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# Small positive values and lower large deviations for supercritical branching processes in random environment

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**Abstract**

Branching Processes in Random Environment (BPREs) $(Z_n : n \geq 0)$ are the generalization of Galton-Watson processes where in each generation the reproduction law is picked randomly in an i.i.d. manner. In the supercritical case, the process survives with positive probability and then almost surely grows geometrically. This paper focuses on rare events when the process takes positive values, but lower than expected.

First, we consider small positive values the process may reach for large times and describe the asymptotic behavior of $\mathbb{P}(1 \leq Z_n \leq k)$ as $n \to \infty$. If the reproduction laws are linear fractional, two regimes appear for the rate of decrease of this probability.

Secondly, we are interested in the lower large deviations of $Z$ and give the rate function under some moment assumptions. This result generalizes the lower large deviation theorem of Bansaye and Berestycki (2009) by considering processes where $\mathbb{P}_1(Z_1 = 0) > 0$ but also weaker moment assumptions.

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## 1 Introduction

A branching process in random environment (BPRE) is a discrete time and discrete size population model going back to [38, 7]. In each generation, an offspring distribution is picked at random, independently from one generation to the other. We can think of a population of plants having a one-year life-cycle. In each year, the outer conditions vary in a random fashion. Given these conditions, all individuals reproduce independently according to the same mechanism. Thus, it satisfies both the Markov and branching properties.

Recently, the problems of rare events and large deviations have attracted attention [32, 10, 13, 33, 11, 26]. In the Galton Watson case, large deviations problems are studied for a long time [6, 8] and fine results have been obtained, see [18, 19, 36, 37].

For the formal definition of a branching process $Z$ in random environment, let $Q$ be a random variable taking values in $\Delta$, the space of all probability measures on $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We denote by

$$m_q = \sum_{k\geq 0} kq(\{k\})$$

the mean number of offsprings of $q \in \Delta$. For simplicity of notation, we will shorten $q(\{\cdot\})$ to $q(\cdot)$ throughout this paper. An infinite sequence $\mathcal{E} = (Q_1, Q_2, \ldots)$ of independent, identically distributed (i.i.d.) copies of $Q$ is called a random environment. Then the integer valued process $(Z_n : n \geq 0)$ is called a branching process in the random environment $\mathcal{E}$ if $Z_0$ is independent of $\mathcal{E}$ and it satisfies

$$\mathcal{L}\big(Z_n \bigm| \mathcal{E},\ Z_0, \ldots, Z_{n-1}\big) \ = \ Q_n^{*Z_{n-1}} \qquad \text{a.s.} \tag{1.1}$$

for every $n \geq 0$, where $q^{*z}$ is the $z$-fold convolution of the measure $q$. We introduce the probability generating function (p.g.f) of $Q_n$, which is denoted by $f_n$ and defined by

$$f_n(s) := \sum_{k=0}^{\infty} s^k Q_n(k), \qquad (s \in [0,1])\ .$$

In the whole paper, we denote indifferently the associated random environment by $\mathcal{E} = (f_1, f_2, ...)$ and $\mathcal{E} = (Q_1, Q_2, ...)$. The characterization (1.1) of the law of $Z$ becomes

$$\mathbb{E}\big[s^{Z_n}|\mathcal{E},\ Z_0, \ldots, Z_{n-1}\big] = f_n(s)^{Z_{n-1}} \qquad \text{a.s.} \qquad (0 \leq s \leq 1).$$

Many properties of $Z$ are mainly determined by the **random walk associated with the environment** $(S_n\ :\ n \in \mathbb{N}_0)$ which is defined by

$$S_0 = 0, \qquad S_n - S_{n-1} = X_n \quad (n \geq 1),$$

where

$$X_n := \log m_{Q_n} = \log f_n'(1)$$

are i.i.d. copies of the logarithm of the mean number of offsprings $X := \log(m_Q) = \log(f'(1))$. Thus, one can check easily that

$$\mathbb{E}[Z_n|Q_1, \ldots, Q_n, Z_0 = 1] \ = \ e^{S_n} \quad \text{a.s.} \tag{1.2}$$

We have the following well-known classification of BPRE [7]. In the subcritical case ($\mathbb{E}[X] < 0$), the population becomes extinct a.s. It also holds in the the critical case ($\mathbb{E}[X] = 0$) if we exclude the degenerated case when $\mathbb{P}_1(Z_1 = 1) = 1$. In the supercritical case ($\mathbb{E}[X] > 0$), the process survives with positive probability under quite general assumptions on the offspring distributions (see [38]). Then $\mathbb{E}[Z_1 \log^+(Z_1)/f'(1)] < \infty$ ensures that the martingale $e^{-S_n}Z_n$ has a positive finite limit on the non-extinction event:

$$\lim_{n\to\infty} e^{-S_n} Z_n = W, \qquad \mathbb{P}(W > 0) = \mathbb{P}(\forall n \in \mathbb{N} : Z_n > 0) > 0.$$

The large deviations are related to the speed of convergence of $\exp(-S_n)Z_n$ to $W$ and the tail of $W$. This latter is directly linked to the existence of moments and harmonic moments of $W$. In the Galton Watson case, we refer to [6] and [37]. For BPRE, Hambly [25] gives the tail of $W$ in 0, whereas Huang & Liu [26, 27] obtain other various results in this direction.

In this paper, we consider the supercritical case and we are interested in the following asymptotic probabilities

$$\mathbb{P}(1 \leq Z_n \leq k_n), \qquad \text{where } k_n = o(\exp(S_n)).$$

As $Z_n \asymp \exp(S_n)$ a.s. on the non-extinction event, this probability goes to 0 and we aim at specifying its speed.

In particular a key role is played by the asymptotic probability to survive but stay bounded, which corresponds to $k_n = k > 0$ constant. For Galton Watson processes, the explicit equivalent of this probability is well known (see e.g. [9] Chapter I, Section 11, Theorem 3) and we have

$$\varrho := \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1(1 \leq Z_n \leq k) = f'(p_e), \qquad \text{where} \quad p_e = \mathbb{P}(\exists n \in \mathbb{N} : Z_n = 0).$$

In Section 2, we characterize this quantity in the random environment framework. In the linear fractional case we can specify the value of $\varrho$ and two regimes appear, similar to the regimes observed for the survival probability in the subcritical case (see [14, 24, 23]). This result is stated as a part of the PhD of one of the authors and can also be found in [12].

Then, in Section 3, we get an expression of the lower rate function for the large deviations of the BPRE, i.e. we specify the exponential rate of decrease of $\mathbb{P}(1 \leq Z_n \leq e^{\theta n})$ for $0 < \theta < \mathbb{E}[X]$. In the Galton Watson case, lower large deviations have been finely studied, see e.g. [18, 19, 36]. In the case of random environment, the rate function has been established in [10] when any individual leaves at least one offspring, i.e. $\mathbb{P}(Z_1 = 0) = 0$. This result is extended here to the situation where $\mathbb{P}(Z_1 = 0) > 0$ and relax the moment assumptions.

In the rest of the paper, the proofs of these results are presented. Section 4 deals with a tree construction due to Geiger, which is used in Section 5 to prove a characterization of $\rho$. Section 6 is dedicated to the proof the second result of this paper : it gives a general upperbound of $\varrho$, which may be reached. In Section 7, we prove the results on large deviations given in Section 3. Finally, in Section 8 the statements for the linear fractional case are proved by using the general results obtained before, whereas in Section 9, we present some details on two examples.

We add that for the problem of upper large deviations, the rate function has been established in [13, 11] and finer results in the case of geometric offspring distribution can be found in [32, 33]. Thus large deviations for BPRE become well understood, even if much work remains to get finer results, deal with weaker assumptions or consider the Bötcher case ($\mathbb{P}(Z_1 \geq 2) = 1$).

# 2 Probability of staying bounded without extinction

The initial population size is denoted by $k$ and the associated probability by $\mathbb{P}_k(\cdot) := \mathbb{P}(\cdot|Z_0 = k)$. For convenience, we write $\mathbb{P}(\cdot)$ when the initial population does not matter and can be taken equal to 1. Let $f_{i,n}$ be the probability generating function of $Z_n$ started in generation $i \leq n$ :

$$f_{i,n} := f_{i+1} \circ f_{i+2,n} \circ \cdots \circ f_n, \quad f_{n,n} = Id \qquad \text{a.s.}$$

We will now specify the asymptotic behavior of $\mathbb{P}_i(Z_n = j)$ for $i, j \geq 1$, which may depend both on $i$ and $j$. One can first observe that some integers $j$ can not be reached by $Z$ starting from $i$ owing to the support of the offspring distribution. The first result below introduce the rate of decrease $\varrho$ of this probability and gives a trajectorial interpretation of the associated rare event $\{Z_n = j\}$. The second one gives a general upperbound of this rate of decrease $\varrho$, which may be reached. This bound corresponds to the environmental stochasticity, which means that the rare event $\{Z_n = j\}$ is explained by rare environments. The third one yields the explicit expression of the rate $\varrho$ in the case of linear fractional offspring distributions, where two supercritical regimes appear.

Let us define

$$\mathcal{I} := \{j \geq 1 \ : \ \mathbb{P}(Q(j) > 0, Q(0) > 0) > 0\}$$

and introduce the set $Cl(\mathcal{I})$ of integers which can be reached from $\mathcal{I}$ by the process $Z$. More precisely,

$$Cl(\mathcal{I}) := \{k \geq 1 : \exists n \geq 0 \text{ and } j \in \mathcal{I} \text{ with } \ \mathbb{P}_j(Z_n = k) > 0\} \ .$$

We observe that $\mathcal{I} \subset Cl(\mathcal{I})$ and if $\mathbb{P}(Q(0)+Q(1) < 1) > 0$ and $\mathbb{P}(Q(0) > 0, \ Q(1) > 0) > 0$, then s $Cl(\mathcal{I}) = \mathbb{N}$.

We are interested in the event $\{Z_n = j\}$ for large $n$. Recall that we focus on the supercritical case $\mathbb{E}[X] > 0$ throughout this paper and thus, the trivial case $\mathbb{P}_1(Z_1 \leq 1) = 1$ is excluded. We also exclude the case $\mathbb{P}_1(Z_1 = 0) = 0$, which is easier and already handled in [10]. Then $Z$ is also nondecreasing and for $k \geq j \in \mathbb{N}$ such that $\mathbb{P}_k(Z_n = j) > 0$ for some $n \geq 0$, we have

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_k(Z_n = j) = -k\varrho.$$

So let us now focus on the supercrtical case, with possible extinction, which ensures that $\mathcal{I}$ is not empty.

**Theorem 2.1.** *We assume that* $\mathbb{E}[X] > 0$ *and* $\mathbb{P}(Z_1 = 0) > 0$. *Then, the following limits exist and coincide for all* $k, j \in Cl(\mathcal{I})$,

$$\varrho := \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_k(Z_n = j) = \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}\big[Q_n(z_0) f_{0,n}(0)^{z_0-1} \Pi_{i=1}^{n-1} f_i'\big(f_{i,n}(0)\big)\big]$$

*where* $z_0$ *is the smallest element of* $\mathcal{I}$. *The common limit* $\varrho$ *belongs to* $[0, \infty)$.

The proof is given in Section 5 and the theorem results from Lemmas 5.1 and 5.2. The right-hand side expression of $\varrho$ shows that the rare events $\{Z_n = j\}$ corresponds to a "spine structure", i.e. one individual survives until generation $n$ and gives birth in the very last generations to the $j$ survivors, whereas the other subtrees become extinct (see forthcoming Lemma 4.2). Moreover, this expression will be used to get some of the forthcoming results.
The proof is easy if we consider the limit of $\frac{1}{n} \log \mathbb{P}_1(Z_n = 1)$ as $n \to \infty$. In this case, a direct calculation of the first derivative of $f_{0,n}$ yields the claim. However, the proof for the general case is more involved. Here, we use probabilistic arguments, which rely on a spine decomposition of the tree via Geiger construction.
We also note that we need to focus on $i, j \in Cl(\mathcal{I})$. Indeed, $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1(Z_n = i)$ and $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1(Z_n = j)$ may both exist and be finite for $i \neq j$, but have different values. Moreover the case $\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_i(Z_n = i) < \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1(Z_n = i)$ with $i > 1$ is also possible. These facts are illustrated by two examples in Section 9 at the end of this paper.

In the Galton Watson case, $f$ is constant, for every $i \geq 0$, $f_i = f$ a.s. and $f_{i,n}(0) \to p_e$ as $n \to \infty$. We recover the classical result [9], which was given in the introduction : $\varrho = -\log f'(p_e)$.

The results and remarks above could lead to the conjecture $\varrho = -\log \mathbb{E}[f'(p(f))]$, where $p(f) = \inf\{s \in [0,1] : f(s) = s\}$. Roughly speaking, it would correspond to integrate the value obtained in the Galton Watson case with respect to the environment. The two following results show that this is not true in general.

First, we give an upperbound of $\varrho$ in terms of the rate function of the random walk $S$. In that view, we assume that:

**Assumption 1.** *There exists an $s > 0$ such that the moment generating function $\mathbb{E}[e^{-sX}] < \infty$. Moreover, the random walk $S$ is non-lattice, i.e. for every $r > 0$, $\mathbb{P}(X \in r\mathbb{Z}) < 1$.*

This assumption assures that a proper rate function $\Lambda$ of the random walk $(S_n : n \in \mathbb{N})$

$$\Lambda(\theta) \quad := \quad \sup_{\lambda \leq 0} \left\{ \lambda\theta - \log(\mathbb{E}[\exp(\lambda X)]) \right\} \tag{2.1}$$

exists. We note that the supremum is taken over $\lambda \leq 0$ and not over all $\lambda \in \mathbb{R}$. As we are only interested in lower deviations here, this definition is more convenient as it implies $\Lambda(\theta) = 0$ for all $\theta \geq \mathbb{E}[X]$. We are also using the following assumptions about the truncated moments of the offspring distributions.

**Assumption 2.** *There exist $\varepsilon > 0$ and $a \in \mathbb{N}$ such that for every $x > 0$,*

$$\mathbb{E}\left[(\log^+ \xi_Q(a))^{\frac{1}{2}+\varepsilon} | X > -x\right] < \infty \ ,$$

*where $\log^+ x := \log(\max(x, 1))$ and $\xi_q(a)$ is the truncated standardized second moment*

$$\xi_q(a) := \sum_{y=a}^{\infty} y^2 q(y)/m_Q^2 \ , \quad a \in \mathbb{N}, q \in \Delta.$$

**Proposition 2.2.** *Assume that $\mathbb{P}(X \geq 0) = 1$ or that both Assumptions 1 and 2 hold. Then*

$$\varrho \leq \Lambda(0).$$

This bound is proved in Section 6 and used both for the proof of the next Corollary and Theorem 3.2. It can be reached and has a natural interpretation. Indeed, one way to keep the population bounded but alive comes from a succession of "critical environments", which means $S_n \approx 0$. Then $\mathbb{E}[Z_n \mid \mathcal{E}] = \exp(S_n)$ is neither small nor large and one can expect that the population is positive but bounded. The event $\{S_n \approx 0\}$ is a large deviation event whose probability decreases exponentially with rate $\Lambda(0)$. This bound is thus directly explained by environmental stochasticity.

Now, we focus on the linear fractional case and derive an explicit expression of $\varrho$. We recall that a probability generating function of a random variable $R$ is linear fractional (LF) if there exist positive real numbers $m$ and $b$ such that

$$f(s) = 1 - \frac{1-s}{m^{-1} + bm^{-2}(1-s)/2} \ ,$$

where $m = f'(1)$ and $b = f''(1)$. This family includes the probability generating function of geometric distributions, with $b = 2m^2$. Thus, LF distributions are geometric laws with a second free parameter $b$ which allows to change the probability of the event $\{R = 0\}$.

**Corollary 2.3.** *If $f$ is a.s. linear fractional and $\mathbb{E}[|X|] < \infty$, then*

$$\varrho \quad = \quad \begin{cases} -\log \mathbb{E}\left[e^{-X}\right] & , \quad \text{if } \mathbb{E}[Xe^{-X}] \geq 0 \\ \Lambda(0) & , \quad \text{else} \end{cases} \ . \tag{2.2}$$

Thus, there are two regimes. For $\mathbb{E}[Xe^{-X}] < 0$, the event $\{1 \leq Z_n \leq k\}$ is a typical event in a suitable exceptional environment, say "critical". This rare event is then explained (only) by the environmental stochasticity. For $\mathbb{E}[Xe^{-X}] \geq 0$, we recover a term analogous to the Galton-Watson case, which is smaller than $\Lambda(0)$ (and thus the probability is larger than $\exp(-\Lambda(0)n + o(n))$). The rare event is then more due to demographical stochasticity.
These two regimes seem to be analoguous to the two regimes in the subcritical case, which deal with the asymtpotic behavior $Z_n > 0$, see e.g. [14, 31, 23]. Thus, the processes may be called respectively *weakly supercritical* and *strongly supercritical*. Such regimes for supercritical branching processes have already been obeserved in [28] in the continuous framework (which essentially represents linear fractional offspring-distributions).

# 3 Lower large deviations

We now introduce the following new rate function defined for $\theta, x \geq 0$ and any nonnegative function $H$

$$\chi(\theta, x, H) \quad = \quad \inf_{t \in [0,1)} \left\{ tx + (1-t)H(\theta/(1-t)) \right\} .$$

To state the large deviation principle, we recall the definition of $\varrho$ and $\Lambda$ from the previous section and we need the following moment assumption:

**Assumption 3.** *For every $\lambda > 0$,*

$$\mathbb{E}\Big[ \Big( \frac{f'(1)}{1-f(0)} \Big)^{\lambda} \Big] < \infty$$

We also denote $k_n \overset{subexp}{\longrightarrow} \infty$ when $k_n \to \infty$ but $k_n / \exp(\theta n) \to 0$ for every $\theta > 0$, as $n \to \infty$.

**Theorem 3.1.** *Under Assumption 3 and $\mathbb{E}[Z_1 \log^+(Z_1)] < \infty$, the following assertions hold for every $\theta \in \big(0, \mathbb{E}[X]\big]$.*
*(i) If $\mathbb{P}_1(Z_1 = 0) > 0$, then for every $i \in Cl(\mathcal{I})$*

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_i(1 \leq Z_n \leq e^{\theta n}) = -\chi(\theta, \varrho, \Lambda)$$

*Moreover, $k_n \overset{subexp}{\longrightarrow} \infty$ ensures that $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_i(1 \leq Z_n \leq k_n) = -\varrho$.*
*(ii) If $\mathbb{P}_1(Z_1 = 0) = 0$, then for every $i \geq 1$,*

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_i(1 \leq Z_n \leq e^{\theta n}) = -\chi(\theta, -i \log \mathbb{E}[Q(1)], \Lambda)$$

*Moreover, $k_n \overset{subexp}{\longrightarrow} \infty$ ensures that $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_i(1 \leq Z_n \leq k_n) = i \log \mathbb{E}[Q(1)]$.*

First, we note that (ii) generalizes Theorem 1 in [10], which required that both the mean and the variance of the reproduction laws were bounded (uniformly with respect to the environment). Moreover, $(i)$ provides an expression of the rate function in the more challenging case which allows extinction ($\mathbb{P}_1(Z_1 = 0) > 0$).
We now try to extend this result and get rid of Assumption 3, before discussing its interpretation and applying it to the linear fractional case. So we now work with a different assumption, which ensures that the tail of the reproduction laws have finite variance.

**Assumption 4.** *There exists a constant $0 < d < \infty$ such that,*

$$M_Q \leq d \cdot [m_Q + (m_Q)^2] \quad a.s.,$$

*where $M_q = \sum_{k\geq 0} k^2 q(k)$ is the second order moment of the probability measure $q$. This condition is equivalent to the fact that $f''(1)/(f'(1) + f'(1)^2)$ is bounded a.s.*

This assumption does not require that $\mathbb{E}[f'(1)^\lambda] < \infty$ for every $\lambda > 0$, contrarily to Assumption 3. But this assumption implies that the second moment of the offspring distributions is a.s. finite. It is e.g. fulfilled for geometric offspring distributions (see [13]). We focus on the case when subcritical environments may occur with positive probability, i.e. $\mathbb{P}_1(Z_1 = 0) > 0$.

**Theorem 3.2.** *We assume that $\mathbb{P}(X < 0) > 0$ and consider a sequence $k_n \overset{subexp}{\longrightarrow} \infty$. Then, for every $i \in Cl(\mathcal{I})$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_i(1 \leq Z_n \leq k_n) = -\varrho.$$

*Under the additional Assumption 4 and $\mathbb{E}[Z_1 \log^+(Z_1)] < \infty$, for every $\theta \in \big(0, \mathbb{E}[X]\big]$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_i(1 \leq Z_n \leq e^{\theta n}) = -\chi(\theta, \varrho, \Lambda)$$

The proof of the upperbound of this result is very different from that of the previous Theorem. Let us now comment the large deviations results obtained by the two previous Theorems.

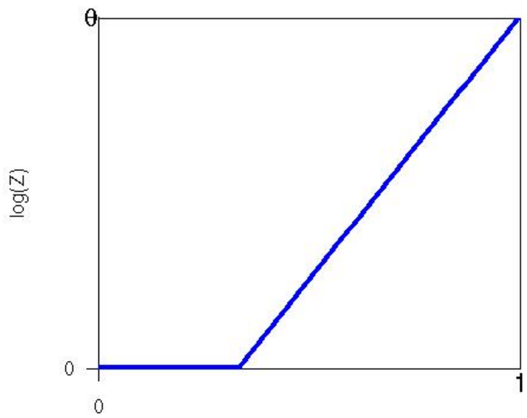


Figure 1: Most probable path for the event $\{1 \leq Z_n \leq e^{\theta n}\}$ with $0 < \theta < \theta^\star$.

We note that $\Lambda$ (and thus $\chi$) is a convex function which is continuous from below and thus has at most one discontinuity. If $\varrho < \Lambda(0)$, there is a phase transition of second order (i.e. there is a discontinuity of the second derivative of $\chi$). In particular, it occurs if $\Lambda(0) > -\log \mathbb{E}[Q(1)]$ since we know from the previous section that $\varrho \leq -\log \mathbb{E}[Q(1)]$. In contrast to the upper deviations [13, 11], there is no general description of this phase transition. It seems to heavily depend on the fine structure of the offspring distributions. In the linear fractional case, we will be able to describe

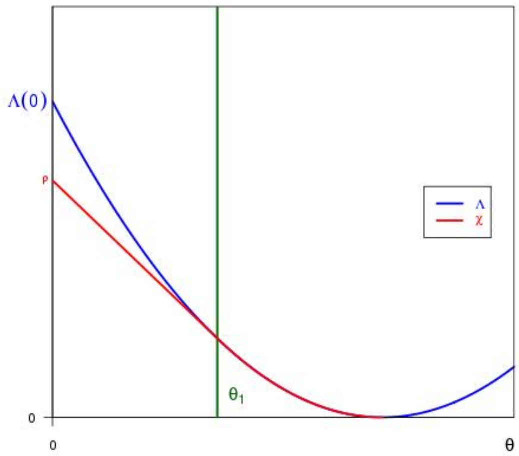


Figure 2: $\chi$ and $\Lambda$ in the case $\theta^\star > 0$.

the phase transition more in detail.

We can also explain the rate function $\chi$ and describe the large deviation event $\{1 \leq Z_n \leq e^{\theta n}\}$ for some $0 < \theta < \mathbb{E}[X]$ and $n$ large. We observe then a population being much smaller than expected, but still alive. A possible path that led to this event looks as follows (see Figure 1).
During a first period, until generation $\lfloor tn \rfloor$ $(0 \leq t \leq 1)$, the population stays small but alive, despite the fact that the process is supercritical. The probability of such an event is exponentially small and of order $\exp(-\varrho\lfloor nt \rfloor + o(n))$. Later, the population grows in a supercritical environment but less favorable than the typical one, i.e. $\{S_n - S_{\lfloor nt \rfloor} \leq \theta n\}$. This atypical environment sequence has also exponentially small probability, of order $\exp(-\Lambda(\theta/(1-t))\lfloor n(1-t) \rfloor + o(n))$. The probability of the large deviation event then results from maximizing the product of these two probabilities.

We also have the following representation of the rate function, whose proof follows exactly Lemma 4 in [11] and is left to the reader. We let $0 \leq \theta^* \leq \mathbb{E}[X]$ be such that

$$\frac{\varrho - \Lambda(\theta^*)}{\theta^*} = \inf_{0 \leq \theta \leq \mathbb{E}[X]} \frac{\varrho - \Lambda(\theta)}{\theta}$$

Then,

$$\chi(\theta, \varrho, \Lambda) = \begin{cases} \rho\big(1 - \frac{\theta}{\theta^*}\big) + \frac{\theta}{\theta^*}\Lambda(\theta^*) & \theta < \theta^* \\ \Lambda(\theta) & \theta \geq \theta^* \end{cases}.$$

We recall that $\varrho$ is known in the LF case from Theorem 2.3, and we derive the following result, which is proved in Section 8.

**Corollary 3.3.** *Assume that $f$ is a.s. linear fractional and either Assumption 3 or Assumption 4 is fulfilled. Then for all $\theta \in (0, \mathbb{E}(X)]$ and $j \geq 1$*

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_j(1 \leq Z_n \leq e^{\theta n}) = \chi(\theta, \varrho, \Lambda) = \min\big\{ -\theta - \log \mathbb{E}\big[e^{-X}\big], \Lambda(\theta)\big\}.$$

*More explicitly,* $\theta^* = \mathbb{E}\big[X\exp(-X)\big]/\mathbb{E}[\exp(-X)]$. *If* $\theta < \theta^*$, *then* $\chi(\theta, \varrho, \Lambda) = -\theta - \log\mathbb{E}\big[e^{-X}\big]$, *otherwise* $\chi(\theta, \varrho, \Lambda) = \Lambda(\theta)$.

Note that if the offspring-distributions are geometric, Assumption 4 is automatically fulfilled (see [13]). Moreover, except for the degenrated case $\mathbb{P}(Z_1 = 0) = 1$, we have $\mathbb{P}(Z_1 = 1) > 0$ in the linear fractional case.

# 4 The Geiger construction for a branching process in varying environment (BPVE)

In this section, we work in a quenched environment, which means that we fix the environment $e := (q_1, q_2, \ldots)$. We consider a branching process in varying environment $e$ and denote by $\mathsf{P}(\cdot)$ (resp. $\mathsf{E}$) the associated probability (resp. expectation), i.e.

$$\mathsf{P}(Z_1 = k_1, \cdots, Z_n = k_n) = \mathbb{P}(Z_1 = k_1, \cdots, Z_n = k_n | \mathcal{E} = e) \ .$$

Thus $(f_1, f_2, \ldots)$ is fixed and and the probability generating function of $Z$ is given by

$$\mathsf{E}\big[s^{Z_n},\ Z_0 = k\big] = f_{0,n}(s)^k \qquad (0 \le s \le 1) \ .$$

We use a construction of $Z$ conditioned on survival, which is due to [21][Proposition 2.1] and extends the spine construction of Galton Watson processes [34]. In each generation, the individuals are labeled by the integers $i = 1, 2, \cdots$ in a breadth-first manner ('from the left to the right'). We follow then the 'ancestral line' of the leftmost individual having a descendant in generation $n$. This line is denoted by $\mathbb{L}$. It means that in generation $k$, the descendance of the individual labeled $\mathbb{L}_k$ survives until time $n$, whereas all the individuals whose label is less than $\mathbb{L}_k$ become extinct before time $n$. The Geiger construction ensures that to the left of $\mathbb{L}$, independent subtrees conditioned on extinction in generation $n$ are growing. To the right of $\mathbb{L}$, independent unconditioned trees are evolving. Moreover the joint distribution of $\mathbb{L}_k$ and the number of offsprings in generation $k$ is known (see e.g. [1]) and for every $k \ge 1$,

$$\mathsf{P}(Z_k = z, \mathbb{L}_k = l | Z_{k-1} = 1, Z_n > 0) = q_k(z) \frac{\mathsf{P}(Z_n > 0\ | Z_k = 1)\mathsf{P}(Z_n = 0 | Z_k = 1)^{l-1}}{\mathsf{P}(Z_n > 0\ | Z_{k-1} = 1)} \ . \qquad (4.1)$$

Note that in [1], $L$ is defined as the number of trees to the left of $\mathbb{L}$, and thus $L = \mathbb{L} - 1$.
Let us now explain this construction in detail. We assume that the process starts with $Z_0 = z$ and denote $\mathsf{P}_z(\cdot) := \mathsf{P}(\cdot | Z_0 = z)$. We define for $0 \le k < n$,

$$\mathsf{p}_{k,n} := \mathsf{P}(Z_n > 0 \mid Z_k = 1) = 1 - f_{k,n}(0), \qquad \mathsf{p}_{n,n} := 1.$$

Let us specify the distribution of the number $Y_k$ of unconditioned trees founded by the ancestral line in generation $k$. In generation 0, for $0 \le i \le z - 1$,

$$\begin{aligned}\mathsf{P}_z(Y_0 = i | Z_n > 0) = \mathsf{P}_z(\mathbb{L}_0 = z - i \mid Z_n > 0) &= \frac{\mathsf{P}(Z_n > 0\ | Z_0 = 1)\mathsf{P}(Z_n = 0 | Z_0 = 1)^{z-i-1}}{\mathsf{P}(Z_n > 0\ | Z_0 = z)} \\ &= \frac{1 - f_{0,n}(0)}{\mathsf{P}_z(Z_n > 0)} f_{0,n}(0)^{z-i-1}. \end{aligned} \qquad (4.2)$$

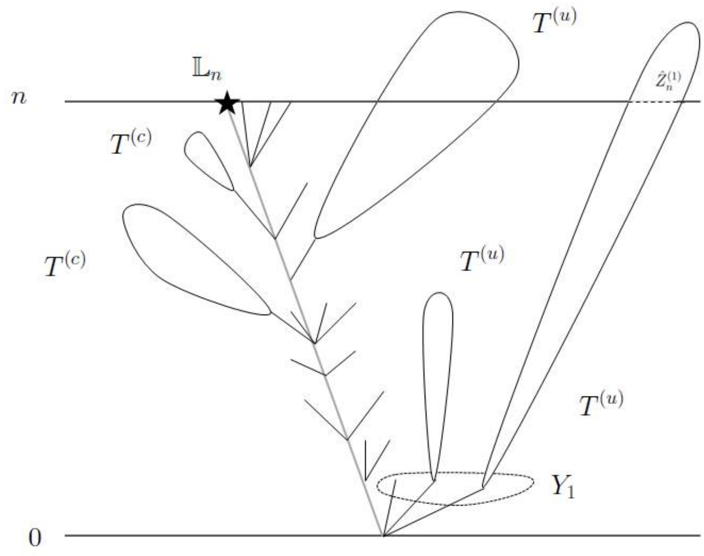


Figure 3: Geiger construction with $T^{(c)}$ trees conditioned on extinction and $T^{(u)}$ unconditioned trees.

More generally, for all $1 \leq k \leq n$ and $i \geq 0$, (4.1) yields

$$\begin{aligned}
\mathsf{P}(Y_k = i | Z_n > 0) &:= \mathsf{P}(Z_k - \mathbb{L}_k = i | Z_n > 0, Z_{k-1} = 1) \\
&= \sum_{j=i+1}^{\infty} \mathsf{P}(Z_k = j, \mathbb{L}_k = j - i | Z_n > 0, Z_{k-1} = 1) \\
&= \sum_{j=i+1}^{\infty} q_k(j) \frac{\mathsf{p}_{k,n} f_{k,n}(0)^{j-i-1}}{\mathsf{p}_{k-1,n}} \\
&= \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} \sum_{j=i+1}^{\infty} q_k(j) f_{k,n}(0)^{j-i-1}. \qquad (4.3)
\end{aligned}$$

Finally, we note that $f_{n,n}(0) = 0$, so for $k = n$, we have $\mathsf{P}(Y_n = i | Z_n > 0) = \frac{q_n(i+1)}{p_{n-1,n}}$.

Here, we do not require the full description of the conditioned tree since we are only interested in the number of individuals at time $n$. Thus we do not have to consider the trees conditioned on extinction, which grow to the left of $\mathbb{L}$. Hence, we can construct the population alive in generation $n$ using the i.i.d random variables $\hat{Y}_0, \hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_n$ whose distribution is specified by (4.2) and (4.3) :

$$\mathbb{P}(\hat{Y}_k) = \mathsf{P}(Y_k = i | Z_n > 0)$$

Let $(\hat{Z}_j^{(k)})_{j \geq 0}$ be independent branching processes in varying environment which are distributed as $Z$ for $j > k$ and satisfy

$$\hat{Z}_j^{(k)} := 0 \text{ for } \quad j < k, \qquad \hat{Z}_k^{(k)} := \hat{Y}_k.$$

More precisely, for all $0 \le k \le n$ and $z_0, \cdots, z_n \ge 0$,

$$\begin{aligned}\mathsf{P}(\hat{Z}_0^{(k)} = 0, \cdots, \hat{Z}_{k-1}^{(k)} = 0, \hat{Z}_k^{(k)} = z_k, \hat{Z}_{k+1}^{(k)} = z_{k+1}, \cdots, \hat{Z}_n^{(k)} = z_n) \\ = \mathsf{P}(\hat{Y}_k = z_k)\mathsf{P}(Z_{k+1} = z_{k+1}, \cdots, Z_n = z_n \mid Z_k = z_k).\end{aligned}$$

The sizes of the independent subtrees generated by the ancestral line in generation $k$, which may survive until generation $n$, are given by $(\hat{Z}_j^{(k)})_{0\le j\le n}$, $0 \le k \le n-1$. In particular,

$$\mathcal{L}(Z_n | Z_n > 0) = \mathcal{L}(\hat{Z}_n^{(0)} + \ldots + \hat{Z}_n^{(n-1)} + \hat{Y}_n + 1)\ . \tag{4.4}$$

**Lemma 4.1.** *The probability that all subtrees emerging before generation $n$ become extinct before generation $n$ is given for $z \ge 1$ by*

$$\mathsf{P}_z(\hat{Z}_n^{(0)} + \ldots + \hat{Z}_n^{(n-1)} = 0) = \prod_{k=0}^{n-1} \mathsf{P}_z(\hat{Z}_n^{(k)} = 0) = \prod_{k=0}^{n-1} \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} f_k'(f_{k,n}(0)),$$

*where we use the following convenient notation $f_0(s) := s^z$, $p_{-1,n} := p_{0,n}^z$.*

*Proof.* First, we compute the probability that the subtree generated by the ancestral line in generation $k$ does not survive until generation $n$, i.e. $\{\hat{Z}_n^{(k)} = 0\}$. By (4.3), for $k \ge 1$,

$$\begin{aligned}\mathsf{P}_z(\hat{Z}_n^{(k)} = 0) &= \sum_{i=0}^{\infty} \mathsf{P}_z(\hat{Y}_k | Z_n > 0)\mathsf{P}(Z_n = 0 | Z_k = i) \\ &= \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} \sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty} q_k(j) f_{k,n}(0)^{j-i-1} \cdot f_{k,n}(0)^i \\ &= \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} \sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty} q_k(j) f_{k,n}(0)^{j-1} \\ &= \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} \sum_{j=1}^{\infty} j q_k(j) f_{k,n}(0)^{j-1} \\ &= \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} f_k'(f_{k,n}(0)).\end{aligned}$$

Similarly, we get from (4.2) that

$$\begin{aligned}\mathsf{P}_z(\hat{Z}_n^{(0)} = 0) &= \sum_{i=0}^{z-1} \mathsf{P}_z(Y_0 = i | Z_n > 0)\mathsf{P}(Z_n = 0 | Z_0 = i) \\ &= \sum_{i=0}^{z-1} \frac{1 - f_{0,n}(0)}{\mathsf{P}_z(Z_n > 0)} f_{0,n}(0)^{z-i-1} f_{0,n}(0)^i \\ &= = \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} z f_{k,n}(0)^{z-1} = \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} f_k'(f_{k,n}(0)).\end{aligned}$$

with the convention $f_0(s) := s^z$. Adding that the subtrees given by $(\hat{Z}_j^{(k)})_{j\ge 0}$ are independent yields the claim. $\square$

For the next lemma, we introduce the last generation before $n$ when the environment allows extinction :

$$\kappa_n := \sup\{1 \le k \le n \ :\ q_k(0) > 0\}, \qquad (\sup \varnothing = 0).$$

**Lemma 4.2.** *Let $z_0 \in \mathcal{I}$ be the smallest element in $\mathcal{I}$. Then,*

$$\mathsf{P}_{z_0}(Z_n = z_0) = \frac{q_{\kappa_n}(z_0)}{\mathsf{p}_{\kappa_n-1,\kappa_n}} \times \prod_{k=0}^{\kappa_n-1} \frac{\mathsf{p}_{k,\kappa_n}}{\mathsf{p}_{k-1,\kappa_n}} f_k'(f_{k,\kappa_n}(0)) \times \prod_{j=\kappa_n+1}^{n} q_j(1)^{z_0},$$

*where we recall the following convenient notation $f_0(s) = s^z$, $p_{-1,n} = p_{0,n}^z$.*

*Proof.* Recall that by definition of $\mathcal{I}$, $q(0) > 0$ implies $q(k) = 0$ for every $1 \le k < z_0$. We first deal with the case $\kappa_n > 0$. Then

$$q_{\kappa_n}(0) > 0, \quad q_{\kappa_n}(k) = 0 \text{ if } 1 \le k < z_0 \ ; \qquad q_{\kappa_n+1}(0) = \cdots = q_n(0) = 0 \ .$$

In particular the number of individuals in generation $\kappa_n$ is at least $z_0$ times the number of individuals in generation $\kappa_n - 1$ who leave at least one offspring in generation $\kappa_n$. Moreover, as extinction is not possible after generation $\kappa_n$, it holds that $Z_{\kappa_n} \le Z_{\kappa_n+1} \le \cdots \le Z_n$.

Let us consider the event $Z_n = z_0 > 0$. Then $Z_{\kappa_n-1} > 0$ and $Z_{\kappa_n} \ge z_0$. Moreover $Z_{\kappa_n} = Z_{\kappa_n+1} = \cdots = Z_n = z_0$ and only a single individual in generation $\kappa_n - 1$ leaves one offspring (or more) in generation $\kappa_n$. This individual lives on the ancestral line. Thus all the subtrees to the right of the ancestral line which are born before generation $\kappa_n$ have become extinct before generation $\kappa_n$, i.e. $\hat{Z}^{(0)}_{\kappa_n} = \ldots = \hat{Z}^{(\kappa_n-1)}_{\kappa_n} = 0$. In generation $\kappa_n - 1$, the individual on the ancestral line has $z_0$ offsprings. After generation $\kappa_n$, all individuals leave exactly one offspring which is the only way to keep the population constant until generation $n$, since $q_{\kappa_n+1}(0) = \cdots = q_n(0) = 0$. Moreover (4.3) simplifies to $\mathsf{P}(\hat{Y}_{\kappa_n} = z_0 - 1) = q_{\kappa_n}(z_0)/\mathsf{p}_{\kappa_n-1,\kappa_n}$. Using the previous lemma, it can be written as

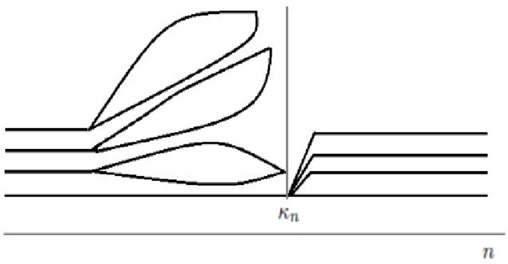


Figure 4: Illustration of the proof of Lemma 4.2.

follows

$$\begin{aligned}
&\mathsf{P}_{z_0}(Z_n = z_0) \\
&\quad = \mathsf{P}_{z_0}(\hat{Z}^{(0)}_{\kappa_n} = \ldots = \hat{Z}^{(\kappa_n-1)}_{\kappa_n} = 0)\mathsf{P}(\hat{Y}_{\kappa_n} = z_0 - 1)\mathsf{P}_{z_0}(\hat{Z}^{(\kappa_n)}_n + \ldots + \hat{Z}^{(n-1)}_n + Y_n + 1 = z_0) \\
&\quad = \mathsf{P}_{z_0}(\hat{Z}^{(0)}_{\kappa_n} = \ldots = \hat{Z}^{(\kappa_n-1)}_{\kappa_n} = 0)\frac{q_{\kappa_n}(z_0)}{\mathsf{p}_{\kappa_n-1,\kappa_n}}\mathsf{P}_{z_0}(\hat{Z}^{(\kappa_n)}_n + \ldots + \hat{Z}^{(n-1)}_n + Y_n + 1 = z_0) \\
&\quad = \frac{q_{\kappa_n}(z_0)}{\mathsf{p}_{\kappa_n-1,\kappa_n}}\left[\prod_{k=0}^{\kappa_n-1}\frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}}f'_k(f_{k,\kappa_n}(0))\right]\mathsf{P}_{z_0}(\hat{Z}^{(\kappa_n)}_n + \ldots + \hat{Z}^{(n-1)}_n + Y_n + 1 = z_0) \\
&\quad = \frac{q_{\kappa_n}(z_0)}{\mathsf{p}_{\kappa_n-1,\kappa_n}}\left[\prod_{k=0}^{\kappa_n-1}\frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}}f'_k(f_{k,\kappa_n}(0))\right]\prod_{j=\kappa_n+1}^{n} q_j(1)^{z_0}.
\end{aligned}$$

Recall that after generation $\kappa_n$, each individual has at least one offspring and thus $\mathsf{p}_{j,n} = \mathsf{p}_{j,\kappa_n}$ for any $j < \kappa_n$. This ends up the proof in the case $\kappa_n > 0$. The case when $\kappa_n = 0$ is easier. Indeed,

$$\mathsf{P}_{z_0}(Z_n = z_0) = \mathsf{P}_{z_0}(Z_1 = \cdots = Z_n = z_0) = \prod_{j=1}^{n} q_j(1)^{z_0}$$

since $q_{\kappa_n+1}(0) = \cdots = q_n(0) = 0$ and $Z$ is nondecreasing until generation $n$. $\square$

# 5 Proof of Theorem 2.1 : the probability of staying positive but bounded

In this section, we prove Theorem 2.1 with the help of two lemmas. The first lemma establishes the existence of a proper 'common' limit.

**Lemma 5.1.** *Assume that $z \geq 1$ satisfies $\mathbb{P}(Q(0) > 0, Q(z) > 0) > 0$. Then for all $k, j \in Cl(\{z\})$, the following limits exist in $[0, \infty)$ and coincide*

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_k(Z_n = j) = \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(Z_n = z).$$

*Moreover, for every sequence $k_n = o(n)$,*

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(Z_n = z) = \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(1 \leq Z_n \leq k_n).$$

*Proof.* Note that for every $k \geq 1$, $\mathbb{P}_k(Z_1 = z) > 0$ since

$$\mathbb{P}_k\big(Z_1 = z \mid Q_1\big) \geq Q_1(0)^{k-1} Q_1(z), \qquad \mathbb{P}(Q(0) > 0, Q(z) > 0) > 0.$$

We know that by Markov property, for all $m, n \geq 1$,

$$\mathbb{P}_z(Z_{n+m} = z) \geq \mathbb{P}_z(Z_n = z)\mathbb{P}_z(Z_m = z). \tag{5.1}$$

Adding that $\mathbb{P}_z(Z_1 = z) > 0$, we obtain that the sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n := -\log \mathbb{P}_z(Z_n = z)$ is finite and subadditive. Then Fekete's lemma ensures that $\lim_{n\to\infty} a_n/n$ exists and belongs to $[0, \infty)$. Next, if $j, k \in Cl(\{z\})$, there exist $l, m \geq 0$ such that $\mathbb{P}_z(Z_l = j) > 0$ and $\mathbb{P}_z(Z_m = k) > 0$. We get

$$\mathbb{P}_k(Z_{n+l+1} = j) \geq \mathbb{P}_k(Z_1 = z)\mathbb{P}_z(Z_n = z)\mathbb{P}_z(Z_l = j)$$

and

$$\mathbb{P}_z(Z_{m+n+1} = z) \geq \mathbb{P}_z(Z_m = k)\mathbb{P}_k(Z_n = j)\mathbb{P}_j(Z_1 = z).$$

Adding that $\mathbb{P}_j(Z_1 = z) > 0$ for $j \in Cl(\{z\})$, we obtain

$$\liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_k(Z_n = j) \geq \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(Z_n = z) \geq \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_k(Z_n = j),$$

which yields the first result.

For the second part of the lemma, we simply observe that $\mathbb{P}_z(Z_n = z) \leq \mathbb{P}_z(1 \leq Z_n \leq k_n)$ for $n$ large enough. To prove the converse inequality, define for $\varepsilon > 0$ the set

$$\mathcal{A}_\varepsilon := \{q \in \Delta | q(0) > \varepsilon,\, q(z) > \varepsilon\}\ .$$

According to the definition of $\mathcal{I}$ and the assumption, $\mathbb{P}(Q \in \mathcal{A}_\varepsilon) > 0$ if $\varepsilon$ is chosen small enough. Thus we get that

$$\begin{aligned}
\mathbb{P}_z(Z_n = z) &\geq \mathbb{P}_z(1 \leq Z_{n-1} \leq k_n) \min_{1\leq j\leq k_n} \mathbb{P}_j(Z_1 = z) \\
&\geq \mathbb{P}_z(1 \leq Z_{n-1} \leq k_n)\mathbb{P}(Q \in \mathcal{A}_\varepsilon) \min_{1\leq j\leq k_n} \mathbb{E}[\mathbb{P}_1(Z_1 = z)\mathbb{P}_1(Z_1 = 0|Q)^{j-1} | Q \in \mathcal{A}_\varepsilon] \\
&\geq \mathbb{P}_z(1 \leq Z_{n-1} \leq k_n)\mathbb{P}(Q \in \mathcal{A}_\varepsilon)\ \varepsilon^{k_n}.
\end{aligned}$$

Taking the logarithm yields

$$\liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(Z_n = z) \geq \limsup_{n\to\infty} \left( \tfrac{1}{n} \log \mathbb{P}_z(1 \leq Z_{n-1} \leq k_n) + \log(\varepsilon) \tfrac{k_n}{n} \right).$$

Adding that $k_n = o(n)$ by assumption gives the claim. $\square$

Next, we will prove a representation of the limit $\rho$ in terms of generating functions. First we treat the case $\mathbb{P}(Q(0) > 0) > 0$ and then $\mathbb{P}(Q(0) = 0) = 1$.

**Lemma 5.2.** *Assume that $\mathbb{P}_1(Z_1 = 0) > 0$. Then for all $i, j \in Cl(\mathcal{I})$,*

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_i(Z_n = j) = \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}\Big[ Q_n(z_0) f_{0,n}(0)^{z_0 - 1} \prod_{i=1}^{n-1} f_i'\big(f_{i,n}(0)\big) \Big],$$

*where $z_0$ is the smallest element in $\mathcal{I}$.*

Note that $\mathbb{P}_1(Z_1 = 0) > 0$ is equivalent to $\mathbb{P}(Q(0) > 0) > 0$ and in view of Lemma 5.1, we only have to prove the result for $k = j = z_0$, where $z_0$ is the smallest element in $\mathcal{I}$. Differentiation of the probability generating function of $Z_n$ yields the result for $z_0 = 1$. The generalization of the result for $z_0 \neq 1$ via higher order derivatives of generating functions appears to be complicated. Instead, we use probabilistic arguments, involving the Geiger construction of the previous section.

*Proof.* First, the result is obvious when $z_0 = 1 \in \mathcal{I}$ since

$$\mathbb{P}(Z_n = 1 | \mathcal{E}) = \frac{d}{ds} f_{0,n}(s)\big|_{s=0} = f_n'(0) \cdot \prod_{i=1}^{n-1} f_i'(f_{i,n}(0)).$$

For the case $z_0 > 1$, we start by proving the lowerbound. Using Lemma 4.1 and (4.2) with a telescope argument and recalling that $\mathbb{P}_{z_0}(Z_n > 0|\mathcal{E}) = p_{-1,n}$, we have

$$\begin{aligned}
\mathbb{P}_{z_0}(Z_n = z_0) &= \mathbb{E}\Big[ \mathbb{P}_{z_0}(Z_n = z_0 | Z_n > 0, \mathcal{E}) \mathbb{P}_{z_0}(Z_n > 0 | \mathcal{E}) \Big] \\
&= \mathbb{E}\Big[ \mathbb{P}_{z_0}(\hat{Z}_n^{(0)} + \cdots + \hat{Z}_n^{(n-1)} + \hat{Y}_n + 1 = z_0 | \mathcal{E}) \mathbb{P}_{z_0}(Z_n > 0 | \mathcal{E}) \Big] \\
&\geq \mathbb{E}\Big[ \mathbb{P}_{z_0}(\hat{Z}_n^{(0)} + \cdots + \hat{Z}_n^{(n-1)} = 0, \hat{Y}_n = z_0 - 1 | \mathcal{E}) \mathbb{P}_{z_0}(Z_n > 0 | \mathcal{E}) \Big] \\
&= \mathbb{E}\Big[ \mathbb{P}_{z_0}(Z_n > 0 | \mathcal{E}) \ \frac{Q_n(z_0)}{\mathsf{p}_{n-1,n}} \prod_{i=0}^{n-1} \frac{\mathsf{p}_{i,n}}{\mathsf{p}_{i-1,n}} f_i'(f_{i,n}(0)) \Big] \\
&= \mathbb{E}\Big[ Q_n(z_0) \prod_{i=0}^{n} f_i'\big(f_{i,n}(0)\big) \Big] \ .
\end{aligned} \tag{5.2}$$

Recalling also that $f_0'(s) = z_0 s^{z_0 - 1}$, we get

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_{z_0}(Z_n = z_0) \geq \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}\Big[ Q_n(z_0) f_{0,n}^{z_0 - 1}(0) \prod_{i=1}^{n} f_i'\big(f_{i,n}(0)\big) \Big].$$

Let us now prove the converse inequality. Following the previous section, $z_0$ is the smallest element in $\mathcal{I}$ and $\kappa_n$ is the (now random) last moment when there is a generation with $Q(0) > 0$. We decompose the event $\{Z_n = z_0\}$ according to $\kappa_n$ and use Lemma 4.2 :

$$\begin{aligned}
\mathbb{P}_{z_0}(Z_n = z_0) &= \mathbb{E}\Big[\mathbb{P}_{z_0}(Z_n = z_0|\mathcal{E}, Z_n > 0)\mathbb{P}_{z_0}(Z_n > 0|\mathcal{E})\Big] \\
&= \sum_{k=0}^{n} \mathbb{E}\Big[\mathbb{P}_{z_0}(Z_n = z_0|\mathcal{E}, Z_n > 0)\mathbb{P}_{z_0}(Z_n > 0|\mathcal{E}); \kappa_n = k\Big] \\
&= \sum_{k=0}^{n} \mathbb{E}\Big[\mathbb{P}_{z_0}(Z_k > 0|\mathcal{E})\frac{Q_k(z_0)}{\mathsf{p}_{k-1,k}} \prod_{i=0}^{k-1} \frac{\mathsf{p}_{i,k}}{\mathsf{p}_{i-1,k}} f_i'(f_{i,k}(0)) \prod_{j=k+1}^{n} Q_j(1)^{z_0} \ ; \ \kappa_n = k\Big] \\
&\le \sum_{k=0}^{n} \mathbb{E}\Big[Q_k(z_0) \prod_{i=0}^{k-1} f_i'(f_{i,k}(0))\Big] \prod_{j=k+1}^{n} \mathbb{E}\big[Q_j(1)^{z_0}\big] \\
&= \sum_{k=1}^{n} \mathbb{E}\Big[Q_k(z_0) \prod_{i=0}^{k-1} f_i'(f_{i,k}(0))\Big]\mathbb{E}\big[Q(1)^{z_0}\big]^{n-k-1} + \mathbb{E}\big[Q(1)^{z_0}\big]^{n} .
\end{aligned}$$

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+$ and $b > 0$. Then, by standard results on the exponential rate of sums, it holds that

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \sum_k a_k b^{n-k} = \max\Big\{ \limsup_{n\to\infty} \tfrac{1}{n} \log a_n, \log b \Big\} .$$

Adding that $f_0'(f_{0,n}(0)) = z_0 f_{0,n}^{z_0-1}(0)$, we get

$$\begin{aligned}
&\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_{z_0}(Z_n = z_0) \\
&\quad\le \max\Big\{ \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}\big[Q_n(z_0) f_{0,n}(0)^{z_0-1} \prod_{i=1}^{n} f_i'(f_{i,n}(0))\big]; \log \mathbb{E}\big[Q(1)^{z_0}\big]\Big\}
\end{aligned}$$

We now prove that the first term always realizes the maximum:

$$\begin{aligned}
\mathbb{E}\Big[Q_n(z_0) f_{0,n}(0)^{z_0-1} \prod_{i=1}^{n-1} f_i'(0)\Big] &= \mathbb{E}\Big[Q_n(z_0)\mathbb{P}_1(Z_n = 0|\mathcal{E})^{z_0-1} \prod_{i=1}^{n-1} Q_i(1)\Big] \\
&\ge \mathbb{E}\Big[Q_n(z_0)\Big(Q_n(0) \prod_{i=1}^{n-1} Q_i(1)\Big)^{z_0-1} \prod_{i=1}^{n-1} Q_i(1)\Big] \\
&\ge \mathbb{E}\big[Q_n(z_0)Q_n(0)^{z_0-1}\big]\mathbb{E}\big[Q(1)^{z_0}\big]^{n} ,
\end{aligned}$$

where by definition of $z_0$, $\mathbb{E}\big[Q_n(z_0)Q_n(0)^{z_0-1}\big] > 0$.
Finally, let us prove the existence of $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\Big[Q_n(z_0) f_{0,n}(0)^{z_0-1} \prod_{i=1}^{n} f_i'\big(f_{i,n}(0)\big)\Big]$. We follow (5.2) to see that

$$\phi_n := z_0\mathbb{E}\Big[Q_n(z_0) f_{0,n}(0)^{z_0-1} \prod_{i=1}^{n} f_i'\big(f_{i,n}(0)\big)\Big] = \mathbb{P}_{z_0}(\hat{Z}_n^{(0)} + \ldots + \hat{Z}_n^{(n-1)} = 0, Z_n = z_0).$$

Starting from $Z_0 = z_0$, it is, up to the factor $z_0$, the probability of having $z_0$-many individuals in generation $n$, where all individuals in generation $n$ have a common ancestor in generation $n-1$. By Markov property, for $k = 1, \ldots, n$

$$\begin{aligned}
&\mathbb{P}_{z_0}(\hat{Z}_n^{(0)} + \ldots + \hat{Z}_n^{(n-1)} = 0, Z_n = z_0) \\
&\quad\ge \mathbb{P}_{z_0}(\hat{Z}_k^{(0)} + \ldots + \hat{Z}_k^{(k-1)} = 0, Z_k = z_0)\mathbb{P}_{z_0}(\hat{Z}_{n-k}^{(0)} + \ldots + \hat{Z}_{n-k}^{(n-k-1)} = 0, Z_{n-k} = z_0).
\end{aligned}$$

The same subadditivity arguments as in the proof of Lemma 5.1 applied to $\phi_n$ yield existence of the limit of $\frac{1}{n} \log \mathbb{E}\Big[Q_n(z_0) f_{0,n}(0)^{z_0-1} \prod_{i=1}^{n} f_i'\big(f_{i,n}(0)\big)\Big]$. This ends up the proof. $\square$

# 6 Upperbound for $\varrho$ : proof of Proposition 2.2

Here, we prove Proposition 2.2, which ensures that $\varrho \leq \Lambda(0)$. It means that small but positive values can always be realized by a suitable exceptional environment, which is "critical". We focus on the nontrivial case when $\Lambda(0) < \infty$. The proof of Proposition 2.2 can be splitted into two subcases, which correspond to the the two following Propositions.

**Proposition 6.1.** *Under Assumptions 1 and 2, $\rho \leq \Lambda(0)$.*

**Proposition 6.2.** *Assume that $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{P}(X = 0) > 0$. Then*

$$\rho \leq -\log \mathbb{P}(X = 0) = \Lambda(0). \tag{6.1}$$

*Proof of Proposition 6.1.* Let $\mathcal{I}$ be defined as in the introduction. For the proof, we use a standard approximation argument (see e.g. [13]) and consider the event $E_{x,n} := \{\min_{i=1,\dots,n} X_i > x\}$ for $x < 0$. Then, $\mathbb{P}(X > x) > 0$ since we are in the supercritcal regime and for every $s \geq 0$ it holds that $\mathbb{E}\big[|X|e^{-sX}|X > x\big] < \infty$. As $\mathbb{P}(X < 0) > 0$, $\mathbb{E}\big[|X|e^{-sX}|X > x\big] < \infty$ tends to infinity as $s \to \infty$. By the preceding arguments, $\mathbb{E}[e^{-sX}|X > x]$ is differentiable with respect to $s$ for $s > 0$ and the expectation and the differentiation may be interchanged by the dominated convergence theorem. Let us call $s = \nu_x$ a point where the minimum is reached, such that $\inf_{s\geq 0} \mathbb{E}[e^{-sX}|X > x] = \mathbb{E}[e^{-\nu_x X}|X > -x]$ and $\frac{d}{ds}\mathbb{E}[e^{-sX}|X > x]\big|_{s=\nu_x} = \mathbb{E}[Xe^{-\nu_x X}|X > x] = 0$ . Now, we follow the arguments of the previous proofs. In view of the second part of Lemma 5.1, for every sequence $k_n = o(n)$,

$$-\rho = \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(Z_n = z) = \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(1 \leq Z_n \leq k_n).$$

Next we will change to the measure $\mathbf{P}$, defined by

$$\mathbf{P}(X \in dy) = \frac{e^{-\nu_x y}\mathbb{P}(X \in dy|E_{x,n})}{\mu} \tag{6.2}$$

where $\mu := \mathbb{E}[e^{-\nu_x X}|X > x]$. Under $\mathbf{P}$, $\mathbf{E}[X] = 0$ and $S$ is a recurrent random walk. Let $c > 0$ be so large such that $\mathbf{P}(L_n \geq 0, S_n \leq c) > 0$ for every $n$. Then

$$\begin{aligned}\mathbb{P}_z(1 \leq Z_n \leq k_n|E_{x,n}) &= \mu^n \mathbf{E}\Big[\mathbb{P}_z(1 \leq Z_n \leq k_n|\mathcal{E})e^{\nu_x S_n}\Big] \\ &\geq \mu^n \mathbf{E}\Big[\mathbb{P}_z(1 \leq Z_n \leq k_n|\mathcal{E}); L_n \geq 0, S_n \leq c\Big] .\end{aligned} \tag{6.3}$$

Note that

$$\mathbb{P}_z(1 \leq Z_n \leq k_n|\mathcal{E}) = \mathbb{P}_z(Z_n > 0|\mathcal{E}) - \mathbb{P}_z(Z_n > k_n|\mathcal{E}) \qquad \text{a.s.}$$

and by Markov inequality,

$$\mathbb{P}_z(Z_n > k|\mathcal{E}) \leq \frac{ze^{S_n}}{k} \qquad \text{a.s.}$$

Using this, we get that

$$\mathbb{P}_z(1 \leq Z_n \leq k_n|\mathcal{E}) \geq \mathbb{P}_z(Z_n > 0|\mathcal{E}) - ze^{S_n}\, k_n^{-1} \qquad \text{a.s.}$$

Plugging this into (6.3) and setting $b_n := \mathbf{P}(L_n \geq 0, S_n \leq c)$, we get

$$\begin{aligned}\mathbb{P}_z(1 \leq Z_n \leq k_n) &\geq \mathbb{P}_z(1 \leq Z_n \leq k_n; E_{x,n}) \\ &\geq\ \mu^n \mathbf{E}_z\Big[\mathbb{1}_{\{Z_n>0\}} - ze^c/k_n\,; L_n \geq 0, S_n \leq c\Big]\mathbb{P}(E_{x,n}) \\ &=\ \mu^n b_n\Big(\mathbf{P}(Z_n > 0|L_n \geq 0, S_n \leq c) - ze^c/k_n\Big)\mathbb{P}(X > x)^n .\end{aligned} \tag{6.4}$$

Assumption 1 together with our construction of $\mathbf{E}$ implies $\mathbf{Var}(X) \le \mathbb{E}[e^{-\nu_x X}]^{-1}\mathbb{E}[X^2 e^{-\nu_x X}] < \infty$. Then from [2, Proposition 2.3], we have $b_n = O(n^{-3/2})$, and thus $\lim_{n\to\infty} \frac{1}{n}\log b_n = 0$. Next let $k_n = n^{-1/2}$. As Assumption 2 holds under $\mathbb{P}$ (and then also under $\mathbf{P}$), we can apply the forthcoming Lemma 6.3 to get that

$$\begin{aligned}\liminf_{n\to\infty} \tfrac{1}{n}\log \mathbb{P}_z(1 \le Z_n \le k_n) &\ge \log\mu + \log\mathbb{P}(X > x)\\ &= \log\mathbb{E}[e^{-\nu_x X}|X > x] + \log\mathbb{P}(X > x)\\ &= -\sup_{s\le 0}\big\{ -\log\mathbb{E}[e^{-sX}; X > x]\big\} \ .\end{aligned}$$

By monotone convergence, we let $x \to -\infty$ and

$$\liminf_{n\to\infty} \tfrac{1}{n}\log \mathbb{P}_z(1 \le Z_n \le k_n) \ge -\sup_{s\le 0}\big\{ -\log\mathbb{E}[e^{-sX}]\big\} = -\Lambda(0).$$

As $k_n = o(n)$, we apply Lemma 5.1 to end up the proof. □

For the proof above, we need the following lemma..

**Lemma 6.3.** *Assume that* $\mathbf{E}[X] = 0$, $\mathbf{Var}(X) < \infty$, *that for every* $n \ge 0$, $\mathbf{P}(L_n \ge 0, S_n \le c) > 0$ *for some* $c > 0$ *and that Assumption 2 holds. Then*

$$\liminf_{n\to\infty} \mathbf{P}(Z_n > 0 | L_n \ge 0, S_n \le c) > 0 \ .$$

*Proof.* Let us briefly explain why Assumption 2 implies $\mathbf{E}[(\log^+ \xi_Q(a))^{\frac{1}{2}+\varepsilon}] < \infty$ which is required in [2]. By definition of $\mathbf{E}$,

$$\begin{aligned}\mathbb{E}\big[(\log^+ \xi_Q(a))^{\frac{1}{2}+\varepsilon}|X > -x\big] &= \mu\mathbb{E}\big[e^{\nu_x X}(\log^+ \xi_Q(a))^{\frac{1}{2}+\varepsilon}\big]\\ &\ge \mu\mathbf{E}\big[e^{-\nu_x x}(\log^+ \xi_Q(a))^{\frac{1}{2}+\varepsilon}\big] \ ,\end{aligned}$$

as $X > -x$ $\mathbf{P}$-a.s.

The proof now follows essentially [2]. Here, we just present the main steps. From Propositions 2.1 and 2.3 in [2], for all $\theta, c > 0$ large enough, there exists an $d > 0$ such that

$$\mathbf{E}\big[e^{-\theta S_n}; L_n \ge 0\big] \sim d\ \mathbf{P}(L_n \ge 0, S_n \le c) \tag{6.5}$$

as $n \to \infty$. Next, we recall the well-known estimate (see e.g. [5][Lemma 2])

$$\mathbf{P}(Z_n > 0 \mid \mathcal{E}) \ge \frac{1}{e^{-S_n} + \sum_{i=0}^{n-1}\eta_{i+1}e^{-S_i}} \quad \text{a.s.},$$

where $\eta_i := \sum_{y=1}^{\infty} y(y-1)Q_i(y)/m_{Q_i}^2$. Moreover, we rewrite the expectation above:

$$\begin{aligned}&\mathbf{E}\Big[\frac{1}{e^{-S_n} + \sum_{i=0}^{n-1}\eta_{i+1}e^{-S_i}}; L_n \ge 0, S_n \le c\Big]\\ &\quad\ge \mathbf{E}\Big[\frac{1}{1 + \sum_{i=0}^{\lfloor n/2\rfloor}\eta_{i+1}e^{-S_i} + e^{-S_{\lfloor n/2\rfloor}}\sum_{i=\lfloor n/2\rfloor+1}^{n-1}\eta_{i+1}e^{S_{\lfloor n/2\rfloor}-S_i}}; L_n \ge 0, S_n \le c\Big]\\ &\quad\ge \mathbf{E}\Big[\frac{(c - S_n)^+ \wedge 1}{1 + \sum_{i=0}^{\lfloor n/2\rfloor}\eta_{i+1}e^{-S_i} + \sum_{i=\lfloor n/2\rfloor+1}^{n-1}\eta_{i+1}e^{S_{\lfloor n/2\rfloor}-S_i}}; L_n \ge 0\Big]\\ &\quad= \mathbf{E}\Big[\varphi(U_n, \tilde{V}_n, S_n); L_n \ge 0\Big]\\ &\quad\ge e^{-c/2}\mathbf{E}\Big[e^{-S_n/2}\varphi(U_n, \tilde{V}_n, S_n); L_n \ge 0\Big] \ ,\end{aligned}$$

where $U_n := \sum_{i=0}^{\lfloor n/2\rfloor} \eta_{i+1} e^{-S_i}$, $\tilde{V}_n := \sum_{i=\lfloor n/2\rfloor+1}^{n-1} \eta_{i+1} e^{S_{\lfloor n/2\rfloor}-S_i}$ and $\varphi(u,v,z) = (1+u+v)^{-1}(c-z)^+ \wedge 1$. Due to monotonicity and Lemma 3.1 in [2], the limits of $U_\infty = \lim_{n\to\infty} U_n$ and $V_\infty = \lim_{n\to\infty} V_n := \sum_{i=0}^{\lfloor n/2\rfloor} \eta_i e^{S_i}$ exist and are finite respectively under the probabilities $\mathbf{P}^+$-a.s. and $\mathbf{P}^-$-a.s. defined in [2]. Thus all conditions of Proposition 2.5 in [2] are met. Applying this proposition with $\theta = 1/2$ and using (6.5), we get that for some measure $\nu_{1/2}$ on $\mathbb{R}^+$ that

$$\begin{aligned}\liminf_{n\to\infty} \mathbf{P}(Z_n > 0 | L_n \geq 0, S_n \leq c) &\geq d^{-1} \lim_{n\to\infty} \frac{e^{-c/2}\mathbf{E}\Big[e^{-S_n/2}\varphi(U_n,\tilde{V}_n,S_n); L_n \geq 0\Big]}{\mathbf{E}\big[e^{-S_n/2}; L_n \geq 0\big]} \\ &= \int_{\mathbb{R}^3_+} \varphi(u,v,-z)\mathbf{P}^+(U_\infty \in du)\mathbf{P}^-(V_\infty \in dv)\nu_{1/2}(dz) > 0\ .\end{aligned}$$

Note that in the function $\varphi$, $z$ is changed to $-z$ for duality reasons (for details see [2]). As the limits of $U_n$ and $V_n$ are a.s. finite with respect to the corresponding measures, this yields the claim. $\square$

**Remark.** The results in [2] are only stated for non-lattice random walks. We expect that similar results are true for the lattice case. The proofs in our paper work for the lattice case, if (6.5) and [2][Proposition 2.5] also hold in the lattice case.

*Proof of Proposition 6.2.* As $\mathbb{P}(X \geq 0) = 1$, we have $\mathbb{P}(S_n = 0) = \mathbb{P}(X = 0)^n$ and $\Lambda(0) = -\log \mathbb{P}(X = 0)$.
If $\mathbb{P}(Q(1) = 1 | X = 0) = 1$, the proof is trivial. So let us assume that $\mathbb{P}(X = 0) > 0$ and $\mathbb{P}(Q(1) = 1 | X = 0) < 1$ and thus, for some suitable $\varepsilon > 0$, $\mathbb{P}(Q(0) \geq \varepsilon | X = 0) > 0$. By conditioning on the environment, we get that for $z \in \mathcal{I}$ that

$$\mathbb{P}_z(Z_n = z) \geq \mathbb{P}(X = 0)^n \cdot \mathbb{P}_z(Z_n = z | X_1 = 0, \ldots, X_n = 0).$$

For simplicity, we define a new measure $\bar{\mathbf{P}}$ on the space of all probability measures on $\mathbb{N}_0$ with expectation 1 for every measurable $A \subset \Delta$:

$$\bar{\mathbf{P}}(Q \in A) := \frac{\mathbb{P}(Q \in A; m_Q = 1)}{\mathbb{P}(m_Q = 1)} = \frac{\mathbb{P}(Q \in A; m_Q = 1)}{\mathbb{P}(X = 0)}\ .$$

Note that $\bar{\mathbf{P}}(X = 0) = 1$ and by definition, for every $z \in \mathcal{I}$ we have $\mathbb{P}(Q(z) > 0, Q(0) > 0) > 0$. It follows that there is a $z \in \mathcal{I}$ such that $\bar{\mathbf{P}}\big(Q(z) > 0, Q(0) > 0\big) > 0$. Applying Lemma 5.1, without loss of generality, we may restrict us to such a $z$. With respect to $\bar{\mathbf{P}}$, $(Z_n : n \in \mathbb{N}_0)$ is still a branching process in random environment and applying Lemma 5.2, there exists a $\bar{\rho} \in [0,\infty)$ such that

$$\begin{aligned}-\bar{\rho} &= \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(Z_n = z | X_1 = 0, \ldots, X_n = 0) \\ &= \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}\Big[Q_n(z) f_{0,n}^{z-1}(0) \prod_{i=1}^{n-1} f_i'\big(f_{i,n}(0)\big) \Big| X_1 = 0, \ldots, X_n = 0\Big].\end{aligned}$$

In the following, we will use convexity arguments. First, for all $i \leq k$ and $s \in [0,1]$, $f_{i,k}(s) \geq 1 - f_{i,k}'(1)(1-s)$. As with respect to $\bar{\mathbf{P}}$, $f_{i,k}'(1) = 1$ a.s., we get that

$$f_{i,k}(s) \geq s \quad \bar{\mathbf{P}} - \text{a.s.} \tag{6.6}$$

By (6.6), we get that for every $a \in \mathbb{N}$ fixed and $n \geq a$

$$\liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}\Big[Q_n(z) f_{0,n}^{z-1}(0) \prod_{i=1}^{n-1} f_i'\big(f_{i,n}(0)\big) \Big| X_1 = 0, \ldots, X_n = 0\Big]$$

$$\geq \liminf_{n\to\infty} \tfrac{1}{n} \log \bar{\mathbf{E}}\Big[\bar{\mathbf{E}}\Big[Q_n(z) f_{n-a,n}^{z-1}(0) \prod_{i=1}^{n-a} f_i'\big(f_{n-a,n}(0)\big) \prod_{i=n-a+1}^{n-1} f_i'\big(f_{i,n}(0)\big) \Big| Q_1, \ldots, Q_{n-a}\Big]\Big] .$$

Next, for $\varepsilon > 0$ fixed, we choose $k = k_\varepsilon \in \mathbb{N}$ large enough such that $\mathbb{P}(Q([1,k]) > \varepsilon | X = 0) \geq 1 - \varepsilon$. Then, conditionally on $\{Q([1,k]) > \varepsilon\}$, $f'(s) \geq \sum_{j=1}^{k} Q(k) s^k \geq \varepsilon s^k$ *a.s.* for $s \in [0,1]$. Using both this inequality and (6.6), we have we get that

$$\bar{\mathbf{E}}\Big[Q_n(z) f_{n-a,n}^{z-1}(0) \prod_{i=1}^{n-a} f_i'\big(f_{n-a,n}(0)\big) \prod_{i=n-a+1}^{n-1} f_i'\big(f_{i,n}(0)\big) \Big| Q_1, \ldots, Q_{n-a}\Big]$$

$$\geq \bar{\mathbf{E}}\Big[Q_n(z) Q_n(0)^{z-1} \prod_{i=1}^{n-a} f_i'\big(f_{n-a,n}(0)\big)$$

$$\times \prod_{i=n-a+1}^{n-1} f_i'\big(Q_n(0)\big); Q_1([1,k]) > \varepsilon, \ldots, Q_{n-a}([1,k]) > \varepsilon \Big| Q_1, \ldots, Q_{n-a}\Big]$$

$$\geq \bar{\mathbf{E}}\Big[Q_n(z) Q_n(0)^{z-1} \prod_{i=1}^{n-a} f_i'\big(f_{n-a,n-1}(0)\big)$$

$$\times \prod_{i=n-a+1}^{n-1} \varepsilon Q_n(0)^k; Q_1([1,k]) > \varepsilon, \ldots, Q_{n-a}([1,k]) > \varepsilon \Big| Q_1, \ldots, Q_{n-a}\Big]$$

$$\geq \bar{\mathbf{E}}\Big[\prod_{i=1}^{n-a} f_i'\big(f_{n-a,n-1}(0)\big); Q_1([1,k]) > \varepsilon, \ldots, Q_{n-a}([1,k]) > \varepsilon \Big| Q_1, \ldots, Q_{n-a}\Big]$$

$$.\bar{\mathbf{E}}\Big[\varepsilon^{a-2} Q_n(z) Q_n(0)^{z-1} Q_n(0)^{(a-2)k+1}\Big] ,$$

where the second expectation is strictly positive as $\bar{\mathbf{P}}(Q(z) > 0, Q(0) > 0) > 0$. By a straightforward computation using that generating functions, as well as all their derivatives are convex, nonnegative and nondrecreasing functions, we see that the product of two generating functions (and thus the product of finitely many) is again convex:

$$(fg)'' = f''g + 2g'f' + fg'' \geq 0.$$

So the product of the derivatives of generating functions is again convex. For more details on the product of nonnegative, convex and nondrecreasing functions, we refer to [35]. Applying Jensen's inequality to the convex function $\Pi_{i=1}^{n-a} f_i'$, the independence of the environments ensures that

$$\bar{\mathbf{E}}\Big[\prod_{i=1}^{n-a} f_i'\big(f_{n-a,n-1}(0)\big) \Big| Q_1, \ldots, Q_{n-a}\Big] \geq \prod_{i=1}^{n-a} f_i'\big(\bar{\mathbf{E}}[f_{n-a,n-1}(0) | Q_1, \ldots, Q_{n-a}]\big) = \prod_{i=1}^{n-a} f_i'\big(\bar{\mathbf{E}}[f_{0,a-1}(0)]\big)$$

$\bar{\mathbf{P}}$ – a.s. Using this inequality multiplied by the indicator function of the event $\{Q_1([1,k]) >$

$\varepsilon, \dots, Q_{n-a}([1,k]) > \varepsilon\}$, we get

$$\begin{aligned}
&\liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}\Big[f_{0,n}^{z-1}(0) \prod_{i=1}^{n} f_i'\big(f_{i,n}(0)\big)\Big| X_1 = 0, \dots, X_n = 0\Big] \\
&\geq \liminf_{n\to\infty} \tfrac{1}{n} \log \Bigg(\bar{\mathbf{E}}\Big[\prod_{i=1}^{n-a} f_i'\big(\bar{\mathbf{E}}\big[f_{0,a-1}(0)\big]\big); Q_1([1,k]) > \varepsilon, \dots, Q_{n-a}([1,k]) > \varepsilon\Big] \\
&\qquad \bar{\mathbf{E}}\Big[\varepsilon^{a-2} Q_n(z) Q_n(0)^{z-1}(0) Q_n(0)^{(a-2)k+1}\Big]\Bigg) \\
&= \liminf_{n\to\infty} \tfrac{1}{n} \log \bar{\mathbf{E}}\Big[f'\big(\bar{\mathbf{E}}\big[f_{0,a-1}(0)\big]\big); Q([1,k]) > \varepsilon\Big]^{n-a} \\
&= \log \bar{\mathbf{E}}\Big[f'\big(\bar{\mathbf{E}}\big[f_{0,a-1}(0)\big]\big); Q([1,k]) > \varepsilon\Big].
\end{aligned}$$

Finally, $Z$ is a critical branching process in random environment under the probability $\bar{\mathbf{P}}$, and thus $\bar{\mathbf{P}}(Z_{a-1} = 0|\mathcal{E}) = f_{0,a-1}(0) \to 1$ $\bar{\mathbf{P}}$-a.s. as $a \to \infty$ (see e.g. [38]). Taking the limit $a \to \infty$ and $\varepsilon \to 0$ and applying dominated convergence yields as $a \to \infty$ and $\varepsilon \to 0$, $k_\varepsilon \to \infty$

$$\log \bar{\mathbf{E}}\Big[f'\big(\bar{\mathbf{E}}\big[f_{0,a-1}(0)\big]\big); Q([1,k_\varepsilon]) > \varepsilon\Big] \to \log \bar{\mathbf{E}}\big[f'(1)\big] = 0$$

and thus

$$\liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}\Big[f_{0,n}^{z-1}(0) \prod_{i=1}^{n-1} f_i'\big(f_{i,n}(0)\big)\Big| X_1 = 0, \dots, X_n = 0\Big] \geq 0 \ .$$

This yields the claim. □

**Remark.** Note that the bound $f'(s) \leq f'(1)$ for $s \in [0,1]$ immediately yields that $\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_z(Z_n = z) \leq \log \mathbb{E}[X]$. In particular, we have proved that for a BPRE with $X = 0$ a.s. that the probability of staying bounded is not exponentially small.

# 7 Proof of lower large deviations

First, we focus on the lower bound, which is easier and can be made under general assumptions (satisfied in both Theorems 3.1 and 3.2). We split then the proof of the upperbounds in two parts, working with Assumption 3.1 in the first part, and then with $P(X > 0) = 0$ and Assumption 4. Finally, we prove the Theorems combining these results.

## 7.1 Proof of the lower bound for Theorems 3.1 and 3.2

First we note that, if the associated random walk has exceptional values, the same is true for the branching process $Z$. The estimation $Z_n \approx \mathbb{E}[Z_n \,|\mathcal{E}] = \exp(S_n)$ gives a lowerbound in the following way. If $\mathbb{E}[Z_1 \log^+(Z_1)] < \infty$, we know from [7] that the limit of the martingale $Z_n \exp(-S_n)$ is non degenerated. Then a direct generalization of [10, Proposition 1], which relies on the same change of measure :

$$\widetilde{\mathbb{P}}(Q \in \mathrm{d}q) := \frac{m(q)^{\lambda_c}}{\mathbb{E}(m(Q)^{\lambda_c})} \mathbb{P}(Q \in \mathrm{d}p),$$

ensures that

$$\liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_j(1 \leq Z_n \leq e^{\theta n} | S_n \leq (\theta+\varepsilon)n) \quad = \quad 0$$

for all $j \geq 1$ and $\varepsilon > 0$. As $\Lambda$ is nonincreasing, continuous from below and convex and thus a right-continuous function, $\Lambda(\theta+\varepsilon) \to \Lambda(\theta)$ as $\varepsilon \to 0$. Then, for every $0 < \theta < \mathbb{E}[X]$ such that $\Lambda(\theta) < \infty$, we have

$$\liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_j(1 \leq Z_n \leq e^{\theta n} | S_n \leq \theta n) \quad = \quad 0. \tag{7.1}$$

Now we can prove the following result

**Lemma 7.1.** *Let $z \geq 1$. We assume that $\mathbb{E}[Z_1 \log^+(Z_1)] < \infty$ and that*

$$\varrho_z = -\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(1 \leq Z_n \leq b)$$

*exists and does not depend on $b$ large enough. Then for every $\theta \in (0, \mathbb{E}(X)]$, we have*

$$\liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(1 \leq Z_n \leq e^{\theta n}) \geq -\chi(\theta, \varrho_z, \Lambda).$$

*Proof.* We decompose the probability following a time $t \in [0,1)$ when the process go beyond $b$. Using the large deviations principle satisfied by the random walk $S$, we have

$$\begin{aligned}
&\mathbb{P}_z(1 \leq Z_n \leq e^{\theta n}) \\
&\geq \mathbb{P}_z(1 \leq Z_{\lfloor tn \rfloor} \leq b) \min_{1\leq k \leq b} \mathbb{P}_k(1 \leq Z_{\lfloor (1-t)n \rfloor} \leq e^{\theta n}; S_{\lfloor (1-t)n \rfloor} \leq e^{\theta n}) \\
&\geq \mathbb{P}_z(1 \leq Z_{\lfloor tn \rfloor} \leq b) e^{-\Lambda(\frac{\theta}{1-t})n(1-t)+o(n)} \min_{1\leq k \leq b} \mathbb{P}_k\Big(1 \leq Z_{\lfloor (1-t)n \rfloor} \leq e^{\frac{\theta}{1-t}n(1-t)} \Big| S_n \leq e^{\frac{\theta}{1-t}n(1-t)}\Big).
\end{aligned}$$

Note that the above inequality is trivially fulfilled if $\Lambda(\theta) = \infty$. The definition of $\varrho_z$ and (7.1) yield with $b$ large enough

$$\liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(1 \leq Z_n \leq e^{\theta n}) \geq -\inf_{t\in[0,1)} \big\{t\rho_z + (1-t)\Lambda\big(\theta/(1-t)\big)\big\} = -\chi(\theta, \varrho_z, \Lambda).$$

It completes the proof. $\square$

## 7.2 Proof of the upper bound for Theorem 3.1 (i) and (ii)

The next lemma ensures that a large population (under the assumption above) grows as its expectation and thus follows the random walk of the environment $S$. The start of the proof of this proposition is in the same vein as [10], but the situation is much more complex since $\mathbb{P}_1(Z_1 = 0)$ may be positive, $f'(1)$ may not be bounded a.s. and the variance of the reproduction laws may be infinite with positive probability.

**Lemma 7.2.** *Under Assumption 3, for every $\varepsilon > 0$ and for every $a > 0$, there exist constants $c, b \geq 1$ such that for every $n \in \mathbb{N}$*

$$\sup_{z\geq b} \mathbb{P}_z(Z_n \leq e^{S_n - n\varepsilon}; Z_1 \geq b, ..., Z_n \geq b) \leq c\ e^{-an}$$

*Proof.* Let us introduce the ratio of the successive sizes of the population

$$R_i := Z_i/Z_{i-1}, \qquad i \in \{1, \ldots, n\}.$$

Recalling that $\log f_i'(1) = X_i$, we can rewrite

$$\frac{e^{S_n - n\varepsilon}}{Z_n} = Z_0^{-1} \prod_{i=1}^n \frac{f_i'(1)}{e^{\varepsilon} R_i}.$$

Then for every $\lambda > 0$, we can use the classical Markov inequality $\mathbb{P}(Y \geq 1) \leq \mathbb{E}[Y^\lambda]$ for any nonnegative random variable $Y$ and get for every $z \geq b$

$$\begin{aligned}
&\mathbb{P}_z(Z_n \leq e^{S_n - n\varepsilon}; Z_1 \geq b, ..., Z_n \geq b) \\
&\qquad\qquad \leq b^{-\lambda}\mathbb{E}\big[\prod_{i=1}^{n}(f_i'(1)/(e^\varepsilon R_i))^\lambda; Z_1 \geq b, ..., Z_n \geq b\big] \\
&\qquad\qquad = b^{-\lambda}\mathbb{E}\big[\prod_{i=1}^{n}(e^\varepsilon R_i/f_i'(1))^{-\lambda}; Z_1 \geq b, ..., Z_n \geq b\big] \ .
\end{aligned}$$

Let us introduce the following random variable, only depending on the environment,

$$M_\lambda(b, g) := \sup_{k \geq b} \mathbb{E}\Big[\Big(e^\varepsilon \frac{\sum_{i=1}^k N_i^g}{kg'(1)}\Big)^{-\lambda}; \sum_{i=1}^k N_i^g > 0\Big]$$

where $N_i^g$ are i.i.d. random variables with p.g.f. $g$. Note that sup and $\mathbb{E}$ may be interchanged as the random variables are nonnegative. As a consequence

$$M_\lambda(b, f_i) = \sup_{k \geq b} \mathbb{E}\Big[\Big(e^\varepsilon \tfrac{Z_{i+1}}{Z_i f_i'(1)}\Big)^{-\lambda}; Z_{i+1} > 0 \ \Big| \ f_i, Z_i = k\Big] = \sup_{k \geq b} \mathbb{E}\Big[\Big(e^\varepsilon \tfrac{R_{i+1}}{f_i'(1)}\Big)^{-\lambda}; Z_{i+1} > 0 \ \Big| \ f_i, Z_i = k\Big].$$

Then, by conditioning on the successive sizes of the population, we obtain

$$\begin{aligned}
&\mathbb{P}_b(Z_n \leq e^{S_n - n\varepsilon}; Z_1 \geq b, ..., Z_n \geq b) \\
&\leq b^{-\lambda}\mathbb{E}\Big[\prod_{i=1}^{n-1}(e^\varepsilon R_i/f_i'(1))^{-\lambda}\mathbb{E}\big[(e^\varepsilon R_n/f_n'(1))^{-\lambda}; Z_n \geq b \mid f_n, Z_{n-1}\big]; Z_1 \geq b, ..., Z_{n-1} \geq b\Big] \\
&\leq b^{-\lambda}\mathbb{E}\big[\prod_{i=1}^{n} M_\lambda(b, f_i)\big] \\
&= b^{-\lambda}\mathbb{E}[M_\lambda(b, f)]^n.
\end{aligned}$$

We now want to prove that for every $\alpha \in (0, 1)$, there exist $\lambda, b > 0$ such that $\mathbb{E}[M_\lambda(b, f)] \leq \alpha$. The idea is that for every $g$, $\sum_{i=1}^k N_i^g/k \to g'(1)$ a.s. as $k \to \infty$ by the law of large numbers. Then we will be able to derive that

$$\mathbb{E}\Big[\Big(e^\varepsilon \frac{\sum_{i=1}^k N_i^g}{kg'(1)}\Big)^{-\lambda}; \sum_{i=1}^k N_i^g > 0\Big] \to e^{-\lambda\varepsilon}$$

as $k \to \infty$ and $M_\lambda(b, f) \to e^{-\lambda\varepsilon}$ a.s. as $b$ goes to infinity. Under suitable conditions, we are then able to prove that $\mathbb{E}[M_\lambda(b, f)] \to e^{-\lambda\varepsilon}$. Finally, considering $\lambda > 0$ such that $e^{-\lambda\varepsilon} < e^{-a}$ and $b$ large enough gives us the result.

Let us now present the details of the proof. First fix a p.g.f. $g$ with $E[N_1^g] = g'(1) < \infty$. Then the law of large numbers ensures that

$$Y_k := \Big(e^\varepsilon \frac{\sum_{i=1}^k N_i^g}{kg'(1)}\Big)^{-\lambda} \stackrel{k \to \infty}{\longrightarrow} e^{-\lambda\varepsilon} \qquad \mathbb{P} - \text{a.s.}$$

Moreover $\sum_{i=1}^k N_i^g$ is stochastically larger than a random variable $B(k, g)$ with binomial distribution of parameters $(k, 1 - g(0))$. Applying the classical large deviations upperbound for Bernoulli random variables (see e.g. [15, 16]) yields

$$\mathbb{P}\Big(Y_k \geq x; \sum_{i=1}^k N_i^g > 0\Big) \leq \mathbb{P}\big(kB(k, g) \leq x^{-1/\lambda}g'(1)e^{-\varepsilon}\big) \leq \exp\big(-k\psi_g(x^{-1/\lambda}g'(1)e^{-\varepsilon})\big)$$

where the function $\psi_g(z)$ is zero if $z \geq 1-g(0)$ and positive for $z < 1-g(0)$. It is specified by the Fenchel Legendre transform of a Bernoulli distribution, i.e.

$$\psi_g(z) = z\log\big(\tfrac{z}{1-g(0)}\big) + (1-z)\log\big(\tfrac{1-z}{g(0)}\big) \ .$$

Moreover $\sum_{i=1}^k N_i^g > 0$ implies $Y_k \leq k^\lambda d$ with $d = (g'(1)e^{-\varepsilon})^\lambda$. So

$$\mathbb{E}\Big[Y_k \mathbb{1}_{Y_k \geq x}; \sum_{i=1}^k N_i^g > 0\Big] \leq dk^\lambda \mathbb{P}\Big(Y_k \geq x; \sum_{i=1}^k N_i^g > 0\Big) \leq 2dk^\lambda \exp\big(-k\psi_g(x^{-1/\lambda}g'(1)e^{-\varepsilon})\big).$$

Let us choose $x$ large enough such that $\psi_g(x^{-1/\lambda}g'(1)e^{-\varepsilon}) > 0$. Then letting $k \to \infty$, the right-hand side of the above equation converges to 0. Moreover we can apply the bounded convergence theorem to $Y_k \mathbb{1}_{Y_k \leq x, \sum_{i=1}^k N_i^g > 0}$ to get

$$\limsup_{k\to\infty} \mathbb{E}\Big[Y_k; \sum_{i=1}^k N_i^g > 0\Big] \leq e^{-\lambda\varepsilon}$$

Recalling that $M_\lambda(b,g)$ decreases with respect to $b$, we get for every $g$

$$\lim_{b\to\infty} M_\lambda(b,g) \leq e^{-\lambda\varepsilon} \ .$$

Second, we apply the bounded convergence theorem again and finish the proof by integrating the previous result with respect to the environment. To check that

$$\mathbb{E}[M_\lambda(1,f)] < \infty,$$

we define for any p.g.f $g$ (note that $g(0) < 1$ a.s.)

$$x_g := \left(e^{-\varepsilon}\frac{2g'(1)}{1-g(0)}\right)^\lambda, \qquad y_g := (ke^{-\varepsilon}g'(1))^\lambda,$$

and note that $x \geq x_g$ implies that $x^{-1/\lambda}g'(1)e^{-\varepsilon} \leq (1-g(0))/2$. Moreover, $\sum_{i=1}^k N_i^g > 0$ implies $Y_k \leq y_g$, so

$$\begin{aligned}
\mathbb{E}\Big[Y_k; \sum_{i=1}^k N_i^g > 0\Big] &= \int_0^{y_g} \mathbb{P}\Big(Y_k \geq x; \sum_{i=1}^k N_i^g > 0\Big)dx \\
&\leq x_g + \int_{x_g}^{dk^\lambda} \exp\big(-k\psi_g(x^{-1/\lambda}g'(1)e^{-\varepsilon})\big)dx \\
&\leq x_g + dk^\lambda \exp\big(-k\psi_g\big(\tfrac{1-g(0)}{2}\big)\big).
\end{aligned}$$

Now we maximize the right-hand side with respect to $k \geq 1$. Using that for $\alpha > 0, x \geq 0$, $x^\lambda e^{-\alpha x} \leq (\lambda/\alpha)^\lambda e^{-\lambda}$ and the definition of $d$, we get that

$$M_\lambda(1,g) \leq x_g + 1 + (e^{-\varepsilon}g'(1))^\lambda \lambda^\lambda e^{-\lambda} \psi_g\big(\tfrac{1-g(0)}{2}\big)^{-\lambda}. \tag{7.2}$$

Finally, we observe that $\psi_g(z)$ is a nonnegative convex function which reaches 0 in $1-g(0)$. Thus $x \leq y \leq 1-g(0)$ implies $\psi_g(x) \geq (x-y)\psi_g'(y)$ and in particular

$$\psi_g\big(\tfrac{1-g(0)}{2}\big) \geq -\tfrac{1-g(0)}{4}\psi_g'\big(\tfrac{1-g(0)}{4}\big).$$

As $\psi_g'(z) = \log\big(\frac{zg(0)}{(1-z)(1-g(0))}\big)$ and $\log(1-x) \leq x$ for $x > 0$, we get that

$$\psi_g\big(\tfrac{1-g(0)}{2}\big) \geq -\tfrac{1-g(0)}{4}\log\big(1 - \tfrac{3}{3+g(0)}\big) \geq \frac{3}{4}\frac{1-g(0)}{3+g(0)} \geq \frac{3(1-g(0))}{16}. \tag{7.3}$$

Combining the inequalities (7.2) and (7.3) yields

$$M_\lambda(1,f) \le 1 + a(\varepsilon,\lambda)\left(\frac{f'(1)}{1-f(0)}\right)^\lambda \qquad \text{a.s.},$$

where $a(\varepsilon,\lambda)$ is a finite constant, only depending on $\varepsilon$ and $\lambda$. Thus the Assumption 3.1 ensures that $\mathbb{E}[M_\lambda(1,f)] < \infty$. Applying the bounded convergence theorem, we get that

$$\lim_{b\to\infty} \mathbb{E}\big[M_\lambda(b,f)\big] = \mathbb{E}\big[\lim_{b\to\infty} M_\lambda(b,f)\big] \le e^{-\lambda\varepsilon}.$$

Then, choosing $b$ large enough,

$$\mathbb{E}\big[M_\lambda(b,f)\big] \le 2e^{-\lambda\varepsilon}.$$

Letting $\lambda$ such that $2e^{-\lambda\varepsilon} \le e^{-a}$ ends up the proof. $\square$

**Lemma 7.3.** *Let $z \ge 1$ and assume that*

$$\varrho_z = -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_z(1 \le Z_n \le b)$$

*exists and does not depend on $b$ large enough. Then, under Assumption 3, for every $\theta \in (0, \mathbb{E}(X)]$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_z\big(1 \le Z_n \le \exp(n\theta)\big) \le -\chi(\theta, \varrho_z, \Lambda).$$

*Proof.* We define the last moment when the process is below $b$ before time $n$ :

$$\sigma_b = \inf\{i \in \mathbb{N} : Z_{i+1} \ge b, \cdots, Z_n \ge b\}, \qquad (\inf \varnothing = \infty)$$

Let $\theta > 0$. Then summing over $i$ leads to

$$\begin{aligned}
&\mathbb{P}_z(1 \le Z_n \le e^{\theta n}) \\
&\quad \le \sum_{i=0}^{n-1} \mathbb{P}_k(1 \le Z_{n(1-t)} \le e^{\theta n}, \sigma_b = i) + \mathbb{P}(1 \le Z_n \le b) \\
&\quad \le \sum_{i=0}^{n-1} \mathbb{P}_z(1 \le Z_i \le b) \sup_{j\ge b} \mathbb{P}_j(1 \le Z_{n-i-2} \le e^{\theta n}, Z_1 \ge b, ..., Z_{n-i-2} \ge b) + \mathbb{P}(1 \le Z_n \le b) \\
&\quad \le \mathbb{P}(1 \le Z_n \le b) + \sum_{i=0}^{n-1} \mathbb{P}_z(1 \le Z_{i-1} \le b)\big[\mathbb{P}(S_{n-i-2} \le \theta n + n\varepsilon) \\
&\qquad\qquad + \sup_{j\ge b} \mathbb{P}_j(Z_{n-i-2} \le e^{\theta n},\ S_{n-i-2} > \theta n + n\varepsilon, Z_1 \ge b, ..., Z_{n-i-2} \ge b)\big].
\end{aligned}$$

First, by assumption, we have for every $t \in [0,1]$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_z(1 \le Z_{\lfloor tn \rfloor} \le b) = -t\varrho_z.$$

For the second term, we use the classical large deviation inequality for the random walk $S$ to get for every $t \in [0,1]$ that

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}(S_{\lfloor (1-t)n \rfloor} \le \theta n + n\varepsilon) = -(1-t)\Lambda\big(\tfrac{\theta+\varepsilon}{1-t}\big).$$

with the convention $0.\infty = 0$. For the last term, we apply Lemma 7.2, which prevents a large population form deviating from the random environment. More precisely, for every $\varepsilon > 0$, we can choose $b$ large enough such that $\sup_{j\geq b}\mathbb{P}_j(Z_{n-i-2}\leq e^{\theta n},\ S_{n-i-2}\geq \theta n+n\varepsilon, Z_1\geq b,...,Z_{n-i-2}\geq b)$ decreases faster than $\exp(-\rho_z(n-i-2))$ as $n$ goes to infinity. Thus, for $b$ large enough and every $t\in[0,1]$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\sup_{j\geq b}\mathbb{P}_j(Z_{n(1-t)}\leq e^{\theta n},\ S_{n(1-t)}>\theta n+n\varepsilon, Z_1\geq b,...,Z_{n(1-t)}\geq b)\leq -\varrho_z(1-t),$$

with the convention $0.x=0$ for any $x\in[0,\infty]$. Combining these upperbounds yields,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}_z\big(1\leq Z_n\leq \exp(n\theta)\big)\leq -\sup_{t\in[0,1]}\big\{t\varrho_z+(1-t)\Lambda\big(\tfrac{\theta+\varepsilon}{1-t}\big)\big\}$$

Letting $\epsilon\to 0$, the right-hand side goes to $\chi(\theta)$ by right-continuity of $\Lambda$. It yields the result since the supremum can be taken over $[0,1)$ with the convention $0\cdot\infty=0$. □

## 7.3 Proof of the upperbound for Theorem 3.2

We assume here that that subcritical environments occur with a positive probability. First, we consider the probability of having less than exponentially many individuals in generation $n$. We prove that decrease is still given by $\varrho$ and obtain the first part of Theorem 3.2. We derive the upperbound of the second part of the Theorem using Assumption 3.1 and an additional lemma.

**Lemma 7.4.** *If $\mathbb{P}(X<0)>0$, then for every $z\in Cl(\mathcal{I})$,*

$$\varrho=\lim_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_z(Z_n=z)=\lim_{\theta\to 0}\lim_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_z(1\leq Z_n\leq e^{\theta n}).$$

*Proof.* As for every $\theta>0$, $\mathbb{P}_z(Z_n=z)\leq\mathbb{P}_z(1\leq Z_n\leq e^{\theta n})$ for $n$ large enough, we have

$$\lim_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_z(Z_n=z)\leq\liminf_{\theta\to 0}\liminf_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_z(1\leq Z_n\leq e^{\theta n}).$$

Let us prove the converse inequality. First, we observe that $m_q<1-\varepsilon$ implies $q(0)>\varepsilon$. Using that $\mathbb{P}(m_Q<1)>0$ by assumption and $z\in\mathcal{I}$, we choose $\varepsilon>0$ and $j_1\geq 1$ so that the sets

$$\mathcal{A}:=\{q\in\Delta: q(0)>\varepsilon,\, q(z)>\varepsilon\},\quad \mathcal{B}:=\{q\in\Delta: m_q<1-\varepsilon,\, q(j_1)>\varepsilon\}$$

satisfy

$$\mathbb{P}(Q_1\in\mathcal{A})>0,\qquad \mathbb{P}(Q_1\in\mathcal{B})>0,\qquad \mathcal{B}\subset\{q\in\Delta: q(0)>\varepsilon,\, q(z)>\varepsilon\}.$$

By Markov property, for every $\theta>0$,

$$\begin{aligned}
\mathbb{P}_z(Z_{n+\lfloor\frac{\theta}{\varepsilon}n\rfloor}=z) &\geq \sum_{k=1}^{\lfloor e^{\theta n}\rfloor}\mathbb{P}_z(Z_n=k)\mathbb{P}_k(Z_{\lfloor\frac{\theta}{\varepsilon}n\rfloor}=z)\\
&\geq \mathbb{P}_z(1\leq Z_n\leq e^{\theta n})\min_{1\leq k\leq e^{\theta n}}\mathbb{P}_k(Z_{\lfloor\frac{\theta}{\varepsilon}n\rfloor}=z)\\
&\geq \mathbb{P}_z(1\leq Z_n\leq e^{\theta n})\min_{1\leq k\leq e^{\theta n}}\mathbb{E}\big[\mathbb{P}_k(Z_{\lfloor\frac{\theta}{\varepsilon}n\rfloor}=z|\mathcal{E}); Q_1,\dots,Q_{\lfloor\frac{\theta}{\varepsilon}n\rfloor-1}\in\mathcal{B}, Q_{\lfloor\frac{\theta}{\varepsilon}n\rfloor}\in\mathcal{A}\big]\\
&\geq \mathbb{P}_z(1\leq Z_n\leq e^{\theta n})\times\\
&\qquad \min_{1\leq k\leq e^{\theta n}}\mathbb{E}\big[\mathbb{P}_{k-1}(Z_{\lfloor\frac{\theta}{\varepsilon}n\rfloor-1}=0|\mathcal{E})\mathbb{P}_1(Z_{\lfloor\frac{\theta}{\varepsilon}n\rfloor}=z|\mathcal{E}); Q_1,\dots,Q_{\lfloor\frac{\theta}{\varepsilon}n\rfloor}\in\mathcal{B}, Q_{\lfloor\frac{\theta}{\varepsilon}n\rfloor}\in\mathcal{A}\big]\ .
\end{aligned}\tag{7.4}$$

Using again the Markov property and the definition of $\mathcal{B}$ and $\mathcal{A}$, we estimate

$$\begin{aligned}
&\mathbb{P}_1(Z_{\lfloor\frac{\theta}{\varepsilon}n\rfloor}=z|Q_1,\dots,Q_{\lfloor\frac{\theta}{\varepsilon}n\rfloor-1}\in\mathcal{B}, Q_{\lfloor\frac{\theta}{\varepsilon}n\rfloor}\in\mathcal{A})\\
&\qquad\geq \mathbb{P}_1(Z_1=j_1|Q_1\in\mathcal{B})\cdot\mathbb{P}_{j_1}(Z_1=j_1|Q_1\in\mathcal{B})^{\lfloor\frac{\theta}{\varepsilon}n\rfloor-2}\cdot\mathbb{P}_{j_1}(Z_1=z|Q_1\in\mathcal{A})\\
&\qquad\geq \varepsilon\cdot\varepsilon^{j_1(\lfloor\frac{\theta}{\varepsilon}n\rfloor-2)}\cdot\varepsilon^{j_1}=\varepsilon^{j_1(\lfloor\frac{\theta}{\varepsilon}n\rfloor-1)+1}\ .
\end{aligned}$$

Using the classical estimates $\mathbb{P}_1(Z_n > 0|\mathcal{E}) \leq \exp(L_n)$, where

$$L_n := \min_{0\leq k\leq n} S_k, \tag{7.5}$$

and $\log(1-x) \leq -x$, $x \in [0,1)$ yields for every $k, n \in \mathbb{N}$

$$\mathbb{P}_k(Z_{\lfloor \frac{\theta}{\varepsilon} n\rfloor} = 0 | Q_1 \in \mathcal{B}, \ldots, Q_n \in \mathcal{B}) \geq \left(1 - e^{\lfloor \frac{\theta}{\varepsilon} n\rfloor \log(1-\varepsilon)}\right)^k \geq \left(1 - e^{-\lfloor \frac{\theta}{\varepsilon} n\rfloor \varepsilon}\right)^k .$$

Inserting the two last inequalities into (7.4), we get that

$$\begin{aligned}
&\mathbb{P}_z(Z_{n+\lfloor \frac{\theta}{\varepsilon} n\rfloor} = z)\mathbb{P}_z(1 \leq Z_n \leq e^{\theta n})^{-1} \\
&\geq \min_{1\leq k\leq e^{\theta n}} \left\{\left(1 - e^{-\varepsilon\lfloor \frac{\theta}{\varepsilon} n - 1\rfloor}\right)^k \varepsilon^{j_1(\lfloor \frac{\theta}{\varepsilon} n\rfloor -1)+1} \mathbb{P}(Q_1 \in \mathcal{B}, \ldots, Q_{\lfloor \frac{\theta}{\varepsilon} n\rfloor -1} \in \mathcal{B}, Q_{\lfloor \frac{\theta}{\varepsilon} n\rfloor} \in \mathcal{A})\right\} \\
&\geq (1 - e^{-\theta n + o(1)})^{e^{\theta n}} \varepsilon^{j_1(\lfloor \frac{\theta}{\varepsilon} n\rfloor -1)+1} \mathbb{P}(Q \in \mathcal{B})^{\lfloor \frac{\theta}{\varepsilon} n\rfloor -1} \mathbb{P}(Q \in \mathcal{A}).
\end{aligned}$$

Taking the logarithm and using the fact that $(1 - 1/x)^x$ is increasing for $x \geq 1$ and bounded

$$\begin{aligned}
&\liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(Z_{n+\lfloor \frac{\theta}{\varepsilon} n\rfloor} = z) \\
&\qquad \geq \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(1 \leq Z_n \leq e^{\theta n}) + \tfrac{j_1\theta}{\varepsilon} \log \varepsilon + \tfrac{\theta}{\varepsilon} \log \mathbb{P}(Q \in \mathcal{B}) .
\end{aligned} \tag{7.6}$$

Thus

$$\limsup_{\theta\to 0} \liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(Z_{n+\lfloor \frac{\theta}{\varepsilon} n\rfloor} = z) \geq \limsup_{\theta\to 0} \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(1 \leq Z_n \leq e^{\theta n}).$$

Using Lemma 5.1, we get

$$\limsup_{\theta\to 0} \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(Z_{n+\lfloor \frac{\theta}{\varepsilon} n\rfloor} = z) = \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(Z_n = z),$$

which completes the proof. □

**Lemma 7.5.** *Under Assumption 4, for every $b > 0$, $n \in \mathbb{N}$ and $r \in (0,1)$, it holds that*

$$\mathbb{P}_b(Z_n \leq r\, e^{S_n} | \mathcal{E}) \leq \left(1 - (1-r)^2 \tfrac{e^{L_n}}{n+2}\right)^b \qquad a.s.$$

*Proof.* Note that $\mathbb{E}[Z_n(Z_n-1)|\mathcal{E}] = f''_{0,n}(1)$. Let us now check briefly that the result of Proposition 1 in [13] still hold, which means that we can replace Assumption 2 in [13] by our Assumption 4. From $f_{0,n} = f_{0,n-1} \circ f_n$, by chain rule for differentiation $f'_{0,n}(1) = f'_{0,n}(1)f'_n(1)$ and $f''_{0,n}(1) = f''_{0,n-1}(1)(f'_n(1))^2 + f'_{0,n-1}(1)f''_n(1)$, we get that

$$\frac{f''_{0,n}(1)}{(f'_{0,n}(1))^2} = \frac{f''_{0,n-1}(1)}{(f'_{0,n-1}(1))^2} + \frac{f''_n(1)}{f'_{0,n-1}(1)(f'_n(1))^2}.$$

Using Assumption 4 yields

$$\frac{f''_n(1)}{f'_{0,n-1}(1)(f'_n(1))^2} \leq d(e^{S_{n-1}} + e^{-S_n})$$

and by a recursion argument

$$\frac{\mathbf{E}[Z_n(Z_n-1)|\Pi]}{\mathbf{E}[Z_n|\Pi]^2} = \frac{f''_{0,n}(1)}{(f'_{0,n}(1))^2} \leq 2d \sum_{k=0}^{n} e^{-S_k} \quad \text{a.s.}$$

Finally we get for every $n \in \mathbb{N}$,

$$\mathbb{E}_1[Z_n(Z_n-1)|\mathcal{E}] \leq 2de^{2S_n} \sum_{k=0}^{n} e^{-S_k} \leq 2d\ (n+2)e^{S_n} e^{S_n - L_n}.$$

Combining this inequality with an inequality due to Paley and Zygmund, which ensures that for any $\mathbb{R}+$ valued random variable $\xi$ with $0 < \mathbb{E}(\xi) < \infty$ and $0 < r < 1$, we have $\mathbb{P}(\xi > r\mathbb{E}(\xi)) \geq (1-r)^2 \mathbb{E}(\xi)^2/\mathbb{E}(\xi^2)$ (see Lemma 4.1 in [30]), we have

$$\begin{aligned}
\mathbb{P}_1(Z_n \geq r\ e^{S_n}|\mathcal{E}) &\geq (1-r)^2 \frac{\mathbb{E}_1[Z_n|\mathcal{E}]^2}{\mathbb{E}_1[Z_n^2|\mathcal{E}]} \\
&\geq (1-r)^2 \frac{e^{2S_n}}{(n+1)e^{S_n}e^{S_n-L_n} + e^{S_n}} = \frac{(1-r)^2}{n+2} e^{L_n}.
\end{aligned}$$

Given $\mathcal{E}$ and starting with $Z_0 = b$, $b$-many subtrees are developing independently. Each has the above probability of being larger than $re^{S_n}$. Thus

$$\begin{aligned}
\mathbb{P}_b(Z_n \leq r\ e^{S_n}|\mathcal{E}) &\leq \mathbb{P}(Z_n \leq r\ e^{S_n}|\mathcal{E})^b \\
&\leq \left(1 - (1-r)^2 \tfrac{e^{L_n}}{n+2}\right)^b \qquad \text{a.s.},
\end{aligned}$$

which is the claim of the lemma. □

**Lemma 7.6.** *If $\mathbb{P}(X < 0) > 0$ and Assumption 4 holds, then for all $z \in Cl(\mathcal{I})$, $\theta \in (0, \mathbb{E}(X)]$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_z(1 \leq Z_n \leq \exp(n\theta)) \leq -\chi(\theta, \varrho, \Lambda).$$

Let $z \in Cl(\mathcal{I})$. For the proof of the upper bound, we will decompose the probability at the first moment when there are at least $n^3$-many individuals for the rest of time. For this, let

$$\sigma_n := \inf\{1 \leq i \leq n : Z_j \geq n^3,\ j = i, \ldots, n\}, \qquad (\inf \emptyset := n).$$

and

$$\tau_n := \inf\left\{0 \leq i \leq n : S_i \leq \min\{S_0, S_1, \ldots, S_n\}\right\}.$$

Let us fix $0 < \theta < \mathbb{E}[X]$. Then by Markov property,

$$\begin{aligned}
\mathbb{P}_z(1 \leq Z_n \leq e^{\theta}) &= \sum_{i=1}^{n} \mathbb{P}_z(\sigma_n = i, 1 \leq Z_n \leq e^{\theta n}) \\
&\leq \sum_{i=1}^{n} \mathbb{P}_z(1 \leq Z_{i-1} < n^3) \max_{k \geq n^3} \mathbb{P}_k(1 \leq Z_{n-i} \leq e^{\theta n}, \quad \forall 1 \leq j \leq n-i:\ Z_j \geq n^3) \\
&= \sum_{i=1}^{n} \mathbb{P}_z(1 \leq Z_{i-1} < n^3) \sum_{j=0}^{n-i} \max_{k \geq n^3} \mathbb{P}_k(1 \leq Z_{n-i} \leq e^{\theta n}; \tau_{n-i} = j, \quad \forall 1 \leq j \leq n-i:\ Z_j \geq n^3) \\
&\leq \sum_{i=1}^{n} \mathbb{P}_z(1 \leq Z_{i-1} < n^3) \sum_{j=0}^{n-i} \mathbb{P}(\tau_j = j) \max_{k \geq n^3} \mathbb{P}_k(1 \leq Z_{n-i-j} \leq e^{\theta n}; L_{n-i-j} \geq 0). \qquad (7.7)
\end{aligned}$$

We treat now the different probabilities separately. First, by Lemma 7.4 for $t, s \in (0,1)$ with $s + t \leq 1$,

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}(1 \leq Z_{\lfloor (1-t-s)n \rfloor - 1} \leq n^3) = -(1-t-s)\rho.$$

As to the second probability, as $\mathbb{P}(\tau_n = n) \le \mathbb{P}(S_n \le 0)$,

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}(\tau_{\lfloor sn\rfloor} = \lfloor sn\rfloor) \le -s\Lambda(0).$$

Next, for every $\varepsilon > 0$,

$$\begin{aligned}
&\min_{k\ge n^3} \mathbb{P}_k(1 \le Z_{\lfloor tn\rfloor} \le e^{\theta n}; L_{\lfloor tn\rfloor} \ge 0)\\
&\quad \le \min_{k\ge n^3} \mathbb{E}\big[\mathbb{P}_k(1 \le Z_{\lfloor tn\rfloor} \le e^{\theta n}|\mathcal{E}); S_{\lfloor tn\rfloor} \ge (\theta+\varepsilon)n, L_{\lfloor tn\rfloor} \ge 0\big] + \mathbb{P}\big(S_{\lfloor tn\rfloor} \le (\theta+\varepsilon)n\big).
\end{aligned}$$

Using Lemma 7.5, for $n$ large enough,

$$\begin{aligned}
&\max_{k\ge n^3} \mathbb{E}\big[\mathbb{P}_k(1 \le Z_{\lfloor tn\rfloor} \le e^{\theta n}|\mathcal{E}); S_{\lfloor tn\rfloor} \ge (\theta+\varepsilon)n, L_{\lfloor tn\rfloor} \ge 0\big]\\
&\quad \le \max_{k\ge n^3} \mathbb{E}\big[\mathbb{P}_k(1 \le Z_n \le e^{-\varepsilon n} e^{S_{\lfloor tn\rfloor}}|\mathcal{E}); S_{\lfloor tn\rfloor} \ge (\theta+\varepsilon)n, L_{\lfloor tn\rfloor} \ge 0\big]\\
&\quad \le \max_{k\ge n^3} \big(1 - (1-e^{-\varepsilon n})^2 \tfrac{1}{\lfloor tn\rfloor+2}\big)^k \mathbb{P}\big(L_{\lfloor tn\rfloor} \ge 0, S_{\lfloor tn\rfloor} \ge (\theta+\varepsilon)n\big)\\
&\quad \le \big(1 - (1-\tfrac{1}{2})^2 \tfrac{1}{\lfloor tn\rfloor+2}\big)^{n^3}.
\end{aligned}$$

Then, for every $t > 0$,

$$\begin{aligned}
&\limsup_{n\to\infty} \tfrac{1}{n} \log \max_{k\ge n^3} \mathbb{E}\big[\mathbb{P}_k(1 \le Z_{\lfloor tn\rfloor} \le e^{\theta n}|\mathcal{E}); S_{\lfloor tn\rfloor} \ge (\theta+\varepsilon)n, L_{\lfloor tn\rfloor} \ge 0\big]\\
&\qquad \le \limsup_{n\to\infty} n^2 \log\big(1 - \tfrac{1}{4}\tfrac{1}{\lfloor tn\rfloor+2}\big) = -\infty.
\end{aligned}$$

Finally, recall that

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}\big(S_{\lfloor tn\rfloor} \le (\theta+\varepsilon)n\big) = -t\Lambda\big((\theta+\varepsilon)/t\big)\ .$$

Applying all this in (7.7) and letting $\varepsilon \to 0$ yields the upper bound, i.e.

$$\begin{aligned}
\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}(1 \le Z_n \le e^{n\theta}) &\le -\inf_{s,t\in[0,1]; s+t\le 1} \big\{(1-s-t)\rho + s\Lambda(0) + t\Lambda((\theta+\varepsilon)/t)\big\}\\
&= -\inf_{t\in[0,1]} \{(1-t)\rho + t\Lambda(\theta/t+)\} = \chi(\theta, \varrho, \lambda).
\end{aligned}$$

In the last step, we used that Proposition 3 guarantees $\Lambda(0) \ge \rho$, $\Lambda(0) \ge \Lambda(x)$ for every $x \ge 0$ and right-continuity of $\Lambda$.

## 7.4 Proof of Theorems 3.1 and 3.2

*Proof of Theorem 3.1 (i).* The second part of Lemma 5.1 ensures that for $b$ large enough,

$$\varrho = -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_z(1 \le Z_n \le b).$$

Then, under Assumption 3, Lemmas 7.1 and 7.3 yield

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}(1 \le Z_n \le e^{n\theta}) = -\chi(\theta, \varrho, \Lambda).$$

The right continuity of $\chi(\theta, \varrho, \Lambda)$ proves the last part of the result. □

*Proof of Theorem 3.1 (ii).* We recall that the monotonicity of $Z$ (see also [10]) ensures that for every $b \ge z$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_z(1 \le Z_n \le b) = z \log \mathbb{E}(Q(1)).$$

Then, under Assumption 3 and $\mathbb{E}(Z_1 \log^+(Z_1)) < \infty$, Lemmas 7.1 and 7.3 yield

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(1 \le Z_n \le e^{n\theta}) = -\chi(\theta, -z\log \mathbb{E}(Q(1)), \Lambda).$$

The right continuity of $\chi(\theta, \varrho, \Lambda)$ proves the last part of the result. □

*Proof of Theorem 3.2.* The first part is given by Lemma 7.4 .
As we assume $\mathbb{P}(X < 0) > 0$, we can use again Lemma 5.1, which ensures that for $b$ large enough,

$$\varrho = -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_z(1 \le Z_n \le b).$$

Then, under Assumption 4 and $\mathbb{E}(Z_1 \log^+(Z_1)) < \infty$, we can combine Lemmas 7.1 and 7.6 to get

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}(1 \le Z_n \le e^{n\theta}) = -\chi(\theta, \varrho, \Lambda).$$

It completes the proof. □

# 8 The linear fractional case

In this section, we restrict ourselves to the case of offspring distributions with generating function of linear fractional form, i.e.

$$f(s) \;=\; 1 - \frac{1-s}{m^{-1} + b\, m^{-2}(1-s)/2} \;,$$

where $m = f'(1)$ and $b = f''(1)$.

Under this assumption, direct calculations with generating functions are feasible, i.e. we can explicitly calculate the generating function of $Z_n$, conditioned on the environment. We define $\eta_k = 1/2\; b_k m_k^{-2}$ and recall that $f_{j,n} = f_{j+1} \circ \ldots \circ f_n$. Then for all $n \in \mathbb{N}$ and $s \in [0,1]$ (see [32, p. 156])

$$f_{j,n}(0) = 1 - \frac{1}{e^{-(S_n - S_j)} + \sum_{k=j+1}^{n} \eta_k e^{-(S_{k-1}-S_j)}}. \tag{8.1}$$

Moreover

$$f_j'(s) = \frac{e^{-X_j}}{(e^{-X_j} + \eta_j(1-s))^2} \tag{8.2}$$

and we can now compute the value of $\varrho$.

*Proof of Corollary 2.3.* By Proposition 6.1, $\rho \le \Lambda(0)$. Then it remains to prove that $\rho = -\log \mathbb{E}\big[e^{-X}\big]$ if $\mathbb{E}[Xe^{-X}] \ge 0$ and $\rho \ge \Lambda(0)$ otherwise. For that purpose, we use the representation of $\rho$ in terms of generating functions. Combining (8.1) and (8.2) we get

$$\begin{aligned}
f_j'\big(f_{j,n}(0)\big) &= e^{-X_j}\big(e^{-X_j} + \tfrac{\eta_j}{e^{-(S_n-S_j)} + \sum_{k=j+1}^{n} \eta_k e^{-(S_{k-1}-S_j)}}\big)^{-2} \\
&= e^{-X_j}\Big(\frac{e^{-(S_n-S_j)} + \sum_{k=j+1}^{n} \eta_k e^{-(S_{k-1}-S_j)}}{e^{-(S_n-S_{j-1})} + \eta_j + \sum_{k=j+1}^{n} \eta_k e^{-(S_{k-1}-S_{j-1})}}\Big)^2 \\
&= e^{-X_j}\Big(\frac{e^{-(S_n-S_j)} + \sum_{k=j+1}^{n} \eta_k e^{-(S_{k-1}-S_j)}}{e^{-(S_n-S_{j-1})} + \sum_{k=j}^{n} \eta_k e^{-(S_{k-1}-S_{j-1})}}\Big)^2 \\
&= e^{-X_j}\Big(\frac{\mathbb{P}(Z_n > 0 | Z_{j-1} = 1, \mathcal{E})}{\mathbb{P}(Z_n > 0 | Z_j = 1, \mathcal{E})}\Big)^2 .
\end{aligned}$$

Thus, noting that $\mathbb{P}(Z_n > 0 | Z_n = 1) = 1$, we get that

$$\mathbb{E}\Big[\prod_{j=1}^{n} f_j'\big(f_{j,n}(0)\big)\Big] = \mathbb{E}\Big[\prod_{j=1}^{n} e^{-X_j} \frac{\mathbb{P}(Z_n > 0 | Z_{j-1} = 1, \mathcal{E})^2}{\mathbb{P}(Z_n > 0 | Z_j = 1, \mathcal{E})^2}\Big]$$

$$= \mathbb{E}\Big[e^{-S_n} \mathbb{P}(Z_n > 0 | \mathcal{E}, Z_0 = 1)^2\Big] \ . \tag{8.3}$$

First, we consider the case $\mathbb{E}[Xe^{-X}] \geq 0$. Note that this condition implies $\mathbb{E}[e^{-X}] < \infty$. Bounding the probability in (8.3) by 1 immediately yields

$$\mathbb{E}\Big[\prod_{j=1}^{n} f_j'\big(f_{j,n}(0)\big)\Big] \leq \mathbb{E}[e^{-S_n}],$$

so $\rho \geq -\log \mathbb{E}[e^{-X}]$. To get the converse inequality, we change to the measure $\hat{\mathbb{P}}$, defined by

$$\hat{\mathbb{P}}(X \in dx) = \frac{e^{-x}\mathbb{P}(X \in dx)}{\mathbb{E}[e^{-X}]}.$$

Then by Jensen's inequality

$$\mathbb{E}\Big[e^{-S_n}\mathbb{P}(Z_n > 0 | \mathcal{E}, Z_0 = 1)^2\Big] = \mathbb{E}\big[e^{-X}\big]^n \hat{\mathbb{E}}\big[\mathbb{P}(Z_n > 0 | \mathcal{E})^2\big] \geq \mathbb{E}[e^{-X}]^{n-1}\hat{\mathbb{P}}(Z_n > 0)^2.$$

We observe that $\hat{\mathbb{E}}[X] = \mathbb{E}[Xe^{-X}] \geq 0$, thus under $\hat{\mathbb{P}}$, $S_n$ is a random walk with nonnegative drift. It ensures that the branching process is still critical or supercritical with respect to $\hat{\mathbb{P}}$, so $\hat{\mathbb{P}}(Z_n > 0) > Cn^{-\beta}$ for some $\beta, C > 0$ as $n \to \infty$ (see e.g. [4] for the critical case, whereas $\mathbb{P}(Z_n > 0)$ has a positive limit in the supercritical case). Letting $n \to \infty$, we get

$$\rho = -\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}\Big[\prod_{j=1}^{n} f_j'\big(f_{j,n}(0)\big)\Big] \leq -\log \mathbb{E}[e^{-X}] \ .$$

Secondly, we consider $\mathbb{E}[Xe^{-X}] < 0$. There exists a $\nu \in (0, 1]$ such that $\mathbb{E}[Xe^{-\nu X}] = 0$ and we change to the measure $\mathbf{P}$ defined in (6.2). Applying this change of measure and the well-known estimate $\mathbb{P}(Z_n > 0|\mathcal{E}) \leq e^{L_n}$ a.s., we get that

$$\mathbb{E}\Big[e^{-S_n}\mathbb{P}(Z_n > 0 | \mathcal{E}, Z_0 = 1)^2\Big] \leq \mathbb{E}[e^{-\nu X}]^n \mathbf{E}\Big[e^{(-1+\nu)S_n + 2L_n}\Big] \ .$$

Note that $L_n \leq -S_n$ and $\nu \in (0, 1]$ imply that $(-1 + \nu)S_n + 2L_n \leq 0$, so

$$\mathbb{E}\Big[e^{-S_n}\mathbb{P}(Z_n > 0 | \mathcal{E}, Z_0 = 1)^2\Big] \leq \mathbb{E}[e^{-\nu X}]^n \ ,$$

which yields $\rho \geq -\log \mathbb{E}[e^{-\nu X}] = \Lambda(0)$. The last line comes from $\Lambda(0) = \sup_{s\leq 0}\{-\log \mathbb{E}[e^{sX}]\}$ and the fact that the condition $\mathbb{E}[Xe^{-\nu X}] = 0$ implies that the supremum is taken in $s = -\nu$. □

Then our theorem on lower large deviations immediately yields the lower rate function in the LF case. If $\mathbb{E}[Xe^{-X}] < 0$, $\chi(\theta) = \Lambda(\theta)$. Otherwise, for $\theta \leq \mathbb{E}[X]$,

$$\chi(\theta) = \inf_{t\in[0,1)} \big\{ - t \log\big(\mathbb{E}\big[e^{-X}\big]\big) + (1 - t)\Lambda\big(\theta/(1 - t)\big)\big\}.$$

Let us now prove the representation of $\chi$ from Corollary 3.3.

*Proof of Corollary 3.3.* Recall that $s \to \mathbb{E}[e^{sX}]$ is the moment generating function of $X$, which is a convex function. The result of the corollary is trivial if $\rho = \Lambda(0)$. Thus we only have to consider the case $\rho = \mathbb{E}[e^{-X}]$ and by Corollary 2.3 and $0 < \mathbb{E}[Xe^{-X}] < \infty$ and thus $\mathbb{E}[e^{-X}] < \infty$. We have

$\varrho = -\log \mathbb{E}[e^{-X}] \leq \sup_{s<0}\{-\log \mathbb{E}[e^{sX}]\} = \Lambda(0)$. Note that $\Lambda(0) = \infty$ is possible.
Let us recall some details of Legendre transforms. It is well-known (see e.g. [16]) that

$$v_\theta(s) := -\theta s - \log \mathbb{E}\big[e^{-sX}\big]$$

is a convex function. The conditions $\mathbb{E}[e^{-X}] < \infty$ and $0 < \mathbb{E}[Xe^{-X}] < \infty$ imply by the dominated convergence theorem that $v$ above is differentiable in $s = 1$ and

$$v'_\theta(1) := -\theta - \mathbb{E}\big[Xe^{-X}\big]/\mathbb{E}\big[e^{-X}\big] \ .$$

Thus by definition of $\theta^*$, the derivative of $v'_{\theta^*}$ vanishes for $s = 1$, i.e. $v^*_\theta$ takes its minimum in $s = 1$. Thus,

$$\Lambda(\theta^*) := -\theta^* - \log \mathbb{E}\big[e^{-X}\big] < \infty$$

and by the theory of Legendre transforms, the tangent $t$ on the graph of $\Lambda$ in $\theta^*$ is described by

$$t(\theta) := -\theta - \log \mathbb{E}\big[e^{-X}\big] \ .$$

As $\Lambda$ is convex and decreasing for $\theta < \mathbb{E}[X]$, we have $\Lambda(\theta) \geq t(\theta)$ for $\theta < \theta^*$. This proves the representation in Corollary 3.3. □

# 9 Examples with two environments : dependence on the initial and final population.

In this section, we focus on the importance of the initial population.

**Example 1 : the limits $\frac{1}{n}\log \mathbb{P}_1(Z_n = i)$ and $\frac{1}{n}\log \mathbb{P}_1(Z_n = j)$ may be both finite but different** Assume that the environment consists of two states $q_1$ and $q_2$ such that

$$r := \mathbb{P}(Q_1 = q_1) = 1 - \mathbb{P}(Q_1 = q_2) > 0; \quad q_1(1) = 1; \quad q_2(0) = p, \ q_2(2) = 1 - p,$$

with $p \in (0, 1)$. Then

$$\tfrac{1}{n}\log \mathbb{P}_1(Z_n = 1) = \log r, \qquad \tfrac{1}{n}\log \mathbb{P}_1(Z_n = 2) \geq \max\big\{\log r; \log[(1-r)2(1-p)p]\big\}.$$

where the term $\log r$ comes from a population which stays equal to 1 in the environment sequence $(q_1, q_1, q_1, \cdots)$ and the last term comes from a population which stays equal to 2 in the environment sequence $(q_2, q_2, q_2, \cdots)$. Thus if $r$ is chosen small enough (i.e. $r < \frac{2(1-p)p}{1+2(1-p)p}$),

$$\lim_{n\to\infty} \tfrac{1}{n}\log \mathbb{P}_1(Z_n = 1) < \lim_{n\to\infty} \tfrac{1}{n}\log \mathbb{P}_1(Z_n = 2) \ .$$

**Example 2 : the situation $\mathbb{P}_1(Z_n = k) \gg \mathbb{P}_k(Z_n = k)$, $k > 1$ is possible** Again, we assume that the environment consists of just two states $q_1$ and $q_2$ and now

$$\begin{aligned}
&r := \mathbb{P}(Q_1 = q_1) = 1 - \mathbb{P}(Q_1 = q_2) > 0,\\
&q_1(1) = p \quad , \quad q_1(a) = 1 - p,\\
&q_2(0) = p \quad , \quad q_2(2) = p \quad , \quad q_2(a) = 1 - 2p,
\end{aligned}$$

with $p \in (0, \frac{1}{2})$ and $a > 2$.
First, we note that $\liminf_{n\to\infty} \log \mathbb{P}_1(Z_n = 2) \geq -rp$, which comes from a population staying equal to 1 in the environment sequence $(q_1, q_1, \ldots)$.

Next, let us estimate the extinction probability, given the environment. We first observe that any BPVE whose environments are either $q_1$ or $q_2$ is stochastically larger that the Galton Watson process with reproduction law (and unique environment) $q_2$. As a consequence,

$$\mathbb{P}_1(Z_n = 0|\mathcal{E}) \leq \mathbb{P}_1(Z_n = 0|Q_1 = \ldots, Q_n = q_2) \leq \mathbb{P}(Z_\infty = 0|Q_1 = q_2, Q_2 = q_2, \ldots) =: s_e \quad \text{a.s.}$$

It is well-known that $s_e$ is given as the fixpoint $< 1$ of the generating function $f_2$ of $q_2$:

$$s_e = f_2(s_e) = p + ps_e^2 + (1 - 2p)s_e^a.$$

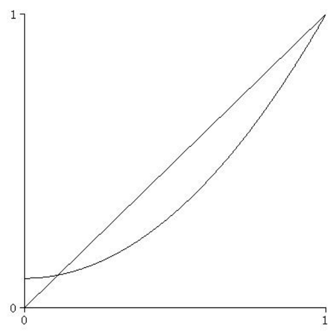


Figure 5: Generating function of a distribution with $q(0) > 0$.

Let us now estimate $s_e$. For for $s = 2p$, we have $2p > f_2(2p) = p + 4p^3 + (1 - 2p)2^a p^a$ if $p < \frac{1}{4}$ and $a$ is large enough. Then $s_e \leq 2p$ (see also Figure 5). We get for $p < \frac{1}{4}$, $a$ large enough and all $i \geq 1$, $k \leq n$,

$$\mathbb{P}(Z_n = 0|\mathcal{E}, Z_k = i) \leq s_{ext}^i \leq (2p)^i.$$

Using this estimate and the explicit law of $\mathbb{P}(Z_{k+1} = . \mid Z_k = 2, Q_k = q_1)$, we obtain

$$\begin{aligned}
&\mathbb{P}_2(Z_n = 2|\mathcal{E}, Q_k = q_1, Z_k = 2) \\
&\quad = p^2\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2) + 2(1-p)p\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 1 + a) \\
&\qquad\qquad + (1-p)^2\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2a) \\
&\quad = \mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2)\Big(p^2 + 2(1-p)p\binom{a+1}{2}\mathbb{P}(Z_n = 0|\mathcal{E}, Z_{k+1} = 1 + a - 2) \\
&\qquad + (1-p)^2\binom{2a}{2}\mathbb{P}(Z_n = 0|\mathcal{E}, Z_{k+1} = 2a - 2)\Big) \\
&\quad \leq \mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2)\Big(p^2 + 2(1-p)p\binom{a+1}{2}(2p)^{a-1} + (1-p)^2\binom{2a}{2}(2p)^{2a-2}\Big)
\end{aligned}$$

If $p$ is small enough, we get

$$\mathbb{P}_2(Z_n = 2|\mathcal{E}, Q_k = q_1, Z_k = 2) \leq \mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2)3p^2 \ .$$

Analogously, if the environment $q_2$ occurs in generation $k$, we get

$$\begin{aligned}
&\mathbb{P}_2(Z_n = 2|\mathcal{E}, Q_k = q_2, Z_k = 2) \\
&\quad = 2p^2\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2) + p^2\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 4) + 2p(1-2p)\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = a) \\
&\qquad + 2p(1-p)\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = a+2) + (1-2p)^2\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2a) \\
&\quad \le \mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2)3p^2 \ .
\end{aligned}$$

Next, note that the population starting from $Z_0 = 2$ is either always $\geq 2$ or extinct. Thus in each generation, there are at least two individuals and we may apply the estimates above for the subtrees emerging in generation $k$. Finally we get that

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_2(Z_n = 2) \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}\big[\mathbb{P}_2(Z_n = 2|\mathcal{E})\big] \le \log(3p^2).$$

If $p$ is now chosen small enough, we get that $3p^2 < rp$ and thus

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_2(Z_n = 2) < \liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_1(Z_n = 2).$$

Note that this example shows that, as in the the case without extinction in [10], the initial population may be of importance for the asymptotic of the probability of staying small, but alive.

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