ECOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES
APPLIQUÉES
UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 41 50. Fax: 01 69 33 30 11

http://www.cmap.polytechnique.fr/

Stochastic models for a chemostat and long time behavior

Pierre Collet, Servet Martínez, Sylvie Méléard, Jaime San Martín

R.I. N^0 755 *June 2012*

Stochastic models for a chemostat and long time behavior

P. Collet[*], Servet Martínez[†], Sylvie Méléard[‡], Jaime San Martín[§]

June 15, 2012

Abstract

We introduce two stochastic chemostat models consisting in a coupled population-nutrient process reflecting the interaction between the nutrient and the bacterias in the chemostat with finite volume. The nutrient concentration evolves continuously but depending on the population size, while the population size is a birth and death process with coefficients depending on time through the nutrient concentration. The nutrient is shared by the bacteria and creates a regulation of the bacterial population size. The latter and the fluctuations due to the random births and deaths of individuals make the population go almost surely to extinction. Therefore, we are interested in the long time behavior of the bacterial population conditioned to the non-extinction. We prove the global existence of the process and its almost sure extinction. The existence of quasi-stationary distributions is obtained based on a general fixed point argument. Moreover, we prove the absolute continuity of the nutrient distribution when conditioned to a fixed number of individuals and the smoothness of the corresponding densities.

---

[*] Centre de Physique Théorique, CNRS UMR 7644, Ecole Polytechnique F-91128 Palaiseau Cedex (France); e-mail: `collet@cpht.polytechnique.fr`

[†] Departamento Ingeniería Matemática and Centro Modelamiento Matemático, Universidad de Chile, UMI 2807 CNRS, Casilla 170-3, Correo 3, Santiago, Chile; `e-mail: smartine@dim.uchile.cl`

[‡] CMAP, Ecole Polytechnique, CNRS, route de Saclay, 91128 Palaiseau Cedex-France; e-mail: `sylvie.meleard@polytechnique.edu`

[§] Departamento Ingeniería Matemática and Centro Modelamiento Matemático, Universidad de Chile, UMI 2807 CNRS, Casilla 170-3, Correo 3, Santiago, Chile; `e-mail: jsanmart@dim.uchile.cl`

1 Introduction

Since some decades and the first work of J. Monod [9] and Novik and Szilar [10], [11], see also [12], biologists have developed procedures which allow to maintain a bacterial population at a stationary finite size while at the same time, the bacteria have a constant individual growth rate. The procedure is based on a chemostat: bacteria live in a growth container of constant volume in which liquid is injected continuously. The entering liquid contains a fixed concentration of nutrient but no bacteria (fresh liquid). In the container, the nutrient is consumed by the bacteria. We assume that the chemostat is well stirred, so that the distribution of bacteria and nutrient are spatially uniform. Since the container has a finite volume and fresh liquid enters permanently, an equal amount of liquid pours out containing both unconsumed nutrients and bacteria.
These chemostats are extremely useful in the study of bacterial population dynamics, in particular in the study of the selection of the fastest growing species or the fixation of advantageous mutations (see [4], [5], [6] or [8]). In the literature, their study is mainly based on deterministic models where both nutrient and bacteria population dynamics are described by a coupled deterministic continuous process. Deterministic approximations for the bacterial population's size are justified by a large population assumption.

In this work, we develop stochastic chemostat models based on a previous work of Crump and O' Young [3] taking into account that the bacteria population may not be large enough so that a deterministic approximation can be justified. We introduce a coupled population-nutrient process reflecting the interaction between the nutrient and the bacterias in the chemostat. The nutrient concentration evolves continuously but depending on the population size, while the population size is a birth and death process with coefficients depending on time through the nutrient concentration. Moreover, the time derivative of the nutrient concentration jumps simultaneously with the population size. We thus take into account the random fluctuations of this population size due to the individual births and deaths. The bacteria need nutrient to reproduce. We will consider two cases. In the first one, the bacterial population dies instantaneously if the nutrient is missing. In the second case bacteria can survive without nutrient by undergoing some kind of hibernation and may wake up once nutrient reappears. The nutrient is shared by the bacteria. This creates an indirect competition between bacteria and leads to a regulation of their population size. In our models, the fluctuations due to the random births and deaths of individuals and the size regulation make the population go almost surely to extinction. Therefore, the long time behavior of the population's size is obvious and the interesting questions concern firstly the rate of extinction and secondly the long time behavior conditioned to the non-extinction which is captured by the notion of quasi-stationary distribution.

To our knowledge, the models introduced in this paper are the first stochastic chemostat models where interaction between bacteria is taken into account leading to extinction. The study of quasi-stationarity gives nevertheless a description of a quasi-stability which can happen in a faster time scale than extinction. This work concerns monotype individuals but could be generalized to a multi-type population.

In Section 2, we describe the two population-nutrient models described above and prove in Section 3 their global existence. We also show the extinction of the population when time increases. The existence of quasi-stationary distributions is obtained in Section 4. Our main theorem is based on a general argument proved in [2]. In Section 5, we prove the absolute continuity of the nutrient distribution when conditioned to a fixed number of individuals and the smoothness of the corresponding densities.

2 The Stochastic Population-Nutrient Process

We consider a stochastic discrete population process describing the dynamics of a bacteria population for which individuals develop and reproduce depending on the quantity of nutrient y in the solution. The dynamics of the nutrient is related to the consumption of the individuals. We assume that the concentration of nutrient in the injected solution (without bacteria) is a constant equal to y^*. The chemostat has a finite volume equal to one. The liquid enters in the chemostat free of bacteria and pours out after being well stirred in the container. The pouring out liquid contains bacteria. The dilution coefficient of nutrient in the fresh liquid per unit of time is D. Since the liquid is well stirred, around $N(t)D$ bacterias will be washed out in the pouring out, when $N(t)$ is the size of the bacteria population at time t. Thus D is also the rate at which an individual will disappear due to the evacuation of liquid.

We consider coupled processes in which the nutrient concentration evolves continuously while the bacteria population size evolves as a time-continuous birth and death process with coefficients depending on the nutrient concentration. We assume that the nutrient is partly consumed during the reproduction of bacteria.

We will denote by $(Y(t), t \geq 0)$ the concentration of nutrient and by $(N(t) : t \geq 0)$ the population size process. The stochastic process $Z = (Z(t) := (N(t), Y(t)) : t \geq 0)$ describes both the population size and the nutrient concentration in the chemostat.

Let us now define the parameters of the model.

If y is the quantity of nutrient, then the birth and death parameters driving the dynamics of the population are as follows.

- The birth rate per individual is $b(y)$, where the function $b : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to be an increasing continuous function and such that $b(0) = 0$ and $b(y) > 0$ for $y > 0$. We assume that b is bounded with an upper-bound b_∞.

A usual example of a function b is, for some constant $K > 0$,

$$b(y) = b_\infty \frac{y}{K + y} .$$

An extra hypothesis that we will add for some results is that b is differentiable and $\frac{db}{dy}(0) > 0$.

- The background death rate per individual is $d(y)$, so it is supposed to be a function of the concentration of nutrient. The function $d : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ is assumed to be continuous non-increasing, strictly positive, and $d(0)$ is the unique value that can be infinite.

- The dilution makes each individual disappear at rate D independently of the birth and death events.

- The per individual rate of consumption of nutrient for reproduction is $\frac{b(y)}{R}$, where R denotes the biomass yield. Furthermore, individuals consume nutrient during their life and the quantity of nutrient consumed per individual will be denoted by $\eta \geq 0$.
- We will consider two cases.

In the first case, individuals need nutrient to survive. Then we will assume that their death is instantaneous as soon as nutrient is missing, therefore $d(0) = \infty$.
In the second case, bacteria enter in some kind of hibernation if nutrient is missing. That means that $d_0(0)$ can be finite.
In both cases, we will set

$$\tilde{b}(y) = \frac{b(y)}{R} + \eta \, \mathbf{1}_{y>0}.$$

Let us now describe the process. In both cases the nutrient concentration process $Y = (Y(t) : t \geq 0)$ evolves according to

$$\frac{dY(t)}{dt} = D(y^* - Y(t)) - \tilde{b}(Y(t))\,N(t)\,. \tag{1}$$

The process Z has the following infinitesimal generator: for $(n, y) \in \mathbb{N} \times \mathbb{R}_+$,

$$\begin{aligned} \mathcal{L}f(n,y) &= b(y)\,n\,f(n+1,y) + (D + d(y))\,n\,f(n-1,y) \\ &\quad -(b(y) + D + d(y))\,n\,f(n,y) \\ &\quad + \Big(D(y^* - y) - n\,\tilde{b}(y)\Big)\ \partial_y f(n,y)\ . \end{aligned} \tag{2}$$

We refer to [1] for the numerical study of similar models.

Below, we show that the hypotheses we gave on the coefficients leading the process, guarantee that Z is well-defined and takes values in $\mathbb{N} \times [0, y^*]$.

Proposition 2.1. *The process Z is well defined and takes values in $\mathbb{N} \times \mathbb{R}_+$ for all positive time $t \in \mathbb{R}_+$. Moreover, $\mathbb{N}_+ \times [0, y^*]$ is an invariant set for the process Z, so $Y(0) \in [0, y^*]$ implies $Y(t) \in [0, y^*]$ for all $t \in \mathbb{R}_+$.*

Proof. The process $N = (N(t) : t \geq 0)$ has a pathwise representation driven by a Point Poisson measure $\mathcal{N}(d\theta, ds)$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$:

$$\begin{aligned} N(t) &= N_0 + \int\limits_{\mathbb{R}_+ \times (0,t]} \mathbf{1}_{\theta \leq b(Y_s)} \mathcal{N}(d\theta, ds) - \int\limits_{\mathbb{R}_+ \times (0,t]} \mathbf{1}_{b(Y_s) \leq \theta \leq b(Y_s) + D + d(Y_s)} \mathcal{N}(d\theta, ds); \\ Y(t) &= Y(0) + \int_0^t \Big(D(y^* - Y(s)) - \tilde{b}(Y(s))\,N_s\Big) ds. \end{aligned} \tag{3}$$

It is obvious that the process N is stochastically upper-bounded by a birth process with individual birth rate b_∞. This latter does not explode, so does N.

Let us now study the nutrient concentration $Y = (Y(t) : t \geq 0)$. Note that the assumptions on the parameters and Equation (1) ensure that $Y(t) \geq 0$. Indeed, the derivative of $Y(t)$ at $y = 0$ cannot be negative.

Let us show that $\mathbb{N} \times [0, y^*]$ is invariant. Take $Y(0) \in [0, y^*]$. A standard comparison theorem yields $Y(t) \leq v(t)$ where $v'(t) = D(y^* - v(t))$; $v(0) = Y(0)$. But in that case, $v(t) = y^* - (y^* - Y(0))e^{-Dt} \leq y^*$ then it remains in $[0, y^*]$ forever proving the invariance.

□

For the initial condition $N(0) \in \mathbb{N}^*$ and $Y(0) \in \mathbb{R}_+ \setminus [0, y^*]$ we have that at time $t = (Y(0) - y^*)/\tilde{b}(y^*)$ the process has already attained the invariant set $[0, y^*] \times \mathbb{R}_+$ or became extinct.

3 Study of Extinction

We are now interested in studying the extinction of the population or the complete consumption of the nutrient or other specific states of the population-nutrient process and the associated hitting times.

Let $\mathcal{B}(\mathbb{N} \times \mathbb{R}_+)$ be the class of Borel sets of $\mathbb{N} \times \mathbb{R}_+$ and $\mathcal{B}(\mathbb{R}_+)$ the class of Borel sets of $\mathbb{R}_+$. For $D \in \mathcal{B}(\mathbb{N} \times \mathbb{R}_+)$ we put by $T^D = \inf\{t \geq 0 : Z(t) \in D\}$ the hitting time of D by the process, with the usual convention $\infty = \inf \emptyset$. The hitting time of the boundaries will be denoted in such a way that the reference to process N will be avoided, the contrary for Y where we will explicit it. More precisely,

$$T_0 = T^{\{0\}\times\mathbb{R}_+} = \inf\{t \geq 0 : N(t) = 0\}, \; T_{Y=0} = T^{\mathbb{N}\times\{0\}} = \inf\{t \geq 0 : Y(t) = 0\}. \tag{4}$$

We will also denote by $T_{\leq m} = T^{\{0,..,m\}\times\mathbb{R}_+} = \inf\{t \geq 0 : N(t) \leq m\}$. Analogously for $T_{<m}$.

Note that the set $\{0\} \times \mathbb{R}_+$ is an absorbing set, that is $N(t) = 0$ for all $t \geq T_0$.

After T_0 the nutrient $Y(t)$ is absorbed linearly at y^*, in fact: $Y(t) = y^*$ for $t \geq T_0 + (y^* - Y(T_0))/D\, y^*$ and $Y(t) = Y(T_0) + D(t - T_0)$ for $t \in [T_0, T_0 + (y^* - Y(T_0))/D]$.

Let us see what happens after $T_{Y=0}$. In the case $\eta \geq Dy^*$ the nutrient $Y(t)$ remains at 0 up to the extinction of the population, so $Y(t) = 0$ when $T_{Y=0} < t < T_0$. The evolution of the nutrient after T_0 was already described. Now assume $n\,\eta > Dy^*$ for some $n \geq 1$, and denote n_0 the minimal of these values, so $(n_0 - 1)\,\eta \leq Dy^* < n_0\,\eta$. In this case $Y(t) = 0$ when $T_{Y=0} < t < T_{<n_0}$. In the case where $d_0(0) = \infty$, we get

$$T_{Y=0} \leq T_0 \Longrightarrow T_{Y=0} = T_0$$

because all individuals die instantaneously.

Let us firstly study the stationary nutrient concentration states at fixed population size.

Lemma 3.1. *Consider the equation*

$$G_n(y) = 0 \;\; \textit{with} \;\; G_n(y) := D(y^* - y) - n\,\tilde{b}(y)\,. \tag{5}$$

Then,

(i) If $\eta = 0$, then for any $n \in \mathbb{N}$, Equation (5) has a unique simple root y_n, which belongs to $[0, y^*]$. In addition, the sequence $(y_n : n \in \mathbb{N}^*)$ of the roots decreases to 0.

(ii) If $\eta > 0$ then Equation (5) has no root for $n > \frac{Dy^*}{\eta}$ and admits a simple root y_n for $n \leq \frac{Dy^*}{\eta}$.

Proof. Let us fix n. Obviously for $n = 0$, Equation (5) has the trivial root $y_0 = y^*$, so we restrict n to be in $\mathbb{N}^*$. By assumption, the function G_n is strictly decreasing so for each $n \in \mathbb{N}^*$ there exists at most one root. Note that for all such n we have $G_n(y^*) < 0$ and so there is no root to (5) in the set $[y^*, \infty)$.

Assume $\eta = 0$. For all n we have $G_n(0) > 0$ then there exists a unique root which is denoted y_n, so it satisfies $G_n(y_n) = 0$. On the other hand, we have

$$G_{n+1}(y_n) = D(y^* - y_n) - (n+1)\,\tilde{b}(y_n) = -\,\tilde{b}(y_n)\,.$$

Then, $G_{n+1}(0) > 0, G_{n+1}(y_n) < 0$. We deduce $0 < y_{n+1} < y_n < y^*$. Let $y_\infty = \lim_{n\to\infty} y_n$. By continuity

$$D(y^* - y_\infty) - n\,\tilde{b}(y_\infty) = \lim_{n\to\infty} D(y^* - y_n) - n\,\tilde{b}(y_n) = 0.$$

Then, necessarily $\tilde{b}(y_\infty) = 0$ and so $y_\infty = 0$, and $b(y_n) \sim \frac{Dy^* R}{n}$ as $n \to \infty$.

Assume $\eta > 0$. Then for all $n > Dy^*/\eta$ we have $G_n(0) < 0$ and so (5) has no solution. Hence the same argument as before gives the existence of a finite set of roots $(y_n : 1 \leq n \leq \lfloor Dy^*/\eta \rfloor)$ decreasing with n, where $\lfloor Dy^*/\eta \rfloor$ the biggest integer that is smaller or equal to Dy^*/η.

Note that when $n_0 = Dy^*/\eta$ then $y_{n_0} = 0$ and if $Dy^* < \eta$, there is no root. □

From Lemma 3.1, we know that the set

$$\mathcal{S} = \{y \in \mathbb{R}_+ : \exists n \in \mathbb{N}^*, D(y^* - y) - \tilde{b}(y)n = 0\}. \tag{6}$$

is a countable set included in $[0, y^*)$. If $\eta = 0$, it is infinite and accumulates at 0 and if $\eta > 0$, it is finite.

In the sequel, when we refer to y_n, we will assume implicitly that it exists, namely we are in the case $\eta = 0$ or $\eta > 0$ but $n \in \{1, ..\lfloor Dy^*/\eta \rfloor\}$.

Corollary 3.2. *(i) The set $\mathbb{N} \times [0, y_1]$ is invariant for the process $Z^{T_0} = (Z(t) : t \leq T_0)$ up to extinction, that is if $Z(0) = (N(0), Y(0)) \in \mathbb{N} \times [0, y_1]$ then $Z(t) = (N(t), Y(t)) \in \mathbb{N} \times [0, y_1]$ for all $t \leq T_0$.*

(ii) The set $\mathbb{N} \times [0, y_n]$ is invariant for the process $Z^{T_{<n}} = (Z(t) : t \leq T_{<n})$.

Proof. Let us show the first part. We have $Z(t) = (N(t), Y(t)) \in \mathbb{N}^* \times [0, y_1]$ for all $t < T_0$ because $dY(t)/dt \leq 0$ when $y \geq y_1$ and $n \geq 1$, and so if the trajectory arrives to y_1 the variable $Y(t)$ immediately decreases. For $n = 0$ it is evident, because we stop the process at T_0, and so $Y(T_0) = Y(T_0^-)$ but $Y(T_0^-) \leq y_1$ since $N(T_0^-) = 1$.

Part (ii) is shown in a similar way as (i). □

Let us state a useful lemma.

Lemma 3.3. *For any* $n_0 \in \mathbb{N}^*$, *there exists* $t_0 > 0$ *such that*

$$\inf_{y \in [0, y^*],\, 1 \leq n \leq n_0} \mathbb{P}_{(n,y)}\big(T_0 < t_0\big) > 0 \ .$$

Proof. This follows at once from the fact that the population process is stochastically dominated by a birth and death process with birth rate $b(y^*)$ and death rate $D + d(y^*)$. □

Theorem 3.4. *We have extinction of the population almost surely, namely for any* $y \in [0, y^*]$ *and any integer* n,

$$\mathbb{P}_{(n,y)}\big(T_0 < \infty\big) = 1 \ .$$

Proof. Remark that an obvious comparison theorem as used in the previous proof cannot be applied. Indeed the birth and death rates $b(y^*)$ and $D+d(y^*)$ could correspond to a supercritical case. The effect of the chemostat through the nutrient is a regulation of the population.
We will exhibit an integer n_0 such that the population process will spend an infinite amount of time below $n_0 + 1$.

Let $\tilde{y} > 0$ be such that $b(\tilde{y}) < D + d(\tilde{y})$. Note that by monotonicity, for all $y \in [0, \tilde{y}]$ we have $b(y) < D + d(y)$.
Let us define n_0 as an integer such that $y_{n_0} < \tilde{y}$ if some exists or equal to $[Dy^*/\tilde{b}(0)] + 1$ otherwise.
Let τ be the random time defined by

$$\sup_{t \geq \tau} N(t) \leq n_0 \ .$$

Then, it follows from Lemma 3.3 and the Markov property that if $\mathbb{P}_{(n,y)}\big(\tau < \infty\big) > 0$, then

$$\mathbb{P}_{(n,y)}\big(T_0 < \infty \,\big|\, \tau < \infty\big) = 1 \ .$$

Assume now $\mathbb{P}_{(n,y)}\big(\tau = \infty\big) = 1$. Let τ' be the random time defined by

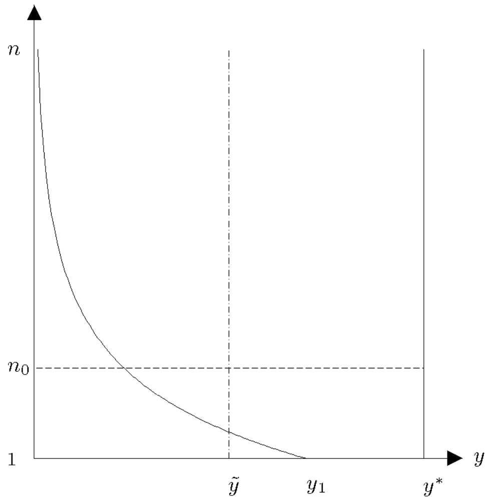
$$\inf_{t \geq \tau'} N(t) > n_0 \ .$$

Assume $\mathbb{P}_{(n,y)}\big(\tau' < \infty\big) > 0$. Let $y_0(t)$ be the solution of the differential equation

$$\frac{dy_0}{dt} = D(y^* - y_0(t)) - (n_0+1)\tilde{b}(y_0(t)) \ ,$$

with initial condition $y_0(0) = y^*$. Let $t_1 > 0$ be the finite solution of

$$y_0(t_1) = \tilde{y} \ .$$



The time t_1 is finite from the choice of n_0. It is easy to verify that for any integer valued measurable function $n(t) \geq n_0 + 1$, the solution $y(t)$ of the equation

$$\frac{dy}{dt} = D(y^* - y(t)) - n(t)\tilde{b}(y(t)) \ ,$$

with initial condition $y(0) \leq y^*$, satisfies

$$y(t) \leq \tilde{y}$$

for any $t \geq t_1$. On the set $\big\{ \tau' < \infty \big\}$, the process $(N(t), t \geq \tau' + t_1)$ is dominated by a linear birth and death process with birth rate $b(\tilde{y})$ and death rate $D + d_0(\tilde{y})$ (from the monotonicity of the functions). This birth and death chain attains n_0 almost surely in finite time since $b(\tilde{y}) < D + d_0(\tilde{y})$, see [13]. Hence on $\big\{ \tau' < \infty \big\}$, the process $(N(t), t \geq \tau' + t_1)$ should also attains n_0 in finite time. This contradicts our assumption $\mathbb{P}_{(n,y)}\big(\tau' < \infty\big) > 0$.

It remains to consider the case $\tau = \tau' = \infty$ almost surely. In this case there exist two infinite sequences of random times

$$T_1 < S_1 < T_2 < S_2 < \cdots$$

such that

$$N(t) \leq n_0 \quad \text{for } t \in [T_j, S_j) \, , \quad N(t) > n_0 \quad \text{for } t \in [S_j, T_{j+1}) \, .$$

Since we visit the set $\{N \leq n_0\}$ infinitely many times, and at each visit we have a uniformly positive probability of extinction, it follows by the Markov property and the Borel Cantelli Lemma that

$$\mathbb{P}_{(n,y)}\big(T_0 < \infty \,\big|\, \tau = \tau' = \infty\big) = 1 \, .$$

□

One of our main objectives of this work is to study the processes up to the moment the population is extinct $Z^{T_0} = (Z(t) : t \leq T_0)$, or before the moment of extinction $Z^{T_0^-} = (Z(t) : t < T_0)$. All the statements related to quasi-stationary distributions depend on $Z^{T_0^-}$.

4 Existence of Quasi-Stationary Distributions

A quasi-stationary distribution ν (with respect to the absorbing time T_0) is a probability measure defined on $\mathbb{N}^* \times \mathbb{R}_+$ that verifies

$$\mathbb{P}_\nu(Z(t) \in D \,|\, T_0 > t) = \nu(D) \, , \ \forall D \in \mathcal{B}(\mathbb{N}^* \times \mathbb{R}_+) \, . \tag{7}$$

It is known that starting from a quasi-stationary distribution, the time of absorption is exponential, that is $\mathbb{P}_\nu(T > t) = e^{-\lambda t}$ where $\lambda = \lambda(\nu) > 0$. Let $\mathcal{M}_b = \{f : \mathbb{N}^* \times \mathbb{R}_+ \to \mathbb{R} \text{ bounded and measurable}\}$. Equation (7) can be written as,

$$\exists \, \lambda > 0 : \ \mathbb{E}_\nu(f(Z(t)), T_0 > t) = e^{-\lambda t} \int_0^\infty f(y)\nu(dy) \ \ \forall t > 0, f \in \mathcal{M}_b. \tag{8}$$

Denote by

$$\kappa_n = \nu(\{n\} \times \mathbb{R}_+) = \mathbb{P}_\nu(N(0) = n) \, ,$$

and the probability measure conditioned to have n individuals by

$$\nu_n(B) = \nu(\{n\} \times B \,|\, \{n\} \times \mathbb{R}_+) = \mathbb{P}_\nu(Y(0) \in B \,|\, N(0) = n) \ \ \forall B \in \mathcal{B}(\mathbb{R}_+) \, .$$

Then $\nu(D) = \sum\limits_{n\in\mathbb{N}^*} \kappa_n \nu_n(D\cap\{n\}\times\mathbb{R}_+)$, $\nu_n(\mathbb{R}_+) = 1 \ \forall n \in \mathbb{N}^*$ and $\sum\limits_{n\in\mathbb{N}^*} \kappa_n = 1$.

In order that the probability measure ν is a quasi-stationary distribution, it must satisfy the infinitesimal condition deduced from (8), which is given by,

$$\exists\, \lambda > 0: \ \sum_{n\in\mathbb{N}^*} \kappa_n \int_0^\infty d\nu_n(y) \Big[\mathcal{L}f(n,y) - \lambda f(n,y)\Big] = 0 \ \ \forall f \in \mathcal{M}_b.$$

In terms of the adjoint operator $\mathcal{L}^\dagger$ of $\mathcal{L}$ defined in (2) the equation to be a quasi-stationary distribution is,

$$\exists\, \lambda > 0: \ \ \mathcal{L}^\dagger \nu = -\lambda\, \nu\,. \tag{9}$$

Theorem 4.1. *Assume there exists* $0 \le \sigma < 1$ *such that* $\limsup\limits_{y\searrow 0} y^\sigma d(y) < \infty$. *Then there exists a quasi-stationary distribution. Moreover there exists at least a quasi-stationary distribution such that* Y *is supported in* $[0, y^*]$.

Proof. It suffices to show the existence of a quasi-stationary distribution. In fact, the last part of the statement follows when the existence proof is applied to the process Z taking values in the invariant set $\mathbb{N} \times [0, y^*]$.

The idea for showing the existence is to use the abstract Theorem 4.2 proved in [2]. We assume $y \in [0, y^*]$. We define a function

$$\varphi_1(y, n) = \mathbf{1}_{n\ge 1}\ .$$

Then a simple computation (it is the same computation as for a birth and death process since φ_1 does not depend on y) leads to

$$\mathcal{L}\varphi_1(y,n) = \begin{cases} \ \ 0 & \text{if } n > 1\ , \\ -D - d(y) & \text{if } n = 1\ . \end{cases}$$

Therefore if d_0 is bounded above by a constant d_0^*, then

$$e^{-t(D+d_0^*)}\varphi_1 \le e^{t\mathcal{L}}\varphi_1 \le \varphi_1\ .$$

If d_0 is not bounded, we will prove the lower bound

$$\inf_{n_0\ge 1,\, y_0\in]0,y^*]} \mathbb{P}_{n_0,y_0}(N(1) \ge 1) \ge Q\ .$$

for a constant $Q > 0$. By the Markov property it suffices to prove that there exists $Q > 0$ such that

$$\inf_{y_0\in[0,y^*]} \mathbb{P}_{1,y_0}(N(t) = 1,\ \forall\, 0 \le t \le 1) \ge Q\ .$$

Since $N = 1$ on the whole time interval $[0, 1]$, $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = D(y^* - y) - \tilde{b}(y) \ , \tag{10}$$

with initial condition y_0. Since there neither birth, nor death on the time interval $[0, 1]$ we get

$$\mathbb{P}_{1,y_0}(N(t) = 1, \ \forall \ 0 \leq t \leq 1) = e^{-\int_0^1 \big(d(y(t))+b(y(t))\big) \, dt} \ .$$

Since $b(y)$ is bounded uniformly in y, the above quantity does not vanish if

$$\int_0^1 d(y(t)) \, dt < \infty \ .$$

It is easy to show that there is a constant $c > 0$ such that for any $y^* \geq y_0 \geq 0$, we have $y(t) \geq ct$ for any $t \in [0, 1]$. Since $d(y) \leq \mathcal{O}(1) y^{-\sigma}$ for y small, we get

$$\int_0^1 d(y(t)) \, dt \leq \mathcal{O}(1) \int_0^1 y(t)^{-\sigma} \, dt < \infty \ .$$

It now follows immediately that

$$Q \ \varphi_1 \leq e^{\mathcal{L}} \varphi_1 \leq \varphi_1 \ .$$

The second function is

$$\varphi_2(y, n) = \mathbf{1}_{n \geq 1} \ e^{a(y)n} \ ,$$

with $a(y) = \alpha y + a_0$, $\alpha > 0$ and $a_0 > 0$. A simple computation for $n > 1$ (it is the same computation as for a birth and death process since φ_2 does not depend on y) leads to,

$$\mathcal{L}\varphi_2(y, n) = n \, e^{a(y)n} \, \Xi(n, y) \ ,$$

where

$$\Xi(n, y) = b(y) \left(e^{a(y)} - 1 - \frac{\alpha n}{R}\right) + (D + d(y)) \left(e^{-a(y)} - 1\right) + D\alpha(y^* - y) - \alpha\eta \mathbf{1}_{y>0} \ .$$

Let us show that there exists $A > 0$ and N_0 such that for all $n > N_0$,

$$\sup_{y \in [0, y^*]} \Xi(n, y) \leq -A.$$

Define $\zeta(y) = (D + d(y)) \left(e^{-a(y)} - 1\right) + D\alpha(y^* - y) - \alpha\eta\mathbf{1}_{y>0}$. We choose a_0 and α such that $D \left(e^{-a_0} - 1\right) + D\alpha y^* < 0$. Then $\zeta(0) < 0$. It follows that $\forall y \in [0, y^*]$, $\zeta(y) < -A$ for some $A > 0$.

Consider $\hat{N}_0$ such that $e^{a(y^*)} - 1 - \frac{\alpha \hat{N}_0}{R} < 0$. We still have for $n > \hat{N}_0$ that $e^{a(y^*)} - 1 - \frac{\alpha n}{R} < 0$.
Then for $n > \hat{N}_0$, we get $\Xi(n, y) \leq -A$. Therefore for any $C > 0$ there exists $N(C)$ such that for any $n > N(C)$,

$$\mathcal{L}\varphi_2(y, n) \leq -C\, \varphi_2(y, n)\ .$$

Therefore, for any $C > 0$ there exists $\Gamma(C) > 0$ (finite) such that

$$\mathcal{L}\varphi_2 \leq -C\ \ \varphi_2 + \Gamma(C)\ \varphi_1\ .$$

(the estimate for $n = 1$ is by direct computation taking $\Gamma(C)$ adequately large enough). Hence,

$$e^{t\mathcal{L}}\varphi_2 \leq e^{-tC}\ \ \varphi_2 + \frac{\Gamma(C)}{C}\varphi_1\ .$$

In the case where d is bounded above by $d^* < \infty$, we now choose $C > D + d^*$ and $t = 1$ and apply Theorem 4.2 in [2].
In the case d unbounded, we choose $C > -\log Q$ and apply Theorem 4.2 in [2].

□

5 Properties of the Quasi-Stationary Distributions

Proposition 5.1. *Any quasi-limiting distribution has support in* $\mathbb{N}^* \times [0, y^*]$.

Proof. We recall that a quasi-limiting distribution ν is a probability measure on $\mathbb{N}^* \times \mathbb{R}_+$ such that for some initial point $(n_0, y_0) \in \mathbb{N}^* \times \mathbb{R}_+$

$$\nu(A) = \lim_{t\to\infty} \mathbb{P}_{(n_0,y_0)}(Z(t) \in A \,|\, T_0 > t)\ ,\ \forall A \in \mathcal{B}(\mathbb{N}^* \times \mathbb{R}_+).$$

If $y_0 \in [0, y^*]$, the assertion follows from Proposition 2.1. Let us now assume that $y_0 > y^*$. We introduce the function $t \to v(t)$ defined by

$$\frac{dv}{dt}(t) = D(y^* - v) - \tilde{b}(y^*)\ ; v(0) = y_0.$$

Let $\tau_0(y_0)$ be defined by $v(\tau_0(y_0)) = y^*$. For $t \leq T_0 \wedge \tau_0(y_0)$, we have $Y(t) \leq v(t)$. It follows from Proposition 2.1 that $\mathbb{P}_{(n_0,y_0)}(Y(t) > y^* | T_0 > t) = 0\ ,\ \forall t \geq \tau_0(y_0)$, which concludes the proof. □

Theorem 5.2. *For all $n \in \mathbb{N}^*$ and any quasi-stationary distribution, the probability measure ν_n is absolutely continuous with respect to the Lebesgue measure, with C^∞-density on the set $\mathbb{R}_+ \setminus \{0, y_n\}$.*

The proof of this theorem is obtained from the following lemmas. Recall that the set $\mathcal{S}$ has been defined in (6).

Lemma 5.3. *For all $n \in \mathbb{N}^*$, the measure ν_n satisfies*

$$d\nu_n = c_0^n \, \delta_0 + \sum_{j \in \mathbb{N}^*} \left(c_j^n \, \delta_{y_j} + u_n(y) \, dy \right) \tag{11}$$

with $c_j^n \geq 0$ for $j \in \mathbb{N}$ and u_n is the density of the absolutely continuous part of ν_n (so it is a non-negative integrable function) and it is a C^∞ function outside $\mathcal{S} \cup \{0\}$.

Proof. The measures ν_n satisfy in the sense of distributions the differential equations

$$\begin{aligned} &\partial_y \left(\left(D(y^* - y) - n \, \tilde{b}(y) \right) \, \nu_n \right) \\ &= b(y) n \, \nu_{n-1} + (D + d(y)) n \, \nu_{n+1} - (b(y) + D + d(y)) n \, \nu_n + \lambda \nu_n. \end{aligned} \tag{12}$$

Since the right hand side is a measure, we conclude by a recursive argument that the measures ν_n have a C^∞ density on $(\mathcal{S} \cup \{0\})^c$. This also shows that these measures have no singular part, and the Lebesgue decomposition theorem gives relation (11). □

Lemma 5.4. *Let I be an open interval included in $(\mathcal{S} \cup \{0\})^c$. If there exists $n \in \mathbb{N}^*$ such that $\nu_n(I) = 0$, then $\nu_j(I) = 0$ for all $j \in \mathbb{N}^*$.*

Proof. Assume that there is an interval I not intersecting $\mathcal{S} \cup \{0\}$ such that for some integer $n \in \mathbb{N}^*$, $\nu_n(I) = 0$ vanishes on I. From (12) we deduce that ν_{n+1} and ν_{n-1} also vanish at I. Therefore, we conclude recursively that $\nu_j(I) = 0$ for all $j \in \mathbb{N}^*$. □

Lemma 5.5. *The probability measure ν_n is absolutely continuous on $\mathbb{R}_+ \setminus \{0, y_n\}$ and its density is bounded on any compact set contained in $\mathbb{R}_+ \setminus \{0, y_n\}$.*

Proof. Let us show that if $j \neq n$, ν_n cannot have a Dirac mass in y_j. We will do it by contradiction, so assume it does. Let f be a C^∞ function with compact support containing y_j and such that its support does not contain any other point of $S \cup \{0\}$ except y_j. By using formula (11) we get,

$$\begin{aligned} & -f'(y_j)\left(D(y^*-y_j) - n\,\tilde{b}(y)\right) - \int f'(y)\left(D(y^*-y) - n\,\tilde{b}(y)\right)\psi_n(y)dy \\ &= \int f(y)\,\big(b(y)\,n\,d\nu_{n-1}(y) + (D+d(y))\,n\,d\nu_{n+1}(y)\big) \\ &\quad + \int f(y)\,\big(\lambda - (b(y)+D+d(y))\,n\big)\,d\nu_n(y)\,. \end{aligned} \tag{13}$$

It is not difficult to construct a sequence of functions $(f_q : q \in \mathbb{N}^*)$ contained in C^∞ with support in a fixed small enough neighborhood of y_j and such that,

$$f_q'(y_j)=1;\ \lim_{q\to\infty}\sup_{y\in\mathbb{R}_+}|f_q(y)|=0;\ \sup_{y\in\mathbb{R}_+}|f_q'(y)|\le 1\ \forall q;\ \lim_{q\to\infty}|f_q'(y)|=0\ \forall y\neq y_j.$$

This leads to a contradiction, when we take $f = f_q$ in (13) and make q tend to infinity.

Now, using again equation (12), we deduce easily the boundedness of the density of ν_n outside a neighborhood of $\{y_n, 0\}$. □

Lemma 5.6. *The probability measure ν_n cannot have a Dirac mass in y_n.*

Proof. Assume it does. Then in a neighborhood of y_n, by using (11) since $c_j^n = 0$ for $j = 0, n$ and by writing $c_n = c_n^n$, we can write

$$d\nu_n = c_0^n\delta_0 + c_n\delta_{y_n} + u_n(y)\,dy\,.$$

Let f be a function C^∞ with compact support containing y_n but that does not contain 0. We have

$$\begin{aligned} & -\int f'(y)\left(D(y^*-y) - n\,\tilde{b}(y)\right)\,u_n(y)\,dy \\ &= \int f(y)\,\big(b(y)\,n\,d\nu_{n-1}(y) + (D+d(y))\,n\,d\nu_{n+1}(y)\big) \\ &\quad + \int f(y)\,\big(\lambda - (b(y)+D+d(y))\,n\big)\,u_n(y)\,dy \\ &\quad + c_n\,f(y_n)\,\big(\lambda - (b(y_n)+D+d(y_n))\,n\big)\,. \end{aligned}$$

We now construct a sequence $(f_q : q \in \mathbb{N}^*)$ of C^∞ functions with support in a fixed small enough neighborhood of y_n such that for some constant C',

$$f_q(y_n) = 1\ \forall q\ ;\ \sup_{y\in\mathbb{R}_+}|f_q(y)| = 1\ ;\ \lim_{q\to\infty}|f_q(y)| = 0\ \forall y\neq y_n$$

and

$$\sup_{y\in\mathbb{R}_+} |y-y_n||f_q'(y)| \le C' \ \forall q \ ; \ \lim_{q\to\infty} (y-y_n)f_q'(y) = 0 \ \forall y \neq y_n .$$

Such a sequence can be easily constructed.

Recall that ν_{n-1} and ν_{n+1} are absolutely continuous with C^∞ density near y_n. We conclude that if $c_n \neq 0$

$$\lambda = \big(b(y_n) + D + d(y_n)\big)\, n \ .$$

In the case d is constant, it is known that the eigenvalue λ satisfies $\lambda = \kappa_n(D+d) < D+d$, and so it is strictly less than $\big(b(y_n)+D+d(y_n)\big)\, n$ and we obtain a contradiction.

If d is not constant, the proof of the contradiction is more intricate.
Coming back to Equation (12) for u_n in a neighborhood of y_n but outside that point we get

$$\partial_y\big(G_n(y)\, u_n(y)\big) = f_n(y)\, G_n(y)\, u_n(y) + h_n(y)$$

with (see Lemma 3.1)

$$G_n(y) = D(y^* - y) - n\,\tilde{b}(y) = \beta_n(y-y_n) + \mathcal{O}\big((y-y_n)^2\big),$$

with $\beta_n < 0$, and

$$f_n(y) = n\,\frac{b(y_n) - b(y) + d(y_n) - d(y)}{G_n(y)} = \mathcal{O}(1) \ ,$$

and

$$h_n(y) = b(y)\, n\, u_{n-1}(y) + \big(D + d(y)\big)\, n\, u_{n+1}(y).$$

and recall that $h_n(y)$ is C^∞ near y_n. The only solution which is integrable near y_n is given by

$$u_n(y) = \frac{1}{G_n(y)} \exp\left(\int_{y_n}^{y} f_n(s) ds\right) \int_{y_n}^{y} e^{-\int_{y_n}^{s} f_n(w)dw}\, h_n(s)\, ds \ .$$

If $h_n > 0$ on a subset of positive measure of a small neighborhood of y_n, we have $u_n < 0$ which is a contradiction.
Therefore, h_n must vanish on both sides of y_n. By the above result on the support, we conclude that u_n vanishes in a neighborhood of y_n as well as all the ν_j with $j \neq n$ (see Lemma 5.4). In particular, if we consider the equation for ν_{n+1} in this neighborhood (see (12)), we get

$$0 = b(y_n)\, c_n\, \delta_{y_n}$$

which contradicts $c_n \neq 0$. □

Theorem 5.7. *On* $(0, y_1)$*, the density of* ν_n *satisfies* $u_n > 0$ *except perhaps in* y_n*.*

The proof uses two lemmas.

Lemma 5.8. *If* ν_{n+1} *or* ν_{n-1} *has a support dense in* $(0, y_1)$ *then* u_n *can be* 0 *only in* y_n*.*

Proof. The function G_n has a simple zero in y_n. Assume that for $z \in (0, y_1)$, $z \neq y_1$, $u_n(z) = 0$ (and $G_n(z) \neq 0$).
Computation as in the proof of Lemma 5.6 gives

$$u_n(y) = \frac{1}{G_n(y)} \exp\left(\int_z^y f_n(s)ds\right) \int_z^y e^{-\int_z^s f_n(w)dw}\, h_n(s)\, ds\ .$$

The conclusion follows. □

Let us show that the process is irreducible up to extinction and before y_1.

Lemma 5.9. *Starting from any initial condition on* $\mathbb{N}^* \times (0, y_1)$ *the process* Z *has dense support on* $\mathbb{N} \times (0, y_1)$*, that is,*

$$\forall m \in \mathbb{N},\ \forall y' \in (0, y_1),\ \forall \gamma > 0,\ \exists t(y') > 0 \text{ such that } \forall t > t(y'),$$
$$\forall (y_0, n_0) \in \mathbb{N}^* \times (0, y_1) :\ \mathbb{P}_{(n_0, y_0)}(N(t) = m, Y(t) \in (y' - \gamma, y' + \gamma)) > 0.$$

Proof. In the proof we will assume $y_0 < y'$ (the case $y_0 \in (y', y_1)$ is shown similarly). Let $\beta > 0$ be smaller than $\min(y_0, y' - y_0)/2$ and let $\tilde{t} > 0$ be fixed. Then, there is $\epsilon = \epsilon(n_0, \beta)$ such that the following event has a strictly positive probability:

− On the interval time $[0, \epsilon]$ there are exactly $n_0 - 1$ deaths and there is no other jump of N, and so $N(t)$ decreases from n_0 to $N(\epsilon) = 1$;

− $Y(\epsilon)$ belongs to $(Y(0) - \beta, Y(0) + \beta)$;

− On the interval of time $[\epsilon, \epsilon + \tilde{t}]$ there is no jump of N (no birth and no death);

− On the interval of time $[\epsilon + \tilde{t}, \epsilon + \tilde{t} + \beta]$ there are exactly $m - 1$ births and no other jump when $m > 1$, there is no jump if $m = 1$, or there is a unique death and no other jump when $m = 0$;

− $|Y(\epsilon + \tilde{t}) + \beta) - Y(\epsilon + \tilde{t})| < \gamma/2$.

For $t \in [\epsilon, \epsilon + \tilde{t})$ we have $N(t) = 1$, then in this interval of time and before the process $Y(t)$ has attained y', the derivate

$$\frac{dY(t)}{dt} = D(y^* - Y(t)) - \tilde{b}(Y(t))$$

is bounded below by $D(y^* - y') - \tilde{b}(y')$. Take $\tilde{t} = (y' - Y(0) + \beta)/(D(y^* - y') - \tilde{b}(y'))$, let us see that the number $t(y') = \epsilon + \tilde{t}'$ makes the job. In fact, we have ensured that in a time smaller or equal to $t(y')$ we have attained $\{m\} \times [y' - \gamma, y' + \gamma]$. For any time bigger than $t(y')$ it suffices to modify slightly the above argument and allow a sequence of jumps up to the moment that $Y(t)$ has negative derivate and in this way we can postpone the time of attaining the set $\{m\} \times [y' - \gamma, y' + \gamma]$ from $t(y')$ to a prescribed time $t > t(y')$. □

Proposition 5.10. *For all $n \in \mathbb{N}^*$, the probability measure ν_n has a support dense in $(0, y_1)$.*

Proof. Denote by $n \otimes \nu_n$ the probability measure defined on $\mathbb{N} \times \mathbb{R}_+$ by $n \otimes \nu_n(\{m\} \times B) = \delta_n(m)\nu_n(B)$ for all $B \in \mathcal{B}(\mathbb{R}_+)$. From Lemma 5.9 and the quasi-stationary distribution property (8) we have

$$\mathbb{P}_{n\otimes\nu_n}(N(0)=m, Y(0)\in(y'-\gamma, y'+\gamma) = e^{-\lambda t}\mathbb{P}_{n\otimes\nu_n}(N(t)=m, Y(t)\in(y'-\gamma, y'+\gamma)$$
$$= e^{-\lambda t}\int \mathbb{P}_{(n,y_0)}(N(t)=m, Y(t)\in(y'-\gamma, y'+\gamma))d\nu_n(y_0) > 0.$$

Then, the result follows. □

6 Bound on the asymptotic survival rate

Theorem 6.1. *Let λ be the exponential extinction rate associated with a quasi-stationary distribution ν. Then*

$$\lambda < \inf_n (n(b(y_n) + D + d(y_n))). \tag{14}$$

Proof. In Section 3 in [2], it was shown that this extinction rate is an eigenvalue of the dual problem associated with the probability measure

$$\mathcal{L}^\dagger \nu = -\lambda\nu.$$

For simplicity, we will prove inequality (14) in case of $n = 1$. The general case proved in a similar way is left to the reader.
We introduce the notation

$$G(y) = D(y^* - y) - \tilde{b}(y)$$

$$H(y) = \big(D + d(y)\big)$$

The quasi-stationary distribution equation for $n=1$ is given by

$$-\frac{d}{dy}\big(G(y)\,u_1(y)\big)-\big(b(y)+H(y)\big)u_1(y)+2H(y)\,u_2(y)+\lambda\,u_1(y)=0\;.$$

Let $a\in]0,y_1[$. We have for $y\in]0,y_1[$

$$\frac{d}{dy}\left(G(y)\,u_1(y)\,e^{\int_a^y (b(s)+H(s)-\lambda)/G_1(s)ds}\right)=e^{\int_a^y (b(s)+H(s)-\lambda)/G(s)ds}\,2\,H(y)\,u_2(y)\;,$$

and integrating between a and y we get

$$\begin{aligned}&\big(G(y)\,u_1(y)\,e^{\int_a^y (b(s)+H(s)-\lambda)/G(s)ds}-G(a)u_1(a)\\&=2\int_a^y e^{\int_a^\sigma (b(s)+H(s)-\lambda)/G(s)ds}\,H(\sigma)\,u_2(w)\,dw\geq 0\;.\end{aligned}$$

Therefore

$$u_1(y)\geq\frac{G(a)u_1(a)}{G(y)}\;e^{-\int_a^y (b(s)+H(s)-\lambda)/G(s)ds}\;.$$

Let us study more carefully the quantity $e^{-\int_a^y (b(s)+H(s)-\lambda)/G(s)ds}$. We recall that G only vanishes at y_1 and that G is decreasing since b is increasing. By a simple computation one gets

$$b(s)+H(s)-\lambda)/G(s)=\frac{b(y_1)+H(y_1)-\lambda}{(s-y_1)\,G'(y_1)}+\mathcal{O}(1)\;,$$

and thus

$$\int_a^y (b(s)+H(s)-\lambda)/G(s)ds=\frac{b(y_1)+H(y_1)-\lambda}{G'(y_1)}\;\log|y-y_1|+\mathcal{O}(1)\;.$$

Finally, since $G(y_1)=0$, and $G'(y_1)<0$, we get for $y\leq y_1$

$$G(y)=(y-y_1)\,G'(y_1)+\mathcal{O}\big((y-y_1)^2\big)=|y-y_1|\big(|G'(y_1)|+\mathcal{O}(|y-y_1|)\big)\;.$$

Therefore

$$\begin{aligned}u_1(y)&\geq\frac{G(a)u_1(a)}{G(y)}\,|y-y_1|^{-(b(y_1)+H(y_1)-\lambda)/G'(y_1)}\;e^{\mathcal{O}(1)}\\&\geq\frac{G(a)u_1(a)}{|y-y_1|\,|G'(y_1|}\,|y-y_1|^{-(b(y_1)+H(y_1)-\lambda)/G'(y_1)}\;e^{\mathcal{O}(1)}\;.\end{aligned}$$

As we have $u_1(a)>0$ by Theorem 5.7, the integrability of u_1 on $[0,y_1]$ implies

$$1+\frac{b(y_1)+H(y_1)-\lambda}{G'(y_1)}<1\;,$$

we finally get

$$\lambda<b(y_1)+H(y_1)\;.$$

□

References

[1] F.Campillo, M.Joannides, I.Larramendy. Stochastic models of the chemostat. arXiv:1011.5108.

[2] P.Collet, S.Méléard, S.Martínez, J.San Martín. Quasi-stationary distributions for structured birth and death processes with mutations. Probab. Theory Related Fields, Volume 151, Issue 1 (2011), Page 191-231.

[3] K.Crump and W.Young. Some stochastic features of bacterial constant growth apparatus. Bulletin of Mathematical Biology, **41** (1979), 53–66.

[4] P. Masci, O. Bernard, F. Grognard, "Continuous selection of the fastest growing species in the chemostat", 17th IFAC World Congress, Seoul, Korea, 2008.

[5] P. Masci, F. Grognard, E. Benoit, O. Bernard, "Competitive exclusion in the generalized Droop model with N species," submitted to Mathematical Biosciences and Engineering, 2009.

[6] D.Dykhuizen and D.Hartl. Selection in chemostats. Microbiol Rev. 1983 June; 47(2): 150-168.

[7] Flegr,J. (1997): Two distinct types of natural selection in turbidostat-like and chemostat-like ecosystems. J. theor. Biol. 188: 121-126.

[8] P. De Leenheer, B. Li, and H.L. Smith, Competition in the chemostat: Some remarks, Can. Appl. Math. Quar. 11. (2003), pp. 229-248

[9] Monod, J.(1950) La technique de culture continue. Ann. Inst. Pasteur 79 : 390 -410

[10] A.Novick, L.Szilard. Experiments with the Chemostat on Spontaneous Mutations of Bacteria. PNAS **36**, 708-719 (1950).

[11] Novick A, Szilard L (1950). "Description of the Chemostat". Science 112 (2920): 715-6

[12] H.Smith, P.Waltman. *The Theory of the Chemostat*. Cambridge University Press, 1995.

[13] E. Van Doorn. Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. Adv. in Appl. Probab. 23 no.4, 683–700 (1991).