ECOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES APPLIQUÉES
UMR CNRS 7641

New global stability estimates for monochromatic inverse acoustic scattering

M.I. Isaev and R.G. Novikov

R.I. 761   *October 2012*

New global stability estimates for monochromatic inverse acoustic scattering

M.I. Isaev and R.G. Novikov

Abstract

We give new global stability estimates for monochromatic inverse acoustic scattering. These estimates essentially improve estimates of [P. Hähner, T. Hohage, SIAM J. Math. Anal., 33(3), 2001, 670-685] and can be considered as a solution of an open problem formulated in the aforementioned work.

1 Introduction

We consider the equation

$$\Delta\psi + \omega^2 n(x)\psi = 0, \quad x \in \mathbb{R}^3, \ \omega > 0, \tag{1.1}$$

where

$$(1 - n) \in \mathbb{W}^{m,1}(\mathbb{R}^3) \text{ for some } m > 3,$$
$$\operatorname{Im} n(x) \geq 0, \quad x \in \mathbb{R}^3, \tag{1.2}$$
$$\operatorname{supp}(1 - n) \subset B_{r_1} \text{ for some } r_1 > 0,$$

where $\mathbb{W}^{m,1}(\mathbb{R}^3)$ denotes the standart Sobolev space on $\mathbb{R}^3$ (see formula (2.11) of Section 2 for details), $B_r = \{x \in \mathbb{R}^3 : |x| < r\}$.

We interpret (1.1) as the stationary acoustic equation at frequency ω in an inhomogeneous medium with refractive index n.

In addition, we consider the Green function $G^+(x, y, \omega)$ for the operator $\Delta + \omega^2 n(x)$ with the Sommerfeld radiation condition:

$$\left(\Delta + \omega^2 n(x)\right) G^+(x, y, \omega) = \delta(x - y),$$
$$\lim_{|x|\to\infty} |x| \left(\frac{\partial G^+}{\partial |x|}(x, y, \omega) - i\omega G^+(x, y, \omega) \right) = 0, \tag{1.3}$$
$$\text{uniformly for all directions } \hat{x} = x/|x|,$$
$$x, y \in \mathbb{R}^3, \ \omega > 0.$$

It is know that, under assumptions (1.2), the function G^+ is uniquely specified by (1.3), see, for example, [9], [6].

We consider, in particular, the following near-field inverse scattering problem for equation (1.1):

Problem 1.1. Given G^+ on $\partial B_r \times \partial B_r$ for some fixed $\omega > 0$ and $r > r_1$, find n on B_{r_1}.

We consider also the solutions $\psi^+(x, k)$, $x \in \mathbb{R}^3$, $k \in \mathbb{R}^3$, $k^2 = \omega^2$, of equation (1.1) specified by the following asymptotic condition:

$$\psi^+(x, k) = e^{ikx} - 2\pi^2 \frac{e^{i|k||x|}}{|x|} f\left(k, |k| \frac{x}{|x|}\right) + o\left(\frac{1}{|x|}\right)$$

$$\text{as} \quad |x| \to \infty \left(\text{uniformly in } \frac{x}{|x|}\right),$$

(1.4)

with some a priory unknown f.

The function f on $\mathcal{M}_\omega = \{k \in \mathbb{R}^3, l \in \mathbb{R}^3 : k^2 = l^2 = \omega^2\}$ arising in (1.4) is the classical scattering amplitude for equation (1.1).

In addition to Problem 1.1, we consider also the following far-field inverse scattering problem for equation (1.1):

Problem 1.2. Given f on $\mathcal{M}_\omega$ for some fixed $\omega > 0$, find n on B_{r_1}.

In [4] it was shown that the near-field data of Problem 1.1 are uniquely determined by the far-field data of Problem 1.2 and vice versa.

Global uniqueness for Problems 1.1 and 1.2 was proved for the first time in [17]; in addition, this proof is constructive. For more information on reconstruction methods for Problems 1.1 and 1.2 see [2], [9], [16], [17], [19], [23] and references therein.

Problems 1.1 and 1.2 can be also considered as examples of ill-posed problems: see [15], [5] for an introduction to this theory.

The main results of the present article consist of the following two theorems:

Theorem 1.1. *Let $C_n > 0$, $r > r_1$ be fixed constants. Then there exists a positive constant C (depending only on m, ω, r_1, r and C_n) such that for all refractive indices n_1, n_2 satisfying $\|1 - n_1\|_{\mathbb{W}^{m,1}(\mathbb{R}^3)}, \|1 - n_2\|_{\mathbb{W}^{m,1}(\mathbb{R}^3)} < C_n$, $\operatorname{supp}(1 - n_1), \operatorname{supp}(1 - n_2) \subset B_{r_1}$, the following estimate holds:*

$$\|n_1 - n_2\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C \left(\ln\left(3 + \delta^{-1}\right)\right)^{-s}, \quad s = \frac{m - 3}{3},$$

(1.5)

where $\delta = \|G_1^+ - G_2^+\|_{\mathbb{L}^2(\partial B_r \times \partial B_r)}$ and G_1^+, G_2^+ are the near-field scattering data for the refractive indices n_1, n_2, respectively, at fixed frequency ω.

Remark 1.1. We recall that if n_1, n_2 are refractive indices satisfying (1.2), then $G_1^+ - G_2^+$ is bounded in $\mathbb{L}^2(\partial B_r \times \partial B_r)$ for any $r > r_1$, where G_1^+ and G_2^+ are the near-field scattering data for the refractive indices n_1 and n_2, respectively, at fixed frequency ω, see, for example, Lemma 2.1 of [9].

Theorem 1.2. *Let $C_n > 0$ and $0 < \epsilon < \frac{m-3}{3}$ be fixed constants. Then there exists a positive constant C (depending only on m, ϵ, ω, r_1 and C_n) such that for all refractive indices n_1, n_2 satisfying $\|1 - n_1\|_{\mathbb{W}^{m,1}(\mathbb{R}^3)}, \|1 - n_2\|_{\mathbb{W}^{m,1}(\mathbb{R}^3)} < C_n$, $\operatorname{supp}(1 - n_1), \operatorname{supp}(1 - n_2) \subset B_{r_1}$, the following estimate holds:*

$$\|n_1 - n_2\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C \left(\ln\left(3 + \delta^{-1}\right)\right)^{-s+\epsilon}, \quad s = \frac{m - 3}{3},$$

(1.6)

where $\delta = ||f_1 - f_2||_{\mathbb{L}^2(\mathcal{M}_\omega)}$ and f_1, f_2 denote the scattering amplitudes for the refractive indices n_1, n_2, respectively, at fixed frequency ω.

For some regularity dependent s but always smaller than 1 the stability estimates of Theorems 1.1 and 1.2 were proved in [9]. Possibility of estimates (1.5), (1.6) with $s > 1$ was formulated in [9] as an open problem, see page 685 of [9]. Our estimates (1.5), (1.6) with $s = \frac{m-3}{3}$ give a solution of this problem. Indeed,

$$ s = \frac{m-3}{3} \to +\infty \quad \text{as} \quad m \to +\infty \tag{1.7} $$

Apparently, using the methods of [21], [22] estimates (1.5), (1.6) can be proved for $s = m-3$. For more information on stability estimates for Problems 1.1 and 1.2 see [9], [11], [24] and references therein. In particular, as a corollary of [11] estimates (1.5), (1.6) can not be fulfilled, in general, for $s > \frac{5m}{3}$.

The proofs of Theorem 1.1 and 1.2 are given in Section 3. These proofs use, in particular:

1. Properties of the Faddeev functions for equation (1.1) considered as the Schrödinger equation at fixed energy $E = \omega^2$, see Section 2.

2. The results of [9] consisting in Lemma 3.1 and in reducing (via Lemma 3.2) estimates of the form (1.6) for Problem 1.2 to estimates of the form (1.5) for Problem 1.1.

In addition in the proofs of Theorem 1.1 and 1.2 we combine some of the aforementioned ingredients in a similar way with the proof of stability estimates of [13].

2 Faddeev functions

We consider (1.1) as the Schrödinger equation at fixed energy $E = \omega^2$:

$$ -\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^3, \tag{2.1} $$

where $v = \omega^2(1 - n)$, $E = \omega^2$.

For equation (2.1) we consider the Faddeev functions G, ψ, h (see [7], [8], [10], [17]):

$$ G(x,k) = e^{ikx}g(x,k), \quad g(x,k) = -(2\pi)^{-3}\int\limits_{\mathbb{R}^3} \frac{e^{i\xi x}d\xi}{\xi^2 + 2k\xi}, \tag{2.2} $$

$$ \psi(x,k) = e^{ikx} + \int\limits_{\mathbb{R}^3} G(x-y,k)v(y)\psi(y,k)dy, \tag{2.3} $$

where $x \in \mathbb{R}^3$, $k \in \mathbb{C}^3$, $k^2 = E$, $\operatorname{Im} k \neq 0$,

$$ h(k,l) = (2\pi)^{-3}\int\limits_{\mathbb{R}^3} e^{-ilx}v(x)\psi(x,k)dx, \tag{2.4} $$

3

where

$$k, l \in \mathbb{C}^3, \ k^2 = l^2 = E, \ \mathrm{Im}\, k = \mathrm{Im}\, l \neq 0. \tag{2.5}$$

One can consider (2.3), (2.4) assuming that

$$v \text{ is a sufficiently regular function on } \mathbb{R}^3$$
$$\text{with suffucient decay at infinity.} \tag{2.6}$$

For example, in connection with Problems 1.1 and 1.2, one can consider (2.3), (2.4) assuming that

$$v \in \mathbb{L}^\infty(B_{r_1}), \quad v \equiv 0 \text{ on } \mathbb{R}^3 \setminus B_{r_1}. \tag{2.7}$$

We recall that (see [7], [8], [10], [17]):

- The function G satisfies the equation

$$(\Delta + E)G(x, k) = \delta(x), \ \ x \in \mathbb{R}^3, \ \ k \in \mathbb{C}^3 \setminus \mathbb{R}^3, \ \ E = k^2; \tag{2.8}$$

- Formula (2.3) at fixed k is considered as an equation for

$$\psi = e^{ikx}\mu(x, k), \tag{2.9}$$

  where μ is sought in $\mathbb{L}^\infty(\mathbb{R}^3)$;

- As a corollary of (2.3), (2.2), (2.8), ψ satisfies (2.1) for $E = k^2$;

- The Faddeev functions G, ψ, h are (non-analytic) continuation to the complex domain of functions of the classical scattering theory for the Schrödinger equation (in particular, h is a generalized "'scattering"' amplitude).

In addition, G, ψ, h in their zero energy restriction, that is for $E = k^2 = 0$, were considered for the first time in [3]. The Faddeev functions G, ψ, h were, actually, rediscovered in [3].

Let

$$\Sigma_E = \left\{ k \in \mathbb{C}^3 : k^2 = k_1^2 + k_2^2 + k_3^2 = E \right\},$$
$$\Theta_E = \left\{ k \in \Sigma_E, \ l \in \Sigma_E : \mathrm{Im}\, k = \mathrm{Im}\, l \right\}, \tag{2.10}$$
$$|k| = (|\mathrm{Re}\, k|^2 + |\mathrm{Im}\, k|^2)^{1/2}.$$

Let

$$\mathbb{W}^{m,q}(\mathbb{R}^3) = \{ w : \ \partial^J w \in \mathbb{L}^q(\mathbb{R}^3), \ |J| \le m \}, \ \ m \in \mathbb{N} \cup 0, \ q \ge 1,$$
$$J \in (\mathbb{N} \cup 0)^3, \ |J| = \sum_{i=1}^3 J_i, \ \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \partial x_2^{J_2} \partial x_3^{J_3}}, \tag{2.11}$$
$$\|w\|_{m,q} = \max_{|J| \le m} \|\partial^J w\|_{\mathbb{L}^q(\mathbb{R}^3)}.$$

Let the assumptions of Theorems 1.1 and 1.2 be fulfilled:

$$(1-n) \in \mathbb{W}^{m,1}(\mathbb{R}^3) \text{ for some } m > 3,$$
$$\operatorname{Im} n(x) \geq 0, \quad x \in \mathbb{R}^3,$$
$$\operatorname{supp}(1-n) \subset B_{r_1}, \tag{2.12}$$
$$\|1 - n\|_{m,1} \leq C_n.$$

Let

$$v = \omega^2(1-n), \quad N = \omega^2 C_n, \quad E = \omega^2. \tag{2.13}$$

Then we have that:

$$\mu(x,k) \to 1 \quad \text{as} \quad |k| \to \infty \tag{2.14}$$

and, for any $\sigma > 1$,

$$|\mu(x,k)| \leq \sigma \quad \text{for} \quad |k| \geq \lambda_1(N,m,\sigma,r_1), \tag{2.15}$$

where $x \in \mathbb{R}^3$, $k \in \Sigma_E$;

$$\hat{v}(p) = \lim_{\substack{(k,l) \in \Theta_E,\ k-l=p \\ |\operatorname{Im} k| = |\operatorname{Im} l| \to \infty}} h(k,l) \quad \text{for any } p \in \mathbb{R}^3, \tag{2.16}$$

$$|\hat{v}(p) - h(k,l)| \leq \frac{c_1(m,r_1)N^2}{(E+\rho^2)^{1/2}} \quad \text{for } (k,l) \in \Theta_E, \quad p = k - l,$$
$$|\operatorname{Im} k| = |\operatorname{Im} l| = \rho, \quad E + \rho^2 \geq \lambda_2(N,m,r_1), \tag{2.17}$$
$$p^2 \leq 4(E+\rho^2),$$

where

$$\hat{v}(p) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ipx} v(x)dx, \quad p \in \mathbb{R}^3. \tag{2.18}$$

Results of the type (2.14), (2.15) go back to [3]. For more information concerning (2.15) see estimate (4.11) of [12]. Results of the type (2.16), (2.17) (with less precise right-hand side in (2.17)) go back to [10]. Estimate (2.17) follows, for example, from formulas (2.3), (2.4) and the estimate

$$\|\Lambda^{-s} g(k)\Lambda^{-s}\|_{\mathbb{L}^2(\mathbb{R}^d) \to \mathbb{L}^2(\mathbb{R}^d)} = O(|k|^{-1})$$
$$\text{as} \quad |k| \to \infty, \quad k \in \mathbb{C}^3 \setminus \mathbb{R}^3, \tag{2.19}$$

for $s > 1/2$, where $g(k)$ denotes the integral operator with the Schwartz kernel $g(x-y,k)$ and Λ denotes the multiplication operator by the function $(1+|x|^2)^{1/2}$. Estimate (2.19) was formulated, first, in [14]. This estimate generilizes, in particular, some related estimate of [25] for $k^2 = E = 0$. Concerning proof of (2.19), see [26].

In addition, we have that:

$$h_2(k,l) - h_1(k,l) = (2\pi)^{-3} \int_{\mathbb{R}^3} \psi_1(x,-l)(v_2(x) - v_1(x))\psi_2(x,k)dx$$
$$\text{for } (k,l) \in \Theta_E, \ |\operatorname{Im} k| = |\operatorname{Im} l| \neq 0, \tag{2.20}$$
$$\text{and } v_1, v_2 \text{ satisfying } (2.6),$$

and, under the assumptions of Theorems 1.1 and 1.2,

$$|\hat{v}_1(p) - \hat{v}_2(p) - h_1(k,l) + h_2(k,l)| \leq \frac{c_2(m,r_1)N\|v_1 - v_2\|_{\mathbb{L}^\infty(B_{r_1})}}{(E + \rho^2)^{1/2}}$$
$$\text{for } (k,l) \in \Theta_E, \quad p = k - l, \quad |\text{Im}\,k| = |\text{Im}\,l| = \rho, \tag{2.21}$$
$$E + \rho^2 \geq \lambda_3(N,m,r_1), \quad p^2 \leq 4(E + \rho^2),$$

where h_j, ψ_j denote h and ψ of (2.4) and (2.3) for $v_j = \omega^2(1 - n_j)$, $j = 1, 2$, $N = \omega^2 C_n$, $E = \omega^2$.

Formula (2.20) was given in [18], [20]. Estimate (2.21) was given e.g. in [13].

3 Proofs of Theorem 1.1 and Theorem 1.2

3.1. Preliminaries. In this section we always assume for simplicity that $r_1 = 1$.

We consider the operators $\hat{S}_j$, $j = 1, 2$, defined as follows

$$(\hat{S}_j\phi)(x) = \int_{\partial B_r} G_j^+(x,y,\omega)\phi(y)dy, \quad x \in \partial B_r, \; j = 1, 2. \tag{3.1}$$

Note that

$$\|\hat{S}_1 - \hat{S}_2\|_{\mathbb{L}^2(\partial B_r)} \leq \|G_1^+ - G_2^+\|_{\mathbb{L}^2(\partial B_r) \times \mathbb{L}^2(\partial B_r)}. \tag{3.2}$$

To prove Theorems 1.1 and 1.2 we use, in particular, the following lemmas (see Lemma 3.2 and proof of Theorem 1.2 of [9]):

Lemma 3.1. *Assume $r_1 = 1 < r < r_2$. Moreover, n_1, n_2 are refractive indices with $\text{supp}\,(1 - n_1), \text{supp}\,(1 - n_2) \subset B_1$. Then, there exists a postive constant c_3 (depending only on ω, r, r_2) such that for all solutions $\psi_1 \in C^2(B_{r_2}) \cap \mathbb{L}^2(B_{r_2})$ to $\Delta\psi + \omega^2 n_1\psi = 0$ in B_{r_2} and all solutions $\psi_2 \in C^2(B_{r_2}) \cap \mathbb{L}^2(B_{r_2})$ to $\Delta\psi + \omega^2 n_2\psi = 0$ in B_{r_2} the following estimate holds:*

$$\left|\int_{B_1} (n_1 - n_2)\psi_1\psi_2 dx\right| \leq c_3\|\hat{S}_1 - \hat{S}_2\|_{\mathbb{L}^2(\partial B_r)}\|\psi_1\|_{\mathbb{L}^2(B_{r_2})}\|\psi_2\|_{\mathbb{L}^2(B_{r_2})}. \tag{3.3}$$

Note that estimate (3.3) is derived in [9] using an Alessandrini type identity, where instead of the Dirichlet-to-Neumann maps the operators $\hat{S}_1$, $\hat{S}_2$ are used, see [1], [9].

Lemma 3.2. *Let $r > r_1 = 1$, $\omega > 0$, $C_n > 0$, $\mu > 3/2$ and $0 < \theta < 1$. Let n_1, n_2 be refractive indices such that $\|(1 - n_j)\|_{\mathbb{H}^\mu(\mathbb{R}^3)} \leq C_n$, $\text{supp}(1 - n_j) \subset B_1$, $j = 1, 2$, where $\mathbb{H}^\mu = \mathbb{W}^{\mu,2}$. Then there exist positive constants T and η such that*

$$\|G_1^+ - G_2^+\|_{\mathbb{L}^2(\partial B_{2r} \times \partial B_{2r})}^2 \leq \eta^2 \exp\left(-\left(-\ln\frac{\|f_1 - f_2\|_{\mathbb{L}^2(\mathcal{M}_\omega)}}{T\eta}\right)^\theta\right) \tag{3.4}$$

for sufficiently small $\|f_1 - f_2\|_{\mathbb{L}^2(\mathcal{M}_\omega)}$, where G_j^+, f_j are near and far field scattering data for n_j, $j = 1, 2$, at fixed frequency ω.

3.2. Proof of Theorem 1.1. Let

$$\mathbb{L}_\mu^\infty(\mathbb{R}^3) = \{u \in \mathbb{L}^\infty(\mathbb{R}^3) : \|u\|_\mu < +\infty\},$$
$$\|u\|_\mu = \text{ess} \sup_{p \in \mathbb{R}^3} (1 + |p|)^\mu |u(p)|, \quad \mu > 0. \tag{3.5}$$

Note that

$$w \in \mathbb{W}^{m,1}(\mathbb{R}^3) \Longrightarrow \hat{w} \in \mathbb{L}_\mu^\infty(\mathbb{R}^3) \cap C(\mathbb{R}^3),$$
$$\|\hat{w}\|_\mu \leq c_4(m)\|w\|_{m,1} \quad \text{for} \quad \mu = m, \tag{3.6}$$

where $\mathbb{W}^{m,1}$, $\mathbb{L}_\mu^\infty$ are the spaces of (2.11), (3.5),

$$\hat{w}(p) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ipx} w(x) dx, \quad p \in \mathbb{R}^3. \tag{3.7}$$

Let

$$N = \omega^2 C_n, \quad E = \omega^2, \quad v_j = \omega^2(1 - n_j), \; j = 1, 2. \tag{3.8}$$

Using the inverse Fourier transform formula

$$w(x) = \int_{\mathbb{R}^3} e^{-ipx} \hat{w}(p) dp, \quad x \in \mathbb{R}^3, \tag{3.9}$$

we have that

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)} \leq \sup_{x \in \bar{B}_1} \left| \int_{\mathbb{R}^3} e^{-ipx} (\hat{v}_2(p) - \hat{v}_1(p)) \, dp \right| \leq$$
$$\leq I_1(\kappa) + I_2(\kappa) \quad \text{for any} \;\; \kappa > 0, \tag{3.10}$$

where

$$I_1(\kappa) = \int_{|p| \leq \kappa} |\hat{v}_2(p) - \hat{v}_1(p)| dp,$$
$$I_2(\kappa) = \int_{|p| \geq \kappa} |\hat{v}_2(p) - \hat{v}_1(p)| dp. \tag{3.11}$$

Using (3.6), we obtain that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \leq 2c_4(m)N(1 + |p|)^{-m}, \quad p \in \mathbb{R}^3. \tag{3.12}$$

Using (3.11), (3.12), we find that, for any $\kappa > 0$,

$$I_2(\kappa) \leq 8\pi c_4(m)N \int_\kappa^{+\infty} \frac{dt}{t^{m-2}} \leq \frac{8\pi c_4(m)N}{m-3} \frac{1}{\kappa^{m-3}}. \tag{3.13}$$

Due to (2.21), we have that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \le |h_2(k,l) - h_1(k,l)| + \frac{c_2(m)N\|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)}}{(E + \rho^2)^{1/2}},$$
$$\text{for } (k,l) \in \Theta_E, \quad p = k - l, \quad |\text{Im}\,k| = |\text{Im}\,l| = \rho, \tag{3.14}$$
$$E + \rho^2 \ge \lambda_3(N,m), \quad p^2 \le 4(E + \rho^2).$$

Let
$$r_2 \text{ be some fixed constant such that } r_2 > r,$$
$$\delta = \|G_1^+ - G_2^+\|_{\mathbb{L}^2(\partial B_r \times \partial B_r)}, \tag{3.15}$$
$$c_5 = (2\pi)^{-3} \int_{B_{r_2}} dx.$$

Combining (2.20), (3.2), (3.3) and (3.8), we get that

$$|h_2(k,l) - h_1(k,l)| \le$$
$$\le c_3 c_5 \omega^2 \|\psi_1(\cdot, -l)\|_{\mathbb{L}^\infty(B_{r_2})} \delta \|\psi_2(\cdot, k)\|_{\mathbb{L}^\infty(B_{r_2})}, \tag{3.16}$$
$$(k,l) \in \Theta_E, \quad |\text{Im}\,k| = |\text{Im}\,l| \ne 0.$$

Using (2.15), we find that

$$\|\psi_j(\cdot, k)\|_{\mathbb{L}^\infty(B_{r_2})} \le \sigma \exp\left(|\text{Im}\,k|r_2\right), \quad j = 1,2, \tag{3.17}$$
$$k \in \Sigma_E, \quad |k| \ge \lambda_1(N,m,\sigma).$$

Here and bellow in this section the constant σ is the same that in (2.15).
    Combining (3.16) and (3.17), we obtain that

$$|h_2(k,l) - h_1(k,l)| \le c_3 c_5 \omega^2 \sigma^2 e^{2\rho r_2}\delta,$$
$$\text{for } (k,l) \in \Theta_E, \quad \rho = |\text{Im}\,k| = |\text{Im}\,l|, \tag{3.18}$$
$$E + \rho^2 \ge \lambda_1^2(N,m,\sigma).$$

Using (3.14), (3.18), we get that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \le c_3 c_5 \omega^2 \sigma^2 e^{2\rho r_2}\delta + \frac{c_2(m)N\|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)}}{(E + \rho^2)^{1/2}}, \tag{3.19}$$
$$p \in \mathbb{R}^3, \ p^2 \le 4(E + \rho^2), \ E + \rho^2 \ge \max\{\lambda_1^2, \lambda_3\}.$$

Let
$$\varepsilon = \left(\frac{3}{8\pi c_2(m)N}\right)^{1/3} \tag{3.20}$$

and $\lambda_4(N,m,\sigma) > 0$ be such that

$$E + \rho^2 \ge \lambda_4(N,m,\sigma) \implies \begin{cases} E + \rho^2 \ge \lambda_1^2(N,m,\sigma), \\ E + \rho^2 \ge \lambda_3(N,m), \\ \left(\dfrac{\varepsilon(E + \rho^2)^{\frac{1}{6}}}{8}\right)^2 \le 4(E + \rho^2). \end{cases} \tag{3.21}$$

Using (3.11), (3.19), we get that

$$I_1(\kappa) \leq \frac{4}{3}\pi\kappa^3\left(c_3c_5\omega^2\sigma^2 e^{2\rho r_2}\delta + \frac{c_2(m)N\|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)}}{(E + \rho^2)^{1/2}}\right),$$
$$\kappa > 0, \ \kappa^2 \leq 4(E + \rho^2),$$
$$E + \rho^2 \geq \lambda_4(N, m, \sigma). \tag{3.22}$$

Combining (3.10), (3.13), (3.22) for $\kappa = \varepsilon(E + \rho^2)^{\frac{1}{6}}$ and (3.21), we get that

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)} \leq c_6(N, m, \omega, \sigma)\sqrt{E + \rho^2}\, e^{2\rho r_2}\delta +$$
$$+ c_7(N, m)(E + \rho^2)^{-\frac{m-3}{6}} + \frac{1}{2}\|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)}, \tag{3.23}$$
$$E + \rho^2 \geq \lambda_4(N, m, \sigma).$$

Let $\tau \in (0, 1)$ and

$$\beta = \frac{1 - \tau}{2r_2}, \quad \rho = \beta \ln\left(3 + \delta^{-1}\right), \tag{3.24}$$

where δ is so small that $E + \rho^2 \geq \lambda_4(N, m, \sigma)$. Then due to (3.23), we have that

$$\frac{1}{2}\|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)} \leq$$
$$\leq c_6(N, m, \omega, \sigma)\left(E + \left(\beta \ln\left(3 + \delta^{-1}\right)\right)^2\right)^{1/2}\left(3 + \delta^{-1}\right)^{2\beta r_2}\delta +$$
$$+ c_7(N, m)\left(E + \left(\beta \ln\left(3 + \delta^{-1}\right)\right)^2\right)^{-\frac{m-3}{6}} = \tag{3.25}$$
$$= c_6(N, m, \omega, \sigma)\left(E + \left(\beta \ln\left(3 + \delta^{-1}\right)\right)^2\right)^{1/2}\left(1 + 3\delta\right)^{1-\tau}\delta^\tau +$$
$$+ c_7(N, m)\left(E + \left(\beta \ln\left(3 + \delta^{-1}\right)\right)^2\right)^{-\frac{m-3}{6}},$$

where τ, β and δ are the same as in (3.24).

Using (3.25), we obtain that

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)} \leq c_8(N, m, \omega, \sigma, \tau)\left(\ln\left(3 + \delta^{-1}\right)\right)^{-\frac{m-3}{3}} \tag{3.26}$$

for $\delta = \|G_1^+ - G_2^+\|_{\mathbb{L}^2(\partial B_r \times \partial B_r)} \leq \delta_1(N, m, \omega, \sigma, \tau)$, where δ_1 is a sufficiently small positive constant. Estimate (3.26) in the general case (with modified c_8) follows from (3.26) for $\delta \leq \delta_1(N, m, \omega, \sigma, \tau)$ and the property that

$$\|v_j\|_{\mathbb{L}^\infty(B_1)} \leq c_9(m)N, \quad j = 1, 2. \tag{3.27}$$

Taking into account (3.8), we obtain (1.5).

3.3. *Proof of Theorem 1.2.* According to the Sobolev embedding theorem, we have that

$$\mathbb{W}^{m,1}(\mathbb{R}^3) \subset \mathbb{H}^{m-3/2}(\mathbb{R}^3), \tag{3.28}$$

where $\mathbb{H}^\mu = \mathbb{W}^{\mu,2}$.

Combining (1.2), (1.5), (3.4) with θ satisfying $\theta\frac{m-3}{3} = \frac{m-3}{3} - \epsilon$, and (3.28), we obtain (1.6) for sufficiently small $\|f_1 - f_2\|_{\mathbb{L}^2(\mathcal{M}_\omega)}$ (analogously with the proof of Theorem 1.2 of [9]). Using also (3.27) and (3.8), we get estimate (1.6) in the general case.

References

[1] G. Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl.Anal. 27, 1988, 153-172.

[2] N.V. Alexeenko, V.A. Burov and O.D. Rumyantseva, *Solution of the three-dimensional acoustical inverse scattering problem. The modified Novikov algorithm*, Acoust. J. 54(3), 2008, 469-482 (in Russian), English transl.: Acoust. Phys. 54(3), 2008, 407-419.

[3] R. Beals and R. Coifman, *Multidimensional inverse scattering and nonlinear partial differential equations*, Proc. Symp. Pure Math., 43, 1985, 45-70.

[4] Yu.M. Berezanskii, *The uniqueness theorem in the inverse problem of spectral analysis for the Schrodinger equation.* (Russian) Trudy Moskov. Mat. Obsc. 7 (1958) 1-62.

[5] L. Beilina, M.V. Klibanov, *Approximate global convergence and adaptivity for coefficient inverse problems*, Springer (New York), 2012. 407 pp.

[6] D. Colton R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 2nd. ed. Springer, Berlin, 1998.

[7] L.D. Faddeev, *Growing solutions of the Schrödinger equation,* Dokl. Akad. Nauk SSSR, 165, N.3, 1965, 514-517 (in Russian); English Transl.: Sov. Phys. Dokl. 10, 1966, 1033-1035.

[8] L.D. Faddeev, *The inverse problem in the quantum theory of scattering. II,* Current problems in mathematics, Vol. 3, 1974, pp. 93-180, 259. Akad. Nauk SSSR Vsesojuz. Inst. Naucn. i Tehn. Informacii, Moscow(in Russian); English Transl.: J.Sov. Math. 5, 1976, 334-396.

[9] P. Hähner, T. Hohage, *New Stability Estimates for the Inverse Acoustic Inhomogeneous Medium Problem and Applications*, SIAM J. Math. Anal., 33(3), 2001, 670-685.

[10] G.M. Henkin and R.G. Novikov, *The $\bar{\partial}$-equation in the multidimensional inverse scattering problem*, Uspekhi Mat. Nauk 42(3), 1987, 93-152 (in Russian); English Transl.: Russ. Math. Surv. 42(3), 1987, 109-180.

[11] M.I. Isaev, *Exponential instability in the inverse scattering problem on the energy interval*, Func. Analiz i ego Pril.(to appear), arXiv:1012.5526.

[12] M.I. Isaev, R.G. Novikov, *Stability estimates for determination of potential from the impedance boundary map*, Algebra i Analiz(to appear), e-print arXiv:1112.3728.

[13] M.I. Isaev, R.G. Novikov *Energy and regularity dependent stability estimates for the Gel'fand inverse problem in multidimensions*, J. of Inverse and III-posed Probl., 2012, Vol. 20, Issue 3, p.313-325.

[14] R.B. Lavine and A.I. Nachman, *On the inverse scattering transform of the n-dimensional Schrödinger* operator Topics in Soliton Theory and Exactly Solvable Nonlinear Equations ed M Ablovitz, B Fuchssteiner and M Kruskal (Singapore: World Scientific), 1987, pp 33-44.

[15] M.M. Lavrent'ev, V.G. Romanov, S.P. Shishatskii, *Ill-posed problems of mathematical physics and analysis*, Translated from the Russian by J. R. Schulenberger. Translation edited by Lev J. Leifman. Translations of Mathematical Monographs, 64. American Mathematical Society, Providence, RI, 1986. vi+290 pp.

[16] A. Nachman, *Reconstructions from boundary measurements*, Ann. Math. 128, 1988, 531-576

[17] R.G. Novikov, *Multidimensional inverse spectral problem for the equation* $-\Delta\psi + (v(x) - Eu(x))\psi = 0$ Funkt. Anal. Prilozhen. 22(4), 1988, 11-22 (in Russian); Engl. Transl. Funct. Anal. Appl. 22, 1988, 263-272.

[18] R.G. Novikov, $\bar{\partial}$-*method with nonzero background potential. Application to inverse scattering for the two-dimensional acoustic equation*, Comm. Partial Differential Equations 21, 1996, no. 3-4, 597-618.

[19] R.G. Novikov, *The $\bar{\partial}$-approach to approximate inverse scattering at fixed energy in three dimensions.* IMRP Int. Math. Res. Pap. 2005, no. 6, 287-349.

[20] R.G. Novikov, *Formulae and equations for finding scattering data from the Dirichlet-to-Neumann map with nonzero background potential*, Inverse Problems 21, 2005, 257-270.

[21] R.G. Novikov, *An effectivization of the global reconstruction in the Gel'fand-Calderon inverse problem in three dimensions*, Contemporary Mathematics, 494, 2009, 161-184.

[22] R.G. Novikov, *New global stability estimates for the Gel'fand-Calderon inverse problem,* Inverse Problems 27, 2011, 015001(21pp).

[23] R.G. Novikov and M. Santacesaria, *Monochromatic Reconstruction Algorithms for Two-dimensional Multi-channel Inverse Problems*, International Mathematics Research Notes, 2012, doi: 10.1093/imrn/rns025.

[24] P. Stefanov, *Stability of the inverse problem in potential scattering at fixed energy* Annales de l'institut Fourier, tome 40, N4 (1990), p.867-884.

[25] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. 125, 1987, 153-169.

[26] R. Weder, *Generalized limiting absorption method and multidimensional inverse scattering theory*, Mathematical Methods in the Applied Sciences, 14, 1991, 509-524.

M.I. Isaev,
Centre de Mathématiques Appliquées,
Ecole Polytechnique, 91128 Palaiseau, France
e-mail: `isaev.m.i@gmail.com`

R.G. Novikov,
Centre de Mathématiques Appliquées,
Ecole Polytechnique, 91128 Palaiseau, France
e-mail: `novikov@cmap.polytechnique.fr`