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Energy and regularity dependent stability estimates for near-field inverse scattering in multidimensions

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Abstract

We prove new global Hölder-logarithmic stability estimates for the near-field inverse scattering problem in dimension $d \geq 3$. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. In addition, a global logarithmic stability estimate for this inverse problem in dimension $d = 2$ is also given.

1 Introduction

We consider the Schrödinger equation

$$L\psi = E\psi, \quad L = -\Delta + v(x), \quad x \in \mathbb{R}^d, \; d \geq 2, \tag{1.1}$$

where

$$\begin{aligned} & v \text{ is real-valued}, \; v \in \mathbb{L}^\infty(\mathbb{R}^d), \\ & v(x) = O(|x|^{-d-\varepsilon}), \; |x| \to \infty, \; \text{ for some } \varepsilon > 0. \end{aligned} \tag{1.2}$$

We consider the resolvent $R(E)$ of the Schrödinger operator L in $\mathbb{L}^2(\mathbb{R}^d)$:

$$R(E) = (L - E)^{-1}, \qquad E \in \mathbb{C} \setminus \sigma(L), \tag{1.3}$$

where $\sigma(L)$ is the spectrum of L in $\mathbb{L}^2(\mathbb{R}^d)$. We assume that $R(x, y, E)$ denotes the Schwartz kernel of $R(E)$ as of an integral operator. We consider also

$$R^+(x, y, E) = R(x, y, E + i0), \quad x, y \in \mathbb{R}^d, \; E \in \mathbb{R}_+. \tag{1.4}$$

We recall that in the framework of equation (1.1) the function $R^+(x, y, E)$ describes scattering of the spherical waves

$$R_0^+(x, y, E) = -\frac{i}{4}\left(\frac{\sqrt{E}}{2\pi|x-y|}\right)^{\frac{d-2}{2}} H^{(1)}_{\frac{d-2}{2}}(\sqrt{E}|x-y|), \tag{1.5}$$

generated by a source at y (where $H^{(1)}_\mu$ is the Hankel function of the first kind of order μ). We recall also that $R^+(x, y, E)$ is the Green function for $L - E$, $E \in \mathbb{R}_+$, with the Sommerfeld radiation condition at infinity.

In addition, the function

$$\begin{aligned} S^+(x, y, E) = R^+(x, y, E) - R_0^+(x, y, E), \\ x, y \in \partial B_r, \; E \in \mathbb{R}_+, \; r \in \mathbb{R}_+, \end{aligned} \tag{1.6}$$

is considered as near-field scattering data for equation (1.1), where B_r is the open ball of radius r centered at 0.

We consider, in particular, the following near-field inverse scattering problem for equation (1.1):

Problem 1.1. Given S^+ on $\partial B_r \times \partial B_r$ for some fixed $r, E \in \mathbb{R}_+$, find v on B_r.

This problem can be considered under the assumption that v is a priori known on $\mathbb{R}^d \setminus B_r$. Actually, in the present paper we consider Problem 1.1 under the assumption that $v \equiv 0$ on $\mathbb{R}^d \setminus B_r$ for some fixed $r \in \mathbb{R}_+$. Below in this paper we always assume that this additional condition is fulfilled.

It is well-known that the near-field scattering data of Problem 1.1 uniquely and efficiently determine the scattering amplitude f for equation (1.1) at fixed energy E, see [4]. Therefore, approaches of [2], [6], [7], [9], [13], [14], [23], [24], [26], [27], [36] can be applied to Problem 1.1 via this reduction.

In addition, it is also known that the near-field data of Problem 1.1 uniquely determine the Dirichlet-to-Neumann map in the case when E is not a Dirichlet eigenvalue for operator L in B_r, see [22], [23]. Therefore, approaches of [1], [6], [16], [18], [21], [23], [28]- [33], [37] can be also applied to Problem 1.1 via this reduction.

However, in some case it is much more optimal to deal with Problem 1.1 directly, see, for example, logarithmic stability results of [12] for Problem 1.1 in dimension $d = 3$. A principal improvement of estimates of [12] was given recently in [17]: stability of [17] efficiently increases with increasing regularity of v.

Problem 1.1 can be also considered as an example of ill-posed problem: see [20], [5] for an introduction to this theory.

In the present paper we continue studies of [12], [17]. We give new global Hölder-logarithmic stability estimates for Problem 1.1 in dimension $d \geq 3$, see Theorem 2.1. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. Results of such a type for the Gel'fand inverse problem were obtained recently in [16] for $d \geq 3$ and in [35] for $d = 2$.

In addition, we give also global logarithmic stability estimates for Problem 1.1 in dimension $d = 2$, see Theorem 2.2.

2 Stability estimates

We recall that if v satisfies (1.2) and $\operatorname{supp} v \subset B_{r_1}$ for some $r_1 > 0$, then

$$S^+(E) \text{ is bounded in } \mathbb{L}^2(\partial B_r \times \partial B_r) \text{ for any } r > r_1, \tag{2.1}$$

where $S^+(E)$ is the near-field scattering data of v for equation (1.1) with $E > 0$, for more details see, for example, Section 2 of [12].

2.1 Estimates for $d \geq 3$

In this subsection we assume for simplicity that

$$\begin{aligned} v \in \mathbb{W}^{m,1}(\mathbb{R}^d) &\text{ for some } m > d, \\ v &\text{ is real-valued}, \\ \operatorname{supp} v \subset B_{r_1} &\text{ for some } r_1 > 0, \end{aligned} \tag{2.2}$$

where

$$\mathbb{W}^{m,1}(\mathbb{R}^d) = \{v: \ \partial^J v \in \mathbb{L}^1(\mathbb{R}^d), \ |J| \le m\}, \ m \in \mathbb{N} \cup 0, \tag{2.3}$$

where

$$J \in (\mathbb{N} \cup 0)^d, \ |J| = \sum_{i=1}^{d} J_i, \ \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}. \tag{2.4}$$

Let

$$||v||_{m,1} = \max_{|J| \le m} ||\partial^J v||_{\mathbb{L}^1(\mathbb{R}^d)}. \tag{2.5}$$

Note that (2.2) $\Rightarrow$ (1.2).

Theorem 2.1. *Let $E > 0$ and $r > r_1$ be given constants. Let dimension $d \ge 3$ and potentials v_1, v_2 satisfy (2.2). Let $||v_j||_{m,1} \le N$, $j = 1, 2$, for some $N > 0$. Let $S_1^+(E)$ and $S_2^+(E)$ denote the near-field scattering data for v_1 and v_2, respectively. Then for $\tau \in (0, 1)$ and any $s \in [0, s^*]$ the following estimate holds:*

$$||v_2 - v_1||_{L^\infty(B_r)} \le C_1 (1 + E)^{\frac{5}{2}} \delta^\tau + C_2 (1 + E)^{\frac{s - s^*}{2}} \left(\ln\left(3 + \delta^{-1}\right)\right)^{-s}, \tag{2.6}$$

where $s^* = \frac{m-d}{d}$, $\delta = ||S_1^+(E) - S_2^+(E)||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}$, and constants $C_1, C_2 > 0$ depend only on N, m, d, r, τ.

Proof of Theorem 2.1 is given in Section 5. This proof is based on results presented in Sections 3, 4.

2.2 Estimates for $d = 2$

In this subsection we assume for simplicity that

$$\begin{aligned} &v \text{ is real-valued}, \quad v \in C^2(\overline{B}_{r_1}), \\ &\operatorname{supp} v \subset B_{r_1} \quad \text{for some } r_1 > 0. \end{aligned} \tag{2.7}$$

Note also that (2.7) $\Rightarrow$ (1.2).

Theorem 2.2. *Let $E > 0$ and $r > r_1$ be given constants. Let dimension $d = 2$ and potentials v_1, v_2 satisfy (2.7). Let $||v_j||_{C^2(B_r)} \le N$, $j = 1, 2$, for some $N > 0$. Let $S_1^+(E)$ and $S_2^+(E)$ denote the near-field scattering data for v_1 and v_2, respectively. Then*

$$||v_1 - v_2||_{L^\infty(B_r)} \le C_3 \left(\ln\left(3 + \delta^{-1}\right)\right)^{-3/4} \left(\ln\left(3 \ln\left(3 + \delta^{-1}\right)\right)\right)^2, \tag{2.8}$$

where $\delta = ||S_1^+(E) - S_2^+(E)||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}$ and constant $C_3 > 0$ depends only on N, m, r.

Proof of Theorem 2.2 is given in Section 7. This proof is based on results presented in Sections 3, 6.

2.3 Concluding remarks

Remark 2.1. The logarithmic stability estimates for Problem 1.1 of [12] and [17] follow from estimate (2.6) for $d = 3$ and $s = s^*$. Apparently, using the methods of [29], [30] it is possible to improve estimate (2.6) for $s^* = m - d$.

Remark 2.2. In the same way as in [12] and [17] for dimesnsion $d = 3$, using estimates (2.6) and (2.8), one can obtain logarithmic stability estimates for the reconstruction of a potential v from the inverse scattering amplitude f for any $d \geq 2$.

Remark 2.3. Actually, in the proof of Theorem 2.1 we obtain the following estimate (see formula (5.20)):

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} \leq C_4(1+E)^2\sqrt{E+\rho^2}\, e^{2\rho(r+1)}\delta + C_5(E+\rho^2)^{-\frac{m-d}{2d}}, \quad (2.9)$$

where constants $C_4, C_5 > 0$ depend only on N, m, d, r and the parameter $\rho > 0$ is such that $E + \rho^2$ is sufficiently large: $E + \rho^2 \geq C_6(N, r, m)$. Estimate of Theorem 2.1 follows from estimate (2.9).

3 Alessandrini-type identity for near-field scattering

In this section we always assume that assumptions of Theorems 2.1 and 2.2 are fulfilled (in the cases of dimension $d \geq 3$ and $d = 2$, respectively).

Consider the operators $\hat{\mathrm{R}}_j$, $j = 1, 2$, defined as follows

$$(\hat{\mathrm{R}}_j\phi)(x) = \int\limits_{\partial B_r} R_j^+(x, y, E)\phi(y)dy, \quad x \in \partial B_r, \quad j = 1, 2. \quad (3.1)$$

Note that

$$\|\hat{\mathrm{R}}_1 - \hat{\mathrm{R}}_2\|_{\mathbb{L}^2(\partial B_r)} \leq \|S_1^+(E) - S_2^+(E)\|_{\mathbb{L}^2(\partial B_r)\times\mathbb{L}^2(\partial B_r)}. \quad (3.2)$$

We recall that (see [12]) for any functions $\phi_1, \phi_2 \in C(\mathbb{R}^d)$, sufficiently regular in $\mathbb{R}^d \setminus \partial B_r$ and satisfying

$$\begin{aligned} &-\Delta\phi + v(x)\phi = E\phi, \quad \text{in } \mathbb{R}^d \setminus \partial B_r, \\ &\lim_{|x|\to+\infty} |x|^{\frac{d-1}{2}}\left(\frac{\partial}{\partial|x|}\phi - i\sqrt{E}\phi\right) = 0, \end{aligned} \quad (3.3)$$

with $v = v_1$ and $v = v_2$, respectively, the following identity holds:

$$\begin{aligned} &\int\limits_{B_r}(v_2 - v_1)\phi_1\phi_2 dx = \\ &= \int\limits_{\partial B_r}\left(\frac{\partial\phi_1}{\partial\nu_+} - \frac{\partial\phi_1}{\partial\nu_-}\right)\left[\left(\hat{\mathrm{R}}_1 - \hat{\mathrm{R}}_2\right)\left(\frac{\partial\phi_2}{\partial\nu_+} - \frac{\partial\phi_2}{\partial\nu_-}\right)\right]dx, \end{aligned} \quad (3.4)$$

where where ν_+ and ν_- are the outward and inward normals to ∂B_r, respectively.

Remark 3.1. The identity (3.4) is similar to the Alessandrini identity (see Lemma 1 of [1]), where the Dirichlet-to-Neumann maps are considered instead of operators $\hat{\mathrm{R}}_j$.

To apply identity (3.4) to our considerations, we use also the following lemma:

Lemma 3.1. *Let $E, r > 0$ and $d \geq 2$. Then, there is a positive constant C_7 (depending only on r and d) such that for any $\phi \in C(\mathbb{R}^d \setminus B_r)$ satisfying*

$$\begin{aligned} -\Delta\phi = E\phi, \quad in\ \mathbb{R}^d \setminus \overline{B}_r, \\ \lim_{|x| \to +\infty} |x|^{\frac{d-1}{2}} \left(\frac{\partial}{\partial |x|}\phi - i\sqrt{E}\phi \right) = 0, \\ \phi|_{\partial B_r} \in \mathbb{H}^1(\partial B_r), \end{aligned} \tag{3.5}$$

the following inequality holds:

$$\left\| \left. \frac{\partial \phi}{\partial \nu_+} \right|_{\partial B_r} \right\|_{\mathbb{L}^2(\partial B_r)} \leq C_7 (1 + E) \left\| \phi|_{\partial B_r} \right\|_{\mathbb{H}^1(\partial B_r)}, \tag{3.6}$$

where $\mathbb{H}^1(\partial B_r)$ denotes the standart Sobolev space on ∂B_r.

The proof of Lemma 3.1 is given in Section 8.

4 Faddeev functions

In dimension $d \geq 3$, we consider the Faddeev functions h, ψ, G (see [10], [11], [13], [23]):

$$h(k, l) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{-ilx} v(x) \psi(x, k) dx, \tag{4.1}$$

where $k, l \in \mathbb{C}^d,\ k^2 = l^2,\ \operatorname{Im} k = \operatorname{Im} l \neq 0$,

$$\psi(x, k) = e^{ikx} + \int\limits_{\mathbb{R}^d} G(x - y, k) v(y) \psi(y, k) dy, \tag{4.2}$$

$$G(x, k) = e^{ikx} g(x, k), \quad g(x, k) = -(2\pi)^{-d} \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 + 2k\xi}, \tag{4.3}$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{C}^d$, $\operatorname{Im} k \neq 0$, $d \geq 3$,

One can consider (4.1), (4.2) assuming that

$$\begin{aligned} v \text{ is a sufficiently regular function on } \mathbb{R}^d \\ \text{with suffucient decay at infinity.} \end{aligned} \tag{4.4}$$

For example, in connection with Theorem 2.1, we consider (4.1), (4.2) assuming that

$$v \in \mathbb{L}^\infty(B_r), \quad v \equiv 0 \text{ on } \mathbb{R} \setminus B_r. \tag{4.5}$$

We recall that (see [10], [11], [13], [23]):

$$(\Delta + k^2)G(x,k) = \delta(x), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{C}^d \setminus \mathbb{R}^d; \tag{4.6}$$

formula (4.2) at fixed k is considered as an equation for

$$\psi = e^{ikx}\mu(x,k), \tag{4.7}$$

where μ is sought in $\mathbb{L}^\infty(\mathbb{R}^d)$; as a corollary of (4.2), (4.3), (4.6), ψ satisfies (1.1) for $E = k^2$; h of (4.1) is a generalized "‘scattering"’ amplitude.

In addition, h, ψ, G in their zero energy restriction, that is for $E = 0$, were considered for the first time in [3]. The Faddeev functions h, ψ, G were, actually, rediscovered in [3].

Let

$$\begin{aligned} \Sigma_E &= \left\{k \in \mathbb{C}^d : k^2 = k_1^2 + \ldots + k_d^2 = E\right\}, \\ \Theta_E &= \{k \in \Sigma_E,\ l \in \Sigma_E : \operatorname{Im} k = \operatorname{Im} l\}, \\ |k| &= (|\operatorname{Re} k|^2 + |\operatorname{Im} k|^2)^{1/2}. \end{aligned} \tag{4.8}$$

Let

$$v \text{ satisfy (2.2)}, \quad \|v\|_{m,1} \le N, \tag{4.9}$$

$$\hat{v}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d, \tag{4.10}$$

then we have that:

$$\mu(x,k) \to 1 \quad \text{as} \quad |k| \to \infty \tag{4.11}$$

and, for any $\sigma > 1$,

$$|\mu(x,k)| + |\nabla\mu(x,k)| \le \sigma \quad \text{for} \quad |k| \ge \lambda_1(N,m,d,r,\sigma), \tag{4.12}$$

where $x \in \mathbb{R}^d$, $k \in \Sigma_E$;

$$\hat{v}(p) = \lim_{\substack{(k,l) \in \Theta_E,\ k-l=p \\ |\operatorname{Im} k| = |\operatorname{Im} l| \to \infty}} h(k,l) \quad \text{for any } p \in \mathbb{R}^d, \tag{4.13}$$

$$\begin{aligned} |\hat{v}(p) - h(k,l)| \le \frac{c_1(m,d,r)N^2}{(E+\rho^2)^{1/2}} \quad &\text{for } (k,l) \in \Theta_E, \quad p = k - l, \\ |\operatorname{Im} k| = |\operatorname{Im} l| = \rho, \quad &E + \rho^2 \ge \lambda_2(N,m,d,r), \\ &p^2 \le 4(E+\rho^2). \end{aligned} \tag{4.14}$$

Results of the type (4.11), (4.12) go back to [3]. For more information concerning (4.12) see estimate (4.11) of [15]. Results of the type (4.13), (4.14) (with less precise right-hand side in (4.14)) go back to [13]. Estimate (4.14) follows, for example, from formulas (4.2), (4.1) and the estimate

$$\begin{aligned} &\|\Lambda^{-s} g(k) \Lambda^{-s}\|_{\mathbb{L}^2(\mathbb{R}^d) \to \mathbb{L}^2(\mathbb{R}^d)} = O(|k|^{-1}) \\ &\qquad \text{as } \ |k| \to \infty, \quad k \in \mathbb{C}^d \setminus \mathbb{R}^d, \end{aligned} \tag{4.15}$$

for $s > 1/2$, where $g(k)$ denotes the integral operator with the Schwartz kernel $g(x-y, k)$ and Λ denotes the multiplication operator by the function $(1+|x|^2)^{1/2}$. Estimate (4.15) was formulated, first, in [19] for $d \geq 3$. Concerning proof of (4.15), see [39].

In addition, we have that:

$$\begin{aligned} h_2(k,l) - h_1(k,l) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \psi_1(x,-l)(v_2(x) - v_1(x))\psi_2(x,k)dx \\ \text{for } (k,l) \in \Theta_E, \ |\mathrm{Im}\, k| = |\mathrm{Im}\, l| \neq 0, \\ \text{and } v_1, \, v_2 \text{ satisfying (4.4)}, \end{aligned} \tag{4.16}$$

and, under assumtions of Theorem 2.1,

$$\begin{aligned} |\hat{v}_1(p) - \hat{v}_2(p) - h_1(k,l) + h_2(k,l)| \leq \frac{c_2(m,d,r)N\|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)}}{(E+\rho^2)^{1/2}} \\ \text{for } (k,l) \in \Theta_E, \quad p = k - l, \quad |\mathrm{Im}\, k| = |\mathrm{Im}\, l| = \rho, \\ E + \rho^2 \geq \lambda_3(N,m,d,r), \quad p^2 \leq 4(E+\rho^2), \end{aligned} \tag{4.17}$$

where h_j, ψ_j denote h and ψ of (4.1) and (4.2) for $v = v_j$, $j = 1, 2$.

Formula (4.16) was given in [25]. Estimate (4.17) was given e.g. in [16].

5 Proof of Theorem 2.1

Let

$$\begin{aligned} &\mathbb{L}^\infty_\mu(\mathbb{R}^d) = \{u \in \mathbb{L}^\infty(\mathbb{R}^d) : \|u\|_\mu < +\infty\}, \\ &\|u\|_\mu = \operatorname*{ess\ sup}_{p \in \mathbb{R}^d} (1+|p|)^\mu |u(p)|, \quad \mu > 0. \end{aligned} \tag{5.1}$$

Note that

$$\begin{aligned} &w \in \mathbb{W}^{m,1}(\mathbb{R}^d) \Longrightarrow \hat{w} \in \mathbb{L}^\infty_\mu(\mathbb{R}^d) \cap C(\mathbb{R}^d), \\ &\|\hat{w}\|_\mu \leq c_3(m,d)\|w\|_{m,1} \quad \text{ for } \quad \mu = m, \end{aligned} \tag{5.2}$$

where $\mathbb{W}^{m,1}$, $\mathbb{L}^\infty_\mu$ are the spaces of (2.3), (5.1),

$$\hat{w}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} w(x)dx, \quad p \in \mathbb{R}^d. \tag{5.3}$$

Using the inverse Fourier transform formula

$$w(x) = \int\limits_{\mathbb{R}^d} e^{-ipx} \hat{w}(p)dp, \quad x \in \mathbb{R}^d, \tag{5.4}$$

we have that

$$\begin{aligned}\|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} \le \sup_{x\in \overline{B}_r} |\int\limits_{\mathbb{R}^d} e^{-ipx}\,(\hat{v}_2(p) - \hat{v}_1(p))\,dp| \le \\ \le I_1(\kappa) + I_2(\kappa) \quad \text{ for any } \ \kappa > 0,\end{aligned} \tag{5.5}$$

where

$$\begin{aligned} I_1(\kappa) &= \int\limits_{|p|\le\kappa} |\hat{v}_2(p) - \hat{v}_1(p)|dp, \\ I_2(\kappa) &= \int\limits_{|p|\ge\kappa} |\hat{v}_2(p) - \hat{v}_1(p)|dp. \end{aligned} \tag{5.6}$$

Using (5.2), we obtain that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \le 2c_3(m,d)N(1+|p|)^{-m}, \quad p \in \mathbb{R}^d. \tag{5.7}$$

Let

$$c_4 = \int\limits_{p\in\mathbb{R}^d, |p|=1} dp. \tag{5.8}$$

Combining (5.6), (5.7), we find that, for any $\kappa > 0$,

$$I_2(\kappa) \le 2c_3(m,d)Nc_4 \int\limits_{\kappa}^{+\infty} \frac{dt}{t^{m-d+1}} \le \frac{2c_3(m,d)Nc_4}{m-d}\frac{1}{\kappa^{m-d}}. \tag{5.9}$$

Due to (4.17), we have that

$$\begin{aligned} |\hat{v}_2(p) - \hat{v}_1(p)| \le |h_2(k,l) - h_1(k,l)| + \frac{c_2(m,d,r)N\|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)}}{(E+\rho^2)^{1/2}}, \\ \text{for } (k,l) \in \Theta_E, \ \ p = k - l, \ \ |\mathrm{Im}\,k| = |\mathrm{Im}\,l| = \rho, \\ E + \rho^2 \ge \lambda_3(N,m,d,r), \ \ p^2 \le 4(E+\rho^2). \end{aligned} \tag{5.10}$$

Let

$$\delta = ||S_1^+(E) - S_2^+(E)||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}. \tag{5.11}$$

Combining (3.2), (3.4) and (4.16), we get that

$$\begin{aligned} |h_2(k,l) - h_1(k,l)| \le \delta \left\| \frac{\partial \phi_1}{\partial \nu_+} - \frac{\partial \phi_1}{\partial \nu_-} \right\|_{\mathbb{L}^2(B_r)} \left\| \frac{\partial \phi_2}{\partial \nu_+} - \frac{\partial \phi_2}{\partial \nu_-} \right\|_{\mathbb{L}^2(B_r)}, \\ (k,l) \in \Theta_E, \ |\mathrm{Im}\,k| = |\mathrm{Im}\,l| \ne 0, \end{aligned} \tag{5.12}$$

where ϕ_j, $j = 1, 2$, denotes the solution of (3.3) with $v = v_j$, satisfying

$$\phi_j(x) = \psi_j(x,k) \quad \text{ for } \quad x \in \overline{B}_r. \tag{5.13}$$

Using (3.6), (4.12) and the fact that $C^1(\partial B_r) \subset \mathbb{H}^1(\partial B_r)$, we find that

$$\left\| \frac{\partial \phi_j}{\partial \nu_+} - \frac{\partial \phi_j}{\partial \nu_-} \right\|_{\mathbb{L}^2(B_r)} \leq \sigma c_5(r,d)(1+E)\exp\Big(|\operatorname{Im} k|(r+1)\Big), \quad k \in \Sigma_E,\ |k| \geq \lambda_1(N,m,d,r,\sigma),\ j=1,2. \tag{5.14}$$

Here and bellow in this section the constant σ is the same that in (4.12).

Combining (5.12) and (5.14), we obtain that

$$\begin{aligned} |h_2(k,l) - h_1(k,l)| &\leq c_5^2 \sigma^2 (1+E)^2 e^{2\rho(r+1)} \delta, \\ \text{for } (k,l) &\in \Theta_E, \quad \rho = |\operatorname{Im} k| = |\operatorname{Im} l|, \\ E + \rho^2 &\geq \lambda_1^2(N,m,d,r,\sigma). \end{aligned} \tag{5.15}$$

Using (5.10), (5.15), we get that

$$\begin{aligned} |\hat{v}_2(p) - \hat{v}_1(p)| \leq c_5^2 \sigma^2 (1+E)^2 e^{2\rho(r+1)} \delta + \\ + \frac{c_2(m,d,r) N \|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)}}{(E+\rho^2)^{1/2}}, \\ p \in \mathbb{R}^d,\ p^2 \leq 4(E+\rho^2),\ E+\rho^2 \geq \max\{\lambda_1^2, \lambda_3\}. \end{aligned} \tag{5.16}$$

Let

$$\varepsilon = \left(\frac{1}{2 c_2(m,d,r) N c_6} \right)^{1/d}, \quad c_6 = \int\limits_{p \in \mathbb{R}^d, |p| \leq 1} dp, \tag{5.17}$$

and $\lambda_4(N,m,d,r,\sigma) > 0$ be such that

$$E + \rho^2 \geq \lambda_4(N,m,d,r,\sigma) \Longrightarrow \begin{cases} E+\rho^2 \geq \lambda_1^2(N,m,d,r,\sigma), \\ E+\rho^2 \geq \lambda_3(N,m,d,r), \\ \Big(\varepsilon (E+\rho^2)^{\frac{1}{2d}}\Big)^2 \leq 4(E+\rho^2). \end{cases} \tag{5.18}$$

Using (5.6), (5.16), we get that

$$\begin{aligned} I_1(\kappa) \leq c_6 \kappa^d \Big(c_5^2 \sigma^2 (1+E)^2 e^{2\rho(r+1)} \delta + \frac{c_2(m,d,r) N \|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)}}{(E+\rho^2)^{1/2}} \Big), \\ \kappa > 0,\ \kappa^2 \leq 4(E+\rho^2), \\ E+\rho^2 \geq \lambda_4(N,m,d,r,\sigma). \end{aligned} \tag{5.19}$$

Combining (5.5), (5.9), (5.19) for $\kappa = \varepsilon (E+\rho^2)^{\frac{1}{2d}}$ and (5.18), we get that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} \leq c_7(N,m,d,r,\sigma)(1+E)^2 \sqrt{E+\rho^2}\, e^{2\rho(r+1)} \delta + \\ + c_8(N,m,d)(E+\rho^2)^{-\frac{m-d}{2d}} + \frac{1}{2} \|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)}, \\ E+\rho^2 \geq \lambda_4(N,m,d,r,\sigma). \end{aligned} \tag{5.20}$$

Let $\tau' \in (0,1)$,

$$\beta = \frac{1-\tau'}{2(r+1)}, \quad \rho = \beta \ln\left(3+\delta^{-1}\right), \tag{5.21}$$

and $\delta_1 = \delta_1(N, m, d, \sigma, r, \tau') > 0$ be such that

$$\delta \in (0,\delta_1) \Longrightarrow \begin{cases} E + \left(\beta \ln\left(3+\delta^{-1}\right)\right)^2 \geq \lambda_4(N,m,d,r,\sigma), \\ E + \left(\beta \ln\left(3+\delta^{-1}\right)\right)^2 \leq (1+E)\left(\beta \ln\left(3+\delta^{-1}\right)\right)^2, \end{cases} \tag{5.22}$$

Then for the case when $\delta \in (0,\delta_1)$, due to (5.20), we have that

$$\begin{aligned} &\frac{1}{2}\|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} \leq \\ &\quad \leq c_7(1+E)^2\left(E + \left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{\frac{1}{2}}\left(3+\delta^{-1}\right)^{2\beta(r+1)}\delta + \\ &\qquad + c_8\left(E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{-\frac{m-d}{2d}} = \\ &\quad = c_7(1+E)^2\left(E + \left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{\frac{1}{2}}(1+3\delta)^{1-\tau'}\delta^{\tau'} + \\ &\qquad + c_8\left(E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{-\frac{m-d}{2d}}. \end{aligned} \tag{5.23}$$

Combining (5.22) and (5.23), we obtain that for $s \in [0, s^*]$, $\tau \in (0, \tau')$ and $\delta \in (0, \delta_1)$ the following estimate holds:

$$||v_2 - v_1||_{L^\infty(B_r)} \leq c_9(1+E)^{\frac{5}{2}}\delta^\tau + c_{10}(1+E)^{\frac{s-s^*}{2}}\left(\ln\left(3+\delta^{-1}\right)\right)^{-s}, \tag{5.24}$$

where $s^* = \frac{m-d}{d}$ and $c_9, c_{10} > 0$ depend only on N, m, d, r, σ, τ' and τ.

Estimate (5.24) in the general case (with modified c_9 and c_{10}) follows from (5.24) for $\delta \leq \delta_1(N, m, d, \sigma, r, \tau')$ and and the property that

$$\|v_j\|_{\mathbb{L}^\infty(B_r)} \leq c_{11}(m,d)N. \tag{5.25}$$

This completes the proof of (2.6)

6 Buckhgeim-type analogs of the Faddeev functions

Let us identify $\mathbb{R}^2$ with $\mathbb{C}$ and use coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, where $(x_1, x_2) \in \mathbb{R}^2$. Following [31]- [34], we consider the functions $G_{z_0}, \psi_{z_0}, \widetilde{\psi}_{z_0}$, δh_{z_0} going back to Buckhgeim's paper [6] and being analogs of the Faddeev functions:

$$\begin{aligned} \psi_{z_0}(z,\lambda) &= e^{\lambda(z-z_0)^2} + \int\limits_{B_r} G_{z_0}(z,\zeta,\lambda)v(\zeta)\psi_{z_0}(\zeta,\lambda)\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \\ \widetilde{\psi}_{z_0}(z,\lambda) &= e^{\bar{\lambda}(\bar{z}-\bar{z}_0)^2} + \int\limits_{B_r} \overline{G_{z_0}(z,\zeta,\lambda)}v(\zeta)\widetilde{\psi}_{z_0}(\zeta,\lambda)\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \end{aligned} \tag{6.1}$$

$$G_{z_0}(z,\zeta,\lambda) = \frac{1}{4\pi^2}\int\limits_{B_r} \frac{e^{-\lambda(\eta-z_0)^2+\bar\lambda(\bar\eta-\bar{z_0})^2} d\mathrm{Re}\eta\, d\mathrm{Im}\eta}{(z-\eta)(\bar\eta-\bar\zeta)}\, e^{\lambda(z-z_0)^2-\bar\lambda(\bar\zeta-\bar{z_0})^2},$$
$$z = x_1 + ix_2,\ z_0 \in B_r,\ \lambda \in \mathbb{C}, \tag{6.2}$$

where v satisfies (2.7);

$$\delta h_{z_0}(\lambda) = \int\limits_{B_r} \widetilde\psi_{z_0,1}(z,-\lambda)\,(v_2(z)-v_1(z))\,\psi_{z_0,2}(z,\lambda)\,d\mathrm{Re}z\, d\mathrm{Im}z, \quad \lambda \in \mathbb{C}, \tag{6.3}$$

where v_1, v_2 satisfy (2.7) and $\widetilde\psi_{z_0,1}$, $\psi_{z_0,2}$ denote $\widetilde\psi_{z_0}$, ψ_{z_0} of (6.1) for $v = v_1$ and $v = v_2$, respectively.

We recall that (see [31], [32]):

- The function G_{z_0} satisfies the equations
$$\begin{aligned} &4\frac{\partial^2}{\partial z\partial\bar z} G_{z_0}(z,\zeta,\lambda) = \delta(z-\zeta),\\ &4\frac{\partial^2}{\partial\zeta\partial\bar\zeta} G_{z_0}(z,\zeta,\lambda) = \delta(z-\zeta),\end{aligned} \tag{6.4}$$
where $z, z_0, \zeta \in B_r$, $\lambda \in \mathbb{C}$ and δ is the Dirac delta function;
- Formulas (6.1) at fixed z_0 and λ are considered as equations for ψ_{z_0}, $\widetilde\psi_{z_0}$ in $L^\infty(B_r)$;
- As a corollary of (6.1), (6.2), (6.4), the functions ψ_{z_0}, $\widetilde\psi_{z_0}$ satisfy (1.1) in B_r for $E = 0$ and $d = 2$;
- The function δh_{z_0} is similar to the right side of (4.16).

Let potentials $v, v_1, v_2 \in C^2(\overline{B}_r)$ and

$$\begin{aligned} &\|v\|_{C^2(\overline{B}_r)} \le N, \quad \|v_j\|_{C^2(\overline{B}_r)} \le N, \quad j = 1,2,\\ &(v_1-v_2)|_{\partial B_r} = 0, \quad \frac{\partial}{\partial\nu}(v_1-v_2)|_{\partial B_r} = 0,\end{aligned} \tag{6.5}$$

then we have that:

$$\psi_{z_0}(z,\lambda) = e^{\lambda(z-z_0)^2}\mu_{z_0}(z,\lambda), \quad \widetilde\psi_{z_0}(z,\lambda) = e^{\bar\lambda(\bar z-\bar z_0)^2}\widetilde\mu_{z_0}(z,\lambda), \tag{6.6}$$

$$\mu_{z_0}(z,\lambda) \to 1, \quad \widetilde\mu_{z_0}(z,\lambda) \to 1 \quad \text{as } |\lambda| \to \infty \tag{6.7}$$

and, for any $\sigma > 1$,

$$|\mu_{z_0}(z,\lambda)| + |\nabla\mu_{z_0}(z,\lambda)| \le \sigma, \tag{6.8a}$$
$$|\widetilde\mu_{z_0}(z,\lambda)| + |\nabla\widetilde\mu_{z_0}(z,\lambda)| \le \sigma, \tag{6.8b}$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$, $z = x_1 + ix_2$, $z_0 \in B_r$, $\lambda \in \mathbb{C}$, $|\lambda| \ge \rho_1(N,r,\sigma)$;

$$v_2(z_0) - v_1(z_0) = \lim_{\lambda\to\infty} \frac{2}{\pi}|\lambda|\delta h_{z_0}(\lambda) \quad \text{for any } z_0 \in B_r, \tag{6.9}$$

$$\left| v_2(z_0) - v_1(z_0) - \frac{2}{\pi}|\lambda|\delta h_{z_0}(\lambda) \right| \le \frac{c_{12}(N,r)\,(\ln(3|\lambda|))^2}{|\lambda|^{3/4}} \tag{6.10}$$
$$\text{for } z_0 \in B_r,\ |\lambda| \ge \rho_2(N,r).$$

Formulas (6.6) can be considered as definitions of μ_{z_0}, $\widetilde{\mu}_{z_0}$. Formulas (6.7), (6.9) were given in [31], [32] and go back to [6]. Estimates (6.8) were proved in [15]. Estimate (6.10) was obtained in [31], [34].

7 Proof of Theorem 2.2

We suppose that $\widetilde{\psi}_{z_0,1}(\cdot,-\lambda)$, $\psi_{z_0,2}(\cdot,\lambda)$, $\delta h_{z_0}(\lambda)$ are defined as in Section 6 but with $v_j - E$ in place of v_j, $j = 1,2$. Note that functions $\widetilde{\psi}_{z_0,1}(\cdot,-\lambda)$, $\psi_{z_0,2}(\cdot,\lambda)$ satisfy (1.1) in B_r with $v = v_j$, $j = 1,2$, respectively. We also use the notation $N_E = N + E$. Then, using (6.10), we have that

$$\left| v_2(z_0) - v_1(z_0) - \frac{2}{\pi}|\lambda|\delta h_{z_0}(\lambda) \right| \le \frac{c_{12}(N_E,r)\,(\ln(3|\lambda|))^2}{|\lambda|^{3/4}} \tag{7.1}$$
$$\text{for } z_0 \in B_r,\ |\lambda| \ge \rho_2(N_E,r).$$

Let

$$\delta = ||S_1^+(E) - S_2^+(E)||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}. \tag{7.2}$$

Combining (3.2), (3.4) and (6.3), we get that

$$|\delta h_{z_0}(\lambda)| \le \delta \left\| \frac{\partial \phi_1}{\partial \nu_+} - \frac{\partial \phi_1}{\partial \nu_-} \right\|_{\mathbb{L}^2(B_r)} \left\| \frac{\partial \phi_2}{\partial \nu_+} - \frac{\partial \phi_2}{\partial \nu_-} \right\|_{\mathbb{L}^2(B_r)}, \tag{7.3}$$
$$(k,l) \in \Theta_E,\ |\mathrm{Im}\, k| = |\mathrm{Im}\, l| \neq 0,$$

where ϕ_j, $j = 1,2$, denotes the solution of (3.3) with $v = v_j$, satisfying

$$\phi_1(x) = \widetilde{\psi}_{z_0,1}(x,-\lambda), \quad \phi_2(x) = \psi_{z_0,2}(x,\lambda), \quad \text{for } x \in \overline{B}_r. \tag{7.4}$$

Using (3.6), (6.8) and the fact that $C^1(\partial B_r) \subset \mathbb{H}^1(\partial B_r)$, we find that:

$$\left\| \frac{\partial \phi_j}{\partial \nu_+} - \frac{\partial \phi_j}{\partial \nu_-} \right\|_{\mathbb{L}^2(B_r)} \le \sigma c_{13}(r)(1+E)\exp\Big(|\lambda|(4r^2+4r)\Big), \tag{7.5}$$
$$\lambda \in \mathbb{C},\quad |\lambda| \ge \rho_1(N_E,r,\sigma),\quad j = 1,2.$$

Here and bellow in this section the constant σ is the same that in (6.8).

Combining (7.3), (7.5), we obtain that

$$|\delta h_{z_0}(\lambda)| \le c_{14}(E,r,\sigma)\exp\Big(|\lambda|(8r^2+8r)\Big)\delta, \tag{7.6}$$
$$\lambda \in \mathbb{C},\quad |\lambda| \ge \rho_1(N_E,r,\sigma).$$

Using (7.1) and (7.6), we get that

$$
\begin{aligned}
|v_2(z_0) - v_1(z_0)| \le c_{14}(E,r,\sigma)\exp\Big(|\lambda|(8r^2+8r)\Big)\delta + \\
+ \frac{c_{12}(N_E,r)\,(\ln(3|\lambda|))^2}{|\lambda|^{3/4}}, \\
z_0 \in B_r,\quad \lambda\in\mathbb{C},\quad |\lambda| \ge \rho_3(N_E,r,\sigma) = \max\{\rho_1,\rho_2\}.
\end{aligned}
\tag{7.7}
$$

We fix some $\tau \in (0,1)$ and let

$$
\beta = \frac{1-\tau}{8r^2+8r},\quad \lambda = \beta\ln\left(3+\delta^{-1}\right), \tag{7.8}
$$

where δ is so small that $|\lambda| \ge \rho_3(N_E,r,\sigma)$. Then due to (7.7), we have that

$$
\begin{aligned}
\|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} &\le c_{14}(E,r,\sigma)\left(3+\delta^{-1}\right)^{\beta(8r^2+8r)}\delta + \\
&+ c_{12}(N_E,r)\frac{\left(\ln\left(3\beta\ln\left(3+\delta^{-1}\right)\right)\right)^2}{\left(\beta\ln\left(3+\delta^{-1}\right)\right)^{\frac{3}{4}}} = \\
&= c_{14}(E,r,\sigma)\left(1+3\delta\right)^{1-\tau}\delta^{\tau} + \\
&+ c_{12}(N_E,r)\beta^{-\frac{3}{4}}\frac{\left(\ln\left(3\beta\ln\left(3+\delta^{-1}\right)\right)\right)^2}{\left(\ln\left(3+\delta^{-1}\right)\right)^{\frac{3}{4}}},
\end{aligned}
\tag{7.9}
$$

where τ,β and δ are the same as in (7.8).

Using (7.9), we obtain that

$$
\|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} \le c_{15}(N,E,r,\sigma)\left(\ln\left(3+\delta^{-1}\right)\right)^{-\frac{3}{4}}\left(\ln\left(3\ln\left(3+\delta^{-1}\right)\right)\right)^2 \tag{7.10}
$$

for $\delta = ||S_1^+(E) - S_2^+(E)||_{\mathbb{L}^2(\partial B_r\times\partial B_r)} \le \delta_2(N_E,r,\sigma)$, where δ_2 is a sufficiently small positive constant. Estimate (7.10) in the general case (with modified c_{15}) follows from (7.10) for $\delta \le \delta_2(N_E,r,\sigma)$ and the property that $\|v_j\|_{\mathbb{L}^\infty(B_r)} \le N$.

This completes the proof of (2.8).

8 Proof of Lemma 3.1

In this section we assume for simplicity that $r = 1$ and therefore $\partial B_r = \mathbb{S}^{d-1}$.

We fix an orthonormal basis in $\mathbb{L}^2(\partial B_r)$:

$$
\begin{aligned}
&\{f_{jp} : j \ge 0;\ 1 \le p \le p_j\}, \\
&f_{jp} \text{ is a spherical harmonic of degree } j,
\end{aligned}
\tag{8.1}
$$

where p_j is the dimension of the space of spherical harmonics of order j,

$$
p_j = \binom{j+d-1}{d-1} - \binom{j+d-3}{d-1}, \tag{8.2}
$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \quad \text{for } n \geq 0 \tag{8.3}$$

and

$$\binom{n}{k} = 0 \quad \text{for } n < 0. \tag{8.4}$$

The precise choice of f_{jp} is irrelevant for our purposes. Besides orthonormality, we only need f_{jp} to be the restriction of a homogeneous harmonic polynomial of degree j to the sphere ∂B_r and so $|x|^j f_{jp}(x/|x|)$ is harmonic pn $\mathbb{R}^d$. In the Sobolev spaces $\mathbb{H}^s(\partial B_r)$ the norm is defined by

$$\left\| \sum_{j,p} c_{jp} f_{jp} \right\|^2_{\mathbb{H}^s(\partial B_r)} = \sum_{j,p} (1+j)^{2s} |c_{jp}|^2. \tag{8.5}$$

The solution ϕ of the exterior Dirichlet problem

$$\begin{aligned} -\Delta\phi &= E\phi, \quad \text{in } \mathbb{R}^d \setminus \overline{B}_r, \\ \lim_{|x|\to+\infty} |x|^{\frac{d-1}{2}} \left(\frac{\partial}{\partial|x|}\phi - i\sqrt{E}\phi \right) &= 0, \\ \phi|_{\partial B_r} &= u \in \mathbb{H}^1(\partial B_r), \end{aligned} \tag{8.6}$$

can be expressed in the following form (see, for example, [4], [8]):

$$\phi = \sum_{j,p} c_{jp}\phi_{jp}, \tag{8.7}$$

where c_{jp} are expansion coefficients of u in the basis $\{f_{jp} : j \geq 0;\ 1 \leq p \leq p_j\}$, and

$$\begin{aligned} &\phi_{jp} \text{ denotes the solution of (8.6) with } u = f_{jp}, \\ &\phi_{jp}(x) = h_{jp}(|x|) f_{jp}(x/|x|), \\ &h_{jp}(|x|) = |x|^{-\frac{d-2}{2}} \frac{H^{(1)}_{j+\frac{d-2}{2}}(\sqrt{E}|x|)}{H^{(1)}_{j+\frac{d-2}{2}}(\sqrt{E})}, \end{aligned} \tag{8.8}$$

where $H^{(1)}_\mu$ is the Hankel function of the first kind. Let

$$\phi^0_{jp}(x) = |x|^{-j-d+2} f_{jp}(x/|x|). \tag{8.9}$$

Note that ϕ^0_{jp} is harmonic in $\mathbb{R}^d \setminus \{0\}$ and

$$\lim_{|x|\to+\infty} |x|^{\frac{d-1}{2}} \left(\frac{\partial}{\partial|x|}\phi^0_{jp} - i\sqrt{E}\phi^0_{jp} \right) = 0 \quad \text{for } j + \frac{d-3}{2} > 0. \tag{8.10}$$

Using the Green formula and the radiation condition for ϕ_{jp}, ϕ^0_{jp}, we get that

$$\begin{aligned} \int\limits_{\mathbb{R}^d\setminus B_r} E\phi_{jp}\phi^0_{jp}dx &= \int\limits_{\mathbb{R}^d\setminus B_r} \left(\Delta\phi^0_{jp}\phi_{jp} - \Delta\phi_{jp}\phi^0_{jp}\right)dx = \\ = \int\limits_{\partial B_r} &\left(\frac{\partial\phi^0_{jp}}{\partial\nu_+}\phi_{jp} - \frac{\partial\phi_{jp}}{\partial\nu_+}\phi^0_{jp}\right)dx \quad \text{for } j+\frac{d-3}{2}>0. \end{aligned} \tag{8.11}$$

Due to (8.8) and (8.9), we have that

$$\left|\int\limits_{\partial B_r} \frac{\partial\phi^0_{jp}}{\partial\nu_+}\phi_{jp}dx\right| = (j+d-2)\int\limits_{\partial B_r} f^2_{jp}dx = j+d-2. \tag{8.12}$$

Using also the following property of the Hankel function of the first kind (see, for example, [38]):

$$|H^{(1)}_\mu(x)| \text{ is a decreasing function of } x \text{ for } x\in\mathbb{R}_+, \mu\in\mathbb{R}, \tag{8.13}$$

we get that

$$\begin{aligned} \left|\int\limits_{\mathbb{R}^d\setminus B_r} \phi_{jp}\phi^0_{jp}dx\right| &= \left|\int\limits_1^{+\infty} t^{-j-d+2}h_{jp}(t)t^{d-1}dt\right| = \\ = \left|\int\limits_1^{+\infty} t^{-j-\frac{d}{2}}\frac{H^{(1)}_{j+\frac{d-2}{2}}(\sqrt{E}t)}{H^{(1)}_{j+\frac{d-2}{2}}(\sqrt{E})}dt\right| &\le \int\limits_1^{+\infty} t^{-j-\frac{d}{2}}dt = \frac{1}{j+\frac{d}{2}-1} \le 2 \\ &\text{for } j+\frac{d-3}{2}>0. \end{aligned} \tag{8.14}$$

Combining (8.8), (8.9), (8.11), (8.12) and (8.14), we obtain that

$$\left|\int\limits_{\partial B_r} \frac{\partial\phi^0_{jp}}{\partial\nu_+}\phi_{jp}dx\right| = \left|\frac{h'_{jp}(r)}{h_{jp}(r)}\right| \le j+d-2+2E \quad \text{for } j+\frac{d-3}{2}>0. \tag{8.15}$$

Let consider the cases when $j+\frac{d-3}{2}\le 0$.

Case 1. $j=0$, $d=2$. Using the property $dH^{(1)}_0(t)/dt = -H^{(1)}_1(t)$, we get that

$$\frac{h'_{jp}(r)}{h_{jp}(r)} = \sqrt{E}\frac{H^{(1)}_1(\sqrt{E})}{H^{(1)}_0(\sqrt{E})}. \tag{8.16}$$

We recall that functions $H^{(1)}_0$ and $H^{(1)}_1$ have the following asymptotic forms (see, for example [38]):

$$\begin{aligned} H_0^{(1)}(t) &\sim \frac{2i}{\pi}\ln(t/2) \quad \text{as} \quad t \to +0, \\ H_0^{(1)}(t) &\sim \sqrt{\frac{2}{\pi t}} e^{i(t-\pi/4)} \quad \text{as} \quad t \to +\infty, \\ H_1^{(1)}(t) &\sim -\frac{i}{\pi}(2/t) \quad \text{as} \quad t \to +0, \\ H_1^{(1)}(t) &\sim \sqrt{\frac{2}{\pi t}} e^{i(t-3\pi/4)} \quad \text{as} \quad t \to +\infty. \end{aligned} \tag{8.17}$$

Using (8.13) and (8.17), we get that for some $c > 0$

$$\frac{H_1^{(1)}(t)}{H_0^{(1)}(t)} \leq c\,(1 + 1/t)\,. \tag{8.18}$$

Combining (8.16) and (8.18), we obtain that for $j = 0$, $d = 2$

$$\left|\frac{h'_{jp}(r)}{h_{jp}(r)}\right| \leq c(1 + \sqrt{E}). \tag{8.19}$$

Case 2. $j = 0$, $d = 3$. We have that

$$H^{(1)}_{j+\frac{d-2}{2}}(t) = \sqrt{\frac{2}{\pi t}} e^{i(t-\pi/2)}. \tag{8.20}$$

Using (8.8) and (8.20), we get that for $j = 0$, $d = 3$

$$\frac{h'_{jp}(r)}{h_{jp}(r)} = -1 + i\sqrt{E}. \tag{8.21}$$

Combining (8.5)-(8.8), (8.15), (8.19) and (8.21), we get that for some constant $c' = c'(d) > 0$

$$\left\| \left.\frac{\partial \phi}{\partial \nu_+}\right|_{\partial B_r} \right\|^2_{\mathbb{L}^2(\partial B_r)} = \sum_{j,p} c_{jp}^2 \left|\frac{h'_{jp}(r)}{h_{jp}(r)}\right|^2 \leq c'(1+E)^2 \sum_{j,p} (1+j)^2 c_{jp}^2. \tag{8.22}$$

Using (8.5) and (8.22), we obtain (3.6)

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