial \phi_i}{\partial n} \right\| = \left\| \frac{\partial \phi_j}{\partial n} \right\|, \tag{31}$$

where the ϕ_i's, $1 \leq i \leq d-1$ are orthonormal eigenfunctions associated with λ.

Then, the main result in [18] is the following.

Theorem 2.8. *Let $d \geq 2$. Then Property $(P_{OZ})_d$ defined above holds true, generically with respect to $\Omega \in \mathbb{D}_3$.*

As an immediate corollary, it is proved in [18] that Property (Simple) holds true generically for domains in $\mathbb{R}^2$. Since we adopt a viewpoint different from [18], we provide below a complete argument. We need to provide the following definition, similar to that of "minimal multiplicity" in [13, page 56].

Definition 2.4. Let $\Omega \in \mathbb{D}_3$ and λ an eigenvalue of (SD_Ω). We use $m_\Omega(\lambda)$ to denote the liminf over the multiplicities $m(\lambda_n)$, where λ_n is an eigenvalue of (SD_{Ω_n}) such that $\Omega_n \to \Omega$ and $\lambda_n \to \lambda$ as n tends to infinity.

Several remarks are immediate with the previous definition.

Remark 2.7. There exists a sequence of domains (Ω_n) in $\mathbb{D}_3$ and a sequence (λ_n), where λ_n is an eigenvalue of (S_{Ω_n}), such that $\Omega_n \to \Omega$, $\lambda_n \to \lambda$ as n tends to infinity and $m(\lambda_n) = m_\Omega(\lambda)$ (and it is also equal to $m_{\Omega_n}(\lambda_n)$).

Remark 2.8. Moreover, Property $(P_{OZ})_d$ for a domain $\Omega \in \mathbb{D}_3$ is clearly equivalent to the fact that, for every λ eigenvalue of (SD_Ω), $m_\Omega(\lambda) \leq d-1$, besides the equality case.

Proof of Theorem 2.8. Fix a domain $\Omega_0 \in \mathbb{D}_3$. We define, for $l \in \mathbb{N}$, the sets

$$A_0 := \mathbb{D}_3(\Omega_0),$$

and, for $l \geq 1$, consider

$$A_l := \{\Omega_0 + u \in A_0,\ u \in W^{4,\infty}(\Omega_0, \mathbb{R}^3),\ m_{\Omega_0}(\lambda) \leq d-1 \text{ for the first } l \text{ first eigenvalues of } (SD_{\Omega_0+u})\}.$$

Set $A := \bigcap_{l\in\mathbb{N}} A_l$. Note that

$$A = \{\Omega_0 + u \in A_0,\ u \in W^{4,\infty}(\Omega_0, \mathbb{R}^3), m_{\Omega_0+u}(\lambda) \leq d-1 \ \text{ if } \lambda \text{ is an eigenvalue of } (SD_{\Omega_0+u})\}.$$

The proof is based on the application of Baire's lemma to the sequence $\{A_l\}_{l\in\mathbb{N}}$. As A_l is open in A_0 for every $l \in \mathbb{N}$, we only need to prove that, for $l \in \mathbb{N}$, A_{l+1} is dense in A_l.

We proceed by contradiction. Assume that A_{l+1} is not dense in A_l. Then, there exists $u \in A_l \setminus A_{l+1}$ and a neighborhood U of u such that $U \subset A_l \setminus A_{l+1}$. Set $\tilde{\Omega} := \Omega_0 + u$ and let λ be the $(l+1)$-th eigenvalue of $(SD_{\tilde{\Omega}})$. For $s \geq 1$, let $\lambda_s(\cdot)$ be the function which associates to $\Omega \in \mathbb{D}_3$ the s-th eigenvalue of (SD_Ω). Recall that $\lambda_s(\cdot)$ is continuous and $\lambda = \lambda_{l+1}(\tilde{\Omega})$. According to the contradiction assumption, one has $m := m_{\tilde{\Omega}}(\lambda) \geq d$ and then $\lambda_l(\tilde{\Omega}) < \lambda$. As a consequence, $m(\lambda_{s+1})$ admits a local maximum at $\Omega = \tilde{\Omega}$ and, if (Ω_n) is the sequence in $\mathbb{D}_3$ considered in Remark 2.7 and associated to $\tilde{\Omega}$, then it has the following additional property: for n large enough, there exists $\varepsilon_n > 0$ such that, for every Ω' with $d(\Omega', \Omega_n) < \varepsilon_n$, one has that

$$m(\lambda_{l+1}(\Omega')) = m \geq d.$$

In particular, $m(\lambda_{l+1}(\cdot))$ is locally constant, equal to $m \geq d$ in an open neighborhood of Ω_n, for n large enough. We will contradict that latter fact, i.e. the existence of a domain Ω_* where $m(\lambda_{l+1}(\cdot))$ is constant and equal to $m \geq d$ in an open neighborhood U_* of Ω_*. For simplicity, λ is used to denote $\lambda_{l+1}(\Omega_*)$ in the remaining part of the argument. Once for all, fix an orthonormal family $v = (v_1, \ldots, v_m)$ of eigenfunctions of (SD_{Ω_*}) associated to λ and define the $m \times m$ matrix

$$M(v) = \Big(\int_{\partial\Omega_*} (u \cdot n) \frac{\partial v_i}{\partial n} \cdot \frac{\partial v_j}{\partial n} \Big)_{1 \leq i,j \leq m}.$$

Note that $M(v)$ is real symmetric. We next perform shape differentiation with respect to the parameter $u \in U_*$. Using the notations of Theorem 2.7, we consider, for every $u \in U_*$, the m analytic branches $t \mapsto (\lambda_i^t(u), \phi_{t,i}(u), p_{t,i}(u))$, for $i = 1, \dots, m$, given by Theorem 2.7. We use $\phi(u) := (\phi_1(u), \dots, \phi_m(u))$ and $(p_1(u), \dots, p_m(u))$ respectively to denote

$$(\phi_{0,1}(u), \dots, \phi_{0,m}(u)), \qquad (p_{0,1}(u), \dots, q_{0,m}(u)),$$

the eigenfunctions and eigenpressures associated to λ (i.e., which correspond to the values of the $\phi_{t,i}(u)$'s and $p_{t,i}(u)$'s at $t = 0$).

Since v and $\phi(u)$ are orthonormal families of eigenfunctions associated to the same eigenvalue λ, then, for every $1 \le i \le m$, there exists m real numbers s_{ij} such that $\phi_i(u) = \sum_{j=1}^m s_{ij} v_j$ and, if $S(u) := (s_{ij})_{1\le i,j\le m}$, then $S(u) \in SO(m)$ and $\phi(u) = vS(u)$ (with the convention that the $\phi_i(u)$'s and the v_i's are viewed as column vectors of $\mathbb{R}^m$). One clearly obtains that

$$M(\phi(u)) = S(u)M(v)S(u)^T. \tag{32}$$

We now need the following standard result whose proof is given in Section B.1 of Appendix.

Lemma 2.9. *Using the notations defined above, then*

$$\mathrm{diag}(\lambda_i'(u))_{1\le i\le m} = -M(\phi(u)) \tag{33}$$

holds for every $u \in W^{4,\infty}(\Omega, \mathbb{R}^3)$.

We next proceed with the proof of Theorem 2.8,

The fact that $m(\lambda_{l+1}(\cdot))$ is constant and equal to m in a neighborhood of $u = 0$ is equivalent to the fact that $\lambda_i^t(u) \equiv \lambda_j^t(u)$, $1 \le i, j \le m$, for t small enough, implying that $\lambda_i'(u)$ takes only one single value μ as i runs from 1 to m. In other words, $M(\phi(u)) = -\mu \mathrm{Id}_m$ and then one gets

$$M(v) = -\mu \mathrm{Id}_m,$$

thanks to Eq. (32). That yields the equations

$$\begin{aligned} \int_{\partial\Omega_*} (u\cdot n)\Big(\left\| \frac{\partial v_i}{\partial n} \right\|^2 - \left\| \frac{\partial v_j}{\partial n} \right\|^2 \Big) &= 0, \text{ for } 1 \le i, j \le m, & (34)\\ \int_{\partial\Omega_*} (u\cdot n) \frac{\partial v_i}{\partial n} \cdot \frac{\partial v_j}{\partial n} &= 0, \text{ for } 1 \le i, j \le d-1,\ i \ne j. & (35) \end{aligned}$$

The integrals in the above equations define linear maps in $(u.n)$ and are equal to zero in an open neighborhood of $u = 0$. It thus implies that, for distinct $1 \le i, j \le m$,

$$\begin{aligned} \left\| \frac{\partial v_i}{\partial n} \right\| - \left\| \frac{\partial v_j}{\partial n} \right\| &\equiv 0 \quad \text{on } \partial\Omega_*, & (36)\\ \frac{\partial v_i}{\partial n} \cdot \frac{\partial v_j}{\partial n} &\equiv 0 \quad \text{on } \partial\Omega_*. & (37) \end{aligned}$$

Moreover, using Lemma (2.5), one has, for $1 \le i \le m$,

$$\frac{\partial v_i}{\partial n} \cdot n \equiv 0, \text{ on } \partial\Omega_*. \tag{38}$$

Therefore, $\dfrac{\partial v_i}{\partial n}$ must be identically equal to zero, for $1 \le i \le m$. Indeed, assume that there exists $x_0 \in \partial\Omega_*$ and an index $i \in \{1, \cdots, m\}$ such that $\dfrac{\partial v_i}{\partial n}(x_0)$ is not zero. According to Eqs. (36), (37) and (38) , the $m+1$ vectors given by $\dfrac{\partial v_j}{\partial n}(x_0)$, $1 \le j \le m$ and $n(x_0)$ are all non zero and two by two perpendicular. This is a contradiction because these vectors belong to a d-dimensional vector space.

Thanks to a unique continuation type of argument due to Osses (cf. [20]), one concludes that the v_i's must also be identically equal to zero, which is in contradiction with the facts that the v_i's have L^2-norm equal to one.

□

Remark 2.9. This argument is an adaptation of the original proof by J. H. Albert in [1] to the Stokes system with Dirichlet boundary conditions, and the perturbation parameters being the domains of $\mathbb{R}^3$. See also [13, Example 4.4] for a more general situation.

3 Proof of Theorem 1.1

We follow the classical strategy initiated by J. H. Albert in [1] for the Laplace operator with Dirichlet boundary conditions. This strategy was in particular applied successfully in [18] for the generic simplicity of the Stokes operator in two space dimensions, and in [7] for other Laplacian-like operators. Fix a domain $\Omega_0 \in \mathbb{D}_3$. We define, for $l \in \mathbb{N}$, the sets

$$A_0 := \mathbb{D}_3(\Omega_0),$$

and, for $l \ge 1$,

$$A_l := \{\Omega_0{+}u,\ u \in W^{4,\infty}(\Omega_0, \mathbb{R}^3),\ \Omega_0{+}u \in A_0 \text{ and the } l \text{ first eigenvalues of } (SD_{\Omega_0+u}) \text{ are simple}\}.$$

Set $A := \bigcap_{l\in\mathbb{N}} A_l$. Note that

$$A = \{u \in W^{4,\infty}(\Omega_0 + u,\ \Omega_0, \mathbb{R}^3),\ \Omega_0 + u \in A_0 \ \text{ and the eigenvalues of } (SD_{\Omega_0+u}) \text{ are simple}\}.$$

Again, the proof of the generic simplicity of (SD_Ω) is based on the application of Baire's lemma to the sequence $\{A_l\}_{l\in\mathbb{N}}$. As A_l is open in $\mathbb{D}_3(\Omega_0)$ for every $l \in \mathbb{N}$, we only need to prove that, for $l \in \mathbb{N}$, A_{l+1} is dense in A_l. We proceed by contradiction. Assume that A_{l+1} is not dense in A_l. Then, there exists $u \in A_l \setminus A_{l+1}$ and a neighborhood U of u such that $U \subset A_l \setminus A_{l+1}$. By Theorem 2.8, we can assume, without loss of generality, that there exists $\Omega := \Omega_0 + u_0$ for some $u_0 \in U$ verifying the following: there exists an open neighborhood $V \subset U$ of 0 such that, for every $u \in V$, then $\Omega + u$ verifies:

(i) the first l eigenvalues $\lambda_1(u), \ldots, \lambda_l(u)$ of $(SD_{\Omega+u})$ are simple;

(ii) the multiplicity of the $(l+1)$-th eigenvalue $\lambda_{l+1}(u)$ of $(SD_{\Omega+u})$ is equal to 2 and, on $\partial\Omega + u$, one has

$$\frac{\partial \phi_i}{\partial n_u} \cdot n_u = 0, \qquad i = 1, 2, \tag{39}$$

$$\frac{\partial \phi_1}{\partial n_u} \cdot \frac{\partial \phi_2}{\partial n_u} = 0 \tag{40}$$

$$\left\| \frac{\partial \phi_1}{\partial n_u} \right\| = \left\| \frac{\partial \phi_2}{\partial n_u} \right\|, \tag{41}$$

where n_u is used to denote the outer unit normal at $\partial\Omega + u$ and (ϕ_1, ϕ_2) is any pair of orthonormal eigenfunctions associated with $\lambda_{l+1}(u)$.

Remark 3.1. These conditions simply state that, for an eigenvalue λ of (SD_Ω) (say the $(l+1)$-th), its multiplicity is larger or equal to 2 and, for every variation v in $W^{4,\infty}(\Omega+u,\mathbb{R}^3)$, there are two equal directionnal derivatives (in the direction of v) of λ_{l+1} at u. This fact actually does not depend on the dimension $d \geq 2$ of the domain Ω. In dimension two, the above conditions immediately yield that

$$\frac{\partial \phi_1}{\partial n_u} \equiv \frac{\partial \phi_2}{\partial n_u} \equiv 0,$$

for any pair of orthonormal eigenfunctions associated with $\lambda_{l+1}(u)$, and one derives at once a contradiction by the unique continuation result of [20], see also [18]. However, in dimension $d \geq 3$, conditions (39), (40), and (41) do not immediately yield a contradiction since three non-zero two-by-two orthogonal vectors may exist in dimension $d \geq 3$.

For the rest of the paper, domains Ω are bounded subsets of $\mathbb{R}^3$ with C^3 boundary, i.e., $d = 3$.

3.1 Shape derivation of Equations (39) (40) and (41)

We begin with the following preliminary result.

Lemma 3.1. *The shape derivative ϕ_i' of ϕ_i in the direction V satisfies*

$$\frac{\partial \phi_i'}{\partial n} = \frac{\partial \phi_i'}{\partial \nu} + \langle \frac{\partial \phi_i}{\partial n}, n' \rangle n + V_n \frac{\partial}{\partial n}\Big((\nabla \phi_i)^T n \Big) + p_i' n, \tag{42}$$

where we use V_n to denote the normal component $V \cdot n$ of the direction V on $\partial\Omega$.

Proof of Lemma 3.1. From the fact that ϕ_i vanishes on $\partial\Omega$ and satisfies $\mathrm{div}(\phi_i) = 0$, one knows that

$$(\nabla \phi_i)^T n = 0. \tag{43}$$

Taking the shape derivative of the two sides of Eq. (43), one gets

$$(\nabla \phi_i')^T\ n + (\nabla \phi_i)^T\ n' + V_n \frac{\partial}{\partial n}\Big((\nabla \phi_i)^T n \Big) = 0.$$

Since $(\nabla \phi_i)^T = \left(\frac{\partial \phi_i}{\partial n}\ n^T \right)^T = n \left(\frac{\partial \phi_i}{\partial n} \right)^T$, it comes that

$$(\nabla \phi_i')^T\ n + \langle \frac{\partial \phi_i}{\partial n}, n' \rangle n + V_n \frac{\partial}{\partial n}\Big((\nabla \phi_i)^T n \Big) = 0,$$

hence

$$(\nabla \phi_i')^T\ n = -\langle \frac{\partial \phi_i}{\partial n}, n' \rangle n - V_n \frac{\partial}{\partial n}\Big((\nabla \phi_i)^T n \Big).$$

The proof is finished once we report this expression in the definition of the co-normal derivative of ϕ_i.

$\square$

Proposition 3.2. *If ϕ_i satisfies (39) and (40), then we have, for $j = 1, 2$,*

$$\begin{aligned}
&\Big\langle \frac{\partial \phi_i'}{\partial \nu}, \frac{\partial \phi_j}{\partial n} \Big\rangle + \Big\langle \frac{\partial \phi_j'}{\partial \nu}, \frac{\partial \phi_i}{\partial n} \Big\rangle \\
= \ & -V_n \Big(\frac{\partial}{\partial n} \Big(\Big\langle \frac{\partial \phi_j}{\partial n}, \frac{\partial \phi_i}{\partial n} \Big\rangle + \Big\langle \frac{\partial}{\partial n} (\nabla \phi_i^T n), \frac{\partial \phi_j}{\partial n} \Big\rangle + \Big\langle \frac{\partial}{\partial n} (\nabla \phi_j^T n), \frac{\partial \phi_i}{\partial n} \Big\rangle \Big). \qquad (44)
\end{aligned}$$

Proof of Proposition 3.2. The shape derivative of Eq. (39) gives

$$\Big\langle \frac{\partial \phi_i'}{\partial n}, \frac{\partial \phi_j}{\partial n} \Big\rangle + \Big\langle \frac{\partial \phi_i}{\partial n}, \frac{\partial \phi_j'}{\partial n} \Big\rangle = -V_n \frac{\partial}{\partial n} \Big(\Big\langle \frac{\partial \phi_j}{\partial n}, \frac{\partial \phi_i}{\partial n} \Big\rangle \Big). \qquad (45)$$

Since $\dfrac{\partial \phi_i}{\partial n} \cdot n = 0$, it comes from Lemma (3.1) that

$$\Big\langle \frac{\partial \phi_i'}{\partial n}, \frac{\partial \phi_j}{\partial n} \Big\rangle = \Big\langle \frac{\partial \phi_i'}{\partial \nu}, \frac{\partial \phi_j}{\partial n} \Big\rangle + V_n \Big\langle \frac{\partial}{\partial n} (\nabla \phi^T n), \frac{\partial \phi_j}{\partial n} \Big\rangle,$$

hence we deduce that

$$\begin{aligned}
&\Big\langle \frac{\partial \phi_i'}{\partial n}, \frac{\partial \phi_j}{\partial n} \Big\rangle + \Big\langle \frac{\partial \phi_j'}{\partial n}, \frac{\partial \phi_i}{\partial n} \Big\rangle \\
= \ & \Big\langle \frac{\partial \phi_i'}{\partial \nu}, \frac{\partial \phi_j}{\partial n} \Big\rangle + \Big\langle \frac{\partial \phi_j'}{\partial \nu}, \frac{\partial \phi_i}{\partial n} \Big\rangle + V_n \Big(\Big\langle \frac{\partial}{\partial n} (\nabla \phi_i^T n), \frac{\partial \phi_j}{\partial n} \Big\rangle + \Big\langle \frac{\partial}{\partial n} (\nabla \phi_j^T n), \frac{\partial \phi_i}{\partial n} \Big\rangle \Big).
\end{aligned}$$

From Eq. (45), we get after identification that

$$\Big\langle \frac{\partial \phi_i'}{\partial \nu}, \frac{\partial \phi_j}{\partial n} \Big\rangle + \Big\langle \frac{\partial \phi_j'}{\partial \nu}, \frac{\partial \phi_i}{\partial n} \Big\rangle = -V_n \Big(\frac{\partial}{\partial n} \Big(\Big\langle \frac{\partial \phi_j}{\partial n}, \frac{\partial \phi_i}{\partial n} \Big\rangle \Big) + \Big\langle \frac{\partial}{\partial n} (\nabla \phi_i^T n), \frac{\partial \phi_j}{\partial n} \Big\rangle + \Big\langle \frac{\partial}{\partial n} (\nabla \phi_j^T n), \frac{\partial \phi_i}{\partial n} \Big\rangle \Big),$$

and this ends the proof of Proposition 3.2.

□

3.2 Special choice of V_n

Let $x \in \partial\Omega$ such that the vectors $\dfrac{\partial \phi_i}{\partial n}(x)$and $\dfrac{\partial \phi_j}{\partial n}(x)$ span the tangent space $T_x(\partial\Omega)$. Let $\mathcal{U}_x$ be a neighborhood of x in $\partial\Omega$ such that, for all y belonging to $\mathcal{U}_x$, the vectors $\frac{\partial \phi_i}{\partial n}(y)$ and $\frac{\partial \phi_j}{\partial n}(y)$ span $T_y(\partial\Omega)$. For $y \in \partial\Omega$ near x, we write the parametrized form of $\partial\Omega$ near x as a graph over the tangent plane at x : if $\eta = P_x(y - x)$ is the projection of $y - x$ onto the tangent plane $T_x(\partial\Omega)$ with η sufficiently small, there exists an open neighborhood $T_x\mathcal{U}_x$ of 0 in $T_x(\partial\Omega)$ such that the map h_x given by

$$\begin{aligned}
h_x : \ T_x\mathcal{U}_x \ &\mapsto \mathcal{U}_x \\
\eta \ &\mapsto y = x + \eta - \nu_x(\eta) n_x,
\end{aligned} \qquad (46)$$

is well-defined and is a diffeomorphism onto its image. For y near x, we have

$$\nu_x(\eta) = \frac{1}{2} \eta^T K_x \eta + O(|\eta|^3), \quad \text{as } \eta \to 0,$$

where K_x is the symmetric matrix representing the curvature operator at x. We fix once for all $\delta > 0$ small enough so that $\mid \eta \mid \leq 2\delta$ implies that $y = x + \eta - \nu_x(\eta) n_x$ belongs to $\mathcal{U}_x$.

We are now ready to define V_n. Let $\varepsilon << \delta$ be a positive real number. We define $x_0 \in \mathcal{U}_x$ as

$$x_0 = x + \eta_0 - \nu_x(\eta_0)n_x,$$

with $\eta_0 \in T_x(\partial\Omega)$ such that

$$\eta_0 = r_0(\cos\theta_0, \sin\theta_0),$$

for $\theta_0 \in S^1$ and $0 < r_0 \leq \varepsilon$. Note that x_0 is an arbitrary point in $\mathcal{U}_x$.

Lemma 3.3. *Let $n_x'(\eta) = \frac{\partial}{\partial\eta} n_x(\eta) \in T_x(\partial\Omega)$. We have*

$$\begin{array}{lllr} i) & n_y & = \dfrac{\nu_x'(\eta) + n_x}{\sqrt{1+|\nu_x'|^2}}, & \\ ii) & \langle n_x, y - x\rangle & = -\dfrac{1}{2}\eta^T K_x \eta + O(|\eta|^3) \quad \text{as } \eta \to 0, & \\ iii) & \langle n_x, n_y\rangle & = \dfrac{1}{\sqrt{1+|\nu_x'|^2}} & (47) \\ & & = 1 - \dfrac{1}{2}|K_x(\eta)\eta|^2) + O(|\eta|^3) \quad \text{as } \eta \to 0. & \end{array}$$

Proof of Lemma 3.3. These equations are easily obtained by standard facts from the theory of surfaces (cf. [5, Chapter 10]) and are explicitly given in [13, p. 146]. □

Remark 3.2. We note that the inverse of the Jacobian of the change of variables $h_x^{-1} : y \to \eta = h_x^{-1}(y)$ from a neighborhood of x on $\partial\Omega$ to a neighborhood of 0 in $\mathbb{R}^2$ is equal to $\langle n_x, n_y\rangle$.

Our choice for V_n will be

$$V_n(y) := (\alpha_\varepsilon \beta_\delta) \circ h_x^{-1}(y),$$

where, for $\eta \in \mathbb{R}^2$,

$$\alpha_\varepsilon(\eta) := \frac{1}{\varepsilon^2}\exp[-\frac{|\ \eta - \eta_0\ |^2}{\varepsilon^2}],$$

and $\beta_\delta(\cdot)$ is a smooth cut-off function equal to 1 on $B(0, 3\delta/2)$ and 0 on $\mathbb{R}^2 \setminus B(0, 2\delta)$.

Lemma 3.4. *If $y = h_x(\eta)$ with $\eta \in B(0,\delta)$, then we have*

$$\nabla V_n(y) = \nabla\alpha_\varepsilon(\eta) = -\frac{2\alpha_\varepsilon(\eta)}{\varepsilon^2}(\eta - \eta_0).$$

In particular, $\langle \nabla V_n(y), n_x\rangle = 0$.

Proof of Lemma 3.4. Since the gradient $n_x'(\eta) = K_x\eta$ is a vector belonging to $T_x(\partial\Omega) \perp n_x$, we deduce after a straightforward chain rule computation that

$$\begin{aligned} \nabla V_n(y) &= [\nabla h_x^{-1}(y)]^T \nabla\alpha_\varepsilon(\eta) = [I - n_x'(\eta))n_x^T]^{-1}\nabla\alpha_\varepsilon(\eta) \\ &= -2\frac{\alpha_\varepsilon}{\varepsilon^2}[I + n_x'(\eta))n_x^T](\eta - \eta_0) \\ &= -2\frac{\alpha_\varepsilon}{\varepsilon^2}(\eta - \eta_0). \end{aligned}$$

□

In the following, the gradient of a scalar function will be considered as a line vector in accordance with the definition of the Jacobian matrix for a vector valued function.

3.3 End of the proof of Theorem 1.1

The main technical result of the paper is summarized in the following proposition. The proof is provided in Section 4.

Proposition 3.5. *With V_n defined above, $x \in \partial\Omega$ and for $j = 1,\ 2$, one has*

$$\begin{aligned} \frac{\partial \phi_j'}{\partial \nu}(x) &= 2\frac{e^{-\bar{r}_0^2}}{\varepsilon^3}\Big(M_2^{A_1}(\bar{r}_0) + M_5^{A_1}(\bar{r}_0) - \bar{r}_0^2 M_3^{A_1}(\bar{r}_0)\Big)\frac{\partial \phi_j}{\partial n}(x) \\ &\quad +2\frac{e^{-\bar{r}_0^2}}{\varepsilon^3} M_4^{A_1}(\bar{r}_0)\langle \bar{\eta}_0, \frac{\partial \phi_j}{\partial n}(x)\rangle \bar{\eta}_0 + O(\frac{1}{\varepsilon^2}), \end{aligned} \tag{48}$$

where $M_k^{A_1}(\cdot)$, $2 \le k \le 5$, are nonzero entire function defined in Eqs. (78), (79), (181) and (180) respectively.

We can now conclude the proof of Theorem 1.1. By conditions (29) and (30), and Proposition 3.5, we have

$$\begin{aligned} &\langle \frac{\partial \phi_1'}{\partial \nu}(x), \frac{\partial \phi_2}{\partial n}(x)\rangle + \langle \frac{\partial \phi_2'}{\partial \nu}(x), \frac{\partial \phi_1}{\partial n}(x)\rangle \\ =& -\frac{e^{-\bar{r}_0^2}}{\pi\varepsilon^3} M_4^{A_1}(\bar{r}_0)\langle \bar{\eta}_0, \frac{\partial \phi_1}{\partial n}(x)\rangle\langle \bar{\eta}_0, \frac{\partial \phi_2}{\partial n}(x)\rangle + O(\frac{1}{\varepsilon^2}) \\ =& -\frac{e^{-\bar{r}_0^2}}{\pi\varepsilon^3} M_4^{A_1}(\bar{r}_0) r_\phi^2 \cos(\theta_1 - \theta_0)\cos(\theta_2 - \theta_0) + O(\frac{1}{\varepsilon^2}), \end{aligned}$$

with $\dfrac{\partial \phi_i}{\partial n}(x) = r_\phi(\cos\theta_i, \sin\theta_i)^T$, for $i = 1, 2$.

However, Proposition 3.2 implies that

$$\langle \frac{\partial \phi_1'}{\partial \nu}(x), \frac{\partial \phi_2}{\partial n}(x)\rangle + \langle \frac{\partial \phi_2'}{\partial \nu}(x), \frac{\partial \phi_1}{\partial n}(x)\rangle = O(\frac{1}{\varepsilon^2}).$$

Therefore, if we now fix $\bar{r}_0 \le 1$ such that $M_4^{A_1}(\bar{r}_0) \neq 0$ and recall that $r_\phi > 0$, we have

$$\cos(\theta_1 - \theta_0)\cos(\theta_2 - \theta_0) = O(\varepsilon). \tag{49}$$

By letting ε tend to zero, we deduce that

$$\cos(\theta_1 - \theta_0)\cos(\theta_2 - \theta_0) = 0, \tag{50}$$

since θ_0 does not depend on ε. Again, by conditions (29) and (30), one has $|\theta_1 - \theta_2| = \pi/2$. Then, by replacing the arbitrary angle θ_0 by $\theta_0 - \theta_1$ in Eq. (49), one derives that

$$\sin 2\theta_0 = 0, \tag{51}$$

holding for arbitrary angle θ_0. This yields the final contradiction and Theorem 1.1 is established.

4 Proof of Proposition 3.5

This section is devoted to the proof of Proposition 3.5. The argument starts by applying (157) to $\phi^\lambda = \phi'_j$, $j = 1, 2$, solution of (160)-(163). The four terms of the rigth-hand side of (157) correspond to four terms W_i^j, $1 \leq i \leq 4$ respectively. Since $\phi'_j = -V_n \dfrac{\partial \phi_j}{\partial n}$ on $\partial\Omega$, it comes that

$$\frac{\partial \phi'_j}{\partial \nu}(x) = W_1^j(x) + W_2^j(x) + W_3^j(x) + W_4^j(x), \tag{52}$$

where we have in coordinates, for $\ell = 1, \ldots, 3$, and $\phi_j = (\phi_j^m)_{m1,\ldots,3}$,

$$\Big[W_1^j(x)\Big]_\ell = -2 \text{ p.v.} \int_{\partial\Omega} \frac{\partial^2 \Gamma^0_{\ell m}(x-y)}{\partial N(x) \partial N(y)} \; V_n(y) \frac{\partial \phi_j^m}{\partial n}(y) \; d\sigma(y), \tag{53}$$

$$\begin{aligned}
\Big[W_2^j(x)\Big]_\ell &= -\Big(\Big(\sum_{k=1}^N \Big[(-2)(K_\Omega^\lambda)^*\Big]^k\Big)\Big[-2 \text{ p.v.} \int_{\partial\Omega} \frac{\partial^2 \Gamma^0_{\ell m}(x-y)}{\partial N(x)\partial N(y)} \; V_n(y) \frac{\partial \phi_j^m}{\partial n}(y)\; d\sigma(y)\Big)(x)\Big] \\
&= \Big(\Big[\sum_{k=1}^N \Big[(-2)(K_\Omega^\lambda)^*\Big]^k\Big] W_1^j\Big)(x),
\end{aligned} \tag{54}$$

$$\Big[W_3^j(x)\Big]_\ell = -\Big(\sum_{k=0}^N \Big[(-2)(K_\Omega^\lambda)^*\Big]^k\Big) \int_{\partial\Omega} \frac{\partial^2 \Delta^\lambda_{\ell m}(x-y)}{\partial N(x)\partial N(y)} \; V_n(y) \frac{\partial \phi_j^m}{\partial n}(y) \; d\sigma(y), \tag{55}$$

and

$$\begin{aligned}
\Big[W_4^j(x)\Big]_\ell &= -\Big[R - \Big(\sum_{k=1}^N \Big[(-2)(K_\Omega^\lambda)^*\Big]^k\Big)\Big]\Big[\text{ p.v.} \int_{\partial\Omega} \frac{\partial^2 \Gamma^0_{\ell m}(x-y)}{\partial N(x)\partial N(y)} \; V_n(y) \frac{\partial \phi_j^m}{\partial n}(y) \; d\sigma(y)\Big] \\
&- \Big[R - \Big(\sum_{k=1}^N \Big[(-2)(K_\Omega^\lambda)^*\Big]^k\Big)\Big] \int_{\partial\Omega} \frac{\partial^2 \Delta^\lambda_{\ell m}(x-y)}{\partial N(x)\partial N(y)} V_n(y) \frac{\partial \phi_j^m}{\partial n}(y) \; d\sigma(y).
\end{aligned} \tag{56}$$

We take $V_n(y) = \dfrac{1}{\varepsilon^2} \exp[-\dfrac{\mid h_x^{-1}(y) - h_x^{-1}(x_0) \mid^2}{\varepsilon^2}]$ and tackle the asymptotic expansion of each term figuring in the right hand side of the equation quoted above. Our strategy is simple: we show that the main term of the expansion is contained in W_1^j, where appears the effect of the hyper-singular operator. Next, we prove that all other terms $W_i^j(x)$, $i = 2, 3, 4$ are actually remainder terms. These are the contents of Proposition 4.1 and Proposition 4.11 respectively given in the next subsections.

4.1 Expansion of W_1^j

The goal of this subsection is to provide the main term in the expansion of $W_1^j(x)$ defined in Eq. (53). More precisely, we prove the following.

Proposition 4.1. *With the notations above, we have, for $\varepsilon > 0$ small enough and $j = 1, 2$,*

$$\begin{aligned}
W_1^j(x) &= 2\frac{e^{-\bar{r}_0^2}}{\varepsilon^3}\Big(M_2^{A_1}(\bar{r}_0) + M_5^{A_1}(\bar{r}_0) - \bar{r}_0^2 M_3^{A_1}(\bar{r}_0)\Big)\frac{\partial \phi_j}{\partial n}(x) \\
&+ 2\frac{e^{-\bar{r}_0^2}}{\varepsilon^3} M_4^{A_1}(\bar{r}_0)\langle \bar{\eta}_0, \frac{\partial \phi_j}{\partial n}(x)\rangle \bar{\eta}_0 + O(\frac{1}{\varepsilon^2}),
\end{aligned} \tag{57}$$

where $M_k^{A_1}(\cdot)$, $2 \leq k \leq 5$, are nonzero entire function defined in Eqs. (78), (79), (181) and (180) respectively.

4.1.1 Computational lemmas

We begin by studying the term $W_1^j(x)$ defined in Eq. (53). We start with the following lemma whose proof is deferred in Appendix. For $u = (u^m)_{m=1,\ldots,3} : \partial\Omega \mapsto \mathbb{R}^3$, we will use $E(u)(x)$ to denote the value at $x \in \partial\Omega$ of the hypersingular operator

$$\Big[E(u)(x)\Big]_\ell = \text{ p.v.} \int_{\partial\Omega} \frac{\partial^2 \Gamma^0_{\ell m}(x-y)}{\partial N(x) \partial N(y)}\, u^m(y)\, d\sigma_y, \qquad \ell = 1, \ldots, 3. \tag{58}$$

Lemma 4.2. *Let $\alpha : \partial\Omega \mapsto \mathbb{R}$ and $\psi : \partial\Omega \mapsto \mathbb{R}^3$ be C^1 functions. One has*

$$4\pi E(\alpha\psi)(x) = \sum_{i=1}^{5} A_i(\alpha, \psi)(x), \tag{59}$$

where

$$A_1(\alpha, \psi)(x) = \text{p.v.} \int_{\partial\Omega} \frac{\langle n_x, n_y \rangle}{|x-y|^3} \Big(\langle \psi(y), x-y \rangle \nabla^T \alpha(y) + (\nabla\alpha(y)(x-y))\psi(y) \Big) d\sigma_y, \tag{60}$$

$$A_2(\alpha, \psi)(x) = \text{p.v.} \int_{\partial\Omega} \frac{\alpha(y) \langle n_x, n_y \rangle}{|x-y|^3} \Big(\nabla\psi(y) + \nabla^T \psi(y) \Big)(x-y) d\sigma_y, \tag{61}$$

$$A_3(\alpha, \psi)(x) = \text{p.v.} \int_{\partial\Omega} \frac{\langle n_x, \psi(y) \rangle \nabla\alpha(y)(x-y) - \langle \psi(y), x-y \rangle \nabla\alpha(y) n_x}{|x-y|^3} n_y d\sigma_y, \tag{62}$$

$$A_4(\alpha, \psi)(x) = \text{p.v.} \int_{\partial\Omega} \frac{\alpha(y) \langle n_x, (\nabla\psi(y) - \nabla^T \psi(y))(x-y) \rangle}{|x-y|^3} n_y d\sigma_y, \tag{63}$$

$$A_5(\alpha, \psi)(x) = \int_{\partial\Omega} l(x,y)[\nabla(\alpha\psi)(y)] d\sigma_y, \tag{64}$$

where $l(\cdot,\cdot)$ is a weakly singular operator of class $C^3_*(1)$ (see Appendix A.2 for a definition).

Lemma 4.2 will be used with $\alpha = V_n$ and $\psi = \dfrac{\partial \phi_j}{\partial n}$, $j = 1, 2$. We will use the change of variables introduced in Section 3.2 and, using these notations, we set

$$\eta := r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \ \eta_0 := r_0 \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix}, \ \psi(x) := r_\psi \begin{pmatrix} \cos\theta_\psi \\ \sin\theta_\psi \end{pmatrix},$$

and

$$\bar{\eta}_0 := \frac{\eta_0}{\varepsilon}, \quad \bar{r}_0 := \frac{r_0}{\varepsilon}.$$

Recall that, with the conventions of Subsection 3.2, one has $\bar{r}_0 \leq 1$. In the sequel, we will provide an asymptotic expansion for each of the A_i, $1 \leq i \leq 5$, using powers in the variable $\dfrac{1}{\varepsilon}$. We will have two types of terms, one of the type $\dfrac{e^{-\bar{r}_0^2}}{\varepsilon^{m_i}} X_i$ (or $\dfrac{1}{\varepsilon^{m_i}} X_i$) and the other one of the type $\dfrac{e^{-\frac{\delta^2}{4\varepsilon^2}}}{\varepsilon^{m_i}} Y_i$, where m_i is an integer and X_i, Y_i are vectors with bounded norms. For each A_i,

$1 \leq i \leq 5$, we will identify the term of the first type (i.e., $\dfrac{e^{-\bar{r}_0^2}}{\varepsilon^{m_i}} X_i$ or $\dfrac{1}{\varepsilon^{m_i}} X_i$) with the largest value of m_i, then gather them and consider all the others terms as a rest. For that purpose, we will use repeateadly the following two lemmas whose proofs are deferred in Sections B.3 and B.4 of Appendix.

Lemma 4.3. *With the notations above and for any non negative integer m, one has*

$$\int_{B(0,\delta)} \frac{\alpha_\varepsilon(\eta)}{\mid \eta \mid^{1-m}} d\eta \leq \frac{C(m)}{\varepsilon^{1-m}}, \tag{65}$$

with $C(m)$ a positive constant only depending on m.

Lemma 4.4. *With the notations above,*

$$\text{p.v.} \int_{\mathbb{R}^2} \frac{\alpha_\varepsilon(\eta)\eta}{\mid \eta \mid^3} d\eta = \frac{e^{-\bar{r}_0^2}}{\varepsilon^2} M_3^{A_1}(\bar{r}_0)\bar{\eta}_0, \tag{66}$$

where $M_3^{A_1}(\cdot)$ is a nonzero entire function defined in (177) or (178) below.

We will provide detailed computations for $A_1(\alpha, \psi)(x)$ in the expansion of $W_1^j(x)$ and will only sketch the main steps for the other terms. In these computations, we will systematically refer to the following procedures. The first one consists of decomposing a C^1 vector-valued function $F(y)$ in two parts as $F(y) = F(x) + G(x)(y - x)$, where G is a continuous matrix-valued function. The second procedure consists of cutting an integral $\int_{\partial\Omega} \cdots d\sigma_y$ as

$$\int_{\partial\Omega} \cdots d\sigma_y = \int_{B(0,2\delta)} \cdots d\eta = \int_{B(0,\delta)} \cdots d\eta + \int_{B(0,2\delta)\setminus B(0,\delta)} \cdots d\eta,$$

and majorizing the second one by $C_i \dfrac{e^{-\frac{\delta^2}{4\varepsilon^2}}}{\varepsilon^{m_i}}$ for appropriate constant C_i and integer m_i. Finally, note that $\langle \psi(x), n_x \rangle = 0$.

4.1.2 Asymptotic expansion of A_1

We give in this paragraph the asymptotic expansion of $A_1(\alpha, \psi)(x)$ with respect to ε. Recall that

$$A_1(\alpha, \psi)(x) = \text{p.v.} \int_{\partial\Omega} \frac{\langle n_x, n_y \rangle}{|x - y|^3} \Big(\langle \psi(y), x - y \rangle \nabla^T \alpha(y) + (\nabla \alpha(y)(x - y))\psi(y) \Big) d\sigma_y.$$

Proposition 4.5. *For $\varepsilon > 0$ small enough, one has*

$$\begin{aligned} A_1(\alpha, \psi)(x) &= 2\frac{e^{-\bar{r}_0^2}}{\varepsilon^3} \Big(M_2^{A_1}(\bar{r}_0) + M_5^{A_1}(\bar{r}_0) - \bar{r}_0^2 M_3^{A_1}(\bar{r}_0) \Big) \psi(x) \\ &+ 2\frac{e^{-\bar{r}_0^2}}{\varepsilon^3} M_4^{A_1}(\bar{r}_0) \langle \bar{\eta}_0, \psi(x) \rangle \bar{\eta}_0 + O(\frac{1}{\varepsilon^2}). \end{aligned} \tag{67}$$

For the sake of clarity, we set $A_1(\alpha,\psi)(x) := A_{1,1}(\alpha,\psi)(x) + A_{1,2}(\alpha,\psi)(x)$ with

$$A_{1,1}(\alpha,\psi)(x) \;:=\; \text{p.v.}\int_{\partial\Omega} \frac{\langle n_x, n_y\rangle}{|x-y|^3}\langle \psi(y), x-y\rangle \nabla^T\alpha(y)d\sigma_y, \tag{68}$$

$$A_{1,2}(\alpha,\psi)(x) \;:=\; \text{p.v.}\int_{\partial\Omega} \frac{\langle n_x, n_y\rangle}{|x-y|^3}(\nabla\alpha(y)(x-y))\psi(y)d\sigma_y. \tag{69}$$

We will establish separately estimates of these two terms in Lemma 4.6 and Lemma 4.8.

Lemma 4.6. *For $\varepsilon > 0$ small enough, one has*

$$A_{1,1}(\alpha,\psi)(x) = 2\frac{e^{-\bar r_0^2}}{\varepsilon^3}\Big(M_4^{A_1}(\bar r_0)\langle \bar\eta_0, \psi(x)\rangle\bar\eta_0 + M_2^{A_1}(\bar r_0)\psi(x)\Big) + O(\frac{1}{\varepsilon^2}), \tag{70}$$

where $M_2^{A_1}(\cdot)$ and $M_4^{A_1}(\cdot)$ are non-zero entire functions defined by (78) and (80) respectively.

Proof of Lemma 4.6. Using the change of variables introduced in Subsection 3.2 and taking into account Lemma 3.4 and Remark 3.2, we have

$$A_{1,1}(\alpha,\psi)(x) = \frac{2}{\varepsilon^2}\ \text{p.v.}\int_{B(0,2\delta)} \frac{\alpha_\varepsilon(\eta)\langle \eta - \nu_x(\eta)n_x, \psi(y)\rangle}{\Big(\mid\eta\mid^2 + \mid\nu_x(\eta)\mid^2\Big)^{\frac{3}{2}}}(\eta-\eta_0)d\eta.$$

Then,

$$A_{1,1}(\alpha,\psi)(x) = I^{A_{1,1}}(\alpha,\psi)(x) + J^{A_{1,1}}(\alpha,\psi)(x) + R^{A_{1,1}}(\alpha,\psi),$$

with

$$I^{A_{1,1}}(\alpha,\psi) \;:=\; \frac{2}{\varepsilon^2}\ \text{p.v.}\int_{B(0,\delta)} \frac{\alpha_\varepsilon(\eta)\langle\eta,\psi(x)\rangle}{|\eta|^3}(\eta-\eta_0)d\eta, \tag{71}$$

$$J^{A_{1,1}}(\alpha,\psi)(x) \;:=\; \frac{2}{\varepsilon^2}\int_{B(0,\delta)} \frac{\alpha_\varepsilon(\eta)O(|\eta|^2)}{|\eta|^3}(\eta-\eta_0)d\eta, \tag{72}$$

$$R^{A_{1,1}}(\alpha,\psi)(x) \;:=\; \int_{B(0,2\delta)\setminus B(0,\delta)} \cdots, \tag{73}$$

where, in $R^{A_{1,1}}(\alpha,\psi)(x)$, one has the same integrand as in $A_1(\alpha,\psi)(x)$. Clearly, there exists a positve constant C_δ only depending on δ such that, for ε small enough with respect to δ, one has

$$\|R^{A_{1,1}}(\alpha,\psi)(x)\| \le C_\delta \frac{e^{-\frac{\delta^2}{\varepsilon^2}}}{\varepsilon^4}. \tag{74}$$

Moreover, one can apply Lemma 4.3 to $J^{A_{1,1}}(\alpha,\psi)(x)$, one gets that

$$\|J^{A_{1,1}}(\alpha,\psi)(x)\| \le \frac{2}{\varepsilon^2}(C(1) + \frac{C(0)r_0}{\varepsilon}),$$

and since $\dfrac{r_0}{\varepsilon} = O(1)$, one finally deduces that there exists a positive constant C_* such that

$$\|J^{A_{1,1}}(\alpha,\psi)(x)\| \le \frac{C_*}{\varepsilon^2}. \tag{75}$$

Note that, for ε small enough the upper bound of (75) is larger than that of (74).

It remains to estimate $I^{A_{1,1}}(\alpha,\psi)(x)$. First of all, notice that the norm of

$$\frac{2}{\varepsilon^2}\int_{\mathbb{R}^2\setminus B(0,\delta)}\frac{\alpha_\varepsilon(\eta)\langle\eta,\psi(x)\rangle}{|\eta|^3}(\eta-\eta_0)d\eta,$$

is clearly less than or equal to $\dfrac{C_\delta e^{-\frac{\delta^2}{4\varepsilon^2}}}{\varepsilon^4}$ for some positive constant C_δ only dependent on δ and ε small enough with respect to δ.

We can therefore estimate, instead of $I^{A_{1,1}}(\alpha,\psi)(x)$, the quantity $\tilde{I}^{A_{1,1}}(\alpha,\psi)(x)$ defined by

$$\tilde{I}^{A_{1,1}}(\alpha,\psi)(x):=\frac{2}{\varepsilon^2}\ \text{p.v.}\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)\langle\eta,\psi(x)\rangle}{|\eta|^3}(\eta-\eta_0)d\eta. \tag{76}$$

By using polar coordinates, one gets

$$\begin{aligned}
&\tilde{I}^{A_{1,1}}(\alpha,\psi)(x)\\
=\;& 2\frac{e^{-\bar r_0^2}}{\varepsilon^4}r_\psi\int_0^\infty \exp(-\frac{r^2}{\varepsilon^2})dr\int_0^{2\pi}\cos(\theta-\theta_\psi)\exp(2\frac{r}{\varepsilon}\bar r_0\cos(\theta-\theta_0))\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}d\theta\\
&-2\frac{e^{-\bar r_0^2}}{\varepsilon^3}r_\psi\begin{pmatrix}\cos\theta_0\\ \sin\theta_0\end{pmatrix}\bar r_0\ \text{p.v.}\int_0^\infty\frac{\exp(-\frac{r^2}{\varepsilon^2})}{r}dr\int_0^{2\pi}\cos(\theta-\theta_\psi)\exp(2\frac{r}{\varepsilon}\bar r_0\cos(\theta-\theta_0))d\theta\\
=\;& 2\frac{e^{-\bar r_0^2}}{\varepsilon^3}r_\psi\begin{pmatrix}M_1^{A_1}(\bar r_0)\cos(\theta_0-\theta_\psi)\cos\theta_0+M_2^{A_1}(\bar r_0)\sin(\theta_0-\theta_\psi)\sin\theta_0\\ M_1^{A_1}(\bar r_0)\cos(\theta_0-\theta_\psi)\sin\theta_0-M_2^{A_1}(\bar r_0)\sin(\theta_0-\theta_\psi)\cos\theta_0\end{pmatrix}\\
&-2\frac{e^{-\bar r_0^2}}{\varepsilon^3}r_\psi\cos(\theta_0-\theta_\psi)\begin{pmatrix}\cos\theta_0\\ \sin\theta_0\end{pmatrix}\bar r_0^2M_3^{A_1^1}(\bar r_0)\\
=\;& 2\frac{e^{-\bar r_0^2}}{\varepsilon^3}r_\psi\begin{pmatrix}[M_1^{A_1}(\bar r_0)-\bar r_0M_3^{A_1^1}(\bar r_0)]\cos(\theta_0-\theta_\psi)\cos\theta_0+M_2^{A_1}(\bar r_0)\sin(\theta_0-\theta_\psi)\sin\theta_0\\ [M_1^{A_1}(\bar r_0)-\bar r_0M_3^{A_1^1}(\bar r_0)]\cos(\theta_0-\theta_\psi)\sin\theta_0-M_2^{A_1}(\bar r_0)\sin(\theta_0-\theta_\psi)\cos\theta_0\end{pmatrix},
\end{aligned}$$

where

$$M_1^{A_1}(\bar r_0) := \int_0^\infty\exp(-r^2)dr\int_0^{2\pi}\cos^2\theta\exp(2r\bar r_0\cos\theta)d\theta, \tag{77}$$

$$M_2^{A_1}(\bar r_0) := \int_0^\infty\exp(-r^2)dr\int_0^{2\pi}\sin^2\theta\exp(2r\bar r_0\cos\theta)d\theta, \tag{78}$$

$$M_3^{A_1}(\bar r_0) := \frac{1}{\bar r_0}\ \text{p.v.}\int_0^\infty\frac{e^{-r^2}}{r}dr\int_0^{2\pi}\cos\theta\exp(2r\bar r_0\cos\theta)d\theta.. \tag{79}$$

The needed information about the functions $M_i^{A_1}(\cdot)$, $i=1,2,3$, is gathered in the following lemma, whose proof is given in Section B.5 in appendix.

Lemma 4.7. *For $i=1,2$, $M_i^{A_1}(\cdot)$ are entire functions. Moreover, the function $M_4^{A_1}(\cdot)$ defined by the relation*

$$M_4^{A_1}(z):=\frac{1}{z^2}(M_1^{A_1}(z)-z^2M_3^{A_1}(z)-M_2^{A_1}(z)) \tag{80}$$

is a nonzero entire function.

Using Lemma 4.7, we further simplify $\tilde{I}_1^{A_{1,1}}$ as follows.

$$\begin{aligned}
&\tilde{I}_1^{A_{1,1}}(\alpha,\psi)\\
= \;& 2\frac{e^{-\bar{r}_0^2}}{\varepsilon^3} r_\psi \begin{pmatrix} [M_1^{A_1}(\bar{r}_0) - \bar{r}_0^2 M_3^{A_1}(\bar{r}_0)]\cos(\theta_0-\theta_\psi)\cos\theta_0 + M_2^{A_1}(\bar{r}_0)\sin(\theta_0-\theta_\psi)\sin\theta_0 \\ [M_1^{A_1}(\bar{r}_0) - \bar{r}_0^2 M_3^{A_1}(\bar{r}_0)]\cos(\theta_0-\theta_\psi)\sin\theta_0 - M_2^{A_1}(\bar{r}_0)\sin(\theta_0-\theta_\psi)\cos\theta_0 \end{pmatrix}\\
= \;& 2\frac{e^{-\bar{r}_0^2}}{\varepsilon^3} r_\psi \bar{r}_0^2 M_4^{A_1}(\bar{r}_0)\cos(\theta_0-\theta_\psi)\begin{pmatrix}\cos\theta_0\\ \sin\theta_0\end{pmatrix} + \; 2\frac{e^{-\bar{r}_0^2}}{\varepsilon^3} r_\psi M_2^{A_1}(\bar{r}_0^2)\begin{pmatrix}\cos\theta_\psi\\ \sin\theta_\psi\end{pmatrix}\\
= \;& 2\frac{e^{-\bar{r}_0^2}}{\varepsilon^3} M_4^{A_1}(\bar{r}_0)\langle\bar{\eta}_0,\psi(x)\rangle\bar{\eta}_0 + \; 2\frac{e^{-\bar{r}_0^2}}{\varepsilon^3} M_2^{A_1}(\bar{r}_0)\psi(x).
\end{aligned} \tag{81}$$

This ends the proof of Proposition 4.5.

□

Lemma 4.8. *With the above notations, for $\varepsilon > 0$ small enough, one has*

$$A_{1,2}(\alpha,\psi)(x) = \frac{2e^{-\bar{r}_0^2}}{\varepsilon^3}(M_5^{A_1}(\bar{r}_0) - \bar{r}_0^2 M_3^{A_1}(\bar{r}_0))\psi(x) + O(\frac{1}{\varepsilon^2}), \tag{82}$$

where $M_5^{A_1}(cdot)$ is the non zero entire function defined as $M_1^{A_1}(\cdot) + M_2^{A_1}(\cdot)$.

Proof of Lemma 4.8. We proceed similarly as in the proof of Lemma 4.6. Besides remainder terms, one must the principal term given by

$$I^{A_{1,2}}(\alpha,\psi)(x) = \text{p.v.}\frac{2}{\varepsilon^2}\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)\langle\eta,\eta-\eta_0\rangle}{|\eta|^3}d\eta\;\psi(x).$$

Using polar coordinates, one gets

$$\begin{aligned}
&I^{A_{1,2}}(\alpha,\psi)(x)\\
= \;& \Big(\frac{2}{\varepsilon^2}\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)}{|\eta|}d\eta - \frac{2}{\varepsilon^2}\;\text{p.v.}\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)\langle\eta,\eta_0\rangle}{|\eta|^3}d\eta\Big)\psi(x)\\
= \;& \Big(\frac{2e^{-\bar{r}_0^2}}{\varepsilon^3}\int_{r=0}^{\infty}\int_0^{2\pi} e^{-r^2}e^{2r\bar{r}_0\cos\theta}\,d\theta\,dr - \frac{2e^{-\bar{r}_0^2}}{\varepsilon^3}\bar{r}_0\;\text{p.v.}\int_{r=0}^{\infty}\int_0^{2\pi} e^{-r^2}e^{2r\bar{r}_0\;\cos\theta}\cos\theta\,d\theta\,\frac{dr}{r}\Big)\psi(x)\\
= \;& \frac{2e^{-\bar{r}_0^2}}{\varepsilon^3}(M_5^{A_1}(\bar{r}_0) - \bar{r}_0^2 M_3^{A_1}(\bar{r}_0))\psi(x),
\end{aligned}$$

where $M_3^{A_1}(\bar{r}_0)$ and $M_5^{A_1}(\bar{r}_0)$ are given respectively by (177) and (180).

□

4.1.3 Asymptotic expansion of A_i for $2 \le i \le 5$

We establish the following proposition for the asymptotic expansion of A_i with $i = 2, \dots, 8$.

Proposition 4.9. *For $i = 2, \dots, 5$ and $\varepsilon > 0$ small enough, one has*

$$A_i(\alpha,\psi)(x) = O(\frac{1}{\varepsilon^2}). \tag{83}$$

Proof of Proposition 4.9. We proceed similarly as in the proof of Lemma 4.6.

For $A_2(\alpha,\psi)(x)$, we only need to estimate the following term:

$$R^{A_2}(\alpha,\psi)(x) := (\nabla\psi(x)+\nabla^T\psi(x))\ \text{p.v.}\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)}{|\eta|^3}\eta d\eta.$$

By Lemma 4.4, one gets

$$R^{A_2}(\alpha,\psi)(x) = \frac{e^{-\bar{r}_0^2}}{\varepsilon^2}M_3^{A_1^1}(\bar{r}_0)(\nabla\psi(x)+\nabla^T\psi(x))\bar{\eta}_0 = O(\frac{1}{\varepsilon^2}). \tag{84}$$

For $A_3(\alpha,\psi)(x)$, we first note that $\nabla\alpha(y)n_x = 0$, and

$$\begin{aligned}\langle n_x,\psi(y)\rangle &= \langle n_x,\psi(x+\eta-n_x(\eta)n_x)\rangle = \langle n_x,\psi(x)+\nabla\psi(x)\eta+O(|\eta|^2)\rangle\\ &= \langle\nabla\psi(x)^T n_x,\eta\rangle + O(|\eta|^2).\end{aligned}$$

Thus, we need to estimate the following integral,

$$R^{A_3}(\alpha,\psi)(x) \ := \ \frac{2}{\varepsilon^2}\int_{\mathbb{R}^2}\alpha_\varepsilon(\eta)\frac{\langle\nabla\psi(x)^T n_x,\eta\rangle}{|\eta|^3}\langle\eta-\eta_0,\eta\rangle d\eta\ n_x.$$

One can clearly apply Lemma 4.3 to $R_1^{A_3}(\alpha,\psi)(x)$ with $m=0,1$ and one gets,

$$\|R^{A_3}(\alpha,\psi)(x)\| \le \frac{2}{\varepsilon^2}(C(1)+\frac{C(0)r_0}{\varepsilon}),$$

and since $\dfrac{r_0}{\varepsilon} = O(1)$, one finally deduces that

$$R^{A_3}(\alpha,\psi)(x) = O(\frac{1}{\varepsilon^2}). \tag{85}$$

For $A_4(\alpha,\psi)(x)$, we only need to estimate the following term:

$$R^{A_4}(\alpha,\psi)(x) := \Big\langle\ \text{p.v.}\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)}{|\eta|^3}\eta d\eta,(\nabla\psi-\nabla^T\psi(x))n_x\Big\rangle n_x.$$

Using Lemma 4.4, one gets

$$R^{A_4}(\alpha,\psi)(x) = \frac{e^{-\bar{r}_0^2}}{\varepsilon^2}M_3^{A_1}(\bar{r}_0)\langle(\nabla\psi-\nabla^T\psi(x))\bar{\eta}_0,n_x\rangle n_x = O(\frac{1}{\varepsilon^2}). \tag{86}$$

For $A_5(\alpha,\psi)(x)$, one gets the estimate

$$R^{A_5}(\alpha,\psi)(x) = O(\frac{1}{\varepsilon^2}), \tag{87}$$

as a consequence of Lemma 4.10.

In summary, for $i=2,\ldots,5$, $A_i(\alpha,\psi)(x) = O(\frac{1}{\varepsilon^2})$, which ends the proof of Proposition 4.9.

□

Lemma 4.10. *With the notations above, consider the function defined for $x \in \partial\Omega$*

$$R(x) = \int_{\partial\Omega} r(x,y) \cdot \nabla(\alpha\psi)(y) d\sigma(y),$$

where $r(\cdot,\cdot)$ is a $C_*^3(1)$ weakly singular kernel and $\cdot$ stands for a linear action of r on the coefficients of $\nabla(\alpha\psi)(\cdot)$. Then, there exists a positive constant C_R such that, for $\varepsilon > 0$ small enough and $x \in \partial\Omega$, one gets

$$\|R(x)\| \leq \frac{C_R}{\varepsilon^2}. \tag{88}$$

Proof of Lemma 4.10. As done for estimating A_1, we use the change of variables introduced in Subsection 3.2 and taking into account Lemma 3.4 and Remark 3.2, it is easy to see that the most 'singular" part corresponds to majorizing

$$\int_{B(0,\delta)} \frac{\nabla\alpha_\varepsilon(\eta)}{|\eta|} d\eta.$$

Thus Eq. (88) follows readily from Lemma 4.3.

□

4.2 Estimates of the remainder terms W_i^j, $i = 2, 3, 4$ and $j = 1, 2$.

In this subsection, we upper bound the remainder terms $W_i^j(x)$, $i = 2, 3, 4$ and $j = 1, 2$, defined respectively in Eqs. (54), (55) and (54). More precisely, we prove that

Proposition 4.11. *With the notations above, we have, for $\varepsilon > 0$ small enough, $j = 1, 2$ and $i = 2, 3, 4$,*

$$W_i^j(x) = O(\frac{1}{\varepsilon^2}). \tag{89}$$

Remark 4.1. One must stress the similarity of our computations with those performed by D. Henry in [13]. More precisely, the terms A_1 and A_2 in $W_1^j(\cdot)$, which are (essentially) the most 'singular" part in the hypersingular operator E defined in Eq. (59), correspond to the operator $J(\cdot)$ defined in Theorem 7.4.1 of [13], page 135, with the specific choice of $Q(x, y, \frac{y-x}{|y-x|}) = V_n(y)\frac{y-x}{|y-x|}$ and $n = 3$. Also notice that our Lemma 4.4 corresponds to an explicit computation of the polynomial $q(\cdot)$ (cf. Theorem 7.4.1 of [13]) and follows the same lines as the strategy proposed in page 137 in [13]. In particular, one gets from Theorem 7.4.1 of [13] that $W_1^j(\cdot)$ extends uniquely to a continuous operator on $\partial\Omega$.

Proof of Proposition 4.11. All the estimates to be established are consequences of (168)-(171) obtained in Corollary B.2. We rewrite it as follows. For u of class C^2 and $x \in \partial\Omega$, one writes $4\pi Eu(x)$ as the sum of two operators,

$$4\pi Eu(x) = Fu(x) + Lu(x) = \text{p.v.} \int_{\partial\Omega} f(x,y) \cdot \nabla u(y) d\sigma(y) + \int_{\partial\Omega} l(x,y) \cdot \nabla u(y) d\sigma(y), \tag{90}$$

where " $\cdot$ " stands for an action of the respective kernels which is linear with respect to $\nabla u(\cdot)$, $l(\cdot,\cdot)$ is a $C_*^3(1)$ kernel (of appropriate matrix size) defined in Appendix A.2 and the kernel $f(\cdot,)$ defining the singular operator F together with its action is given by

$$f(x,y) \cdot M(y) := [M(y) + M^T(y)]\frac{x-y}{|x-y|^3} + n_x n_x^T [M(y) - M^T(y)]\frac{x-y}{|x-y|^3}, \tag{91}$$

for $x \neq y$, points on $\partial\Omega$ and M is C^1 matrix-valued function defined on $\partial\Omega$.

In order to handle the remainder terms W_i^j's, $i = 2, 3, 4$, one must handle the evaluation at $V_n \frac{\partial \phi_j}{\partial n}$ of the operators obtained as the composition of K_Ω^λ defined in (152) and its iterations with W_1^j. In fact, we will show next that all remainder terms W_i^j's, $i = 2, 3, 4$ are $O(\frac{1}{\varepsilon^2})$ and to proceed, we will be only interested in the contribution of the "most" singular part in each term W_i^j's, $i = 2, 3, 4$. For that purpose, we will perform several (and standard) reductions. The first one consists in considering the operator K_Ω^0 instead of K_Ω^λ since $K_\Omega^\lambda - K_\Omega^0$ admits a C^1 kernel. Lemma 4.10 already handles the term W_3^j. Next, recall K_Ω^0 is a weakly singular operator of class $C_*^3(1)$ (see Appendix A.2 for a definition). To handle the terms W_2^j and W_4^j, we first need the following result.

Lemma 4.12. *The operator defined on $C^1(\partial\Omega)$ as the composition of K_Ω^0 and F is a weakly singular operator of class $C_*^3(1)$.*

Thanks to the above lemma, the first term in the summation (54) is controlled as $O(\frac{1}{\varepsilon^2})$. For the other terms, it is now enough to see that they correspond to compositions of iterates of K_Ω^0 with $K_\Omega^0 \circ F$ and thus we can apply Theorem A.4 given below on the composition of weakly singular operators of class $C_*^3(\gamma)$ with $\gamma > 0$. We deduce at once that every term appearing in the summation (54) corresponds to the evaluation at $\nabla(V_n \frac{d\phi_j}{dn})(\cdot)$ of a weakly singular operator of class $C_*^3(\gamma)$, with $\gamma \geq 1$, and is therefore controlled as $O(\frac{1}{\varepsilon^2})$. The term W_3^j is handled in a similar way and Proposition 4.11 is established.

□

We now give the proof of Lemma 4.12.

Proof of Lemma 4.12. The argument given below is already contained in Section 7.6 of [13], which considers a more general situation (see, more particularly, the proof of Theorem 7.6.3 page 147, [13]). For sake of clarity, we reproduce the main lines. Let M be a C^1 matrix-valued function defined on $\partial\Omega$. Then, the composition $(K_\Omega^0 \circ F)[M](\cdot)$ is defined, for $x \in \partial\Omega$, as the sum

$$(K_\Omega^0 \circ F)[M](x) = R_1(x) + R_2(x),$$

where

$$R_1(x) = \frac{3}{4\pi} \text{ p.v.} \iint_{\partial\Omega\times\partial\Omega} \frac{\langle x-z, n(z)\rangle}{\mid z-x\mid^5} (x-z)(x-z)^T [M(y)+M^T(y)] \frac{z-y}{\mid z-y\mid^3} \, d\sigma_y \, d\sigma_z, \quad (92)$$

and

$$R_2(x) = \frac{3}{4\pi} \text{ p.v.} \iint_{\partial\Omega\times\partial\Omega} \frac{\langle x-z, n(z)\rangle^2}{\mid z-x\mid^4} \frac{(x-z)}{\mid z-x\mid} \frac{\langle n(z), [M(y) - M^T(y)](z-y)\rangle}{\mid z-y\mid^3} \, d\sigma_y \, d\sigma_z. \quad (93)$$

Thanks to (155), the operator R_2 is clearly more regular than R_1. In the sequel, we only provide details for R_1 and only give the estimate for R_2.

We next develop in coordinates the above expressions and obtain that, for $i = 1, 2, 3$,

$$\begin{aligned}\Big(R_1(x)\Big)_i &= \frac{3}{4\pi} \sum_{k,l=1}^{3} \text{p.v.} \int_{\partial\Omega} (M(y))_{kl} \, d\sigma_y \\ &\int_{\partial\Omega} \Big[\frac{\langle x-z, n(z)\rangle (x-z)_i (x-z)_k}{\mid z-x\mid^5} \frac{(z-y)_l}{\mid z-y\mid^3} \\ &+ \frac{\langle x-z, n(z)\rangle (x-z)_i (x-z)_l}{\mid z-x\mid^5} \frac{(z-y)_k}{\mid z-y\mid^3} \Big] \, d\sigma_z, \end{aligned} \quad (94)$$

The integrand of (94) shows that R_1 is the contraction of $M(\cdot)$ and a tensor field of order $(2,1)$ defined (in coordinates) by the interior integral in (94). In order to describe R_1 as a convolution, we prefer to rewrite (94) in a more elementary way, as follows,

$$\Big(R_1(x)\Big)_i = \frac{3}{4\pi}\text{p.v.}\int_{\partial\Omega} \operatorname{Tr}(M(y)c_i(x,y))\, d\sigma_y,$$

where the kernel $c_i(x,y)$ is defined for $x \neq y$ and $i = 1,2,3$, as

$$c_i(x,y) := \text{p.v.}\int_{\partial\Omega} \frac{\langle x-z, n(z)\rangle (x-z)_i}{\mid z-x\mid^5}\Big[\frac{(x-z)(z-y)^T}{\mid z-y\mid^3} + \frac{(z-y)(x-z)^T}{\mid z-y\mid^3}\Big]\, d\sigma_z. \tag{95}$$

Let $(e_i)_{i=1,2,3}$ be the canonical basis of $\mathbb{R}^3$. Then one has $c_i(x,y) = d_i(x,y) + d_i(x,y)^T$, where

$$d_i(x,y) := \text{p.v.}\int_{\partial\Omega} k^0(x,z)[g^i(z,y)]\, d\sigma_z, \tag{96}$$

i.e., $d_i(x,y)$ is the kernel corresponding to the convolution of K^0_Ω with kernel $k^0(\cdot,\cdot)$ given by

$$k^0(x,y) := \frac{1}{\mid x-y\mid}\frac{\langle x-y, n(y)\rangle}{\mid x-y\mid^2}\;\frac{(x-y)}{\mid x-y\mid}\frac{(x-y)^T}{\mid x-y\mid},$$

and the singular operator G^i with kernel $g^i(\cdot,\cdot)$ given by

$$g^i(x,y) := \frac{e_i(x-y)^T}{\mid x-y\mid^3}.$$

To perform that analysis, one writes (96) in the chart h_x defined in (46) and only considers the most "singular" term of the composition, which is given by

$$\text{p.v.}\int_{B(0,\delta)} \frac{\eta^T K_x \eta}{\mid\eta\mid^5}\eta\eta^T\frac{\tilde{e}_i(\eta-\eta_y)^T}{\mid\eta-\eta_y\mid^3}d\eta. \tag{97}$$

Here, $\tilde{e}_i$ is the (orthogonal) projection of e_i onto $T_x\partial\Omega$. In (97), one clearly recognizes the convolution between the kernels $\dfrac{\eta^T K_x\eta}{\mid\eta\mid^5}\eta\eta^T$ and $\dfrac{\tilde{e}_i\eta^T}{\mid\eta\mid^3}$. The first kernel can also be written as $\frac{1}{|\eta|}Q(\frac{\eta}{|\eta|})$ where the components of Q are homogeneous polynomials of degree four defined on S^1. According to [13, Th. 7.3.1 p. 128] (which refers to [26] for more complete computations), the Fourier transforms of these kernels are respectively equal to

$$F.T.(\frac{\eta^T K_x\eta}{\mid\eta\mid^5}\eta\eta^T)(\xi) = \frac{1}{\mid\xi\mid}\tilde{Q}(\frac{\xi}{\mid\xi\mid}),$$

and

$$F.T.(\frac{\eta}{\mid\eta\mid^3})(\xi) = \gamma_1\frac{\xi}{\mid\xi\mid},$$

where γ_1 is a positive constant and the components of $\tilde{Q}$ are homogeneous polynomials of degree four. We get that the Fourier transform of the operator whose kernel is given by (97) is equal the product of the two Fourier transforms written previously and, as a consequence, that operator is weakly singular of class $C^3_*(1)$. The same conclusion holds true as well for R_1. A similar line of reasoning shows that R_2 is weakly singular of class $C^3_*(2)$ and Lemma 4.10 is finally proved.

□

5 Proof of Theorem 1.2

In this section, we establish in full generality the Foias-Saut conjecture in 3D as stated in [10]. First of all, notice that there is a countable number of resonance relations as defined in Definition 1.1. To see that, simply remark that, for every positive integer N, there exists a finite number of resonance relations of the type $\lambda_k = \sum_{j=1}^{\ell} m_j\lambda_j$, with $\lambda_1 \leq \cdots \leq \lambda_\ell \leq \lambda_k$, so that $k + \sum_{j=1}^{\ell} m_j \leq N$. We use $(RR)_n$, $n \geq 1$, to denote these resonances relations.

Fix a domain $\Omega_0 \in \mathbb{D}_3$. We define, for $n \in \mathbb{N}$, the sets

$$A_0 := \mathbb{D}_3(\Omega_0),$$

and, for $n \geq 1$,

$$\begin{aligned} A_n := \{\Omega_0 + u,\ u \in W^{4,\infty}(\Omega_0, \mathbb{R}^3),\ \Omega_0 + u \in A_0 \\ \text{and the} \quad n \text{ first resonance relations } (RR)_j,\ 1 \leq j \leq n, \text{ are not satisfied}\}. \end{aligned}$$

Set $A := \bigcap_{l\in\mathbb{N}} A_n$. Note that

$$A = \{\Omega_0 + u,\ u \in W^{4,\infty}(\Omega_0, \mathbb{R}^3),\ \Omega_0 + u \in A_0 \quad (SD_{\Omega_0+u}) \text{ is not resonant}\}.$$

For $n \geq 0$, each set A_n is open and one must show that A_{n+1} is dense in A_n. Reasoning by contradiction, assume that there exists $n \in \mathbb{N}$ so that A_{n+1} is not dense in A_n and fix $(RR)_{n+1}$ to be equal to $\lambda_k = \sum_{j=1}^{\ell} m_j\lambda_j$, for some integers $k, l, m_1 \cdots, m_l$. With no loss of generality, we assume that there exists $\Omega \in \mathbb{D}_3$ and $\varepsilon > 0$ so that, for $u \in W^{4,\infty}$ with $\|u\|_{4,\infty} < \varepsilon$, we have

(i) the k first eigenvalues $\lambda_1(u), \dots, \lambda_k(u)$ of $(SD_{\Omega+u})$ are simple;

(ii) the resonance condition holds true:

$$\lambda_k(u) = \sum_{j=1}^{\ell} m_j\lambda_j(u). \tag{98}$$

By Condition (i) and Eq. (33), one has, for $u \in W^{4,\infty}$ with $\|u\|_{4,\infty} < \varepsilon$ and $1 \leq j \leq k$,

$$\lambda_j'(u) = -\int_{\partial\Omega} \langle u, n\rangle \|\frac{\partial\phi_j}{\partial n}\|^2, \tag{99}$$

where ϕ_j is the orthonormal eigenfuncion associated to the eigenvalue λ_j of (SD_Ω).

Taking the shape derivative of Eq. (98), we have

$$\int_{\partial\Omega} \langle u, n\rangle \|\frac{\partial\phi_k}{\partial n}\|^2 = \int_{\partial\Omega} \langle u, n\rangle \sum_{j=1}^{\ell} m_j \|\frac{\partial\phi_j}{\partial n}\|^2. \tag{100}$$

Since Eq. (100) holds true for all u small enough, we obtain

$$\|\frac{\partial\phi_k}{\partial n}\|^2 - \sum_{j=1}^{\ell} m_j \|\frac{\partial\phi_j}{\partial n}\|^2 = 0 \quad \text{on } \partial\Omega. \tag{101}$$

Continuing the argument by contradiction, we assume that Eq. (101) holds true in a neighborhood of Ω and take again the shape derivative. By Proposition 3.2, we have, on $\partial\Omega$,

$$\Big\langle \frac{\partial \phi_k'}{\partial \nu}, \frac{\partial \phi_k}{\partial n}\Big\rangle - \sum_{j=1}^{\ell} m_j \Big\langle \frac{\partial \phi_j'}{\partial \nu}, \frac{\partial \phi_j}{\partial n}\Big\rangle = -\langle u, n\rangle \Big[\langle \frac{\partial}{\partial n}\frac{\partial \phi_k}{\partial N}, \frac{\partial \phi_k}{\partial n}\rangle - \sum_{j=1}^{\ell} m_j \langle \frac{\partial}{\partial n}\frac{\partial \phi_j}{\partial N}, \frac{\partial \phi_j}{\partial n}\rangle\Big]. \quad (102)$$

We choose a variation u such that $\langle u, n\rangle = V_n$ with V_n defined in Section 3.2. Using Proposition 3.5 together with Eq. (101), since $\bar{\eta}_0$ is an arbitrary unitary vector of $\mathbb{R}^2$, we obtain, on $\partial\Omega$,

$$\frac{\partial \phi_k}{\partial n}\Big(\frac{\partial \phi_k}{\partial n}\Big)^T - \sum_{j=1}^{\ell} m_j \frac{\partial \phi_j}{\partial n}\Big(\frac{\partial \phi_j}{\partial n}\Big)^T = 0. \quad (103)$$

From now on, fix $x \in \partial\Omega$ such that $\frac{\partial \phi_k}{\partial n}(x) \neq 0$. Recall that such an x exists by the result of Osses in [20]. According to Eq. (103), there exists an open neighborhood O_x of x on $\partial\Omega$ such that, for $j = 1, \ldots, \ell$, there is a C^2 function μ_j such that

$$\frac{\partial \phi_j}{\partial n} = \mu_j \frac{\partial \phi_k}{\partial n}, \quad \text{on } O_x. \quad (104)$$

In addition, one has,

$$1 - \sum_{j=1}^{\ell} m_j \mu_j^2 = 0, \quad \text{on } O_x. \quad (105)$$

It is clear that all the equations from (99) to (105) were obtain by assuming that Eq. (98) holds true in an open neighborood of $u = 0$. As a consequence, these equations must also hold true in an open neighborood of $u = 0$ and thus, one can take the shape derivatives of Equations (103) at $u = 0$ along any variation. We will perform such a shape derivation along the variations V_n defined in Section 3.2, with this time the real number $\delta > 0$ chosen so that the support of V_n is contained in O_x. Using Lemma 3.1, the shape derivative of Eq. (103) is equal to

$$\begin{aligned}
& \frac{\partial \phi_k'}{\partial \nu}\Big(\frac{\partial \phi_k}{\partial n}\Big)^T + \frac{\partial \phi_k}{\partial n}\Big(\frac{\partial \phi_k'}{\partial \nu}\Big)^T - \sum_{j=1}^{\ell} m_j \Big[\frac{\partial \phi_j'}{\partial \nu}\Big(\frac{\partial \phi_j}{\partial n}\Big)^T + \frac{\partial \phi_j}{\partial n}\Big(\frac{\partial \phi_j'}{\partial \nu}\Big)^T\Big] \\
+ & \Big(p_k' + \langle \frac{\partial \phi_k}{\partial n}, n'\rangle\Big)\Big[n\Big(\frac{\partial \phi_k}{\partial n}\Big)^T + \frac{\partial \phi_k}{\partial n} n^T\Big] - \sum_{j=1}^{\ell} m_j \Big(p_j' + \langle \frac{\partial \phi_j}{\partial n}, n'\rangle\Big)\Big[n\Big(\frac{\partial \phi_j}{\partial n}\Big)^T + \frac{\partial \phi_j}{\partial n} n^T\Big] \\
= & -V_n\Big(\frac{\partial}{\partial n}\frac{\partial \phi_k}{\partial N}\Big(\frac{\partial \phi_k}{\partial n}\Big)^T + \frac{\partial \phi_k}{\partial n}\Big(\frac{\partial}{\partial n}\frac{\partial \phi_k}{\partial N}\Big)^T - \sum_{j=1}^{\ell} m_j \mu_j \Big[\frac{\partial}{\partial n}\frac{\partial \phi_j}{\partial N}\Big(\frac{\partial \phi_k}{\partial n}\Big)^T + \frac{\partial \phi_k}{\partial n}\Big(\frac{\partial}{\partial n}\frac{\partial \phi_j}{\partial N}\Big)^T\Big]\Big),
\end{aligned} \quad (106)$$

where the above equation holds on $\partial\Omega$.

Moreover, on O_x, one deduces that

$$\begin{aligned}
\frac{\partial}{\partial n}\frac{\partial \phi_j}{\partial n} &= \nabla(\mu_j(\nabla \phi_k n))n = \frac{\partial \mu_j}{\partial n}\frac{\partial \phi_k}{\partial n} + \mu_j \nabla^2 \phi_k(n, n). \\
\frac{\partial}{\partial n}\nabla^T \phi_j n &= \nabla(\mu_j(\nabla^T \phi_k n))n = \frac{\partial \mu_j}{\partial n}\nabla^T \phi_k n + \mu_j \nabla(\nabla^T \phi_k n)n = \mu_j \nabla(\nabla^T \phi_k n)n.
\end{aligned}$$

This implies that, on O_x,

$$\frac{\partial}{\partial n}\frac{\partial \phi_j}{\partial N} = \frac{\partial \mu_j}{\partial n}\frac{\partial \phi_k}{\partial n} + \mu_j v_k, \tag{107}$$

with $v_k := \nabla^2\phi_k(n,n) + \nabla(\nabla^T\phi_k n)n$. Therefore, one has on O_x,

$$\begin{aligned}
&\frac{\partial}{\partial n}\frac{\partial \phi_k}{\partial N}\Big(\frac{\partial \phi_k}{\partial n}\Big)^T + \frac{\partial \phi_k}{\partial n}\Big(\frac{\partial}{\partial n}\frac{\partial \phi_k}{\partial N}\Big)^T - \sum_{j=1}^{\ell} m_j\mu_j\Big[\frac{\partial}{\partial n}\frac{\partial \phi_j}{\partial N}\Big(\frac{\partial \phi_k}{\partial n}\Big)^T + \frac{\partial \phi_k}{\partial n}\Big(\frac{\partial}{\partial n}\frac{\partial \phi_j}{\partial N}\Big)^T\Big]\\
=\;& (1-\sum_{j=1}^{\ell} m_j\mu_j^2)\big[v_k\Big(\frac{\partial \phi_k}{\partial n}\Big)^T + \frac{\partial \phi_k}{\partial n}v_k^T\big] - 2\sum_{j=1}^{\ell} m_j\mu_j\frac{\partial \mu_j}{\partial n}\frac{\partial \phi_k}{\partial n}\Big(\frac{\partial \phi_k}{\partial n}\Big)^T\\
=\;& -2\sum_{j=1}^{\ell} m_j\mu_j\frac{\partial \mu_j}{\partial n}\frac{\partial \phi_k}{\partial n}\Big(\frac{\partial \phi_k}{\partial n}\Big)^T.
\end{aligned} \tag{108}$$

Plugging Eqs. (104), (105), and (108) into Eq. (106), we obtain on O_x that

$$\begin{aligned}
&\frac{\partial \phi_k'}{\partial \nu}\Big(\frac{\partial \phi_k}{\partial n}\Big)^T + \frac{\partial \phi_k}{\partial n}\Big(\frac{\partial \phi_k'}{\partial \nu}\Big)^T - \sum_{j=1}^{\ell} m_j\mu_j\Big[\frac{\partial \phi_j'}{\partial \nu}\Big(\frac{\partial \phi_k}{\partial n}\Big)^T + \frac{\partial \phi_k}{\partial n}\Big(\frac{\partial \phi_j'}{\partial \nu}\Big)^T\Big]\\
&+\Big(p_k' - \sum_{j=1}^{\ell} m_j\mu_j p_j'\Big)\Big[n\Big(\frac{\partial \phi_k}{\partial n}\Big)^T + \frac{\partial \phi_k}{\partial n}n^T\Big]\\
=\;& 2V_n\sum_{j=1}^{\ell} m_j\mu_j\frac{\partial \mu_j}{\partial n}\frac{\partial \phi_k}{\partial n}\Big(\frac{\partial \phi_k}{\partial n}\Big)^T.
\end{aligned} \tag{109}$$

On O_x, set

$$d_k := \sum_{j=1}^{\ell} m_j\mu_j(\mu_j\frac{\partial \phi_k'}{\partial \nu} - \frac{\partial \phi_j'}{\partial \nu}). \tag{110}$$

The main part of the rest of the argument consists in deriving the main term of the asymptotic expansion of d_k at x, in terms of the powers of $\frac{1}{\varepsilon}$. The first step will be to establish that $d_k(x) = O(\frac{1}{\varepsilon})$ and, in the second step, we will compute precisely the coefficient d^1 defined as

$$d_k(x) = \frac{d^1}{\varepsilon} + O(1),$$

where the coefficient d^1 will depend on the parameters involved in the special variation V_n. Once this is performed, we will resume the contradiction argument using the information contained in d^1.

To prepare these computations, we first rewrite Eq. (109) using again Eq. (105) as

$$\begin{aligned}
&d_k\Big(\frac{\partial \phi_k}{\partial n}\Big)^T + \frac{\partial \phi_k}{\partial n}d_k^T + \Big(p_k' - \sum_{j=1}^{\ell} m_j\mu_j p_j'\Big)\Big[n\Big(\frac{\partial \phi_k}{\partial n}\Big)^T + \frac{\partial \phi_k}{\partial n}n^T\Big]\\
=\;& 2\big[\sum_{j=1}^{\ell} m_j\mu_j\frac{\partial \mu_j}{\partial n}\big]V_n\frac{\partial \phi_k}{\partial n}\Big(\frac{\partial \phi_k}{\partial n}\Big)^T.
\end{aligned} \tag{111}$$

Multiplying Eq. (111) from the left by $(\frac{\partial \phi_k}{\partial n})^T$ and from the right by $\frac{\partial \phi_k}{\partial n}$, we obtain the following scalar equation which will be used to achieve a contradiction.

$$\langle \frac{\partial \phi_k}{\partial n}, d_k \rangle = [\sum_{j=1}^{\ell} m_j \mu_j \frac{\partial \mu_j}{\partial n}] V_n. \tag{112}$$

We now prove the following lemma.

Lemma 5.1. *With the notations above, one has* $d_k(x) = O(\frac{1}{\varepsilon})$.

Proof of Lemma 5.1. Set $\psi := \frac{\partial \phi_k}{\partial n}$ and, for $y \in O_x$,

$$\beta(y) := \sum_{j=1}^{\ell} m_j \mu_j(x)(\mu_j(y) - \mu_j(x)). \tag{113}$$

Then, one has $\beta(y) = O(|x-y|^2)$. More precisely, if we use the parameterization defined in Eq. (46), we obtain

$$\beta(y) = \frac{1}{2}\Big(\frac{\partial \beta}{\partial n}(x)\eta^T K_x \eta + \eta^T H_x \eta\Big) + O(|\eta|^3) := \frac{1}{2}\eta^T F_x \eta + O(|\eta|^3), \tag{114}$$

where H_x denotes the Hessian matrix of β at x. Note that by taking twice tangent derivatives of Eq.(105), we know that H_x is a negative semi-definite matrix.

Consider now the representation formula of d_k as described in Eq. (157). Note that two contributions give rise to the term of order of $O(\frac{1}{\varepsilon})$, namely $b^{(0)}$ and $e^{(\lambda)}$.

The term corresponding to $b^{(0)}$ in that equation is equal to

$$E(\beta V_n \frac{\partial \phi_k}{\partial n})(x). \tag{115}$$

Thanks to the estimate of β in (114), it is clear, by proceeding as in Subsection 4.2, that all the other terms of the representation formula of d_k are indeed of the type $O(\frac{1}{\varepsilon})$. Therefore, one has only to determine the asymptotic expansion of the term given in Eq. (115). According to Lemma 4.2, it amounts now to estimate the five terms A_i, $1 \le i \le 5$, and after elementary or standard computations using systematically Eq. (114) , we obtain

$$A_1(\alpha, \beta\psi)(x) = \frac{1}{4\pi\varepsilon^2}\int_{\mathbb{R}^2} \frac{\alpha_\varepsilon(\eta)}{|\eta|^3}\eta^T F_x \eta \Big(\langle \psi(x), \eta\rangle(\eta - \eta_0) + \langle \eta - \eta_0, \eta\rangle \psi(x)\Big) d\eta + O(1), \tag{116}$$

$$A_2(\alpha, \beta\psi)(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^2} \frac{\alpha_\varepsilon(\eta)}{|\eta|^3}\Big(\psi(x)\eta^T F_x \eta + \langle \psi(x), \eta\rangle F_x \eta\Big) d\eta + O(1), \tag{117}$$

and, for $3 \le j \le 5$, $A_j(\alpha, \beta\psi)(x) = O(1)$.

Let us now treat the terms given by $e^{(\lambda)}$. Note that the presence of these terms reflects the fact that ϕ_j and ϕ_k correspond to different eigenvalues of the Stokes operator. Using Lemma A.1, we define the operator $\Delta^\lambda(\alpha, \psi)$ as follows

$$\Delta^\lambda(\alpha, \psi)(x) := -\frac{\lambda}{8\pi}\int_{\mathbb{R}^2} \frac{\alpha_\varepsilon(\eta)\langle \psi(x), \eta/|\eta|\rangle}{|\eta|}\frac{\eta}{|\eta|} d\eta. \tag{118}$$

It is clear that

$$\Delta^{\lambda}(\alpha,\psi)(x) = O(\frac{1}{\varepsilon}).$$

Therefore, with above notations, we have

$$\begin{aligned}
d_k(x) &= A_1(\alpha,\psi)(x) + A_2(\alpha,\psi)(x) + \sum_{j=1}^{\ell} m_j\mu_j(\mu_j\Delta^{\lambda_k}(\alpha,\psi)(x) - \Delta^{\lambda_j}(\alpha,\mu_j\psi)(x)) + O(1) \\
&= O(\frac{1}{\varepsilon}). \qquad (119)
\end{aligned}$$

This ends the proof of Lemma 5.1.

□

Let us now pursue the proof of Theorem 1.2. Since the value of the right-hand side of Eq. (112) at x is given by the following expression

$$[\sum_{j=1}^{\ell} m_j\mu_j(x)\frac{\partial\mu_j}{\partial n}(x)]\frac{e^{-\bar{r}_0^2}}{\varepsilon^2} = O(\frac{1}{\varepsilon^2}),$$

we conclude that

$$\sum_{j=1}^{\ell} m_j\mu_j(x)\frac{\partial\mu_j}{\partial n}(x) = 0, \qquad \text{i.e.,} \quad \frac{\partial\beta}{\partial n}(x) = 0, \qquad (120)$$

which implies that

$$\langle\frac{\partial\phi_k}{\partial n}(x), d_k(x)\rangle = 0, \qquad (121)$$

In order to get additional information from Eq. (121), we compute explicitly the numerical coefficient in front of $\frac{1}{\varepsilon}$ in the asymptotic expansion of $d_k(x)$. It is enough to have a closer look at the representation formula of d_k as described in Eq. (157). From Eq. (119), we have

$$d_k(x) = a_1 + a_2 + \rho a_3 + O(1), \qquad (122)$$

where

$$\begin{aligned}
a_1 &:= \frac{1}{4\pi\varepsilon^2}\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)}{|\eta|^3}\eta^T F_x\eta\Big(\langle\psi(x),\eta\rangle(\eta-\eta_0) + \langle\eta-\eta_0,\eta\rangle\psi(x)\Big)d\eta, \qquad (123)\\
a_2 &:= -\frac{1}{4\pi}\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)}{|\eta|^3}\Big(\psi(x)\eta^T F_x\eta + \langle\psi(x),\eta\rangle F_x\eta\Big)d\eta, \qquad (124)\\
a_3 &:= -\frac{1}{8\pi}\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)\langle\psi(x),\eta/|\eta|\rangle}{|\eta|}\frac{\eta}{|\eta|}d\eta, \qquad (125)\\
\rho &:= \sum_{j=1}^{\ell} m_j\mu_j^2(\lambda_k-\lambda_j). \qquad (126)
\end{aligned}$$

Notice that $\rho > 0$ since $\lambda_k > \lambda_j$ for $1 \le j \le l$ and at least one of the integers m_j is positive.

We now compute the coefficients a_i, $1 \le i \le 3$. For $\theta_0 \in S^1$, we set

$$R_{\theta_0} := \begin{pmatrix}\cos\theta_0 & -\sin\theta_0\\ \sin\theta_0 & \cos\theta_0\end{pmatrix}, \qquad F_{\theta_0} := R_{\theta_0}^T F_x R_{\theta_0} := (F_{\theta_0}^{ij})_{i,j=1,2}.$$

For $i = 6, \ldots, 10$, we define the functions M_i as follows.

$$
\begin{aligned}
M_6(z) &:= \int_0^{\infty} e^{-r^2} r^2 dr \int_0^{2\pi} e^{2rz\cos\theta} d\theta, &(127)\\
M_7(z) &:= \int_0^{\infty} e^{-r^2} r^2 dr \int_0^{2\pi} e^{2rz\cos\theta} \cos^2\theta d\theta, &(128)\\
M_8(z) &:= \int_0^{\infty} e^{-r^2} r^2 dr \int_0^{2\pi} e^{2rz\cos\theta} \cos^4\theta d\theta, &(129)\\
M_9(z) &:= \int_0^{\infty} e^{-r^2} r dr \int_0^{2\pi} e^{2rz\cos\theta} \cos\theta d\theta, &(130)\\
M_{10}(z) &:= \int_0^{\infty} e^{-r^2} r dr \int_0^{2\pi} e^{2rz\cos\theta} \cos^3\theta d\theta. &(131)
\end{aligned}
$$

The expressions of a_1, a_2, and a_3 are summarized in the following lemma whose proof is postponed in Section B.6 of Appendix.

Lemma 5.2. *We have*

$$
\begin{aligned}
a_1 &= \frac{1}{4\pi}\frac{e^{-\bar r_0^2}}{\varepsilon}\Big(\\
&\quad \big[2F_{\theta_0}^{22}M_6(\bar r_0) + (2F_{\theta_0}^{11} - 3F_{\theta_0}^{22})M_7(\bar r_0) - (F_{\theta_0}^{11} - F_{\theta_0}^{22})(M_8(\bar r_0) + M_{10}(\bar r_0)) - F_{\theta_0}^{22}M_9(\bar r_0)\big]\psi(x)\\
&- \big[\big(F_{\theta_0}^{22}M_9(\bar r_0) + (F_{\theta_0}^{11} - F_{\theta_0}^{22})(M_{10}(\bar r_0) - 2M_8(\bar r_0)) + (F_{\theta_0}^{11} - 3F_{\theta_0}^{22})M_7(\bar r_0)\\
&\quad + F_{\theta_0}^{22}M_6(\bar r_0)\big)\langle\psi(x), \bar\eta_0\rangle + 2F_{\theta_0}^{12}(M_9(\bar r_0) - M_{10}(\bar r_0))\langle\psi(x), \bar\eta_0^{\perp}\rangle\big]\bar\eta_0\\
&- 2F_{\theta_0}^{12}(M_7(\bar r_0) - M_8(\bar r_0))\psi(x)^{\perp} + 4F_{\theta_0}^{12}(M_7(\bar r_0) - M_8(\bar r_0))\langle\psi(x), \bar\eta_0\rangle\bar\eta_0^{\perp}\Big), \qquad (132)\\
a_2 &= -\frac{1}{4\pi}\frac{e^{-\bar r_0^2}}{\varepsilon}\Big\{\Big[F_{\theta_0}^{22}M_5^{A_1}(\bar r_0) + (F_{\theta_0}^{11} - F_{\theta_0}^{22})M_1^{A_1}(\bar r_0)\Big]\psi(x)\\
&\quad + F_x\Big((M_5^{A_1}(\bar r_0) - M_1^{A_1}(\bar r_0))\psi(x) + (2M_1^{A_1}(\bar r_0) - M_5^{A_1}(\bar r_0))\langle\psi(x), \bar\eta_0\rangle\bar\eta_0\Big)\Big\}, \qquad (133)\\
a_3 &= -\frac{1}{8\pi}\frac{e^{-\bar r_0^2}}{\varepsilon}\Big[(M_5^{A_1}(\bar r_0) - M_1^{A_1}(\bar r_0))\psi(x) + (2M_1^{A_1}(\bar r_0) - M_5^{A_1}(\bar r_0))\langle\psi(x), \bar\eta_0\rangle\bar\eta_0\Big]. \qquad (134)
\end{aligned}
$$

Let us now finish the proof of Theorem 1.2. We choose $\bar\eta_0 \perp \psi(x)$. Without loss of generality, we also assume that $\bar\eta_0 = (1,0)^T$ and $\psi(x)/|\psi(x)| = (0,1)^T$. Recall that we have chosen x such that $\psi(x) \neq 0$. Then, we deduce from Eq. (122) and Lemma 5.2 that

$$
d_k(x) = \frac{e^{-\bar r_0^2}}{4\pi\varepsilon}\Big(\alpha_1\psi(x) + \alpha_2\psi(x)^{\perp} + \alpha_3 F_x\psi(x)\Big), \qquad (135)
$$

where,

$$
\begin{aligned}
\alpha_1 &:= 2F_x^{22}M_6(\bar r_0) + (2F_x^{11} - 3F_x^{22})M_7(\bar r_0) - (F_x^{11} - F_x^{22})(M_8(\bar r_0) + M_{10}(\bar r_0)) - F_x^{22}M_9(\bar r_0)\\
&\quad - \big((F_x^{22} + \frac{\rho}{2})M_5^{A_1}(\bar r_0) + (F_x^{11} - F_x^{22} - \frac{\rho}{2})M_1^{A_1}(\bar r_0)\big),\\
\alpha_2 &:= 2F_x^{12}(M_9(\bar r_0) + M_8(\bar r_0) - M_{10}(\bar r_0) - M_7(\bar r_0)),\\
\alpha_3 &:= -\big(M_5^{A_1}(\bar r_0) - M_1^{A_1}(\bar r_0)\big).
\end{aligned}
$$

Plugging Eq. (135) into Eq. (121), we obtain

$$\alpha_1 + F_x^{22}\alpha_3 = 0. \tag{136}$$

The final contradiction will be obtained by showing that the two non zero entire functions of $\bar{r}_0$ given by α_1 and α_2 cannot satisfy Eq. ((136)). For that purpose, we need to get more explicit expressions. Eq. ((136)) writes

$$\begin{aligned}
0 &= 2F_x^{22}M_6(\bar{r}_0) + (2F_x^{11} - 3F_x^{22})M_7(\bar{r}_0) - (F_x^{11} - F_x^{22})(M_8(\bar{r}_0) + M_{10}(\bar{r}_0)) - F_x^{22}M_9(\bar{r}_0) \\
&\quad -\big((2F_x^{22} + \frac{\rho}{2})M_5^{A_1}(\bar{r}_0) + (F_x^{11} - 2F_x^{22} - \frac{\rho}{2})M_1^{A_1}(\bar{r}_0)\big) \\
&= F_x^{11}\big(2M_7(\bar{r}_0) - M_8(\bar{r}_0) - M_{10}(\bar{r}_0) - M_1^{A_1}(\bar{r}_0)\big) \\
&+ F_x^{22}\big(2M_6(\bar{r}_0) - 3M_7(\bar{r}_0) + M_8(\bar{r}_0) + M_{10}(\bar{r}_0) - M_9(\bar{r}_0) - 2M_5^{A_1}(\bar{r}_0) + 2M_1^{A_1}(\bar{r}_0)\big) \\
&+ \frac{\rho}{2}\big(M_1^{A_1}(\bar{r}_0) - M_5^{A_1}(\bar{r}_0)\big).
\end{aligned} \tag{137}$$

Since the right-hand side of Eq. (137) is an entire function, we deduce that all the coefficients in its power series expansion are equal to zero. We need the following lemma whose proof is deferred in Section B.7 of Appendix.

Lemma 5.3. *The entire functions involved in Eq.* (137) *have the following power series expansions*

$$\begin{aligned}
&2M_7(z) - M_8(z) - M_{10}(z) - M_1^{A_1}(z) \\
= &\sum_{p=0}^{\infty} \frac{2^{2p+1}}{(2p)!}\Gamma(p+\frac{1}{2})I_{2p+2}\frac{2p^2+4p-\frac{3}{2}}{2p+3}z^{2p} - \sum_{p=0}^{\infty}\frac{2^{2p+2}}{(2p+1)!}\Gamma(p+\frac{3}{2})I_{2p+4}\ z^{2p+1}, \\
&2M_6(z) - 3M_7(z) + M_8(z) + M_{10}(z) - M_9(z) - 2M_5^{A_1}(z) + 2M_1^{A_1}(z) \\
= &\sum_{p=0}^{\infty} \frac{2^{2p+1}}{(2p)!}\Gamma(p+\frac{1}{2})I_{2p}\frac{6p^2+16p+\frac{7}{2}}{(2p+2)(2p+4)}z^{2p} - \sum_{p=0}^{\infty}\frac{2^{2p+2}}{(2p+1)!}\Gamma(p+\frac{3}{2})I_{2p+2}\frac{1}{2p+4}z^{2p+1}.
\end{aligned}$$

Using Lemma 5.3 and considering the coefficients of the odd powers of z in the power series expansion of the right-hand side of Eq. (137), we deduce that

$$F_x^{11}(2p+3) + F_x^{22} = 0, \qquad \text{for all } p \in \mathbb{N}. \tag{138}$$

This implies that

$$F_x^{11} = F_x^{22} = 0. \tag{139}$$

Therefore, using Eq. (137), we have

$$\frac{\rho}{2}\big(M_1^{A_1}(\bar{r}_0) - M_5^{A_1}(\bar{r}_0)\big) = 0. \tag{140}$$

Recall that $M_5^{A_1}(z) - M_1^{A_1}(z)$ is equal to $M_2^{A_1}(z)$, then not identically equal to zero. Then Eq. (140) yields that

$$\rho = 0,$$

which is in contradiction with the fact that

$$\rho = \sum_{j=1}^{\ell} m_j\mu_j^2(\lambda_k - \lambda_j) > 0.$$

Theorem 1.2 is finally proved.

A Layer potentials and representation formulas

Most of the material presented here is borrowed from [3], [16] and [13]. Let λ be a non negative real number. Consider ϕ and p satisfying the eigenvalue problem associated to the following Stokes system

$$\begin{cases} (\Delta+\lambda)\phi - \nabla p &= h \text{ in } \Omega \\ \text{div } \phi &= 0 \text{ in } \Omega \\ \phi &= g \text{ on } \partial\Omega \\ \displaystyle\int_\Omega p &= 0, \end{cases}$$

under the compatibility condition

$$\int_\Gamma \phi \cdot n \, ds = 0, \tag{141}$$

where n is the outward unit normal to $\partial\Omega$. Recall that, for such a pair of fields, the conormal derivative denoted by $\dfrac{\partial \phi}{\partial \nu}$ was defined in (11).

A.1 Layer potentials

We denote by ∂_i the operator $\frac{\partial}{\partial x_i}$ and by $\sqrt{-\lambda}$ the complex number $i\sqrt{\lambda}$.

Fundamental tensors We define the fundamental tensors $\Gamma^\lambda = (\Gamma^\lambda_{ij})^3_{i,j=1}$ and $F = (F_i)^3_{i=1}$ as

$$\begin{cases} \Gamma^\lambda_{i,j} &= -\dfrac{\delta_{ij} e^{\sqrt{-\lambda}|x|}}{4\pi \mid x \mid} - \dfrac{1}{4\pi\lambda}\partial_i\partial_j \dfrac{e^{\sqrt{-\lambda}|x|}-1}{\mid x \mid}, \\ F_i(x) &= -\dfrac{x_i}{4\pi \mid x \mid^3}. \end{cases} \tag{142}$$

In the sense of distributions, straightforward computations of the fundamental solution of Helmholtz operator $\Delta + \lambda$ allow to get

$$(\Delta+\lambda)\Gamma^\lambda_{ij} - \partial_j F_i = \delta_{ij}\delta(x), \text{ and } \partial_i \Gamma^\lambda_{ij} = 0,$$

where we use $\delta(x)$ to denote the delta distribution based at $x \in \mathbb{R}^3$. The tensor Γ^0, which corresponds to the standard Stokes system, is defined as

$$\Gamma^0_{ij}(x) := -\frac{1}{8\pi}\left(\frac{\delta_{ij}}{\mid x \mid} + \frac{x_i x_j}{\mid x \mid}\right),$$

and one has, uniformly on compact subsets of $\mathbb{R}^3$,

$$\Gamma^\lambda_{ij}(x) = \Gamma^0_{ij}(x) - \frac{\delta_{ij}\sqrt{-\lambda}}{6\pi} + O(\lambda). \tag{143}$$

We also denote

$$\Delta^\lambda_{ij}(x) := \Gamma^\lambda_{ij}(x) - \Gamma^0_{ij}(x). \tag{144}$$

Note that

$$\Delta^\lambda_{ij}(x) = -\frac{\delta_{ij}\sqrt{-\lambda}}{6\pi} - \frac{\lambda}{32\pi}\Delta_{ij}(x) + O(|x|^2). \tag{145}$$

with $\Delta_{ij}(\cdot)$ defined by

$$\Delta_{ij}(x) := 3\delta_{ij}|x| - \frac{x_i x_j}{|x|}. \tag{146}$$

After simple computations, one gets the following useful result.

Lemma A.1. *We have*

$$\frac{\partial^2 \Delta^\lambda(x-y)}{\partial N(x)\partial N(y)} = -\frac{\lambda}{8\pi}\Big[\frac{\langle n_x, n_y\rangle}{|x-y|}\frac{(x-y)(x-y)^T}{|x-y|^2} + \frac{n_y n_x^T}{|x-y|}\Big] + T_\lambda, \tag{147}$$

where T_λ is a kernel of class C^1. Here, the definition of $\frac{\partial}{\partial N}$ given in (10)*.*

Single and double boundary layers In the sequel, we use the Einstein convention for summation signs, i.e., we omit them for indices appearing twice. Let $\phi = (\phi^1, \phi^2, \phi^3) \in L^2(\partial\Omega)^3$. The *single-layer potential* pair $(S_\Omega^\lambda, \mathcal{F}_\Omega)$ with density ϕ is defined, for $x \in \Omega$, as

$$\begin{cases} S_\Omega^\lambda[\phi]_i(x) &= \displaystyle\int_{\partial\Omega} \Gamma_{ij}^\lambda(x-y)\phi^j(y)\, d\sigma_y, \quad 1 \le i \le 3, \\ \mathcal{F}_\Omega[\phi](x) &= \displaystyle\int_{\partial\Omega} F_j(x-y)\phi^j(y)\, d\sigma_y, \end{cases} \tag{148}$$

while the *double hydrodynamic potential* pair $(D_\Omega^\lambda, \mathcal{V}_\Omega)$ with density ϕ is defined by

$$\begin{cases} D_\Omega^\lambda[\phi]_i(x) &= \displaystyle\int_{\partial\Omega} \left(\frac{\partial \Gamma_{ij}^\lambda}{\partial N(y)}(x-y) + F_i(x-y)n_j(y)\right)\phi_j(y)\, d\sigma_y, \quad 1 \le i \le 3, \\ \mathcal{V}_\Omega[\phi](x) &= -2\displaystyle\int_{\partial\Omega} \frac{\partial F_j}{\partial x_l}(x-y)\phi^j(y)\, n_l(y)\, d\sigma_y. \end{cases} \tag{149}$$

Recall that

$$\frac{\partial \Gamma_{ij}^\lambda}{\partial N(y)}(x-y) = \left(\frac{\partial \Gamma_{ij}^\lambda(x-y)}{\partial y_l} + \frac{\partial \Gamma_{il}^\lambda(x-y)}{\partial y_j}\right) n_l(y),$$

according to the definition of $\frac{\partial}{\partial N}$ given in (10).

Some background results about the layer potential representations From [3], we quote the following integral equations satisfied by ϕ^λ and the associated pressure p^λ. First, we have the following representation formulas,

$$\begin{cases} \phi^\lambda(x) &= -S_\Omega^\lambda[\dfrac{\partial \phi^\lambda}{\partial \nu}](x) + D_\Omega^\lambda[\phi^\lambda](x), \qquad x \in \Omega, \\ p^\lambda(x) &= -\mathcal{F}_\Omega[\dfrac{\partial \phi^\lambda}{\partial \nu}](x) + \mathcal{V}_\Omega[\phi^\lambda](x), \qquad x \in \Omega. \end{cases} \tag{150}$$

Applying the trace stress operators and taking into account the single layer potential as well as the jump relations for the double layer potential across the boundary, we get for ϕ belonging to $L^2(\partial\Omega)^3$ the following relations,

$$\begin{cases} D_\Omega^\lambda[\phi](x) &= (\frac{1}{2}I + K_\Omega^\lambda)[\phi](x), \qquad \text{a.e. on } \partial\Omega, \\ \dfrac{\partial}{\partial \nu} S_\Omega^\lambda[\phi](x) &= (-\frac{1}{2}I + (K_\Omega^\lambda)^*)[\phi](x), \qquad \text{a.e. on } \partial\Omega, \end{cases} \tag{151}$$

where the kernel $K_\Omega^\lambda[\phi]$ is defined a.e. on $\partial\Omega$ by its components,

$$K_\Omega^\lambda[\phi]_i(x) := \text{p.v.} \int_{\partial\Omega} \frac{\partial \Gamma_{ij}^\lambda}{\partial N(y)}(x-y)\phi^j(y)\ d\sigma_y + \text{p.v.} \int_{\partial\Omega} F_i(x-y)\phi^j(y) n_j(y)\ d\sigma_y. \tag{152}$$

Here, the notation " p.v." indicates the Cauchy principal value when the integrand is singular at x, more precisely

$$\text{p.v.} \int_\Gamma \ldots = \lim_{\varepsilon \to 0} \int_{\Omega \setminus B(x,\varepsilon)} \ldots$$

where $B(x,\varepsilon)$ is the ball centered at x of radius ε. The adjoint operator ${K_\Omega^\lambda}^*$ of K_Ω^λ is defined similarly a.e. on $\partial\Omega$ by its components

$${K_\Omega^\lambda}^*[\phi]_i(x) = \text{p.v.} \int_{\partial\Omega} \frac{\partial \Gamma_{ij}^\lambda}{\partial N(x)}(x-y)\phi^j(y)\ d\sigma_y - \text{p.v.} \int_{\partial\Omega} F_i(x-y)\phi^j(y) n_j(x)\ d\sigma_y, \tag{153}$$

for all functions ϕ belonging to $L^2(\partial\Omega)^3$. Let us recall that in the case of the standard Stokes system ($\lambda = 0$), we have

$$K_\Omega^0[\phi](x) = -\frac{3}{4\pi} \int_{\partial\Omega} (x-y) \frac{\langle x-y, n(y)\rangle\ \langle x-y, \phi(y)\rangle}{\mid x-y \mid^5}\ d\sigma_y. \tag{154}$$

An important fact is that the single and double layer potentials S_Ω^λ and D_Ω^λ are compact perturbations of the single and double layer potentials corresponding to the standard Stokes problem.

From the C^3 regularity of the boundary Γ, it comes that

$$|\langle x-y, \phi(y)\rangle| \leq C|x-y|^2, \tag{155}$$

hence, we deduce (cf. [16]) that the mapping $K_\Omega^\lambda[\phi] : C^\alpha(\partial\Omega) \mapsto C^{\alpha+1}(\partial\Omega)$ is in fact continuous. That shows that $K_\Omega^\lambda[\phi]$ has a weakly singular kernel and then that it is a compact operator on $L^2(\partial\Omega)^3$. According to (143), the operators $S_\Omega^\lambda - S_\Omega^0$ and $D_\Omega^\lambda - D_\Omega^0$ are smoothing operators.

Thanks to the integral representations provided in the preceding paragraph, we can use the trace and the stress operators to deduce the second boundary integral equation satisfied by the conormal derivative. Indeed, by using the same arguments of jump relations and the integral equations satisfied by ϕ^λ, we get

$$\begin{aligned} \Big(\frac{1}{2}I + (K_\Omega^\lambda)^*\Big)\Big[\frac{\partial\phi}{\partial\nu}\Big]_i(x) &= \Big[\frac{\partial D_\Omega^\lambda[\phi]}{\partial\nu}(x)\Big]_i \\ &= \text{p.v.} \int_{\partial\Omega} \frac{\partial^2 \Gamma_{ij}^\lambda(x-y)}{\partial N(x)\partial N(y)}\phi^j(y)\ d\sigma_y \\ . \qquad\qquad &= \text{p.v.} \int_{\partial\Omega} \frac{\partial^2 \Gamma_{ij}^0(x-y)}{\partial N(x)\partial N(y)}\phi^j(y)\ d\sigma_y \\ &+ \int_{\partial\Omega} \frac{\partial^2 \Delta_{ij}^\lambda(x-y)}{\partial N(x)\partial N(y)}\phi^j(y)\ d\sigma_y. \end{aligned} \tag{156}$$

We cannot deduce directly the Neumann data (conormal derivative) since the operator $\Big(\frac{1}{2}I + (K_\Omega^\lambda)^*\Big)$ is not invertible. We give, in the next paragraph, the recipes to get the solution of the system by using the projector methods.

A.2 Weakly singular integral operators of exponent $\alpha > 0$

The rest of the paragraph follows Section 7.2 of [13]. Recall that the conormal derivative is solution of $Tx = b$ where $T = I + 2(K_{\Omega}^{\lambda})^*$ is a Fredholm operator with a nontrivial kernel. We use $\mathcal{R}(T)$ to denote its closed image and $\mathcal{N}(T)$ its finite dimensional null space. We can therefore find projections P and Q of finite rank such that there exists a unique operator S satisfying $TS = I - Q$ and $PS = 0$. Hence, the equation $Tx = f$ has a solution if and only if $Qf = 0$. In our context, we have $T = I - C$ with $C = -2(K_{\Omega}^{\lambda})^*$, which is a compact operator. From the projector theory recalled above, we can find S such that the equation $Tx = b$ has a solution $x = Sf$ when $Q = 0$. To proceed, we need some regularity assumptions on the operator T. For that purpose, we recall the following definition [13, Definition 7.1.1, p117].

Definition A.1. Let A be an open set in $\mathbb{R}^3$. A function $K(x, y)$ defined for $x \neq y$ in $A \times A$ is a kernel of class $C_*^r(\alpha)$ in A (r non negative integer, and $\alpha > 0$) if it is C^r for $x \neq y$ and for any $\delta > 0$ and $|i| + |j| + |k| \leq r$, one has

$$\partial_x^i \partial_y^j (\partial_x + \partial_y)^k K(x, y) = O(1 + |x - y|^{\alpha - m - |i| - |j| - \delta}),$$

uniformly for $x \neq y$ in compact subsets of A. If $\alpha > m + |i| + |j|$, we require $\partial_x^i \partial_y^j (\partial_x + \partial_y)^k K(x, y)$ to extend continuously to $\{x = y\}$.

Assume now that T is an integral operator with kernel $C_*^r(\alpha)$ for some $\alpha > 0$. We may choose the projections P and Q to be integral operators with C^r kernels so that, if S is the operator such that

$$\begin{aligned} TS &= I, \\ S &= I + R, \end{aligned}$$

then the resolvent kernel R is an integral operator with $C_*^r(\alpha)$ kernel. Then $R - (C + C^2 + \cdots + C^j)$ has kernel of class $C_*^r((j+1)\alpha)$ for each $j \geq 1$. Hence, for N sufficiently large, the operator $R - \sum_{k=1}^N C^j$ has a smooth kernel of class C^r. In summary, one has the following result.

Theorem A.2 (Theorem 7.2.3, page 125 in [13])**.** *We suppose Ω regular of class C^{r+1}, for some $r > 0$. If K a kernel of class $\mathcal{K}(\alpha, r)$, then we may choose the kernels P and Q of the projections to be of class C^r and such that the resolvent kernel R belongs to $\mathcal{K}(\alpha, r)$. Furthermore N can be chosen sufficiently large so that the kernel of $R - (K + K^2 + \dots K^N)$ is a C^r kernel.*

We return to the study of Eq. (156). We introduce the vectors $b^{(0)}(x) = (b_i^{(0)}(x))_i$ and $e^{(\lambda)}(x) = (e_i^{(\lambda)}(x))$ where

$$b_i^{(0)}(x) = \int_{\partial\Omega} \frac{\partial^2 \Gamma_{ij}^0(x - y)}{\partial N(x) \partial N(y)} \phi_j^{\lambda}(y) \, d\sigma_y,$$

and where

$$e_i^{(\lambda)}(x) = \int_{\partial\Omega} \frac{\partial^2 \Delta_{ij}^{\lambda}(x - y)}{\partial N(x) \partial N(y)} \phi_j^{\lambda}(y) \, d\sigma_y.$$

Hence it comes that

$$\left[\frac{\partial \phi^\lambda}{\partial \nu}\right] = b^{(0)} + (\sum_{k=1}^N K^k) b^{(0)} + (\sum_{k=0}^N K^k) e^{(\lambda)} + (R - \sum_{k=1}^N K^k)(b^{(0)} + e^{(\lambda)}). \tag{157}$$

Note that $(I+R)e^{(\lambda)}$ is actually a weakly singular operator acting on the Dirichlet data ϕ^λ of class $C^3_*(1)$.

In our study, we will apply (157) to the case $K = -2K^\lambda_\Omega$. Moreover, with our specific choice of Dirichlet data, we will show that $N = 1$ is sufficient in our context and that all the other terms in the sum will be also absorbed by the remainder.

A.3 Composition of weakly singular kernels

For applications to our result on generic perturbation of the boundary, we need to give an explicit representation of the conormal derivative or at least, of its principal and subprincipal parts as it is treated in the case of the Laplacian (for more details in the Laplacian case, one can refer to [28]). Some preliminaries are required in order to study the resolvent kernels and their regularity. We begin by recalling some results due to D. Henry (cf. [13]). It concerns kernels $K(x,y)$ of the form

$$K(x,y) :=\mid x-y\mid^{\alpha-2} Q\Big(x,y,\frac{x-y}{\mid x-y\mid}\Big) \tag{158}$$

where $Q(x,y,s)$ is of class C^r $(r>0)$ on $\mathbb{R}^2$. We will denote by $\mathcal{K}(r,\alpha)$ the set of such kernels, which is a subclass of $C^r_*(\alpha)$.

These kernels are in fact smoothing operators and we recall the main result of [13].

Theorem A.3 (Theorem 7.1.2 in [13])**.** *Given a kernel K belonging to the class $\mathcal{K}(\alpha,r)$, $\alpha, r>0$, we denote by $\tilde{K}$ the corresponding integral operator*

$$\tilde{K}u(x) = \int_{\mathbb{R}^2} K(x,y)u(y)\ d\sigma_y.$$

Then we have

- $\tilde{K}:\ W^{j,p} \mapsto W^{k,p}$ *is a compact operator if* $j-\frac{m}{p}\alpha > k - \frac{m}{q}$*;*
- $\tilde{K}:\ C^{j,\sigma} \mapsto C^{k,\tau}$ *is a compact operator if* $j+\sigma+\alpha > k+\tau,\quad k<r$ *and* $k<j+\alpha$*.*

As it was mentioned in [13], the above result can be summarized by the fact the operator $\tilde{K}$ is smoothing of order α. By analogy with the pseudo-differential operator theory, such an operator is said of order α. We will also need a result on the composition of certain weakly singular operators. For that purpose, we first define the composition of corresponding kernels as follows.

Definition A.2. Let K and L be kernels belonging to $\mathcal{K}(\alpha,r)$ and $\mathcal{K}(\beta,r)$ respectively with $\alpha,\beta,r>0$. Then $K\circ L$ is defined by

$$K\circ L(x,y) = \int_{\mathbb{R}^2} K(x,z)L(x,z)\ dz \tag{159}$$

Then, one has the following property.

Theorem A.4 (Theorem 7.1.3, p. 119 in [13])**.** *Let K and L be kernels belonging to $\mathcal{K}(\alpha,r)$ and $\mathcal{K}(\beta,r)$ respectively, with $\alpha,\beta,r>0$. Then $K\circ L$ is kernel of compact support belonging to $\mathcal{K}(\alpha+\beta,r)$. Furthermore, if $\alpha+\beta>r+2$, then $K\circ L$ is of class C^r.*

To these kernels, are associated integral operators $u \mapsto \int_{\partial\Omega} K(x,y)\ dS(y)$ where dS is the surface area measure on $\partial\Omega$. In a first step, we begin to work in $\mathbb{R}^2$. To transfer all the results to $\partial\Omega$ (in particular, those provided above), one has to follow the classical steps: construct a partition of unity and then define the integral by a local change of variables as it is precisely performed in [13, Section 7.1].

B Proofs of computational lemmas

B.1 Proof of Lemma 2.9

From Eq. (28), we get the following system

$$\begin{aligned} -(\Delta+\lambda)\phi_i'(u)+\nabla p_i'(u) &= \lambda_i'(u)\phi_i(u) \quad \text{in } \Omega, && (160)\\ \text{div } \phi_i'(u) &= 0 \quad \text{in } \Omega, && (161)\\ \phi_i'(u)+(u\cdot n)\frac{\partial \phi_i(u)}{\partial n} &= 0 \quad \text{on } \partial\Omega, && (162)\\ p_i'(u)+\text{div } (up_i(u)) &\in L^2_0(\Omega). && (163) \end{aligned}$$

Multiplying (160) by $\phi_k(u)$ with $1\le k\le m$, integrating over Ω and using Corollary 2.4, we have

$$\lambda_i'(u)\delta_{ik} = -\int_\Omega \phi_k(u)[(\Delta+\lambda)\phi_i'(u)-\nabla p_i'(u)] = \int_{\partial\Omega}\phi_i'(u)\frac{\partial\phi_k(u)}{\partial\nu}.$$

Hence, it comes that

$$\lambda_i'(v)\delta_{ik} = -\int_{\partial\Omega}(u\cdot n)\frac{\partial\phi_i(u)}{\partial n}\cdot\frac{\partial\phi_k(u)}{\partial\nu}. \tag{164}$$

Moreover, by Lemma (2.5), we have

$$\begin{aligned} \frac{\partial\phi_i(u)}{\partial n}\cdot\frac{\partial\phi_k(u)}{\partial\nu} &= \frac{\partial\phi_i(u)}{\partial n}\cdot(\frac{\partial\phi_k(u)}{\partial n}+\nabla^T\phi_k(u)n-p_k(u)n)\\ = \frac{\partial\phi_i(u)}{\partial n}\cdot\frac{\partial\phi_k(u)}{\partial n}&+\frac{\partial\phi_i(u)}{\partial n}\cdot(\frac{\partial\phi_k(u)}{\partial n}n^T)^T n = \frac{\partial\phi_i(u)}{\partial n}\cdot\frac{\partial\phi_k(u)}{\partial n}. \end{aligned}$$

Therefore, we immediately get Eq. (33).

B.2 Proof of Lemma 4.2

Lemma 4.2 is derived from [14, Lemma 2.2.3 Formula (2.2.34) and Lemma 2.3.1] by straightforward computations. For the reader's convenience, we first summarize these results in the following lemma and then give the proof of Lemma 4.2.

Lemma B.1. *Let $\partial\Omega$ be of class C^1 and $u=(u^\ell)_{\ell=1,2,3}$ be a Hölder continuously differentiable function. Then the operator E defined in (58) can be expressed as follows*

$$\begin{aligned} Eu(x) &= -\frac{1}{4\pi}(n_x\times\nabla_x)\cdot\int_{\partial\Omega}\frac{1}{|x-y|}(n_y\times\nabla_y)u(y)d\sigma(y) && (165)\\ &\quad -\frac{1}{2\pi}\mathcal{M}(\partial_x,n_x)\int_{\partial\Omega}\frac{(x-y)(x-y)^T}{|x-y|^3}\mathcal{M}(\partial_y,n_y)u(y)d\sigma(y)) && (166)\\ &\quad +\frac{1}{4\pi}\Big(\sum_{l,k=1}^{3}m_{lk}(\partial_x,n_x)\int_{\partial\Omega}\frac{1}{|x-y|}(m_{kj}(\partial_y,n_y)u^\ell)(y)d\sigma(y)\Big)_{j=1,2,3}, && (167) \end{aligned}$$

where the ℓ^{th} column of the matrix $(n_y \times \nabla_y)u(y)$ is given by the vector $n_y \times \nabla_y u^\ell(y)$, and the Günter derivatives $\mathcal{M}$ is given by the following matrix of differential operators

$$\mathcal{M}(\partial_x, n_x) = (m_{jk}(\partial_x, n_x))_{j,k=1,2,3} := (n_{x,k}\partial_{x_j} - n_{x,j}\partial_{x_k})_{j,k=1,2,3},$$

with $n_x = (n_{x,j})_{j=1,2,3}$.

Corollary B.2. *Under the assumptions of Lemma B.1, we have*

$$\begin{aligned}
4\pi E u(x) &= \text{p.v.}\int_{\partial\Omega} \frac{\langle n_x, n_y\rangle}{|x-y|^3}\Big(\nabla u(y) + \nabla^T u(y)\Big)(x-y)d\sigma_y && (168)\\
&+ \text{p.v.}\int_{\partial\Omega} \frac{\langle n_x, (\nabla u(y) - \nabla^T u(y))(x-y)\rangle}{|x-y|^3} n_y d\sigma_y && (169)\\
&- \int_{\partial\Omega} \frac{\langle x-y, n_y\rangle}{|x-y|^3}\Big(\nabla u(y) + \nabla^T u(y)\Big)n_x d\sigma_y && (170)\\
&+ \int_{\partial\Omega} \frac{\langle x-y, n_y\rangle}{|x-y|^3}\Big(I - 3\frac{(x-y)(x-y)^T}{|x-y|^2}\Big)\mathcal{M}(\partial_y, n_y)u(y)d\sigma_y. && (171)
\end{aligned}$$

Proof of Corollary B.2. For (165), we get

$$\begin{aligned}
&(n_x \times \nabla_x)\cdot \int_{\partial\Omega} \frac{1}{|x-y|}(n_y \times \nabla_y u^\ell(y))d\sigma(y)\\
&= \text{p.v.}\int_{\partial\Omega} (n_x \times \nabla_x \frac{1}{|x-y|})\cdot (n_y \times \nabla_y u^\ell(y))d\sigma(y).
\end{aligned}$$

For $x \neq y$, one has

$$\begin{aligned}
&(n_x \times \nabla_x \frac{1}{|x-y|})\cdot (n_y \times \nabla_y u^\ell(y))\\
&= (n_x^T n_y)(\nabla_x \frac{1}{|x-y|}\nabla_y^T u^\ell(y)) - (\nabla_y u^\ell(y) n_x)(\nabla_x \frac{1}{|x-y|} n_y)\\
&= -\frac{n_x^T n_y}{|x-y|^3}\nabla_y u^\ell(y)(x-y) + \frac{(x-y)^T n_y}{|x-y|^3}\nabla_y u^\ell(y) n_x.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&(n_x \times \nabla_x)\cdot \int_{\partial\Omega} \frac{1}{|x-y|}(n_y \times \nabla_y)u(y)d\sigma(y)\\
&= -\text{p.v.}\int_{\partial\Omega} \frac{\langle n_x, n_y\rangle}{|x-y|^3}\nabla_y u(y)(x-y)d\sigma_y + \text{p.v.}\int_{\partial\Omega} \frac{\langle x-y, n_y\rangle}{|x-y|^3}\nabla_y u(y) n_x d\sigma_y. \quad (172)
\end{aligned}$$

We compute now the second piece of (166) and obtain for $x \neq y$

$$
\begin{aligned}
&\mathcal{M}(n_x, \partial_x)\frac{(x-y)(x-y)^T}{|x-y|^3} = \Big(\sum_{k=1}^{3} m_{ik}(n_y, \partial_y)\frac{(x_k-y_k)(x_j-y_j)}{|x-y|^3}\Big)_{i,j=1,2,3}\\
= &\Big(\sum_{k=1}^{3}(n_{x,k}\partial_{x_i} - n_{x,i}\partial_{x_k})\frac{(x_k-y_k)(x_j-y_j)}{|x-y|^3}\Big)_{i,j=1,2,3}\\
= &\Big(\sum_{k=1}^{3} n_{x,k}[-3\frac{(x_i-y_i)(x_k-y_k)(x_j-y_j)}{|x-y|^3} + \frac{\delta_{ik}(x_j-y_j)}{|x-y|^3} + \frac{\delta_{ij}(x_k-y_k)}{|x-y|^3}]\\
&-n_{x,i}[-3\frac{(x_k-y_k)^2(x_j-y_j)}{|x-y|^3} + \frac{(x_j-y_j)}{|x-y|^3} + \frac{\delta_{kj}(x_k-y_k)}{|x-y|^3}]\Big)_{i,j=1,2,3}\\
= &-3\frac{\langle n_x, x-y\rangle}{|x-y|^5}(x-y)(x-y)^T + \frac{n_x(x-y)^T}{|x-y|^3} + \frac{\langle n_x, x-y\rangle}{|x-y|^3}I_3\\
&+3\frac{n_x(x-y)^T}{|x-y|^3} - 3\frac{n_x(x-y)^T}{|x-y|^3} - \frac{n_x(x-y)^T}{|x-y|^3}\\
= &\frac{\langle n_x, x-y\rangle}{|x-y|^3}\Big(I_3 - 3\frac{(x-y)(x-y)^T}{|x-y|^2}\Big).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\mathcal{M}(\partial_x, n_x)\int_{\partial\Omega}\frac{(x-y)(x-y)^T}{|x-y|^3}\mathcal{M}(\partial_y, n_y)u(y)d\sigma(y)\\
= \;&\text{p.v.}\int_{\partial\Omega}\frac{\langle n_x, x-y\rangle}{|x-y|^3}\Big(I_3 - 3\frac{(x-y)(x-y)^T}{|x-y|^2}\Big)\mathcal{M}(\partial_y, n_y)u(y)d\sigma_y, \qquad (173)
\end{aligned}
$$

keeping in mind that there is no principal value if one uses (155).

We finally turn to (167). One has, for $x \neq y$,

$$
\begin{aligned}
&\Big(\sum_{\ell,k=1}^{3}(m_{lk}(\partial_x, n_x)\frac{1}{|x-y|})(m_{kj}(\partial_y, n_y)u^\ell(y))\Big)_{j=1,2,3}\\
= &\Big(\sum_{\ell,k=1}^{3}\Big(-n_{x,k}\frac{x_\ell-y_\ell}{|x-y|^3} + n_{x,l}\frac{x_k-y_k}{|x-y|^3}\Big)\Big(n_{y,j}\partial_{y_k}u^\ell(y) - n_{y,k}\partial_{y_j}u^\ell(y)\Big)\Big)_{j=1,2,3}\\
= &\frac{\langle n_x, (\nabla u(y) - \nabla^T u(y))(x-y)\rangle}{|x-y|^3}n_y - \frac{\langle x-y, n_y\rangle}{|x-y|^3}\nabla^T u(y)n_x\\
&+\frac{\langle n_x, n_y\rangle}{|x-y|^3}\nabla^T u(y)(x-y).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\Big(\sum_{\ell,k=1}^{3} m_{lk}(\partial_x, n_x)\int_{\partial\Omega}\frac{1}{|x-y|}(m_{kj}(\partial_y, n_y)u^\ell)(y)d\sigma(y)\Big)_{j=1,2,3}\\
= \;&\text{p.v.}\int_{\partial\Omega}\frac{\langle n_x, (\nabla u(y) - \nabla^T u(y))(x-y)\rangle}{|x-y|^3}n_y d\sigma_y - \text{p.v.}\int_{\partial\Omega}\frac{\langle x-y, n_y\rangle}{|x-y|^3}\nabla^T u(y)n_x d\sigma_y\\
&+\text{p.v.}\int_{\partial\Omega}\frac{\langle n_x, n_y\rangle}{|x-y|^3}\nabla^T u(y)(x-y)d\sigma_y. \qquad (174)
\end{aligned}
$$

Gathering (172), (173), and (174), Corollary B.2 is proved.

□

Proof of Lemma 4.2. Recall that $u = \alpha\psi$ with $\alpha : \partial\Omega \mapsto \mathbb{R}$ and $\psi : \partial\Omega \mapsto \mathbb{R}^3$. We note that

$$\nabla(\alpha\psi) \quad = \quad \alpha\nabla\psi + \psi\nabla\alpha, . \tag{175}$$

and

$$\begin{aligned}
\mathcal{M}(\partial_y, n_y)(\alpha\psi)(y) &= \Big(\sum_{k=1}^{3} m_{ik}(n_y, \partial_y)(\alpha(y)\psi_k(y))\Big)_{i=1,2,3} \\
&= \alpha(y)\mathcal{M}(\partial_y, n_y)\psi(y) + \Big(\sum_{k=1}^{3}(n_{y,k}\partial_{y_i}\alpha(y) - n_{y,i}\partial_{y_k}\alpha(y))\psi_k(y)\Big)_{i=1,2,3} \\
&= \alpha(y)\mathcal{M}(\partial_y, n_y)\psi(y) + \langle n_y, \psi(y)\rangle\nabla^T\alpha(y) - (\nabla\alpha(y)\psi(y))n_y \\
&= \alpha(y)\mathcal{M}(\partial_y, n_y)\psi(y) - (\nabla\alpha(y)\psi(y))n_y.
\end{aligned} \tag{176}$$

Then, the expressions of A_i, $1 \le i \le 4$, simply result from developping ∇u in (168) and (169) of Corollary B.2 and A_5 collects (170) and (171) as a weakly singular operator of class $C^3_*(1)$. Hence Lemma 4.2 follows.

□

B.3 Proof of Lemma 4.3

Using polar coordinates, we have

$$\begin{aligned}
&\int_{B(0,\delta)} \frac{\alpha_\varepsilon(\eta)}{\mid \eta \mid^{1-m}} d\eta = \frac{1}{\varepsilon^2}\int_{B(0,\delta)} \frac{e^{-\frac{|\eta-\eta_0|^2}{\varepsilon^2}}}{|\eta|^{1-m}} d\eta = \frac{e^{-\bar{r}_0^2}}{\varepsilon^2}\int_0^\delta\int_0^{2\pi} \exp\big(-\frac{r^2}{\varepsilon^2} + 2\frac{r}{\varepsilon}\bar{r}_0\cos\theta\big) r^m dr d\theta \\
&= \frac{e^{-\bar{r}_0^2}}{\varepsilon^{1-m}}\int_0^{\delta/\varepsilon}\int_0^{2\pi} \exp\big(-r^2 + 2r\bar{r}_0\cos\theta\big) r^m dr d\theta \\
&\le \frac{e^{-\bar{r}_0^2}}{\varepsilon^{1-m}}\int_0^{\infty}\int_0^{2\pi} \exp\big(-r^2 + 2r\bar{r}_0\cos\theta\big) r^m dr d\theta \le \frac{2\pi}{\varepsilon^{1-m}}\int_0^\infty \exp-(r-\bar{r}_0)^2 r^m dr
\end{aligned}$$

As $\bar{r}_0 \le 1$, there exists a constant $C(m) > 0$ depending only on m such that (65) holds true.

B.4 Proof of Lemma 4.4

We use polar coordinates and get

$$\begin{aligned}
\text{p.v.}\int_{\mathbb{R}^2} \frac{\alpha_\varepsilon(\eta)\eta}{\mid \eta \mid^3} d\eta &= \frac{e^{-\bar{r}_0^2}}{\varepsilon^2}\,\text{p.v.}\int_0^\infty \frac{e^{-r^2}}{r} dr \int_0^{2\pi} \exp(2r\bar{r}_0\cos(\theta-\theta_0)) \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} d\theta \\
&= \frac{e^{-\bar{r}_0^2}}{\varepsilon^2}\,\text{p.v.}\int_0^\infty \frac{e^{-r^2}}{r} dr \int_0^{2\pi} \cos\theta\exp(2r\bar{r}_0\cos\theta) d\theta \begin{pmatrix}\cos\theta_0 \\ \sin\theta_0\end{pmatrix} \\
&= \frac{e^{-\bar{r}_0^2}}{\varepsilon^2} M_3^{A_1^1}(\bar{r}_0)\bar{\eta}_0,
\end{aligned}$$

where we recall that $\bar{\eta}_0 = \bar{r}_0 \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix}$ and where we have set

$$M_3^{A_1}(z) := \frac{1}{z}\ \text{p.v.} \int_0^\infty \frac{e^{-r^2}}{r} dr \int_0^{2\pi} \cos\theta \exp(2rz\cos\theta) d\theta. \tag{177}$$

Standard computations yield that

$$\begin{aligned} M_3^{A_1}(z) &= \frac{1}{z}\ \text{p.v.} \int_0^\infty \frac{e^{-r^2}}{r} dr \sum_{k=0}^\infty \frac{(2r)^k z^k}{k!} \int_0^{2\pi} \cos^{k+1}\theta d\theta \\ &= \frac{4}{z}\ \text{p.v.} \int_0^\infty \frac{e^{-r^2}}{r} dr \sum_{p=0}^\infty \frac{(2r)^{2p+1} z^{2p+1}}{(2p+1)!} I_{2p+2} \\ &= 2\sum_{p=0}^\infty \frac{2^{2p+1}}{(2p+1)!} I_{2p+2} \Gamma(p+\frac{1}{2}) z^{2p}, \end{aligned}$$

where $I_k := \int_0^{\pi/2} \cos^k \theta d\theta$ is the Wallis integral and $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt$ is the Gamma function. Using the fact that $I_{2p} = \frac{(2p)!}{2^{2p}(p!)^2}\frac{\pi}{2}$, we have

$$M_3^{A_1}(z) = \pi \sum_{p=0}^\infty \frac{\Gamma(p+\frac{1}{2})}{p!(p+1)!} z^{2p}. \tag{178}$$

The radius of convergence of $M_3^{A_1}$ is clearly infinite, since

$$\lim_{p\to\infty} \frac{\Gamma(p+\frac{1}{2})(p+1)!(p+2)!}{\Gamma(p+\frac{3}{2})p!(p+1)!} = \frac{(p+1)(p+2)}{p+\frac{1}{2}} = \infty,$$

where we have used the standard fact that $\Gamma(z+1) = z\Gamma(z)$ for $\Re(z) > 0$. Lemma 4.4 is thus established.

B.5 Proof of Lemma 4.7

One has

$$\begin{aligned} & M_1^{A_1}(z) \\ = & \int_0^\infty e^{-r^2} dr \int_0^{2\pi} \cos^2\theta \exp(2rz\cos\theta) d\theta = \int_0^\infty e^{-r^2} dr \int_0^{2\pi} \cos^2\theta \sum_{k=0}^\infty \frac{(2r)^k z^k}{k!} \cos^k \theta d\theta \\ = & \int_0^\infty e^{-r^2} dr \sum_{k=0}^\infty \frac{(2r)^k z^k}{k!} \int_0^{2\pi} \cos^{k+2}\theta d\theta = \sum_{p=0}^\infty \frac{2^{2p} z^{2p}}{(2p)!} I_{2p+2} \int_0^\infty e^{-r^2} r^{2p} dr \\ = & \sum_{p=0}^\infty \frac{2^{2p+1}}{(2p)!} I_{2p+2} \Gamma(p+\frac{1}{2}) z^{2p}. \end{aligned}$$

Using the fact that $I_{2p} = \frac{(2p)!}{2^{2p}(p!)^2}\frac{\pi}{2}$, we have

$$M_1^{A_1}(z) = \frac{\pi}{4} \sum_{p=0}^\infty \frac{(2p+2)(2p+1)}{((p+1)!)^2} \Gamma(p+\frac{1}{2}) z^{2p}. \tag{179}$$

The radius of convergence of $M_1^{A_1}$ is infinite since

$$\lim_{p\to+\infty} \frac{(2p+2)(2p+1)}{(2p+4)(2p+3)} \frac{((p+2)!)^2}{((p+1)!)^2} \frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+\frac{1}{2}+1)} = +\infty.$$

Let $M_5^{A_1}(z)$ be defined by

$$M_5^{A_1}(z) := \int_0^\infty \exp(-r^2)dr \int_0^{2\pi} \exp(2rz\cos\theta)d\theta. \tag{180}$$

We have

$$\begin{aligned}
M_5^{A_1}(z) &= \int_0^\infty \exp(-r^2)dr \int_0^{2\pi} \exp(2rz\cos\theta)d\theta = \int_0^\infty e^{-r^2}dr \sum_{k=0}^\infty \frac{(2r)^k z^k}{k!} \int_0^{2\pi} \cos^k\theta d\theta \\
&= \sum_{p=0}^\infty \frac{2^{2p+1}}{(2p)!} I_{2p}\Gamma(p+\frac{1}{2}) z^{2p} = \pi \sum_{p=0}^\infty \frac{\Gamma(p+\frac{1}{2})}{(p!)^2} z^{2p}.
\end{aligned}$$

It is clear that the radius of convergence of $M_5^{A_1}(\cdot)$ is infinite. Since $M_2^{A_1}(z) = M_5^{A_1}(z) - M_1^{A_1}(z)$, the radius of convergence of $M_2^{A_1}(z)$ is also infinite.

We now prove that $z \mapsto M_4^{A_1}(z)$ is well-defined and not identically equal to zero. Indeed,

$$\begin{aligned}
& M_1^{A_1}(z) - z^2 M_3^{A_1^1}(z) - M_2^{A_1^1}(z) \\
= \;& 2M_1^{A_1}(z) - \pi \sum_{p=0}^\infty \frac{\Gamma(p+\frac{1}{2})}{(p!)^2} z^{2p} - z^2 M_3^{A_1^1}(z) \\
= \;& \pi \sum_{p=0}^\infty [\frac{(p+1)(2p+1)}{((p+1)!)^2} - \frac{1}{(p!)^2}]\Gamma(p+\frac{1}{2}) z^{2p} - \pi \sum_{p=1}^\infty \frac{p}{(p!)^2}\Gamma(p-\frac{1}{2}) z^{2p} \\
= \;& -\frac{3\pi}{2} \sum_{p=1}^\infty \frac{p\Gamma(p-\frac{1}{2})}{(p+1)(p!)^2} z^{2p},
\end{aligned}$$

Then, the function $z \mapsto M_4^{A_1}(z)$ is defined by

$$M_4^{A_1}(z) = -\frac{3\pi}{2} \sum_{p=0}^\infty \frac{(p+1)\Gamma(p+\frac{1}{2})}{(p+2)((p+1)!)^2} z^{2p}, \tag{181}$$

which is clearly a non zero entire function.

B.6 Proof of Lemma 5.2

We give in this section explicit expressions of a_1, a_2, and a_3 defined respectively in Eqs. (123), (124), and (125). The computations are lengthy but straightforward.

We start by computing a_1.

$$
\begin{aligned}
& \int_{\mathbb{R}^2} \frac{\alpha_\varepsilon(\eta)}{|\eta|^3} \eta^T F_x \eta \langle \psi(x), \eta \rangle \eta d\eta \\
= \quad & \varepsilon e^{-\bar{r}_0^2} R_{\theta_0} \int_0^\infty e^{-r^2} r^2 dr \int_0^{2\pi} e^{2r\bar{r}_0 \cos\theta} (F_{\theta_0}^{11} \cos^2\theta + 2F_{\theta_0}^{12} \cos\theta \sin\theta + F_{\theta_0}^{22}(1-\cos^2\theta)) \\
& \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & 1-\cos^2\theta \end{pmatrix} d\theta \; R_{\theta_0}^T \psi(x) \\
= \quad & \varepsilon e^{-\bar{r}_0^2} R_{\theta_0} \int_0^\infty e^{-r^2} r^2 dr \int_0^{2\pi} e^{2r\bar{r}_0 \cos\theta} \\
& \begin{pmatrix} F_{\theta_0}^{22} \cos^2\theta + (F_{\theta_0}^{11} - F_{\theta_0}^{22}) \cos^4\theta, & 2F_{\theta_0}^{12} \cos^2\theta (1-\cos^2\theta) \\ 2F_{\theta_0}^{12} \cos^2\theta(1-\cos^2\theta), & F_{\theta_0}^{22} + (F_{\theta_0}^{11} - 2F_{\theta_0}^{22}) \cos^2\theta - (F_{\theta_0}^{11} - F_{\theta_0}^{22}) \cos^4\theta \end{pmatrix} d\theta \\
& R_{\theta_0}^T \psi(x).
\end{aligned}
$$

The functions $M_6(\cdot)$, $M_7(\cdot)$, and $M_8(\cdot)$ were defined in Eqs. (127), (128) and (129) respectively. Then, we have

$$
\int_{\mathbb{R}^2} \frac{\alpha_\varepsilon(\eta)}{|\eta|^3} \eta^T F_x \eta \langle \psi(x), \eta \rangle \eta d\eta = \varepsilon e^{-\bar{r}_0^2} R_{\theta_0} \mathcal{M}(\bar{r}_0) R_{\theta_0}^T \psi(x), \tag{182}
$$

with

$$
\begin{aligned}
& \mathcal{M}(\bar{r}_0) \\
:= \quad & \begin{pmatrix} F_{\theta_0}^{22} M_7(\bar{r}_0) + (F_{\theta_0}^{11} - F_{\theta_0}^{22}) M_8(\bar{r}_0), & 2F_{\theta_0}^{12}(M_7(\bar{r}_0) - M_8(\bar{r}_0)) \\ 2F_{\theta_0}^{12}(M_7(\bar{r}_0) - M_8(\bar{r}_0)), & F_{\theta_0}^{22} M_6(\bar{r}_0) + (F_{\theta_0}^{11} - 2F_{\theta_0}^{22}) M_7(\bar{r}_0) - (F_{\theta_0}^{11} - F_{\theta_0}^{22}) M_8(\bar{r}_0) \end{pmatrix}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
& R_{\theta_0} \mathcal{M}(\bar{r}_0) R_{\theta_0}^T \\
= \quad & \frac{\mathcal{M}_{11} + \mathcal{M}_{22}}{2} I_2 + \frac{\mathcal{M}_{11} - \mathcal{M}_{22}}{2} \begin{pmatrix} \cos 2\theta_0 & \sin 2\theta_0 \\ \sin 2\theta_0 & -\cos 2\theta_0 \end{pmatrix} + \mathcal{M}_{12} \begin{pmatrix} -\sin 2\theta_0 & \cos 2\theta_0 \\ \cos 2\theta_0 & \sin 2\theta_0 \end{pmatrix},
\end{aligned}
$$

with

$$
\begin{aligned}
\frac{\mathcal{M}_{11} + \mathcal{M}_{22}}{2} &= \frac{1}{2} \Big(F_{\theta_0}^{22} M_6(\bar{r}_0) + (F_{\theta_0}^{11} - F_{\theta_0}^{22}) M_7(\bar{r}_0) \Big), \\
\frac{\mathcal{M}_{11} - \mathcal{M}_{22}}{2} &= \frac{1}{2} \Big(2(F_{\theta_0}^{11} - F_{\theta_0}^{22}) M_8(\bar{r}_0) - (F_{\theta_0}^{11} - 3F_{\theta_0}^{22}) M_7(\bar{r}_0) - F_{\theta_0}^{22} M_6(\bar{r}_0) \Big), \\
\mathcal{M}_{12} &= 2F_{\theta_0}^{12}(M_7(\bar{r}_0) - M_8(\bar{r}_0)).
\end{aligned}
$$

We also note that

$$
\begin{aligned}
\bar{\eta}_0 \bar{\eta}_0^T &= \frac{1}{2} I_2 + \frac{1}{2} \begin{pmatrix} \cos 2\theta_0 & \sin 2\theta_0 \\ \sin 2\theta_0 & -\cos 2\theta_0 \end{pmatrix}, \\
\bar{\eta}_0^\perp \bar{\eta}_0^T &= \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -\sin 2\theta_0 & \cos 2\theta_0 \\ \cos 2\theta_0 & \sin 2\theta_0 \end{pmatrix}.
\end{aligned}
$$

We get $R_{\theta_0}\mathcal{M}(\bar r_0)R_{\theta_0}^T = \mathcal{M}_{22}I_2 + (\mathcal{M}_{11}-\mathcal{M}_{22})\bar\eta_0\bar\eta_0^T - \mathcal{M}_{12}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} + 2\mathcal{M}_{12}\bar\eta_0^\perp\bar\eta_0^T$, which implies that

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)}{|\eta|^3}\eta^T F_x\eta\langle\psi(x),\eta\rangle\eta d\eta \qquad (183)\\
=\ & \varepsilon e^{-\bar r_0^2}\Big(\mathcal{M}_{22}\psi(x) + (\mathcal{M}_{11}-\mathcal{M}_{22})\langle\psi(x),\bar\eta_0\rangle\bar\eta_0 - \mathcal{M}_{12}\psi(x)^\perp + 2\mathcal{M}_{12}\langle\psi(x),\bar\eta_0\rangle\bar\eta_0^\perp\Big).
\end{aligned}
$$

On the other hand, one has

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)}{|\eta|^3}\eta^T F_x\eta\langle\psi(x),\eta\rangle d\eta\\
=\ & e^{-\bar r_0^2}\psi^T(x)R_{\theta_0}\int_0^\infty e^{-r^2}rdr\int_0^{2\pi}e^{2r\bar r_0\cos\theta}\begin{pmatrix}F_{\theta_0}^{22}\cos\theta + (F_{\theta_0}^{11}-F_{\theta_0}^{22})\cos^3\theta\\ 2F_{\theta_0}^{12}\cos\theta(1-\cos^2\theta)\end{pmatrix}d\theta.
\end{aligned}
$$

The functions $M_9(\cdot)$ and $M_{10}(\cdot)$ were defined in Eqs. (130) and (131) respectively. Then, we have

$$
\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)}{|\eta|^3}\eta^T F_x\eta\langle\psi(x),\eta\rangle d\eta = e^{-\bar r_0^2}\psi^T(x)R_{\theta_0}\begin{pmatrix}F_{\theta_0}^{22}M_9(\bar r_0) + (F_{\theta_0}^{11}-F_{\theta_0}^{22})M_{10}(\bar r_0)\\ 2F_{\theta_0}^{12}(M_9(\bar r_0) - M_{10}(\bar r_0))\end{pmatrix}.
$$

Since $\psi^T(x)R_{\theta_0} = (\langle\psi(x),\bar\eta_0\rangle,\ \langle\psi(x),\bar\eta_0^\perp\rangle)$, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)}{|\eta|^3}\eta^T F_x\eta\langle\psi(x),\eta\rangle d\eta &= e^{-\bar r_0^2}\Big(\big[F_{\theta_0}^{22}M_9(\bar r_0) + (F_{\theta_0}^{11}-F_{\theta_0}^{22})M_{10}(\bar r_0)\big]\langle\psi(x),\bar\eta_0\rangle \qquad (184)\\
&+\ 2F_{\theta_0}^{12}(M_9(\bar r_0) - M_{10}(\bar r_0))\langle\psi(x),\bar\eta_0^\perp\rangle\Big).
\end{aligned}
$$

One also gets

$$
\begin{aligned}
\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)}{|\eta|}\eta^T F_x\eta d\eta &= \varepsilon e^{-\bar r_0^2}\int_0^\infty e^{-r^2}r^2dr\int_0^{2\pi}e^{2r\bar r_0\cos\theta}(F_{\theta_0}^{11}\cos^2\theta + F_{\theta_0}^{22}\sin^2\theta)d\theta\\
&= \varepsilon e^{-\bar r_0^2}\int_0^\infty e^{-r^2}r^2dr\int_0^{2\pi}e^{2r\bar r_0\cos\theta}(F_{\theta}^{22} + (F_{\theta_0}^{11}-F_{\theta}^{22})\cos^2\theta)d\theta\\
&= \varepsilon e^{-\bar r_0^2}(F_{\theta_0}^{22}M_6(\bar r_0) + (F_{\theta_0}^{11}-F_{\theta_0}^{22})M_7(\bar r_0)).
\end{aligned}
$$

Finally, one derives

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)}{|\eta|^3}(\eta^T F_x\eta)\eta d\eta\\
=\ & e^{-\bar r_0^2}R_{\theta_0}\int_0^\infty e^{-r^2}rdr\int_0^{2\pi}e^{2r\bar r_0\cos\theta}\begin{pmatrix}F_{\theta_0}^{22}\cos\theta + (F_{\theta_0}^{11}-F_{\theta_0}^{22})\cos^3\theta\\ 2F_{\theta_0}^{12}(\cos\theta-\cos^3\theta)\end{pmatrix}d\theta\\
=\ & e^{-\bar r_0^2}R_{\theta_0}\begin{pmatrix}F_{\theta_0}^{22}M_9(\bar r_0) + (F_{\theta_0}^{11}-F_{\theta_0}^{22})M_{10}(\bar r_0)\\ 2F_{\theta_0}^{12}(M_9(\bar r_0) - M_{10}(\bar r_0))\end{pmatrix}.
\end{aligned}
$$

Since $\bar\eta_0^T R_{\theta_0} = (1,0)$, we have

$$
\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)}{|\eta|^3}(\eta^T F_x\eta)\langle\eta_0,\eta\rangle d\eta = \varepsilon e^{-\bar r_0^2}\big(F_{\theta_0}^{22}M_9(\bar r_0) + (F_{\theta_0}^{11}-F_{\theta_0}^{22})M_{10}(\bar r_0)\big). \qquad (185)
$$

In summary, we get

$$\begin{aligned}
a_1 &= \frac{1}{4\pi}\frac{e^{-\bar{r}_0^2}}{\varepsilon}\Big(\mathcal{M}_{22}\psi(x) + (\mathcal{M}_{11} - \mathcal{M}_{22})\langle\psi(x), \bar{\eta}_0\rangle\bar{\eta}_0 - \mathcal{M}_{12}\psi(x)^{\perp} + 2\mathcal{M}_{12}\langle\psi(x), \bar{\eta}_0\rangle\bar{\eta}_0^{\perp} \\
&- \big[\big(F^{22}_{\theta_0}M_9(\bar{r}_0) + (F^{11}_{\theta_0} - F^{22}_{\theta_0})M_{10}(\bar{r}_0)\big)\langle\psi(x), \bar{\eta}_0\rangle + 2F^{12}_{\theta_0}(M_9(\bar{r}_0) - M_{10}(\bar{r}_0))\langle\psi(x), \bar{\eta}_0^{\perp}\rangle\big]\bar{\eta}_0 \\
&+ (F^{22}_{\theta_0}(M_6(\bar{r}_0) - M_9(\bar{r}_0)) + (F^{11}_{\theta_0} - F^{22}_{\theta_0})(M_7(\bar{r}_0) - M_{10}(\bar{r}_0))\psi(x)\Big) \\
&= \frac{1}{4\pi}\frac{e^{-\bar{r}_0^2}}{\varepsilon}\Big(\\
&\quad \big[2F^{22}_{\theta_0}M_6(\bar{r}_0) + (2F^{11}_{\theta_0} - 3F^{22}_{\theta_0})M_7(\bar{r}_0) - (F^{11}_{\theta_0} - F^{22}_{\theta_0})(M_8(\bar{r}_0) + M_{10}(\bar{r}_0)) - F^{22}_{\theta_0}M_9(\bar{r}_0)\big]\psi(x) \\
&- \big[\big(F^{22}_{\theta_0}M_9(\bar{r}_0) + (F^{11}_{\theta_0} - F^{22}_{\theta_0})(M_{10}(\bar{r}_0) - 2M_8(\bar{r}_0)) + (F^{11}_{\theta_0} - 3F^{22}_{\theta_0})M_7(\bar{r}_0) \\
&\quad + F^{22}_{\theta_0}M_6(\bar{r}_0)\big)\langle\psi(x), \bar{\eta}_0\rangle + 2F^{12}_{\theta_0}(M_9(\bar{r}_0) - M_{10}(\bar{r}_0))\langle\psi(x), \bar{\eta}_0^{\perp}\rangle\big]\bar{\eta}_0 \\
&- 2F^{12}_{\theta_0}(M_7(\bar{r}_0) - M_8(\bar{r}_0))\psi(x)^{\perp} + 4F^{12}_{\theta_0}(M_7(\bar{r}_0) - M_8(\bar{r}_0))\langle\psi(x), \bar{\eta}_0\rangle\bar{\eta}_0^{\perp}\Big).
\end{aligned}$$

Let us now compute a_2. Using the computations performed for the term a_1, one has

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{\alpha_\varepsilon(\eta)}{|\eta|^3}\eta^T F_x \eta d\eta &= \frac{e^{-\bar{r}_0^2}}{\varepsilon}\int_0^\infty e^{-r^2}dr\int_0^{2\pi} e^{2r\bar{r}_0\cos\theta}(F^{11}_{\theta_0}\cos^2\theta + F^{22}_{\theta_0}(1 - \cos^2\theta))d\theta \\
&= \frac{e^{-\bar{r}_0^2}}{\varepsilon}\Big[F^{22}_{\theta_0}M_5^{A_1}(\bar{r}_0) + (F^{11}_{\theta_0} - F^{22}_{\theta_0})M_1^{A_1}(\bar{r}_0)\Big].
\end{aligned}$$

The other contribution in a_2 is given by the following expression.

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{\alpha_\varepsilon(\eta)}{|\eta|^3}\langle\psi(x), \eta\rangle F_x \eta d\eta \\
&= F_x R_{\theta_0}\frac{e^{-\bar{r}_0^2}}{\varepsilon}\int_0^\infty e^{-r^2}dr\int_0^{2\pi} e^{2r\bar{r}_0\cos\theta}\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & 1 - \cos^2\theta\end{pmatrix}d\theta\ R^T_{\theta_0}\psi(x) \\
&= \frac{e^{-\bar{r}_0^2}}{\varepsilon}F_x R_{\theta_0}\begin{pmatrix}M_1^{A_1}(\bar{r}_0) & 0 \\ 0 & M_5^{A_1}(\bar{r}_0) - M_1^{A_1}(\bar{r}_0)\end{pmatrix}\ R^T_{\theta_0}\psi(x) \\
&= \frac{e^{-\bar{r}_0^2}}{\varepsilon}F_x\Big(\frac{1}{2}M_5^{A_1}(\bar{r}_0)I_2 + (M_1^{A_1}(\bar{r}_0) - \frac{1}{2}M_5^{A_1}(\bar{r}_0))\begin{pmatrix}\cos 2\theta_0 & \sin 2\theta_0 \\ \sin 2\theta_0 & -\cos 2\theta_0\end{pmatrix}\Big)\ \psi(x) \\
&= \frac{e^{-\bar{r}_0^2}}{\varepsilon}F_x\Big((M_5^{A_1}(\bar{r}_0) - M_1^{A_1}(\bar{r}_0))I_2 + (2M_1^{A_1}(\bar{r}_0) - M_5^{A_1}(\bar{r}_0))\bar{\eta}_0\bar{\eta}_0^T\Big)\psi(x).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
a_2 = -\frac{1}{4\pi}\frac{e^{-\bar{r}_0^2}}{\varepsilon}\Big\{&\Big[F^{22}_{\theta_0}M_5^{A_1}(\bar{r}_0) + (F^{11}_{\theta_0} - F^{22}_{\theta_0})M_1^{A_1}(\bar{r}_0)\Big]\psi(x) \\
&+ F_x\Big((M_5^{A_1}(\bar{r}_0) - M_1^{A_1}(\bar{r}_0))\psi(x) + (2M_1^{A_1}(\bar{r}_0) - M_5^{A_1}(\bar{r}_0))\langle\psi(x), \bar{\eta}_0\rangle\bar{\eta}_0\Big)\Big\}.
\end{aligned}$$

Finally, a_3 is computed as follows.

$$
\begin{aligned}
a_3 &= -\frac{1}{8\pi}\int_{\mathbb{R}^2}\frac{\alpha_\varepsilon(\eta)\langle\psi(x),\eta/|\eta|\rangle}{|\eta|}\frac{\eta}{|\eta|}d\eta \\
&= -\frac{1}{8\pi}\frac{e^{-\bar r_0^2}}{\varepsilon}R_{\theta_0}\int_0^\infty e^{-r^2}dr\int_0^{2\pi}e^{2r\bar r_0\cos\theta}\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & (1-\cos^2\theta)\end{pmatrix}d\theta R_{\theta_0}^T\psi(x) \\
&= -\frac{1}{8\pi}\frac{e^{-\bar r_0^2}}{\varepsilon}R_{\theta_0}\begin{pmatrix}M_1^{A_1}(\bar r_0) & 0\\ 0 & M_5^{A_1}(\bar r_0)-M_1^{A_1}(\bar r_0)\end{pmatrix}R_{\theta_0}^T\psi(x) \\
&= -\frac{1}{8\pi}\frac{e^{-\bar r_0^2}}{\varepsilon}\Big[(M_5^{A_1}(\bar r_0)-M_1^{A_1}(\bar r_0))\psi(x)+(2M_1^{A_1}(\bar r_0)-M_5^{A_1}(\bar r_0))\langle\psi(x),\bar\eta_0\rangle\bar\eta_0\Big].
\end{aligned}
$$

This ends the proof of Lemma 5.2.

B.7 Proof of Lemma 5.3

Recall that

$$
\int_0^{2\pi}\cos^{2p}\theta d\theta = 4I_{2p},\ \int_0^\infty e^{-r^2}r^{2p}dr = \frac{1}{2}\Gamma(p+\frac{1}{2}).
$$

Then, one gets

$$
\begin{aligned}
M_6(z) &= \int_0^\infty e^{-r^2}r^2dr\int_0^{2\pi}e^{2rz\cos\theta}d\theta = \sum_{k=0}^\infty\frac{2^k}{k!}\big[\int_0^\infty e^{-r^2}r^{k+2}dr\int_0^{2\pi}\cos^k\theta d\theta\big]z^k \\
&= \sum_{p=0}^\infty\frac{2^{2p+1}}{(2p)!}\Gamma(p+\frac{3}{2})I_{2p}\ z^{2p},
\end{aligned}
$$

$$
\begin{aligned}
M_7(z) &= \int_0^\infty e^{-r^2}r^2dr\int_0^{2\pi}e^{2rz\cos\theta}\cos^2\theta d\theta = \sum_{k=0}^\infty\frac{2^k}{k!}\big[\int_0^\infty e^{-r^2}r^{k+2}dr\int_0^{2\pi}\cos^{k+2}\theta d\theta\big]z^k \\
&= \sum_{p=0}^\infty\frac{2^{2p+1}}{(2p)!}\Gamma(p+\frac{3}{2})I_{2p+2}\ z^{2p},
\end{aligned}
$$

$$
\begin{aligned}
M_8(z) &= \int_0^\infty e^{-r^2}r^2dr\int_0^{2\pi}e^{2rz\cos\theta}\cos^4\theta d\theta = \sum_{k=0}^\infty\frac{2^k}{k!}\big[\int_0^\infty e^{-r^2}r^{k+2}dr\int_0^{2\pi}\cos^{k+4}\theta d\theta\big]z^k \\
&= \sum_{p=0}^\infty\frac{2^{2p+1}}{(2p)!}\Gamma(p+\frac{3}{2})I_{2p+4}\ z^{2p},
\end{aligned}
$$

$$
\begin{aligned}
M_9(z) &= \int_0^\infty e^{-r^2}rdr\int_0^{2\pi}e^{2rz\cos\theta}\cos\theta d\theta = \sum_{k=0}^\infty\frac{2^k}{k!}\big[\int_0^\infty e^{-r^2}r^{k+1}dr\int_0^{2\pi}\cos^{k+1}\theta d\theta\big]z^k \\
&= \sum_{p=0}^\infty\frac{2^{2p+2}}{(2p+1)!}\Gamma(p+\frac{3}{2})I_{2p+2}\ z^{2p+1},
\end{aligned}
$$

and finally

$$\begin{aligned} M_{10}(z) &:= \int_0^\infty e^{-r^2} r dr \int_0^{2\pi} e^{2rz\cos\theta} \cos^3\theta d\theta = \sum_{k=0}^\infty \frac{2^k}{k!} \big[\int_0^\infty e^{-r^2} r^{k+1} dr \int_0^{2\pi} \cos^{k+3}\theta d\theta \big] z^k \\ &= \sum_{p=0}^\infty \frac{2^{2p+2}}{(2p+1)!} \Gamma(p+\frac{3}{2}) I_{2p+4}\, z^{2p+1}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} & 2M_7(z) - M_8(z) - M_{10}(z) - M_1^{A_1}(z) \\ = & \sum_{p=0}^\infty \frac{2^{2p+1}}{(2p)!} \Gamma(p+\frac{1}{2}) I_{2p+2} \frac{2p^2+4p-\frac{3}{2}}{2p+3} z^{2p} - \sum_{p=0}^\infty \frac{2^{2p+2}}{(2p+1)!} \Gamma(p+\frac{3}{2}) I_{2p+4}\, z^{2p+1}, \end{aligned}$$

$$\begin{aligned} & 2M_6(z) - 3M_7(z) + M_8(z) + M_{10}(z) - M_9(z) - 2M_5^{A_1}(z) + 2M_1^{A_1}(z) \\ = & \sum_{p=0}^\infty \frac{2^{2p+1}}{(2p)!} \Gamma(p+\frac{1}{2}) I_{2p} \frac{6p^2+16p+\frac{7}{2}}{(2p+2)(2p+4)} z^{2p} - \sum_{p=0}^\infty \frac{2^{2p+2}}{(2p+1)!} \Gamma(p+\frac{3}{2}) I_{2p+2} \frac{1}{2p+4} z^{2p+1}. \end{aligned}$$

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