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Abstract

We consider continuous state branching processes (CSBP's for short) with additional multiplicative jumps, which we call catastrophes. Informally speaking, the dynamics of the CSBP is perturbed by independent random catastrophes which cause negative (or positive) jumps to the original process. These jumps are described by a Lévy process with paths of bounded variation. Conditionally on these jumps, the process still enjoys the branching property.
We construct this class of processes as the unique solution of a SDE and characterize their Laplace exponent as the solution of a backward ODE. We can then study their asymptotic behavior and establish whether the process becomes extinct. For a class of processes for which extinction and absorption coincide (including the α-stable CSBP plus a drift), we determine the speed of extinction of the process. Then, three subcritical regimes appear, as in the case for branching processes in random environments. To prove this, we study the asymptotic behavior of a certain divergent exponential functional of Lévy processes. Finally, we apply these results to a cell infection model, which was a motivation for considering such CSBP's with catastrophes.

Key words. Continuous State Branching Processes, Lévy processes, Poisson Point Processes, Stochastic Differential Equation, random environment

A.M.S. Classification. 60J80, 60J25, 60G51, 60H10, 60G55, 60K37.

1 Introduction

Continuous state branching processes (or CSBP's for simplicity) are the analogues of Bienaymé-Galton-Watson processes in continuous time and continuous state space. Such

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classes of processes have been introduced by Jirina [Jir58] and studied by many authors included Bingham [Bin76], Grey [Gre74], Grimvall [Gri74], Lamperti [Lam67a, Lam67b], to name but a few. A continuous state branching process $Z = (Z_t, t \geq 0)$ is a strong Markov process taking values in $[0, \infty]$, where 0 and ∞ are two absorbing states and $(\mathbb{P}_x,\, x > 0)$ is the law of the process starting from x. Moreover, Z satisfies the branching property; that is to say, for any $x, y \geq 0$, $\mathbb{P}_{x+y}$ is equal in law to the convolution of $\mathbb{P}_x$ and $\mathbb{P}_y$. Since the pioneering work of Lamperti [Lam67b], it is known that continuous-state branching processes are the only possible scaling limits of Bienaymé-Galton-Watson branching processes and that every CSBP can be realized in this way. Thus CSBP may be model for the evolution of (renormalized) large populations which evolve during a large time window (see for instance [BT11] for parasite infection).

The branching property implies that the Laplace transform of Z_t is of the form

$$\mathbb{E}_x\Big[\exp(-\lambda Z_t)\Big] = \exp\{-x u_t(\lambda)\}, \qquad \text{for } \lambda \geq 0, \tag{1}$$

for some non negative function u_t. According to Silverstein [Sil68], this function is determined by the integral equation

$$\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(u)} \mathrm{d}u = t, \tag{2}$$

where ψ satisfies the celebrated Lévy-Khintchine formula

$$\psi(\lambda) = a\lambda + \sigma^2\lambda^2 + \int_{(0,\infty)} \big(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}\big)\mu(\mathrm{d}x),$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and μ is a σ-finite measure on $(0, \infty)$ such that $\int_{(0,\infty)} \big(1 \wedge x^2\big)\mu(\mathrm{d}x)$ is finite. The function ψ is known as the branching mechanism of Z.

It is important to note that the first moment of Z_t can be obtained by differentiating (1) with respect to λ. More precisely, note that $\partial_\lambda u_t(\lambda)|_{\lambda=0} = e^{-\psi'(0^+)t}$ and $\mathbb{E}_x(Z_t) = xe^{-\psi'(0^+)t}$. Hence, in respective order, a CSBP is called supercritical, critical or subcritical depending on $\psi'(0^+) < 0$, $\psi'(0^+) = 0$ or $\psi'(0^+) > 0$. Adding that

$$\mathbb{P}_x\left(\lim_{t\to\infty} Z_t = 0\right) = e^{-\eta x},$$

where η is the largest root of the branching mechanism ψ, the sign of $\psi'(0+)$ yields the criterion for a.s. extinction. Finally, we know that a CSBP Z with branching mechanism ψ is absorbed at 0 in a finite time with positive probability if and only if

$$\int^{\infty} \frac{\mathrm{d}u}{\psi(u)} < \infty.$$

In this case, $\mathbb{P}_x\left(\exists t \geq 0 : Z_t = 0\right) = \mathbb{P}_x\left(\lim_{t\to\infty} Z_t = 0\right) = \exp(-\eta x)$.

In this paper, we focus on the case when Z has finite expectation, which is equivalent to

$$g := -\psi'(0+) = \int_1^{\infty} x\mu(dx) - a < \infty,$$

or equivalent to $\int_0^{\infty}(x \wedge x^2)\mu(dx) < \infty$. Then we can write

$$\psi(\lambda) = -g\lambda + \sigma^2\lambda^2 + \int_0^{\infty} \left(e^{-\lambda x} - 1 + \lambda x\right)\mu(\mathrm{d}x),$$

and the CSBP is characterized by the triplet (g, σ, μ). Moreover a CSBP can also be defined as the unique non-negative strong solution of a SDE, which will be useful here. More precisely, from [FL10],

$$Z_t = Z_0 + \int_0^t gZ_s \mathrm{d}s + \int_0^t \sqrt{2\sigma^2 Z_s} \mathrm{d}B_s + \int_0^t \int_0^\infty \int_0^{Z_{s-}} z\widetilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \tag{3}$$

where $B = (B_t, t \geq 0)$ is a standard Brownian motion, $N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is a Poisson random measure with intensity $\mathrm{d}s\mu(\mathrm{d}z)\mathrm{d}u$ independent of B and $\widetilde{N}_0$ is the compensated measure of N_0.

The stable case is of special interest and will be studied in more detail in Section 3. It is motivated by applications (see Section 4) and provide us the key step to more general cases. It corresponds to the (only possible) scaling limit of Galton Watson processes (GW) when the reproduction law is fixed. Then $\psi(\lambda) = g\lambda + c\lambda^\alpha$ and two natural classes appear:

- If $\alpha = 2$ (Feller diffusion), we necessarily have $\mu = 0$. Thus we get the continuous diffusion
$$Z_t = Z_0 + \int_0^t gZ_s \mathrm{d}s + \int_0^t \sqrt{2\sigma^2 Z_s} \mathrm{d}B_s.$$
- If $\alpha \in (1, 2)$, we have that $\sigma = 0$ and $\mu(\mathrm{d}x) = c\alpha x^{-\alpha+1}\mathrm{d}x/\Gamma(2-\alpha)$. Then the process has (only) positive jumps with infinite intensity [Lam07]. Moreover, we have
$$Z_t = Z_0 + \int_0^t gZ_s \mathrm{d}s + \int_0^t Z_{s^-}^{1/\alpha} \mathrm{d}X_s,$$
where X is an α-stable spectrally positive Lévy process.

In the stable case, we can solve equation (2). In particular, we know precisely the speed of absorption, i.e. the asymptotic behavior of $\mathbb{P}(Z_t > 0)$.

In this work, we want to take into account catastrophes which occur randomly and kill each individual with some probability (depending on the catastrophe). In the scaling limit of large population (the continuous state setting), it amounts to let the process make a negative jump and multiply Z by a random fraction. The process we obtain is still Markovian if the catastrophes happen without memory and the fractions are i.i.d., in other words catastrophes are described by a Poisson Point Process. Conditionally on the times and the effects of the catastrophes, the process Z also satisfies the branching property, as detailed below. It yields then a particular class of CSBP's in random environment, which can also be obtained as scaling limit of GW in random environment (see [BS47]).

More precisely, we consider here CSBP's with catastrophes described by an independent Poisson Point Process. Such processes are motivated in particular by cell division models. For example, in [BT11], the case of Feller diffusion with catastrophes describes the evolution of parasites in a cell line. The parasites grow following a Feller diffusion and the division of the cell causes a negative jump of this quantity by splitting the parasites into two parts.

More generally, we can also take into account positive jumps due to favorable environments. They can represent immigration events proportional to the size of the population. One possible application comes from the aggregation behavior of some species. Indeed, many species form aggregates, what may allow them to reduce predation, make the food search easier, or improve the mate choice (see respectively chapters 12.1, 12.2.2, 12.2.3

and 12.3.4 in [DGC08]). For convenience, we still call the processes obtained CSBP with catastrophes.

The remainder of the paper is structured as follows. In the next section, we study the CSBP with catastrophes, in particular we verify that these kinds of processes are well defined by a SDE. We also characterize their Laplace exponent via an ODE which allows us to describe their long time behavior. In particular, we determine when $(Y_t, t \geq 0)$, a CSBP with catastrophes becomes extinct. That is, we specify when $\mathbb{P}(\lim_{t\to\infty} Y_t = 0) = 1$ and the asymptotic behavior when the process survives, under some moment assumption.

Section 3 is devoted to the speed of extinction. We take special attention to the stable case (Section 3.1), where the extinction event coincides with the absorption event. In particular, we obtain an explicit expression for the Laplace exponent in terms of an exponential functional of a Lévy process whose jump structure is given by the catastrophes. This expression allows us to determine the speed of absorption of the process when it is subcritical or critical (see Section 5).

From this result in the stable case, we can deduce the speed of extinction for a large class of CSBP's with catastrophes, when extinction and absorption coincide. In the subcritical case, three regimes appear (see Theorem 5 and Corollary 6). It means that the law of the catastrophes (and not only their mean effect) can change the exponential decrease of the survival probability. This result is closely related to the discrete framework via asymptotic behavior of functionals of random walks. More precisely, we use in our arguments local limit theorems for semi direct product [LPP97, GL01] and some analytical results on random walks [Koz76, Hir98] (see Section 5). In the same vein, we refer to [BH12] for different subcritical regimes for Feller diffusions which evolve "in Brownian environment".

This result can be related to discussions in ecology about the role of environmental and demographical stochasticity. Such topics are fundamental in ecology and conservation biology, as discussed for instance in Chapter 1 in [LES03]. In our model, the survival may be either due to the randomness of the individual reproduction, which is given by the parameters σ and μ of the CSBP, or to the randomness (rate, size) of the catastrophes which is linked to the environment. For a study of relative effects of environmental and demographical stochasticity, one can read [Lan93] and references therein.

In Section 4, we apply our results to Feller diffusion with catastrophes and derive the different regimes for the speed of propagation of an infection in the cell division model [BT11].

Section 5 is devoted to the asymptotic behavior of exponential functionals of Lévy processes with paths of bounded variation. More precisely, we are interested in the asymptotic behavior at ∞ of

$$\mathbb{E}\left[F\left(\int_0^t e^{-\beta K_s}\mathrm{d}s\right)\right],$$

where $\beta > 0$, F belongs to a given class of functions (see (26)) and K is a Lévy process with paths of bounded variation that does not drift to ∞ and has some exponential positive moments. We find four different regimes for its behavior that depend on the shape of the Laplace exponent of the process K. Up to our knowledge, this situation has been studied only by Carmona et al. [CPY97], Lemma 4.7, where they provide only one precise regime. The case when the process K drifts to ∞ has been deeply studied by many authors, see for instance Bertoin and Yor [BY05] and references therein.

Finally, Section 6 contains some technical results which are used in the proofs. We establish such results at the end of the paper for the convenience of the reader.

2 CSBP with catastrophes

We consider a CSBP $Z = (Z_t, t \geq 0)$ defined by (3) characterized by the triplet (g, σ, μ), where we recall that μ satisfies

$$\int_0^\infty (x \wedge x^2)\mu(\mathrm{d}x) < \infty. \tag{4}$$

The catastrophes are independent of the process Z and are given by a Poisson Point Process $(\delta_{e_{t_i}, t_i})_{i \in I}$ on $[0, \infty) \times [0, \infty)$ with intensity $\mathrm{d}t\nu(\mathrm{d}x)$ such that

$$\int_{(0,\infty)} (1 \wedge |x-1|)\nu(\mathrm{d}x) < \infty. \tag{5}$$

The associated Poisson random measure $N_1 = \sum_{i \in I} \delta_{e_{t_i}, t_i}$ is independent of B and N_0, and the jump process

$$\Delta_t = \int_0^t \int_{(0,\infty)} \log(x) N_1(\mathrm{d}s, \mathrm{d}x) = \sum_{s \leq t} \log(e_s),$$

is a Lévy process with paths of bounded variation.

The CSBP (g, σ, μ) with catastrophes ν is then given by the following SDE

$$\begin{aligned} Y_t = Y_0 + \int_0^t gY_s \mathrm{d}s + \int_0^t \sqrt{2\sigma^2 Y_s} \mathrm{d}B_s \quad &+ \int_0^t \int_{[0,\infty)} \int_0^{Y_{s-}} z\widetilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &+ \int_0^t \int_{[0,\infty)} \Big(z-1\Big) Y_{s-} N_1(\mathrm{d}s, \mathrm{d}z), \end{aligned} \tag{6}$$

where $Y_0 > 0$ a.s. In other words, when a catastrophe occurs at time s, the population size is multiplied by e_s which is distributed according to the measure ν.

Theorem 1. *The SDE (6) has a unique non-negative strong solution for $g \in \mathbb{R}, \sigma \geq 0$, μ and ν satisfying the conditions (4) and (5), respectively.*

Then, the process $Y = (Y_t, t \geq 0)$ defined by the SDE (6) is a càdlàg Markov process satisfying the branching property conditionally on $\Delta = (\Delta_t, t \geq 0)$ and with generator given by

$$\begin{aligned} \mathcal{A}f(x) = gxf'(x) + \sigma^2 x f''(x) &+ \int_0^\infty \Big(f(xz) - f(x)\Big)\nu(dz) \\ &+ \int_0^\infty \Big(f(x+z) - f(x) - zf'(x)\Big) x\mu(dz). \end{aligned} \tag{7}$$

Moreover, for every $t \geq 0$,

$$\mathbb{E}_y \left[\exp\Big\{ -\lambda \exp\big\{ -gt - \Delta_t \big\} Y_t \Big\} \Big| \, \Delta \right] = \exp\Big\{ -yv_t(0, \lambda, \Delta) \Big\} \qquad a.s.,$$

where for every (λ, δ), $v_t : s \mapsto v_t(s, \lambda, \delta)$ is the unique solution of the following backward differential equation :

$$\frac{\partial}{\partial s} v_t(s, \lambda, \delta) = e^{gs+\delta_s} \psi_0\big(e^{-gs-\delta_s} v(s, \lambda, \delta)\big), \quad 0 \leq s \leq t, \qquad v_t(t, \lambda, \delta) = \lambda, \tag{8}$$

where

$$\psi_0(\lambda) = \psi(\lambda) - \lambda\psi'(0) = \sigma^2\lambda^2 + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x)\mu(dx). \tag{9}$$

Proof. We first study the SDE given by

$$Y_t = Y_0 + \int_0^t gY_s \mathrm{d}s + \int_0^t \sqrt{2\sigma^2 Y_s}\mathrm{d}B_s \quad + \quad \int_0^t \int_{[0,\infty)} \int_0^{Y_{s-}} z\widetilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$$
$$+ \quad \int_0^t \int_{(0,2)} \Big(z-1\Big)Y_{s-}N_1(\mathrm{d}s, \mathrm{d}z). \quad (10)$$

The above SDE satisfies the conditions of Theorems 3.2 and 5.1 of [FL10]. In particular, to verify condition (5.b) we can simply let $V_n = [1/n, \infty) \times (0, \infty)$, for $n \geq 1$. Hence from Theorems 3.2 and 5.1 in [FL10], we have pathwise uniqueness and the existence of a unique non-negative strong solution for (10).
Now, note from (5) that $\nu([2, \infty)) < \infty$. Thus, Proposition 2.2 in [FL10] implies pathwise uniqueness and the existence of a unique non-negative strong solution for the SDE (6). By Itô's formula (see for instance [IW89] Th.5.1) one sees that the solution $(Y_t, t \geq 0)$ solves the following martingale problem: for every $f \in C_b^2(\mathbb{R}_+)$,

$$f(Y_t) = f(Y_0) + \text{ loc. mart. } + g\int_0^t f'(Y_s)Y_s\mathrm{d}s$$
$$+ \sigma^2 \int_0^t f''(Y_s)Y_s\mathrm{d}s + \int_0^t \int_0^\infty Y_s\Big(f(Y_s+z) - f(Y_s) - f'(Y_s)z\Big)\mu(\mathrm{d}z)\mathrm{d}s$$
$$+ \int_0^t \int_0^\infty \Big(f(zY_{s-}) - f(Y_{s-})\Big)\nu(\mathrm{d}z)\mathrm{d}s,$$

where the local martingale is given by

$$\text{loc. mart.} \quad = \quad \int_0^t f'(Y_s)\sqrt{2\sigma^2 Y_s}\mathrm{d}B_s + \int_0^t \int_0^\infty \Big(f(zY_{s-}) - f(Y_{s-})\Big)\widetilde{N}_1(\mathrm{d}s, \mathrm{d}z) \quad (11)$$
$$+ \int_0^t \int_0^\infty \int_0^{Y_{s-}} \Big(f(Y_{s-} + z) - f(Y_{s-})\Big)\widetilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

and $\widetilde{N}_1$ is the compensated measure of N_1. From pathwise uniqueness, we deduce that the solution of (6) is a strong Markov process whose generator is given by (7).

The branching property of Y, conditionally on the jumps, is inherited from the branching property of the CSBP and the fact that the additionnal jumps are multiplicative.

In order to prove the second part of the theorem, let us now work conditionally on Δ. By applying Itô's formula to the process $\widetilde{Z}_t = Y_t e^{-gt-\Delta_t}$, Lemma 15 in Section 6 ensures that for every $F \in C_b^{1,2}$, $F(t, \widetilde{Z}_t)$ is also a local martingale if and only if for every $t \geq 0$,

$$\int_0^t \frac{\partial^2}{\partial x^2}F(s, \widetilde{Z}_s)\sigma^2 \tilde{Z}_s e^{-gs-\Delta_s}\mathrm{d}s + \int_0^t \frac{\partial}{\partial t}F(s, \widetilde{Z}_s)\mathrm{d}s$$
$$+ \int_0^t \int_0^\infty \tilde{Z}_s\Big(\Big[F(s, \widetilde{Z}_s + ze^{-gs-\Delta_s}) - F(s, \widetilde{Z}_s)\Big]e^{gs+\Delta_s} - \frac{\partial}{\partial x}F(s, \widetilde{Z}_s)z\Big)\mu(\mathrm{d}z)\mathrm{d}s = 0.$$

In the vein of [IW89, BT11], we choose $F(s, x) := \exp\{-xv_t(s, \lambda, \Delta)\}$, where $v_t(s, \lambda, \Delta)$ is differentiable with respect to the variable s, non negative and such that $v_t(t, \lambda, \Delta) = \lambda$, for $\lambda \geq 0$. We observe that F is bounded. Therefore, from the above computations $(\exp\{-\tilde{Z}_s v_t(s, \lambda, \Delta)\}, 0 \leq s \leq t)$ will be a martingale if and only if

$$\frac{\partial}{\partial s}v_t(s, \lambda, \Delta) = e^{gs+\Delta_s}\psi_0\left(e^{-gs-\Delta_s}v_t(s, \lambda, \Delta)\right), \quad 0 \leq s \leq t,$$

where ψ_0 is defined in (9). Proposition 16 in Section 6 ensures that a.s. the solution of this backward ODE exists and is unique, which essentially comes from local Lipschitz property of ψ_0 and paths of bounded variation of Δ. Thus the process $(\exp\{-\tilde{Z}_s v_t(s,\lambda,\Delta)\}, 0 \leq s \leq t)$ is a martingale conditionally on Δ. We get

$$\mathbb{E}_y\left[\exp\Big\{-\tilde{Z}_t v_t(t,\lambda,\Delta)\Big\}\Big|\ \Delta\right] = \mathbb{E}_y\left[\exp\Big\{-\tilde{Z}_0 v_t(0,\lambda,\Delta)\Big\}\Big|\Delta\right],$$

which implies

$$\mathbb{E}_y\left[\exp\Big\{-\lambda\tilde{Z}_t\Big\}\Big|\ \Delta\right] = \exp\Big\{-y v_t(0,\lambda,\Delta)\Big\}, \tag{12}$$

and ends up the proof. □

Now we derive from Theorem 1 an extinction criterion for CSBP's with catastrophes. First, we recall that a Lévy process $L = (L_t, t \geq 0)$ has only three types of asymptotic behaviors: either it drifts to ∞, $-\infty$, or oscillates a.s. The latter means that $\limsup_{t\to\infty} L_t = -\liminf_{t\to\infty} L_t = \infty$. We refer to Theorem 7.2 in [Kyp06] for a criterion for such types of behaviors.

Corollary 2. *We have the three following regimes.*

i) If $(\Delta_t + gt)_{t\geq 0}$ drifts to $-\infty$, then $Y_t \to 0$ a.s.

ii) If $(\Delta_t + gt)_{t\geq 0}$ oscillates, then $\liminf_{t\to\infty} Y_t = 0$ a.s.

iii) If $(\Delta_t + gt)_{t\geq 0}$ drifts to $+\infty$ and there exists $\varepsilon > 0$, such that

$$\int_0^\infty x\log(1+x)^{1+\epsilon}\mu(dx) < \infty, \tag{13}$$

then $\mathbb{P}\big(\liminf_{t\to\infty} Y_t > 0\big) > 0$ and there exists a non negative finite r.v. W such that

$$e^{-gt-\Delta_t}Y_t \xrightarrow[t\to\infty]{} W \quad a.s., \qquad \{W=0\} = \Big\{\lim_{t\to\infty} Y_t = 0\Big\}.$$

It is important to note, according to Theorem 7.2 in [Kyp06], that whenever Δ_1 has a finite first moment, then the sign of $\mathbb{E}(\Delta_1)+g$ yields the regimes above. Hence, whenever its Laplace exponent, $\phi(\lambda) = \log\mathbb{E}[\exp(\lambda\Delta_1)]$, is well defined for some positive λ, then cases (i), (ii) and (iii) will be equivalent to $g+\phi'(0)<0$, $g+\phi'(0)=0$ and $g+\phi'(0)>0$ respectively. This will be the case in the rest of the paper.

Moreover, let us note that in the regime (ii), we can have $\liminf_{t\to\infty} Y_t = 0$ a.s. but Y_t a.s. does not tend to zero. For example, if $\mu=0$ and $\sigma=0$, $Y_t = \exp(gt+\Delta_t)$, hence $\limsup_{t\to\infty} Y_t = \infty$.

Assumption (iii) of the corollary does not imply that $\{\lim_{t\to\infty} Y_t = 0\} = \{\exists t : Y_t = 0\}$. For example if $\mu(dx) = x^{-2}\mathbf{1}_{[0,1]}dx$, it is not difficult to show that $\psi(u) \sim u\log u$ as $u\to\infty$, and according to Remark 2.2 in [Lam08] we necessarily have $\mathbb{P}(\exists t : Y_t = 0) = 0 < \mathbb{P}(\lim_{t\to\infty} Y_t = 0) < 1$.

Proof. We know from (12) with $F(s,x)=x$, that $\exp\{-gt-\Delta_t\}Y_t$ is a non negative local martingale, thus it is a non negative supermartingale and it converges a.s. to a non negative finite random variable W. This leads to the result for the first two cases.

In the case when $(\Delta_t + gt)_{t\geq 0}$ drifts to $+\infty$, we should now prove that $\mathbb{P}(W > 0 \mid \Delta) > 0$ a.s. According to Lemma 18 in Section 6, there exists a non negative increasing function h on $\mathbb{R}^+$ such that for all $\lambda > 0$,

$$\psi_0(\lambda) \leq \lambda h(\lambda) \quad \text{and} \quad c(\Delta) := \int_0^\infty h\Big(e^{-(gt+\Delta_t)}\Big) dt < \infty \quad \text{a.s.}$$

We can now show that conditionnally on Δ there exists a positive lower bound for $v_t(0,1,\Delta)$, for $t \geq 0$. For every $(t,\lambda) \in (\mathbb{R}_+^*)^2$, the solutions of (8) are non decreasing on $[0,t]$, thus for all $s \in [0,t]$, $v_t(s,1,\Delta) \leq 1$, and

$$\begin{aligned}\psi_0(e^{-gs-\Delta_s} v_t(s,1,\Delta)) &\leq e^{-gs-\Delta_s} v_t(s,1,\Delta) h(e^{-gs-\Delta_s} v_t(s,1,\Delta)) \\ &\leq e^{-gs-\Delta_s} v_t(s,1,\Delta) h(e^{-gs-\Delta_s}).\end{aligned}$$

Then

$$\frac{\partial}{\partial s} v_t(s,1,\Delta) \leq v_t(s,1,\Delta) h(e^{-gs-\Delta_s}).$$

This clearly implies,

$$-\ln(v_t(0,1,\Delta)) \leq \int_0^t h(e^{-gs-\Delta_s}) ds \leq c(\Delta) < \infty, \qquad \text{a.s.}$$

Hence, for every $t \geq 0$, $v_t(0,1,\Delta) \geq e^{-c(\Delta)} > 0$. Then from (12),

$$\mathbb{E}_y(\exp(-\lambda W) \mid \Delta) = \exp\Big(- y \lim_{t\to\infty} v_t(0,1,\Delta)\Big) < 1$$

and $\mathbb{P}(W > 0 \mid \Delta) > 0$.

Moreover, since Y satisfies the branching property conditionally on Δ, we can show (see Lemma 19 in Section 6) that

$$\{W = 0\} = \Big\{ \lim_{t\to\infty} Y_t = 0 \Big\} \qquad \text{a.s.},$$

which completes the proof. □

We now derive a central limit theorem in the supercritical regime that only requires a second moment assumption.

Corollary 3. *Assume that $(\Delta_t + gt)_{t\geq 0}$ drifts to $+\infty$ and (13) is satisfied. Then, under the additional assumption*

$$\int_{(0,e^{-1}]\cup[e,\infty)} (\log(x))^2 \nu(\mathrm{d}x) < \infty, \tag{14}$$

conditionally on $\{W > 0\}$,

$$\frac{\log(Y_t) - \boldsymbol{m} t}{\rho\sqrt{t}} \xrightarrow[t\to\infty]{d} N(0,1),$$

where $\xrightarrow{d}$ means convergence in distribution,

$$\boldsymbol{m} := g + \int_{\{|\log(x)|\geq 1\}} \log(x)\nu(\mathrm{d}x) < \infty, \qquad \rho^2 := \int_0^\infty (\log(x))^2 \nu(\mathrm{d}x) < \infty,$$

and $N(0,1)$ denotes a centered gaussian random variable with variance equals 1.

Proof. We first establish a central limit theorem for the Lévy process $(gt + \Delta_t, t \geq 0)$ under the assumption (14) which relies on Theorem 3.5 in Doney and Maller [DM02] (the details can be found in Section 6.5). We get

$$\frac{gt + \Delta_t - \mathbf{m}t}{\rho\sqrt{t}} \xrightarrow[t\to\infty]{d} N(0,1). \tag{15}$$

From Corollary 2 part *iii*), on the event $\{W > 0\}$, we then have

$$\log(Y_t) - (gt + \Delta_t) \xrightarrow[t\to\infty]{a.s.} \log(W) \in (-\infty, \infty),$$

and we conclude by using (15).

□

3 Speed of extinction of CSBP with catastrophes

In this section, we first consider the stable CSBP with a deterministic growth $g \in \mathbb{R}$, before deriving the result for some general class of CSBP.

3.1 The stable case

We assume in this section that

$$\psi(\lambda) = g\lambda + c_+\lambda^{\beta+1}, \tag{16}$$

for some $\beta \in (0,1]$, $c_+ > 0$ and g in $\mathbb{R}$. In this particular case, the backward differential equation (8) can be solved and we get

Proposition 4. *For all $x_0 > 0$ and $t \geq 0$:*

$$\mathbb{P}_{x_0}(Y_t > 0) = 1 - \mathbb{E}\left[\exp\left\{-x_0\left(c_+\int_0^t e^{-\beta(\Delta_s+gs)}\mathrm{d}s\right)^{-1/\beta}\right\}\right]. \tag{17}$$

Moreover,

$$\mathbb{P}_{x_0}\big(\textit{there exists } t > 0;\ Y_t = 0\big) = 1,$$

if and only if the process $(\Delta_t + gt, t \geq 0)$ does not drift to $+\infty$.

Proof. We solve equation (8) with $\psi(\lambda) = g\lambda + c_+\lambda^{\beta+1}$. As $\psi_0(\lambda) = c_+\lambda^{\beta+1}$, a direct integration yields

$$v_t(u,t,\lambda) = \left[c_+\int_u^t e^{-\beta(\Delta_s+gs)}\mathrm{d}s + \lambda^{-\beta}\right]^{-1/\beta},$$

so that

$$\mathbb{E}_{x_0}\left[e^{-\lambda \tilde{Z}_t}\right] = \mathbb{E}_{x_0}\left[\exp\left\{-x_0\left(c_+\int_0^t e^{-\beta(\Delta_s+gs)}\mathrm{d}s + \lambda^{-\beta}\right)^{-1/\beta}\right\}\right]. \tag{18}$$

The expression for the absorption probability is a direct application of (18). Indeed letting λ goes to ∞, we get

$$\mathbb{P}_{x_0}(Y_t = 0) = \mathbb{E}_{x_0}\left[\exp\left\{-x_0\left(c_+\int_0^t e^{-\beta(\Delta_s+gs)}\mathrm{d}s\right)^{-1/\beta}\right\}\right].$$

The mapping $t \mapsto \int_0^t e^{-\beta(\Delta_s+gs)}\mathrm{d}s$ is a.s. increasing. Thus,

$$\int_0^\infty e^{-\beta(\Delta_s+gs)}\mathrm{d}s \in \mathbb{R}_+ \cup \{\infty\},$$

is well defined. Hence, the process $(\tilde{Z}_t, t \geq 0)$ converges in distribution as $t \to \infty$ towards the r.v. W (already defined in the case *iii*) of the Corollary 2) whose distribution is specified by

$$\mathbb{E}_{x_0}\left[e^{-\lambda W}\right] = \mathbb{E}_{x_0}\left[\exp\left\{-x_0\left(c_+ \int_0^\infty e^{-\beta(\Delta_s+gs)}\mathrm{d}s + \lambda^{-\beta}\right)^{-1/\beta}\right\}\right].$$

Letting $\lambda \to \infty$, we get by monotone convergence

$$\mathbb{P}_{x_0}(W=0) = \mathbb{E}_{x_0}\left[\exp\left\{-x_0\left(c_+ \int_0^\infty e^{-\beta(\Delta_s+gs)}\mathrm{d}s\right)^{-1/\beta}\right\}\right]. \tag{19}$$

Finally, according to Theorem 1 in [BY05] we have $\int_0^\infty e^{-\beta(\Delta_s+gs)}\mathrm{d}s = \infty$, a.s. if and only if the process $(\Delta_t + gt, t \geq 0)$ does not drift to $+\infty$. Adding that $\mathbb{P}_{x_0}(\exists t \geq 0; Y_t = 0) = \lim_{t\to\infty} \mathbb{P}_{x_0}(Y_t = 0)$ and using (17) and (19), we get that

$$1 = \mathbb{P}_{x_0}(W=0) = \mathbb{P}_{x_0}(\exists t \geq 0; Y_t = 0)$$

if and only if the process $(\Delta_t + gt, t \geq 0)$ does not drift to $+\infty$. □

In what follows, we assume that the Laplace exponent of the Lévy process Δ is well-defined for some positive real number, i.e.

$$\phi(\lambda) = \log \mathbb{E}[e^{\lambda \Delta_1}] \qquad \text{for } \lambda \in [0, \theta_{max}),$$

where $\theta_{max} = \sup\{\lambda > 0,\ \phi(\lambda) < \infty\}$. In other words, $\int_{[e,\infty)} x^\lambda \nu(\mathrm{d}x) < \infty$ if $\lambda \in [0, \theta_{max})$.

We get then the main result of this paper, where we recall that ϕ' is non decreasing.

Theorem 5. *We assume that ψ satisfies (16).*

a/ If $\phi'(0) + g < 0$ (subcritical case) and $\theta_{max} > 1$, then we have the following three asymptotic regimes

(i) If $\phi'(1) + g < 0$ (strongly subcritical regime), then there exists $c_1 := d_1(\beta,\nu)c_+^{-1/\beta} > 0$ such that for every $x_0 > 0$,

$$\mathbb{P}_{x_0}(Y_t > 0) \sim c_1 x_0 e^{t(\phi(1)+g)}, \qquad \textit{as} \quad t \to \infty.$$

(ii) If $\phi'(1) + g = 0$ (intermediate subcritical regime), then there exists $c_2 := d_2(\beta,\nu)c_+^{-1/\beta} > 0$ such that for every $x_0 > 0$,

$$\mathbb{P}_{x_0}(Y_t > 0) \sim c_2 x_0 t^{-1/2} e^{t(\phi(1)+g)}, \qquad \textit{as} \quad t \to \infty.$$

(iii) If $\phi'(1) + g > 0$ (weakly subcritical regime) and $\theta_{max} > \beta + 1$, then for every $x_0 > 0$, there exists $c_3 := c_3(x_0, \psi, \nu) > 0$ such that

$$\mathbb{P}_{x_0}(Y_t > 0) \sim c_3 t^{-3/2} e^{t(\phi(\tau)+g\tau)}, \qquad \textit{as} \quad t \to \infty,$$

where τ is the root of $\phi' + g$ on $]0,1[$.

b/ If $\phi'(0) + g = 0$[1] *(critical case) and $\theta_{max} > \beta$, then for every $x_0 > 0$, there exists $c_4 := c_4(x_0, \psi, \nu) > 0$ such that*

$$\mathbb{P}_{x_0}(Y_t > 0) \sim c_4 t^{-1/2}, \qquad \textit{as} \quad t \to \infty.$$

Proof. From the Proposition 4 we know that

$$\mathbb{P}_{x_0}(Y_t > 0) = 1 - \mathbb{E}\left[\exp\left\{-x_0\left(c_+ \int_0^t e^{-\beta(\Delta_s + gs)}\mathrm{d}s\right)^{-1/\beta}\right\}\right] = \mathbb{E}\left[F\left(\int_0^t e^{-\beta K_s}\mathrm{d}s\right)\right],$$

where $F(x) = 1 - \exp\{-x_0(c_+ x)^{-1/\beta}\}$ and $K_s = \Delta_s + gs$.
Since F satisfies (26), the result is a consequence of Proposition 8 whose statement and proof are deferred to Section 5. □

In the case of the CSBP without any catastrophes, the subcritical regime is reduced to (i). The critical case differs from b/, since then the asymptotic behavior is given by $1/t$.
This result can be compared to the regimes which appear in the literature of discrete (time and space) branching processes in random environment (see e.g. [GL01, GKV03, AGKV05]), even if the proofs do not use directly the corresponding results.
Besides, in the continuous framework, such asymptotic regimes for the survival probability of Feller diffusion whose drift varies following a Brownian motion appear in [BH12].

In the strongly and intermediate subcritical cases (i) and (ii), $\mathbb{E}(Y_t)$ gives the good exponential rate for the decrease of the survival probability, $\phi(1) + g$. Moreover the probability of non-extinction is proportional to the initial state x_0 of the population, and to $c_+^{-1/\beta}$. This observation follows from the following equivalence

$$\mathbb{E}\left[F\left(\int_0^t \exp(-\beta(\Delta_s + gs))\right)\right] \sim \mathbb{E}\left[x_0\left(c_+ \int_0^t \exp(-\beta(\Delta_s + gs))\right)^{-1/\beta}\right], \quad \text{as} \quad t \to \infty,$$

and we refer to the proof of Lemma 11 and Section 5.4 for details.

In the weakly subcritical case (iii), the exponential rate of decrease of the survival probability is $\phi(\tau) + g\tau$. The latter is strictly smaller than $\phi(1) + g$, indeed

$$\phi(\tau) + g\tau = \min_{0<s<1}\{\phi(s) + gs\} < \phi(1) + g.$$

In fact, as it appears in the proof of this theorem (see Section 5), the quantity which determines the scale of the asymptotic behavior in all cases is linked to $\mathbb{E}[e^{I_t}]$, where $I_t := \inf_{s\in[0,t]}(\Delta_s + gs)$.
Let us note also that c_3 and c_4 may not depend linearly on x_0. We refer to [Ban09] for a result in this vein for discrete branching processes in random environment.

[1] We exclude the degenerated case $\nu = 0$, $g = 0$.

3.2 Beyond the stable case.

In this section, we want a similar result as in Theorem 5 for CSBP's with catastrophes when the branching mechanism ψ_0 is not stable. In order to do so, we may compare ψ_0 with a stable branching mechanism. For some technical reasons, we assume that the Brownian coefficient is positive and the associated Lévy measure μ satisfies a second moment condition. It allows us to obtain the following result from the Feller diffusion case in Theorem 5, i.e $\beta = 1$.

Corollary 6. *Assume that $\int_{(0,\infty)} x^2\mu(dx) < \infty$ and $\sigma^2 > 0$.*

a/ In the subcritical case, i.e. $\phi'(0) + g < 0$, if $\theta_{max} > 1$, we have the following three asymptotic regimes

(i) If $\phi'(1) + g < 0$, there exist $0 < c_1 \le c_1' < \infty$ such that for every x_0,

$$c_1 x_0 e^{t(\phi(1)+g)} \le \mathbb{P}_{x_0}(Y_t > 0) \le c_1' x_0 e^{t(\phi(1)+g)} \quad (t \ge 0).$$

(ii) If $\phi'(1) + g = 0$, there exist $0 < c_2 \le c_2' < \infty$ such that for every x_0,

$$c_2 x_0 t^{-1/2} e^{t(\phi(1)+g)} \le \mathbb{P}_{x_0}(Y_t > 0) \le c_2' x_0 t^{-1/2} e^{t(\phi(1)+g)} \quad (t \ge 0).$$

(iii) If $\phi'(1) + g > 0$ and $\theta_{max} > \beta + 1$, for every x_0, there exist $0 < c_3(x_0) \le c_3'(x_0) < \infty$ such that

$$c_3(x_0) t^{-3/2} e^{t(\phi(\tau)+g\tau)} \le \mathbb{P}_{x_0}(Y_t > 0) \le c_3'(x_0) t^{-3/2} e^{t(\phi(\tau)+g\tau)} \quad (t \ge 0),$$

where τ is the root of $\phi' + g$ on $]0, 1[$.

b/ In the critical case, i.e. $\phi'(0) + g = 0$[2], if $\theta_{max} > \beta$, then for every x_0, there exist $0 < c_4(x_0) < c_4(x_0)' < \infty$ such that

$$c_4(x_0) t^{-1/2} \le \mathbb{P}_{x_0}(Y_t > 0) \le c_4'(x_0) t^{-1/2} \quad (t \ge 0).$$

We observe that in the proof, the assumption $\sigma^2 > 0$ is required only for the upper bounds in the previous inequalities.

Proof. We recall that the branching mechanism associated with the CSBP Z satisfies, for every $\lambda \ge 0$

$$\psi(\lambda) = -g\lambda + \sigma^2\lambda^2 + \int_0^\infty \left(e^{-\lambda x} - 1 + \lambda x\right) \mu(\mathrm{d}x).$$

So

$$\psi''(\lambda) = 2\sigma^2 + \int_{(0,\infty)} x^2 e^{-\lambda x} \mu(dx),$$

and for every $\lambda \ge 0$

$$2\sigma^2 \le \psi''(\lambda) \le 2\sigma^2 + \int_{(0,\infty)} x^2 e^{-\lambda x} \mu(dx).$$

[2] We exclude the degenerated case $\nu = 0$, $g = 0$.

Since $c := \int_0^\infty x^2\mu(dx) < \infty$, then ψ'' is continuous over $[0,\infty)$. By Taylor Lagrange theorem, we get for every $\lambda \geq 0$, $\psi_-(\lambda) \leq \psi(\lambda) \leq \psi_+(\lambda)$, where

$$\psi_-(\lambda) = \lambda\psi'(0) + \sigma^2\lambda^2 \quad \text{and} \quad \psi_+(\lambda) = \lambda\psi'(0) + (\sigma^2 + c/2)\lambda^2.$$

We first consider the case $\nu(0,\infty) < \infty$, so that Δ has a finite number of jumps on each compact interval a.s. We introduce the CSBP's with catastrophes Y_- and Y_+ which are both associated with the same catastrophes as Y, Δ, but respectively with the CSBP $(g,\sigma^2,0)$ and $(g,\sigma^2+c/2,0)$. We denote $u_{-,t}$ and $u_{+,t}$ their Laplace exponent, i.e. for all $(\lambda,t) \in \mathbb{R}_+^2$,

$$\mathbb{E}\Big[\exp\{-\lambda Y_t^-\}\Big] = \exp\{-u_{-,t}(\lambda)\}, \qquad \mathbb{E}\Big[\exp\{-\lambda Y_t^+\}\Big] = \exp\{-u_{+,t}(\lambda)\}.$$

Thus conditionally on Δ, for every time t such that $\Delta_t = \Delta_{t-}$, by Theorem 1,

$$u'_{-,t}(\lambda) = -\psi_-(u_{-,t}), \qquad u'_{+,t}(\lambda) = -\psi_+(u_{+,t}), \qquad u'_t(\lambda) = -\psi(u_t).$$

Moreover for every t such that $\theta_t = \exp\{\Delta_t - \Delta_{t-}\} > 0$, it is clear that

$$\frac{u_{-,t}(\lambda)}{u_{-,t-}(\lambda)} = \frac{u_t(\lambda)}{u_{t-}(\lambda)} = \frac{u_{+,t}(\lambda)}{u_{+,t-}(\lambda)} = \theta_t,$$

and $u_{-,0}(\lambda) = u_0(\lambda) = u_{+,0}(\lambda) = \lambda$. So for all t,λ,

$$u_{+,t}(\lambda) \leq u(t,\lambda) \leq u_{-,t}(\lambda).$$

Now we generalize this inequality to the case $\nu(0,\infty) \in [0,\infty]$ by successive approximations. With this purpose, let $A^{\epsilon_1,\epsilon_2} = (0,1-\varepsilon_1)\cup(1+\varepsilon_2,\infty)$, where $0 < 1-\varepsilon_1 < 1 < 1+\varepsilon_2$ and define the Poisson random measure $N_1^{\varepsilon_1,\varepsilon_2}$ as the restriction of N_1 to $A^{\epsilon_1,\epsilon_2}\times\mathbb{R}^+$. We denote by $\mathrm{d}t\nu^{\varepsilon_1,\varepsilon_2}(\mathrm{d}x)$ its intensity measure, where $\nu^{\varepsilon_1,\varepsilon_2}(\mathrm{d}x) = \mathbf{1}_{\{x\in A^{\epsilon_1,\epsilon_2}\}}\nu(\mathrm{d}x)$, and the corresponding Lévy process $\Delta^{\varepsilon_1,\varepsilon_2}$ defined by

$$\Delta_t^{\varepsilon_1,\varepsilon_2} = \int_0^t \int_{(0,\infty)} \log(x) N_1^{\varepsilon_1,\varepsilon_2}(\mathrm{d}s,\mathrm{d}x).$$

We also consider the CSBP's $Y^{\varepsilon_1,\varepsilon_2}$ (resp $Y^{\varepsilon_1,\varepsilon_2,-}$ and $Y^{\varepsilon_1,\varepsilon_2,+}$) with branching mechanism ψ (resp. ψ_- and ψ_+) and catastrophes $\Delta^{\varepsilon_1,\varepsilon_2}$ via (6). Since $\nu^{\varepsilon_1,\varepsilon_2}(0,\infty) < \infty$, from the first step we have $u_{+,t}^{\varepsilon_1,\varepsilon_2}(\lambda) \leq u^{\varepsilon_1,\varepsilon_2}(t,\lambda) \leq u_{-,t}^{\varepsilon_1,\varepsilon_2}(\lambda)$, where as expected $\mathbb{E}[\exp\{-\lambda Y_t^{\varepsilon_1,\varepsilon_2,*}\}] = \exp\{-u_{*,t}^{\varepsilon_1,\varepsilon_2}(\lambda)\}$ for each $* \in \{+,\emptyset,-\}$.

Similarly, let $A^{\epsilon_1} = (0,1-\varepsilon_1)\cup(1,\infty)$ and define the Poisson random measure $N_1^{\varepsilon_1}$ as the restriction of N_1 to $A^{\epsilon_1}\times\mathbb{R}^+$ whose intensity measure is given by $\mathrm{d}t\nu^{\varepsilon_1}(\mathrm{d}x)$, where $\nu^{\varepsilon_1}(\mathrm{d}x) = \mathbf{1}_{\{x\in A^{\epsilon_1}\}}\nu(\mathrm{d}x)$. Let us fix t in $\mathbb{R}_+^*$, and define Y^{ε_1} the unique strong solution of

$$\begin{aligned} Y_t^{\varepsilon_1} = Y_0 + \int_0^t gY_s^{\varepsilon_1}\mathrm{d}s + \int_0^t \sqrt{2\sigma^2 Y_s^{\varepsilon_1}}\mathrm{d}B_s + \int_0^t\int_{[0,\infty)}\int_0^{Y_{s-}^{\varepsilon_1}} z\widetilde{N}_0(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\ + \int_0^t\int_{[0,\infty)}\Big(z-1\Big)Y_{s-}^{\varepsilon_1}N_1^{\varepsilon_1}(\mathrm{d}s,\mathrm{d}z). \end{aligned} \tag{20}$$

We already know from Theorem 1 that equation (20) has a unique non negative strong solution. Moreover, from Theorem 5.5 in [FL10] and the fact that $N_1^{\varepsilon_1}$ has the same jumps as $N_1^{\varepsilon_1,\varepsilon_2}$ plus additional jumps greater than one, we know that

$$Y_t^{\varepsilon_1,\varepsilon_2} \leq Y_t^{\varepsilon_1}, \qquad \text{a.s.}$$

Using assumption (5), we can apply Gronwall Lemma to the non negative function $t \mapsto \mathbb{E}[Y_t^{\varepsilon_1} - Y_t^{\varepsilon_1,\varepsilon_2}]$ and obtain

$$\mathbb{E}\Big[\big|Y_t^{\varepsilon_1,\varepsilon_2} - Y_t^{\varepsilon_1}\big|\Big] \xrightarrow[\varepsilon_2 \to 0]{} 0.$$

Adding that $Y^{\varepsilon_1,\varepsilon_2}$ is decreasing with ε_2, we finally get, $Y_t^{\varepsilon_1,\varepsilon_2} \xrightarrow{a.s.} Y_t^{\varepsilon_1}$, as $\varepsilon_2 \to 0$. Using similar arguments as above for $Y^{\varepsilon_1,\varepsilon_2,+}$ and $Y^{\varepsilon_1,\varepsilon_2,-}$, we deduce

$$u_{+,t}^{\varepsilon_1}(\lambda) \leq u^{\varepsilon_1}(t,\lambda) \leq u_{-,t}^{\varepsilon_1}(\lambda).$$

Thus, when ε_1 goes to 0, we finally obtain $u_{+,t}(\lambda) \leq u(t,\lambda) \leq u_{-,t}(\lambda)$. This implies, taking $\lambda \to \infty$, that

$$\mathbb{P}(Y_t^+ > 0) \leq \mathbb{P}(Y_t > 0) \leq \mathbb{P}(Y_t^- > 0).$$

The result now follows from the asymptotic behavior of $\mathbb{P}(Y_t^- > 0)$ and $\mathbb{P}(Y_t^+ > 0)$. $\square$

4 Application to a cell division model

When the reproduction law has a finite second moment, the scaling limit of the Galton Watson process is a Feller diffusion with growth g and diffusion part σ^2. It yields the stable case with $\beta = 1$ and additional drift term g. Such process is also the scaling limit of birth and death process. It gives a natural model for populations which die and multiply fast, randomly, without interaction. Such a model is considered in [BT11] for parasites growing in (dividing) cells. In this model, the cell divides at constant rate r and a random fraction Θ in $(0,1)$ of parasites goes in the first daughter cell, whereas the rest goes in the second daughter cell. Following the infection in a cell line, the parasites grow as a Feller diffusion process and undergo a catastrophe (at finite rate) when the cell divides. If there is one infected cell at time 0, the numbers N_t of cells and N_t^* of infected cells at time t satisfy $\mathbb{E}[N_t] = e^{rt}$ and $\mathbb{E}[N_t^*] = e^{rt}\mathbb{P}(Y_t > 0)$, where

$$Y_t = 1 + \int_0^t gY_s ds + \int_0^t \sqrt{2\sigma^2 Y_s} dB_s + \int_0^t \int_0^1 (\theta - 1) Y_{s_-} \rho(ds, d\theta). \tag{21}$$

Here B is a Brownian motion and $\rho(ds, d\theta)$ a Poisson Point measure with intensity $2rds\mathbb{P}(\Theta \in d\theta)$. Then Theorem 5 and Corollary 2 directly ensure the following result.

Corollary 7. *a/ We assume that $g < 2r\mathbb{E}\left[\log(1/\Theta)\right]$. Then there exist positive constants c_1, c_2, c_3 such that*

(i) If $g < 2r\mathbb{E}\left[\Theta \log(1/\Theta)\right]$, then

$$\mathbb{E}\left[N_t^*\right] \sim c_1 e^{gt}, \quad as \quad t \to \infty.$$

(ii) If $g = 2r\mathbb{E}\left[\Theta \log(1/\Theta)\right]$, then

$$\mathbb{E}\left[N_t^*\right] \sim c_2 t^{-1/2} e^{gt}, \quad as \quad t \to \infty.$$

(iii) If $g > 2r\mathbb{E}\left[\Theta \log(1/\Theta)\right]$, then

$$\mathbb{E}\left[N_t^*\right] \sim c_3 t^{-3/2} e^{\alpha t}, \quad as \quad t \to \infty.$$

where $\alpha = \min_{\lambda \in [0,1]}\{g\lambda + 2r(\mathbb{E}[\Theta^\lambda] - 1/2)\} < g$.

b/ We assume now $g = 2r\mathbb{E}[\log(1/\Theta)]$, then there exists $c_4 > 0$ such that,

$$\mathbb{E}[N_t^*] \sim c_4 t^{-1/2} e^{rt}, \quad as \quad t \to \infty.$$

c/ Finally, if $g > 2r\mathbb{E}[\log(1/\Theta)]$, then there exists $0 < c_5 < 1$ such that,

$$\mathbb{E}[N_t^*] \sim c_5 e^{rt}, \quad as \quad t \to \infty.$$

Hence if $g > 2r\mathbb{E}[\log(1/\Theta)]$ (supercritical case c/), the mean number of infected cells is equivalent to $\exp(rt)$, which is the mean number of cells. In the critical case (b/), there are a bit less infected cells, owing to the additional square root term. In the strongly subcritical regime (a/ (i)), the mean number of infected cells is of the same order as the number of parasites. It let us think that parasites do not accumulate in some infected cells. The asymptotic behavior in the two remaining cases is more complex.

We stress on the fact that fixing the growth rate g of parasites but making the law of the repartition Θ vary make change the asymptotic behavior of the number of infected cells (when $\mathbb{E}[\Theta\log(1/\Theta)]$ goes beyond $g/2r$). For example, let us fix the growth rate of parasites g and the rate of division r of the cells such that $g < r\log 2$. It yields $g < 2r\mathbb{E}[\Theta\log(1/\Theta)]$ for $\Theta = 1/2$ a.s. We make now increase the entropy of the repartition of parasites $\mathbb{E}[\Theta\log(1/\Theta)]$ (the function $x \mapsto x\ln(1/x)$ is strictly concave on $[0,1]$ and null at the boundaries). In the domain $g < 2r\mathbb{E}[\Theta\log(1/\Theta)]$, the growth rate of the number of infected cells is unchanged. A threshold appears when $\mathbb{E}[\Theta\log(1/\Theta)] = g/2r$. Then, this growth rate decreases and the number of infected cells grows slower and slower. Such phenomena have already been observed in the discrete time, discrete space framework in [Ban08].

The results can also be extend to a growth of parasites which follows a stable CSBP. Such model appears when renormalizing a discrete model where the reproduction law of parasites has an heavy tail.

5 Local limit theorem for some functionals of Lévy processes

We consider a Lévy process $K = (K_t, t \geq 0)$ of the form

$$K_t = \gamma t + \sigma_t^{(+)} - \sigma_t^{(-)}, \qquad t \geq 0, \tag{22}$$

where γ is a real constant, $\sigma^{(+)}$ and $\sigma^{(-)}$ are two independent subordinators without drift. We denote by Π, $\Pi^{(+)}$ and $\Pi^{(-)}$ for the associated Lévy measures of K, $\sigma^{(+)}$ and $\sigma^{(-)}$.

Our aim, in this section, is to determine the asymptotic behavior of the distribution of the exponential functional associated to K,

$$\int_0^t \exp\{-\beta K_s\}\mathrm{d}s,$$

where β belongs to $(0,1]$.
Let us define the Laplace exponents of K, $\sigma^{(+)}$ and $\sigma^{(-)}$ by

$$\phi_K(\lambda) = \log\mathbb{E}[e^{\lambda K_1}], \quad \phi_+(\lambda) = \log\mathbb{E}\Big[e^{\lambda\sigma_1^{(+)}}\Big] \quad \text{and} \quad \phi_-(\lambda) = \log\mathbb{E}\Big[e^{-\lambda\sigma_1^{(-)}}\Big], \tag{23}$$

and assume that

$$\theta_{max} = \sup\left\{\lambda \in \mathbb{R}^+, \int_{[1,\infty)} e^{\lambda x}\Pi^{(+)}(\mathrm{d}x) < \infty\right\} > 0. \tag{24}$$

From the Lévy-Khintchine formula, we deduce

$$\phi_K(\lambda) = \gamma\lambda + \int_{(0,\infty)} \Big(e^{\lambda x} - 1\Big)\Pi^{(+)}(\mathrm{d}x) + \int_{(0,\infty)} \Big(e^{-\lambda x} - 1\Big)\Pi^{(-)}(\mathrm{d}x).$$

Finally, we assume that $\mathbb{E}(K_1^2) < \infty$, which is equivalent to

$$\int_{(-\infty,\infty)} x^2\Pi(dx) < \infty. \tag{25}$$

We are interested in the asymptotic behavior at infinity of

$$a_F(t) := \mathbb{E}\left[F\left(\int_0^t \exp\{-\beta K_s\}\mathrm{d}s\right)\right],$$

where F belongs to a particular class of functions on $\mathbb{R}_+$. We will focus on functions which decrease polynomially at infinity (with exponent $-1/\beta$) in order to get a fine estimate which yields the four asymptotic regimes in Theorem 5.

Proposition 8. *Assume that F is a positive non increasing function such that for $x \geq 0$*

$$F(x) = C_F(x+1)^{-1/\beta}\Big[1 + (1+x)^{-\varsigma}h(x)\Big], \tag{26}$$

where $\varsigma \geq 1$, C_F is a positive constant, and h is a Lipschitz function which is bounded. Then we have the four following regimes

a/ (Subcritical case) If $\phi'_K(0) < 0$

(i) If $\theta_{max} > 1$ and $\phi'_K(1) < 0$, then there exists a positive constant c_1 such that

$$a_F(t) \sim c_1 e^{t\phi_K(1)}, \quad \textit{as} \quad t \to \infty.$$

(ii) If $\theta_{max} > 1$ and $\phi'_K(1) = 0$, then there exists a positive constant c_2 such that

$$a_F(t) \sim c_2 t^{-1/2} e^{t\phi_K(1)}, \quad \textit{as} \quad t \to \infty.$$

(iii) If $\theta_{max} > \beta + 1$ and $\phi'_K(1) > 0$, then there exists a positive constant c_3 such that

$$a_F(t) \sim c_3 t^{-3/2} e^{t\phi_K(\tau)}, \quad \textit{as} \quad t \to \infty,$$

where τ is the root of ϕ'_K on $]0,1[$.

b/ (Critical case) If $\theta_{max} > \beta$ and $\phi'_K(0) = 0$, then there exists a positive constant c_4 such that

$$a_F(t) \sim c_4 t^{-1/2}, \quad \textit{as} \quad t \to \infty.$$

This result is proved in Section 5.4. It generalizes Lemma 4.7 in Carmona et al. [CPY97] in the case when the process K has paths of bounded variation. We also note that the result of Carmona et al. does not have a precise asymptotic behavior when $\phi'_K(1) \geq 0$.

The behavior of $a_F(t)$ is determined by the shape of ϕ_K, which is the Laplace exponent of K. This function is convex and zero at the origin and four different regimes appear according to the sign of ϕ_K' on $[0,1]$.

The assumption on the tail of F as $x \to \infty$ is finely used to get the asymptotic behavior of $a_F(t)$. Lemma 21 gives the properties of F which are required in the proof.

The strongly subcritical case (case (i)) is proven using a continuous time change of measure. For the other cases, we divide the proof in three steps. The first step consists in discretizing the integral $\int_0^t e^{-\beta K_s} \mathrm{d}s$ by introducing for every $(p,q) \in \mathbb{N} \times \mathbb{N}^*$ the random variable

$$A_{p,q} = \sum_{i=0}^{p} \exp\{-\beta K_{i/q}\} = \sum_{i=0}^{p} \prod_{j=0}^{i-1} \exp\Big\{ -\beta\big(K_{(j+1)/q} - K_{j/q}\big)\Big\}. \tag{27}$$

Secondly, we study the asymptotic behavior of the discretized expectation :

$$F_{p,q} := \mathbb{E}\Big[F\Big(A_{p,q}/q\Big)\Big] \quad (q \in \mathbb{N}^*), \tag{28}$$

when p goes to infinity. This step relies on Theorem 2.1 in [GL01], which is a limit theorem for random walks on an affine group and generalizes theorems A and B in [LPP97].
Finally, we prove that the limit of $F_{\lfloor qt \rfloor,q}$, when $q \to \infty$, and $a_F(t)$ both have the same asymptotic behavior when t goes to infinity.

Remark 1. Let us mention limit behaviors in the discrete setting, which have comparable forms as our results. A BPRE $(X_n, n \in \mathbb{N})$ is an integer valued branching process, specified by a sequence of generating functions $(f_n, n \in \mathbb{N})$. Conditionally on the environment, individuals reproduce independently of each other and the offsprings of an individual at generation n has generating function f_n. We present the results of Theorem 1.1 in [GK00] and Theorems 1.1, 1.2 and 1.3 in [GKV03] (to lighten the presentation, we do not specify here the moment conditions).
In the *subcritical case*, i.e. $\mathbb{E}[\log(f_0'(1))] < 0$, we have the following three asymptotic regimes when n goes to ∞,

$$\mathbb{P}(X_n > 0) \sim c a_n, \qquad \text{as} \qquad n \to \infty,$$

for some positive constant c and

$$a_n = \mathbb{E}\Big[f_0'(1)\Big]^n, \quad a_n = n^{-1/2}\mathbb{E}\Big[f_0'(1)\Big]^n \quad \text{and} \quad a_n = n^{-3/2}\left(\min_{0<s<1} \mathbb{E}\Big[(f_0'(1))^s\Big]\right)^n,$$

if respectively $\mathbb{E}[f_0'(1)\log(f_0'(1))]$ is negative, zero or positive.
In the *critical case*, i.e. $\mathbb{E}[\log(f_0'(1))] = 0$, we have

$$\mathbb{P}(X_n > 0) \sim c n^{-1/2}, \qquad \text{as} \qquad n \to \infty,$$

for some positive constant c.

5.1 Discretization of the Lévy process

The following result, which is a direct consequence from the definition of Lévy processes, allows us to concentrate our attention on $A_{p,q}$, which was defined in (27).

Lemma 9. *Let $t \geq 1$ and $q \in \mathbb{N}^*$. There exist two random variables $C_{\lfloor qt \rfloor, q}$ and $D_{\lfloor qt \rfloor -1, q}$ such that*

$$\frac{1}{q} e^{-\beta|\gamma|/q} D_{\lfloor qt \rfloor -1, q} \leq \int_0^t e^{-\beta K_s} \mathrm{d}s \leq \frac{1}{q} e^{\beta|\gamma|/q} C_{\lfloor qt \rfloor, q},$$

and for every $(p,q) \in \mathbb{N} \times \mathbb{N}^*$,

$$D_{p,q} \overset{(d)}{=} U_{1/q}^{\beta} A_{p,q} \qquad \textit{and} \qquad C_{p,q} \overset{(d)}{=} V_{1/q}^{\beta} A_{p,q},$$

where the couple of r.v. $(U_{1/q}, V_{1/q})$ is independent of $A_{p,q}$ and has the same distribution as $(e^{-\sigma^{(+)}_{1/q}}, e^{\sigma^{(-)}_{1/q}})$.

Proof. Let (p,q) be in $\mathbb{N} \times \mathbb{N}^*$ and s in $[\frac{p}{q}, \frac{p+1}{q}]$. Then

$$K_s \leq K_{p/q} + |\gamma|/q + [\sigma^{(+)}_{(p+1)/q} - \sigma^{(+)}_{p/q}] \quad \text{and} \quad K_s \geq K_{p/q} - |\gamma|/q - [\sigma^{(-)}_{(p+1)/q} - \sigma^{(-)}_{p/q}]. \tag{29}$$

Now introduce

$$K^{(1)}_{p/q} = K_{p/q} + [\sigma^{(+)}_{(p+1)/q} - \sigma^{(+)}_{p/q}] - \sigma^{(+)}_{1/q} \quad \text{and} \quad K^{(2)}_{p/q} = K_{p/q} - [\sigma^{(-)}_{(p+1)/q} - \sigma^{(-)}_{p/q}] + \sigma^{(-)}_{1/q}.$$

Then, we have for all $(p,q) \in \mathbb{N} \times \mathbb{N}^*$

$$(K_0, K_{1/q}, ..., K_{p/q}) \overset{(d)}{=} (K^{(1)}_0, K^{(1)}_{1/q}, ..., K^{(1)}_{p/q}) \overset{(d)}{=} (K^{(2)}_0, K^{(2)}_{1/q}, ..., K^{(2)}_{p/q}).$$

Moreover, $(K^{(1)}_0, K^{(1)}_{1/q}, ..., K^{(1)}_{p/q})$ is independent of $\sigma^{(+)}_{1/q}$ and $(K^{(2)}_0, K^{(2)}_{1/q}, ..., K^{(2)}_{p/q})$ is independent of $\sigma^{(-)}_{1/q}$. Finally, the inequalities in (29) lead to,

$$\frac{1}{q} e^{-\beta(|\gamma|/q + \sigma^{(+)}_{1/q})} \sum_{i=0}^{\lfloor qt \rfloor -1} e^{-\beta K^{(1)}_{i/q}} \leq \int_0^t e^{-\beta K_s} \mathrm{d}s \leq \frac{1}{q} e^{\beta(|\gamma|/q + \sigma^{(-)}_{1/q})} \sum_{i=0}^{\lfloor qt \rfloor} e^{-\beta K^{(2)}_{i/q}},$$

which ends the proof. □

5.2 Asymptotical behavior of the discretized process

We first recall Theorem 2.1 in [GL01], in the case where test functions do not vanish. This result is the key to obtain the asymptotical behavior of the discretized process.

Theorem 10 (Giuvarc'h, Liu 01). *Let $(a_n, b_n)_{n \geq 0}$ be a $(\mathbb{R}_+^*)^2$ valued sequence of iid random variables such that $\mathbb{E}[\log(a_0)] = 0$. Assume that $b_0/(1-a_0)$ is not a.s. constant and define $A_0 = 1$, $A_n = \prod_{k=0}^{n-1} a_k$ and $B_n = \sum_{k=0}^{n-1} A_i b_i$, for $n \geq 1$. Let η, κ, λ be three positive numbers such that $\kappa < \lambda$, and $\tilde{\phi}$ and $\tilde{\psi}$ be two positive continuous functions on $\mathbb{R}_+$ such that they do not vanish and for a constant $C > 0$ and for every $a > 0$, $b \geq 0$, $b' \geq 0$, we have*

$$\tilde{\phi}(a) \leq C a^{\kappa}, \quad \tilde{\psi}(b) \leq \frac{C}{(1+b)^{\lambda}}, \quad \textit{and} \quad |\tilde{\psi}(b) - \tilde{\psi}(b')| \leq C|b-b'|^{\eta}.$$

Moreover, assume that

$$\mathbb{E}\left[a_0^{\kappa}\right] < \infty, \quad \mathbb{E}\left[a_0^{-\eta}\right] < \infty, \quad \mathbb{E}\left[b_0^{\eta}\right] < \infty \quad \textit{and} \quad \mathbb{E}\left[a_0^{-\eta} b_0^{-\eta}\right] < \infty.$$

Then there exist two positive constants $c(\tilde{\phi}, \tilde{\psi})$ and $c(\tilde{\psi})$ such that

$$\lim_{n \to \infty} n^{3/2} \mathbb{E}\left[\tilde{\phi}(A_n) \tilde{\psi}(B_n)\right] = c(\tilde{\phi}, \tilde{\psi}) \qquad \textit{and} \qquad \lim_{n \to \infty} n^{1/2} \mathbb{E}\left[\tilde{\psi}(B_n)\right] = c(\tilde{\psi}).$$

Recall the definition of $A_{p,q}$ and $F_{p,q}$ in (26) to (28). The three following Lemmas study the asymptotic behavior of their expectations in the regimes (ii), (iii) and b/.

Lemma 11. *Assume that $|\phi_K'(0+)|<\infty$, $\theta_{max}>1$ and $\phi_K'(1)=0$. Then there exists a positive and finite constant $c_2(q)$ such that,*

$$F_{p,q} \sim C_F c_2(q)(p/q)^{-1/2}e^{(p/q)\phi_K(1)}, \quad \textit{as} \quad p\to\infty, \tag{30}$$

and

$$\mathbb{E}\left[(A_{p,q}/q)^{-1/\beta}\right] \sim c_2(q)(p/q)^{-1/2}e^{(p/q)\phi_K(1)}, \quad \textit{as} \quad p\to\infty. \tag{31}$$

Proof. Let introduce the exponential change of measure known as the Escheer transform

$$\left.\frac{\mathrm{d}\mathbb{P}^{(\lambda)}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\lambda K_t - \phi_K(\lambda)t} \qquad \text{for } \lambda\in[0,\theta_{max}[, \tag{32}$$

where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated by K which is naturally completed.

The following equality in law

$$A_{p,q} = e^{-\beta K_{p/q}}\Big(\sum_{i=0}^{p} e^{\beta(K_{p/q}-K_{i/q})}\Big) \overset{(d)}{=} e^{-\beta K_{p/q}}\Big(\sum_{i=0}^{p} e^{\beta K_{i/q}}\Big),$$

leads to $e^{-(p/q)\phi_K(1)}\mathbb{E}\left[A_{p,q}^{-1/\beta}\right] = \mathbb{E}^{(1)}\left[\tilde{A}_{p,q}^{-1/\beta}\right]$, where $\tilde{A}_{p,q}=\sum_{i=0}^{p} e^{\beta K_{i/q}}$. Let $\varepsilon>0$ such that (47) holds and observe that $\tilde{A}_{p,q}\geq 1$ for every (p,q) in $\mathbb{N}\times\mathbb{N}^*$. Thus,

$$\mathbb{E}^{(1)}\left[\tilde{A}_{p,q}^{-(1+\varepsilon)/\beta}\right] \leq \mathbb{E}^{(1)}\left[\tilde{A}_{p,q}^{-1/\beta}\right] \leq \mathbb{E}^{(1)}\left[\inf_{i\in[0,p]\cap\mathbb{N}} e^{-K_{i/q}}\right].$$

Since $\phi_K'(1)=0$ and $\mathbb{E}[K_{1/q}^2]<\infty$, Theorem A in [Koz76] yields

$$\mathbb{E}^{(1)}\left[\inf_{i\in[0,p]\cap\mathbb{N}} e^{-K_{i/q}}\right] \sim C_q(p/q)^{-1/2}, \quad \text{as} \quad p\to\infty,$$

where C_q is a positive finite constant. Define for $z\geq 1$

$$D_q(z,p) = (p/q)^{1/2}\mathbb{E}^{(1)}\left[\tilde{A}_{p,q}^{-z/\beta}\right].$$

Note that there exists $p_0\in\mathbb{N}$ such that for $p\geq p_0$, $D_q(1,p)\leq 2C_q$.

Our aim is to prove that $D_q(1,p)$ converges to a finite positive constant $d_2(q)$. Then, we introduce an arbitrary $x\in(0,(C_F/M)^{1/\varepsilon}q^{-1/\beta})$ and apply Theorem 10 with

$$\tilde{\psi}(z)=F(z), \quad \tilde{\phi}(z)=z^{1/(2\beta)}, \qquad (\eta,\kappa,\lambda)=(1,1/(2\beta),1/\beta).$$

Under the probability measure $\mathbb{P}^{(1)}$, $(a_n,b_n)_{n\geq 0} = (e^{\beta(K_{(n+1)/q}-K_{n/q})}, x^{-\beta}q^{-1})_{n\geq 0}$, is an iid sequence of random variables such that $\mathbb{E}^{(1)}[\log(a_0)]=0$, as $\phi_K'(1)=0$. A simple computation gives us

$$\mathbb{E}^{(1)}[a_0^{-1}] = e^{(\phi_K(1-\beta)-\phi_K(1))/q} < \infty.$$

This implies that the moment conditions of Theorem 10 are satisfied. In this case,

$$B_n = q^{-1}x^{-\beta}\sum_{i=0}^{n-1} e^{\beta K_{i/q}}, \quad n\in\mathbb{N}^*.$$

Thus, there exists a positive real number $b(q,x)$ such that

$$(p/q)^{1/2}\mathbb{E}^{(1)}\left[F\Big(x^{-\beta}\tilde{A}_{p,q}/q\Big)\right]\to b(q,x),\quad \text{as}\quad p\to\infty.$$

Now, we define $\liminf_{n\to\infty} D_q(1,n)=\underline{D}_q$ and $\limsup_{n\to\infty} D_q(1,n)=\overline{D}_q$. Taking expectation in (47) yields

$$\left|(p/q)^{1/2}\mathbb{E}^{(1)}\left[F\Big(x^{-\beta}\tilde{A}_{p,q}/q\Big)\right]-C_F x q^{1/\beta}D_q(1,p)\right|\le Mx^{1+\varepsilon}q^{(1+\varepsilon)/\beta}D_q(1+\varepsilon,p).$$

Let n_k and m_k be two increasing subsequences in $\mathbb{N}$ such that,

$$D(1,n_k)\underset{k\to\infty}{\longrightarrow}\underline{D}_q\qquad\text{and}\qquad D_q(1,m_k)\underset{k\to\infty}{\longrightarrow}\overline{D}_q.$$

As $D_q(z,p)$ is decreasing with respect to z, we have for all k in $\mathbb{N}$,

$$(C_F x q^{1/\beta}+Mx^{1+\varepsilon}q^{(1+\varepsilon)/\beta})D_q(1,n_k)\ge (n_k/q)^{1/2}\mathbb{E}^{(1)}\left[F\Big(x^{-\beta}\tilde{A}_{n_k,q}/q\Big)\right],$$

and

$$(C_F x q^{1/\beta}-Mx^{1+\varepsilon}q^{(1+\varepsilon)/\beta})D_q(1,m_k)\le (m_k/q)^{1/2}\mathbb{E}^{(1)}\left[F\Big(x^{-\beta}\tilde{A}_{m_k,q}/q\Big)\right].$$

This implies, taking $k\to\infty$, that

$$\underline{D}_q\ge\frac{b(q,x)}{C_F x q^{1/\beta}+Mx^{1+\varepsilon}q^{(1+\varepsilon)/\beta}}>0,\quad \overline{D}_q\le\frac{b(q,x)}{C_F x q^{1/\beta}-Mx^{1+\varepsilon}q^{(1+\varepsilon)/\beta}}<\infty,$$

and

$$\overline{D}_q-\underline{D}_q\le\frac{4MC_q x^{\varepsilon}q^{(1+\varepsilon)/\beta}}{C_F}.$$

Finally, letting $x\to 0$, we get that $D_q(1,p)$ converges to a finite positive constant $d_2(q)$, which gives (31).

Using (47), we get

$$\mathbb{E}\Big|F_{p,q}-C_F\left(A_{p,q}/q\right)^{-1/\beta}\Big|\le\mathbb{E}\left[(A_{p,q}/q)^{-(1+\varepsilon)/\beta}\right],$$

so (30) will be proved as soon as

$$\mathbb{E}\Big[A_{p,q}^{-(1+\varepsilon)/\beta}\Big]=o\left(\mathbb{E}\left[A_{p,q}^{-1/\beta}\right]\right),\quad\text{as}\quad p\to\infty.$$

From the Escheer transform (32), with $\lambda=1+\varepsilon$, and the independent increments of K, we have

$$\begin{aligned}
\mathbb{E}\Big[A_{p,q}^{-(1+\varepsilon)/\beta}\Big] &= e^{(p/q)\phi_K(1)}\mathbb{E}^{(1)}\Big[\Big(\sum_{i=0}^{p}e^{-\beta K_{i/q}}\Big)^{-\varepsilon/\beta}\Big(\sum_{i=0}^{p}e^{\beta(K_{p/q}-K_{i/q})}\Big)^{-1/\beta}\Big]\\
&\le e^{(p/q)\phi_K(1)}\mathbb{E}^{(1)}\Big[\inf_{0\le i\le\lfloor p/3\rfloor}e^{\varepsilon K_{i/q}}\inf_{\lfloor 2p/3\rfloor\le j\le p}e^{-(K_{p/q}-K_{j/q})}\Big]\\
&= e^{(p/q)\phi_K(1)}\mathbb{E}^{(1)}\Big[\inf_{0\le i\le\lfloor p/3\rfloor}e^{\varepsilon K_{i/q}}\Big]\mathbb{E}^{(1)}\Big[\inf_{0\le j\le\lfloor p/3\rfloor}e^{-K_{j/q}}\Big].
\end{aligned}$$

Using (25), we observe that $\mathbb{E}^{(1)}(K_{1/q}) = 0$ and $\mathbb{E}^{(1)}(K_{1/q}^2) < \infty$. We can then apply Theorem A in [Koz76] to the random walks $(-K_{i/q})_{i\geq 1}$ and $(\varepsilon K_{i/q})_{i\geq 1}$. Therefore, there exists $c > 0$ such that

$$\mathbb{E}\left[A_{p,q}^{-(1+\varepsilon)/\beta}\right] \leq (c/p)e^{(p/q)\phi_K(1)} = o\left(\mathbb{E}\left[A_{p,q}^{-1/\beta}\right]\right), \quad \text{as} \quad p \to \infty.$$

Taking $c_2(q) = d_2(q)q^{1/\beta}$ leads to the result. □

Remark 2. In the particular case when $\beta = 1$, it is enough to apply Theorem 1.2 in [GKV03] to a geometric BPRE $(X_n, n \geq 0)$ whose p.g.f's satisfy

$$f_n(s) = \sum_{k=0}^{\infty} p_n q_n^k s^k = \frac{p_n}{1 - q_n s},$$

with $1/p_n = 1 + \exp\left\{\beta\left(K_{(n+1)/q} - K_{n/q}\right)\right\}$, $q_n = 1 - p_n$. Using $\mathbb{E}(A_{p,q}^{-1}) = \mathbb{P}(X_p > 0)$ and $\log f_0'(1) = K_{1/q}$, allows to get the asymptotic behavior of $\mathbb{E}(A_{p,q}^{-1})$ from the speed of extinction of BPRE in the case of geometric reproduction law (but we need the extra assumption $\phi_K(2) < \infty$).

Recall that τ is the root of ϕ_K' on $]0, 1[$, i.e. $\phi_K(\tau) = \min_{0<s<1}\phi_K(s)$.

Lemma 12. *Assume that $\phi_K'(0+) < 0$, $\phi_K'(1) > 0$ and $\beta + 1 < \theta_{max}$. Then there exist two positive and finite constants $d(q)$ and $c_3(q)$ such that*

$$F_{p,q} \quad \sim \quad c_3(q)(p/q)^{-3/2}e^{(p/q)\phi_K(\tau)}, \quad \text{as} \quad p \to \infty \tag{33}$$

and

$$\mathbb{E}\left[(A_{p,q}/q)^{-1/\beta}\right] \quad \sim \quad d(q)(p/q)^{-3/2}e^{(p/q)\phi_K(\tau)}, \quad \text{as} \quad p \to \infty. \tag{34}$$

Proof. To prove these asymptotic behaviors, we apply Theorem 10 with,

$$\tilde{\psi}(z) = F(z), \quad \tilde{\phi}(z) = z^{\tau/\beta} \quad z \geq 0, \qquad (\eta, \kappa, \lambda) = (1, \tau/\beta, 1/\beta).$$

Under the probability measure $\mathbb{P}^{(\tau)}$, $(a_n, b_n)_{n\geq 0} = (e^{-\beta(K_{(n+1)/q} - K_{n/q})}, q^{-1})_{n\geq 0}$, is an iid sequence of random variables such that $\mathbb{E}^{(\tau)}[\log(a_0)] = 0$, as $\phi_K'(\tau) = 0$. The moment conditions

$$\mathbb{E}^{(\tau)}\left[a_0^{\tau/\beta}\right] = e^{-\phi_K(\tau)/q} < \infty \qquad \text{and} \qquad \mathbb{E}^{(\tau)}\left[a_0^{-1}\right] = e^{(\phi_K(\beta+\tau) - \phi_K(\tau))/q} < \infty,$$

enable us to apply Theorem 10. In this case,

$$B_n = q^{-1}\sum_{i=0}^{n-1} e^{-\beta K_{i/q}}, \quad n \in \mathbb{N}^*$$

Then there exists $c_3(q) > 0$ such that

$$\mathbb{E}\left[F(A_{p,q}/q)\right]e^{-(p/q)\phi_K(\tau)} = \mathbb{E}^{(\tau)}\left[F(A_{p,q}/q)e^{-\tau K_{p/q}}\right] \sim \quad c_3(q)(p/q)^{-3/2}, \quad \text{as} \quad p \to \infty.$$

This gives (33).

In order to prove

$$\mathbb{E}\left[(A_{p,q}/q)^{-1/\beta}\right] \sim d(q)(p/q)^{-3/2}e^{\frac{p}{q}\phi_K(\tau)}, \quad \text{as} \quad p \to \infty$$

for $d(q) > 0$, we follow the same arguments as those used in the proof of Lemma 11. In other words, we define for $z \geq 1$,

$$D_q(z,p) = (p/q)^{3/2} e^{-(p/q)\phi_K(\tau)} \mathbb{E}\left[A_{p,q}^{-z/\beta}\right],$$

which is decreasing with respect to z. We obtain the same type of inequalities as in the Lemma 11, for the random variable A instead of $\tilde{A}$.

Again we take $\varepsilon > 0$ such that (47) holds. Then from Lemma 7 in [Hir98], we know that there exists $C_q > 0$ such that for p large enough,

$$\mathbb{E}\left[A_{p,q}^{-(1+\varepsilon)/\beta}\right] \leq \mathbb{E}\left[A_{p,q}^{-1/\beta}\right] \leq \mathbb{E}\left[\inf_{i\in[0,p]\cap\mathbb{N}} e^{-K_{i/q}}\right] \sim C_q p^{-3/2} e^{(p/q)\phi_K(\tau)},$$

then we use Theorem 10 to get $0 < \liminf_{n\to\infty} D_q(1,n) = \limsup_{n\to\infty} D_q(1,n) < \infty$, which ends the proof. □

Lemma 13. *Assume that $\phi'_K(0+) = 0$ and $\beta < \theta_{max}$. Then there exist two positive and finite constants $b(q)$ and $c_4(q)$ such that*

$$F_{p,q} \sim c_4(q)(p/q)^{-1/2}, \quad \textit{as} \quad p \to \infty, \tag{35}$$

and

$$\mathbb{E}\left[(A_{p,q}/q)^{-1/\beta}\right] \sim b(q)(p/q)^{-1/2}, \quad \textit{as} \quad p \to \infty. \tag{36}$$

Proof. The proof is almost the same as for the Lemma 12. We first apply Theorem 10 to the same function $\tilde{\psi}$ and sequence $(a_n, b_n)_{n\geq 0}$ defined in Lemma 12 but with the probability $\mathbb{P}$ instead of $\mathbb{P}^{(\tau)}$. Then, we get

$$\mathbb{E}\Big[F(A_{p,q}/q)\Big] \sim c_4(q)(p/q)^{-1/2}, \quad \text{as} \quad p \to \infty.$$

We then define for $z \geq 1$,

$$D_q(z,p) = (p/q)^{1/2} \mathbb{E}\Big[A_{p,q}^{-z/\beta}\Big],$$

and from Theorem A in [Koz76] and Theorem 10 , we obtain that $D_q(1,p)$ has a positive finite limit when p goes to infinity. □

5.3 From the discretized process to the continuous process

Up to now, the asymptotic behaviors of the processes depend on the step size $1/q$. By letting q tend to infinity, we obtain our results in continuous time. Recalling the notations (30) to (36), we prove the following limits :

Lemma 14. *There exist five positive finite constants b, d, c_2, c_3 and c_4 such that*

$$(b(q), d(q), c_2(q), c_3(q), c_4(q)) \longrightarrow (b, d, c_2, c_3, c_4), \quad \textit{as} \quad q \to \infty. \tag{37}$$

Proof. Let us first prove the convergence of $d(q)$. From Lemma 9, we know that for every $n \in \mathbb{N}^*$

$$\frac{e^{(\phi_-(1)-|\gamma|)/q}\mathbb{E}\Big[\Big(A_{nq,q}/q\Big)^{-1/\beta}\Big]}{n^{-3/2}e^{n\phi_K(\tau)}} \leq \frac{\mathbb{E}\Big[\Big(\int_0^n e^{-\beta K_u}\mathrm{d}u\Big)^{-1/\beta}\Big]}{n^{-3/2}e^{n\phi_K(\tau)}}$$
$$\leq \frac{e^{(\phi_+(1)+|\gamma|)/q}\mathbb{E}\Big[\Big(A_{nq-1,q}/q\Big)^{-1/\beta}\Big]}{n^{-3/2}e^{n\phi_K(\tau)}}.$$

A direct application of the Lemma 22 with

$$a(q) = d(q), \quad c^-(q) = e^{(\phi_-(1)-|\gamma|)/q}, \quad \text{and} \quad c^+(q) = e^{(\phi_+(1)+|\gamma|)/q},$$

yields that $d(q)$ converges as $q \to \infty$.

Similar arguments lead to the convergence of $b(q)$ and we now prove the convergence of $c_2(q)$, $c_3(q)$ and $c_4(q)$. Again the proofs of the three cases are very similar, so we only prove the second one. From Lemmas 9 and 12, we know that for every $(n,q) \in (\mathbb{N}^*)^2$,

$$\mathbb{E}\left[F\Big(e^{\beta|\gamma|/q}V_{1/q}^{\beta}A_{nq,q}/q\Big)\right] \le a_F(n) \le \mathbb{E}\left[F\Big(e^{-\beta|\gamma|/q}U_{1/q}^{\beta}A_{nq-1,q}/q\Big)\right].$$

Using (50) and dividing by $n^{-3/2}\exp(n\phi_K(\tau))$, we obtain

$$\begin{aligned}\frac{F_{nq,q} + M\mathbb{E}\left[e^{-|\gamma|/q}V_{1/q}^{-1} - 1\right]\mathbb{E}\left[\left(\frac{A_{nq,q}}{q}\right)^{-\frac{1}{\beta}}\right]}{n^{-3/2}e^{n\phi_K(\tau)}} &\le \frac{a_F(n)}{n^{-3/2}e^{n\phi_K(\tau)}} \\ &\le \frac{F_{nq-1,q} + M\mathbb{E}\left[e^{|\gamma|/q}U_{1/q}^{-1} - 1\right]\mathbb{E}\left[\left(\frac{A_{nq-1,q}}{q}\right)^{-\frac{1}{\beta}}\right]}{n^{-3/2}e^{n\phi_K(\tau)}},\end{aligned}$$

Lemmas 12, 22 and equation (50), where $a(q) = c_3(q)$,

$$c^-(q) = 1 - \frac{Md(q)(e^{(\phi_-(1)-|\gamma|)/q} - 1)}{c_3(q)} \quad \text{and} \quad c^+(q) = 1 + \frac{Md(q)(e^{(\phi_+(1)+|\gamma|)/q} - 1)}{c_3(q)}$$

yield the result. □

5.4 Proof of Proposition 8

Proof of Proposition 8 a/ (i). Recall from Lemma II.2 in [BLG00] that the process $(K_t - K_{(t-s)^-}, 0 \le s \le t)$ has the same law as $(K_s, 0 \le s \le t)$. Then

$$\int_0^t e^{-\beta K_s}ds = \int_0^t e^{-\beta K_{(t-s)}}ds = e^{-\beta K_t}\int_0^t e^{\beta K_t - \beta K_{(t-s)}}ds \overset{(d)}{=} e^{-\beta K_t}\int_0^t e^{\beta K_s}ds.$$

We first note that for every $q \in \mathbb{N}^*$ and $t \ge 2/q$, Lemma 9 leads to

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t e^{-\beta K_s}ds\right)^{-1/\beta}\right] &\le \mathbb{E}\left[\left(\int_0^{2/q} e^{-\beta K_s}ds\right)^{-1/\beta}\right] \\ &\le q^{1/\beta}e^{|\gamma|/q}\mathbb{E}\Big(U_{1/q}^{-1/\beta}A_{1,q}^{-1/\beta}\Big) \\ &= q^{1/\beta}\exp\Big(\frac{\phi_K(1) + |\gamma| + \phi_K^+(1)}{q}\Big) < \infty,\end{aligned}$$

where ϕ_K^+ was defined in (23). Hence using the change of measure (32), with $\lambda = 1$, we have

$$\mathbb{E}\left[\left(\int_0^t e^{-\beta K_s}ds\right)^{-1/\beta}\right] = \mathbb{E}\left[e^{K_t}\left(\int_0^t e^{\beta K_s}ds\right)^{-1/\beta}\right] = e^{t\phi_K(1)}\mathbb{E}^{(1)}\left[\left(\int_0^t e^{\beta K_s}ds\right)^{-1/\beta}\right].$$

The above identity implies that the decreasing function $t \mapsto \mathbb{E}^{(1)}[(\int_0^t e^{\beta K_s}ds)^{-1/\beta}]$ is finite for all $t > 0$. So it converges to a non negative and finite limit c_1, as t increases. This limit is positive, since under the probability $\mathbb{P}^{(1)}$, K is still a Lévy process with negative mean $\mathbb{E}^{(1)}(K_1) = \phi_K'(1)$ and according to Theorem 1 in [BY05], we have

$$\int_0^\infty e^{\beta K_s} ds < \infty, \qquad \mathbb{P}^{(1)}\text{-a.s.}$$

It remains to prove that

$$a_F(t) \quad \sim \quad C_F \mathbb{E}\Big[\Big(\int_0^t e^{-\beta K_s} ds\Big)^{-1/\beta}\Big], \quad \text{as} \quad t \to \infty.$$

Recall that $\theta_{max} > 1$ and $\phi_K'(1) < 0$. So we can chose $\varepsilon > 0$ such that (47) holds, $1+\varepsilon < \theta_{max}$, $\phi_K(1+\varepsilon) < \phi_K(1)$ and $\phi_K'(1+\varepsilon) < 0$. Then

$$\left| F\left(\int_0^t e^{-\beta K_s} ds\right) - \left(\int_0^t e^{-\beta K_s} ds\right)^{-1/\beta} \right| \le M \left(\int_0^t e^{-\beta K_s} ds\right)^{-(1+\varepsilon)/\beta}.$$

Thus, we just need to show that

$$\mathbb{E}\left[\left(\int_0^t e^{-\beta K_s} ds\right)^{-(1+\varepsilon)/\beta}\right] = o(e^{t\phi_K(1)}), \quad \text{as} \quad t \to \infty.$$

It is achieved by a new change of measure (32), with $\lambda = 1+\varepsilon$,

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t e^{-\beta K_s} ds\right)^{-(1+\varepsilon)/\beta}\right] &= \mathbb{E}\left[e^{(1+\varepsilon)K_t}\left(\int_0^t e^{\beta K_s} ds\right)^{-(1+\varepsilon)/\beta}\right] \\
&= e^{t\phi_K(1+\varepsilon)} \mathbb{E}^{(1+\varepsilon)}\left[\left(\int_0^t e^{\beta K_s} ds\right)^{-(1+\varepsilon)/\beta}\right].
\end{aligned}$$

Again using Lemma 9, we obtain for $t \ge q/2$,

$$\mathbb{E}\left[\left(\int_0^t e^{-\beta K_s} ds\right)^{-\frac{1+\varepsilon}{\beta}}\right] \le q^{(1+\varepsilon)/\beta} \exp\Big(\frac{\phi_K(1+\varepsilon) + |\gamma|(1+\varepsilon) + \phi_K^+(1+\varepsilon)}{q}\Big) < \infty,$$

which ensures that the decreasing function $t \mapsto \mathbb{E}^{(1+\varepsilon)}[(\int_0^t e^{\beta K_s} ds)^{-(1+\varepsilon)/\beta}]$ is finite for all $t > 0$ and gives the result. $\square$

Remark 3. In the particular case when $\beta = 1$, it is enough to apply Theorem 1.1 in [GKV03] to the geometric BPRE $(X_n, n \ge 0)$ defined in Remark 2 to obtain the result.

Proof of Proposition 8 a/ (ii), (iii), and b/. Proofs are similar for the different regimes, so we only focus in the proof of the regime in a/(iii).

Let $\varepsilon > 0$ and $q \in \mathbb{N}^*$ such that $q \ge 1/\varepsilon$ and $(1-\varepsilon)c_3 \le c_3(q) \le (1+\varepsilon)c_3$. Then for every $t \ge 1$,

$$\begin{aligned}
F_{\lfloor qt \rfloor, q} + \mathbb{E}\Big[F(C_{\lfloor qt \rfloor, q} e^{\beta|\gamma|/q}/q)\Big] - F_{\lfloor qt \rfloor, q} &\le a_F(t) \\
&\le F_{\lfloor qt \rfloor - 1, q} + \mathbb{E}\Big[F(D_{\lfloor qt \rfloor - 1, q} e^{-\beta|\gamma|/q}/q)\Big] - F_{\lfloor qt \rfloor - 1, q}.
\end{aligned}$$

Applying (50), we obtain :

$$\left|\mathbb{E}\Big[F(C_{\lfloor qt\rfloor,q}e^{\beta|\gamma|/q}/q)\Big]-F_{\lfloor qt\rfloor,q}\right|\leq(1-e^{-\varepsilon(|\gamma|-\phi_-(1))})M\mathbb{E}\Big[(A_{\lfloor qt\rfloor,q}/q)^{-1/\beta}\Big],$$
$$\left|\mathbb{E}\Big[F(D_{\lfloor qt\rfloor-1,q}e^{-\beta|\gamma|/q}/q)\Big]-F_{\lfloor qt-1\rfloor,q}\right|\leq(e^{\varepsilon(|\gamma|+\phi_+(1))}-1)M\mathbb{E}\Big[(A_{\lfloor qt\rfloor-1,q}/q)^{-1/\beta}\Big].$$

When t goes to infinity, we can bound both terms by

$$h(\varepsilon)t^{-3/2}e^{t\phi_K(\tau)}=\Big[2Md(e^{\varepsilon(|\gamma|+\phi_+(1))}-e^{-\varepsilon(|\gamma|-\phi_-(1))})e^{-\varepsilon\phi_K(\tau)}\Big]t^{-3/2}e^{t\phi_K(\tau)}\tag{38}$$

where ϕ_- and ϕ_+ are defined in (23), and $h(\varepsilon)$ goes to 0 with ε. On the other hand, for t large enough

$$(1-2\varepsilon)c_3t^{-3/2}e^{t\phi_K(\tau)}\leq F_{\lfloor qt\rfloor,q}\leq F_{\lfloor qt\rfloor-1,q}\leq(1+2\varepsilon)c_3t^{-3/2}e^{t\phi_K(\tau)},$$

which ends the proof of Proposition 8. □

6 Auxiliary results

This section is devoted to the technical results which are necessary for the previous proofs.

6.1 A local martingale

To prove the expression of the Laplace exponent of $\widetilde{Z}$ in Theorem 1, we used a local martingale, which is determined below via Itô's formulae.

Lemma 15. *For every $F\in C^{1,2}(\mathbb{R}^+,\mathbb{R})$, conditionally on Δ,*

$$F(t,\widetilde{Z}_t)-F(0,X_0)-\int_0^t\frac{\partial^2}{\partial x^2}F(s,\widetilde{Z}_s)\sigma^2e^{-gs-\Delta_s}\widetilde{Z}_s\mathrm{d}s-\int_0^t\frac{\partial}{\partial t}F(s,\widetilde{Z}_s)\mathrm{d}s$$
$$-\int_0^t\int_0^\infty Y_s\Big(F(s,\widetilde{Z}_s+ze^{-gs-\Delta_s})-F(s,\widetilde{Z}_s)-\frac{\partial}{\partial x}F(s,\widetilde{Z}_s)ze^{-gs-\Delta_s}\Big)\mu(\mathrm{d}z)\mathrm{d}s,$$

is a local martingale.

Proof. First, we apply Itô's formula to $\widetilde{Z}_t$ and

$$\begin{aligned}\widetilde{Z}_t=&\,Y_0+\int_0^te^{-gs-\Delta_s}\sqrt{2\sigma^2Y_s}\mathrm{d}B_s+g\int_0^tY_se^{-gs-\Delta_s}\mathrm{d}s\\&+\int_0^t\int_0^\infty\int_0^{Y_{s-}}e^{-gs-\Delta_{s-}}z\widetilde{N}_0(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)-g\int_0^tY_se^{-gs-\Delta_s}\mathrm{d}s\\&+\int_0^t\int_{[0,\infty)}\Big(ze^{-gs-\Delta_{s-}-\log(z)}-e^{-gs-\Delta_{s-}}\Big)Y_{s-}N_1(\mathrm{d}s,\mathrm{d}z)\\=&\,Y_0+\int_0^te^{-gs-\Delta_s}\sqrt{2\sigma^2Y_s}\mathrm{d}B_s+\int_0^t\int_0^\infty\int_0^{Y_{s-}}e^{-gs-\Delta_{s-}}z\widetilde{N}_0(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u),\end{aligned}$$

which is a local martingale.

Now, again from Itô's formula, we have

$$\begin{aligned}
F(t,\widetilde{Z}_t) &= F(0,Y_0) + \int_0^t \frac{\partial}{\partial x}F(s,\widetilde{Z}_s)e^{-gs-\Delta_s}\sqrt{2\sigma^2 Y_s}\mathrm{d}B_s \\
&+ \int_0^t \frac{\partial^2}{\partial x^2}F(s,\widetilde{Z}_s)\sigma^2 e^{-gs-\Delta_s}\widetilde{Z}_s\mathrm{d}s + \int_0^t \frac{\partial}{\partial t}F(s,\widetilde{Z}_s)\mathrm{d}s \\
&+ \int_0^t\int_0^\infty\int_0^{Y_{s-}} \Big(F(s,\widetilde{Z}_{s-}+ze^{-gs-\Delta_{s-}}) - F(s,\widetilde{Z}_{s-})\Big)\widetilde{N}_0(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\
&+ \int_0^t\int_0^\infty Y_s\Big(F(s,\widetilde{Z}_s+ze^{-gs-\Delta_s}) - F(s,\widetilde{Z}_s) - \frac{\partial}{\partial x}F(s,\widetilde{Z}_s)ze^{-gs-\Delta_s}\Big)\mu(\mathrm{d}z)\mathrm{d}s.
\end{aligned}$$

It yields the expected local martingale, conditionnaly on Δ, and ends up the proof. □

6.2 Existence and uniqueness of the backward ODE

The Laplace exponent of $\widetilde{Z}$ in Theorem 1 is the solution of a backward ODE. The existence and uniqueness of this latter are stated and proved below.

Proposition 16. *Let δ be a cadlag function on $\mathbb{R}^+$ of bounded variation. Then the backward ODE (8) admits a unique solution.*

The proof relies on a classical approximation of the solution of (8) and the Cauchy-Lipschitz theorem. When there is no accumulation of jumps, the latter ensures the existence and uniqueness of the solution between two successive jump times of δ. The problem remains on the times where accumulation of jumps occurs. Let us define the family of functions δ^n by deleting the small jumps of δ,

$$\delta_t^n = \delta_t - \sum_{s\le t}\Big(\delta_s - \delta_{s-}\Big)\mathbf{1}_{\{|\delta_s-\delta_{s-}|<1/n\}}.$$

We note that ψ_0 is continuous, and $s\mapsto e^{gs+\delta_s^n}$ is piecewise $C^1(\mathbb{R}^+)$ (with a finite number of discontinuities). From the Cauchy-Lipschitz Theorem, for every $n\in\mathbb{N}^*$ we can define a solution $v_t^n(.,\lambda,\delta)$ continuous with càdlàg first derivative of the backward differential equation:

$$\frac{\partial}{\partial s}v_t^n(s,\lambda,\delta) = e^{gs+\delta_s^n}\psi_0\big(e^{-gs-\delta_s^n}v_t^n(s,\lambda,\delta)\big),\quad 0\le s\le t,\qquad v_t^n(t,\lambda,\delta)=\lambda.$$

We want to show that the sequence $(v_t^n(.,\lambda,\delta))_{n\ge1}$ converges to a function $v_t(.,\lambda,\delta)$ solution of (8).
Then the proof of this proposition follows from the next result. Let t be fixed and

$$S := \sup_{s\in[0,t],n\in\mathbb{N}^*}\Big\{e^{gs+\delta_s^n}, e^{-gs-\delta_s^n}\Big\}. \tag{39}$$

Lemma 17. *For every $\lambda>0$,*

(i) we have

$$I := \inf_{0\le s\le t,\,1\le n} v_t^n(s,\lambda,\delta) > 0, \tag{40}$$

(ii) there exists a positive finite constant C such that for all $IS^{-1} < \eta \le \kappa \le \lambda S$,

$$0 \le \psi_0(\kappa) - \psi_0(\eta) \le C(\kappa - \eta). \tag{41}$$

Proof. First, we observe that S is finite. Now, using that $x \mapsto e^{-x} + x$ is increasing on $\mathbb{R}_+^*$ and the Taylor-Lagrange's formula to $x \mapsto e^{-x}$, we can check that for all $0 < \eta < \kappa$ and $x \ge 0$

$$0 \le \frac{e^{-\kappa x} - e^{-\eta x} + (\kappa - \eta)x}{(\kappa - \eta)x} \le 1 + \frac{\kappa - \eta}{2\eta e}. \tag{42}$$

In order to prove (i), we first note that $\psi_0(\lambda) \ge 0$ for all $\lambda \ge 0$. Moreover, if there exists $0 \le s_0 < t$, such that $v_t^n(s_0, \lambda, \delta) = 0$, then $v_t^n(s, \lambda, \delta)$ equals 0 for every $s \in [s_0, t]$ since $\psi_0(0) = 0$. Hence $v_t^n(., \lambda, \delta)$ is non decreasing and

$$v_t^n(s, \lambda, \delta) \in (0, \lambda] \qquad \text{for all} \quad s \in [0, t] \text{ and } n \ge 1. \tag{43}$$

Moreover ψ_0 is increasing and for all $s \in [0, t]$, and $n \ge 1$:

$$e^{gs + \delta_s^n} \psi_0\big(e^{-gs - \delta_s^n} v_t^n(s, \lambda, \delta)\big) \le S\psi_0(Sv_t^n(s, \lambda, \delta)),$$

where S is the constant defined in (39). On the other hand, note that for all $x \ge 0$,

$$0 \le e^{-x} - 1 + x \le x \wedge x^2, \tag{44}$$

then for all $0 \le \upsilon \le \lambda S$, we get

$$\psi_0(\upsilon) \le \Big[\lambda S\Big(\int_0^1 x^2 \mu(dx) + \sigma^2\Big) + \int_1^\infty x\mu(dx)\Big]\upsilon := B\upsilon.$$

Putting all the pieces together, we deduce that for all $0 \le s \le t$,

$$\frac{\partial}{\partial s} v_t^n(s, \lambda, \delta) \le SBv_t^n(s, \lambda, \delta),$$

and since $v_t^n(t, \lambda, \delta) = \lambda$, we have

$$v_t^n(s, \lambda, \delta) \ge \lambda e^{SB(s-t)}, \quad \text{for all } s \in [0, t].$$

We get (i) by defining $I = \lambda e^{-SBt}$.

Finally, we note that for all $IS^{-1} < \eta < \kappa \le \lambda S$,

$$\begin{aligned}
&\psi_0(\kappa) - \psi_0(\eta) \\
&\quad = \sigma^2(\kappa^2 - \eta^2) + \int_1^\infty \Big(e^{-\kappa x} - e^{-\eta x} + (\kappa - \eta)x\Big)\mu(dx) \\
&\qquad + (\kappa - \eta)\int_0^1 x(1 - e^{-\eta x})\mu(dx) + \int_0^1 \Big(e^{-(\kappa - \eta)x} - 1 + (\kappa - \eta)x\Big)e^{-\eta x}\mu(dx) \\
&\quad \le \sigma^2(\kappa^2 - \eta^2) + (\kappa - \eta)\Big(1 + \frac{(\kappa - \eta)}{2\eta e}\Big)\int_1^\infty x\mu(dx) \\
&\qquad\qquad + (\kappa - \eta)\eta\int_0^1 x^2\mu(dx) + (\kappa - \eta)^2\int_0^1 x^2\mu(dx) \\
&\quad \le \Big[2\lambda S\sigma^2 + 2\lambda S\int_0^1 x^2\mu(dx) + (1 + \frac{\lambda S^2}{2Ie})\int_1^\infty x\mu(dx)\Big](\kappa - \eta),
\end{aligned}$$

which proves part (ii). $\square$

We can now prove the result of existence and uniqueness.

Proof of Proposition 16. We now prove that $(v_t^n(s,\lambda,\delta), s \in [0,t])_{n\geq 0}$ is a Cauchy sequence. For sake of simplicity, in what follows we denote $v^n(s) = v_t^n(s,\lambda,\delta)$, and for all $v \geq 0$:

$$\psi^n(s,v) = e^{gs+\delta_s^n}\psi_0\big(e^{-gs-\delta_s^n}v\big) \quad \text{and} \quad \psi^\infty(s,v) = e^{gs+\delta_s}\psi_0\big(e^{-gs-\delta_s}v\big).$$

We have for any $0 \leq s \leq t$ and $m,n \geq 1$:

$$\begin{aligned}
&|v^n(s) - v^m(s)| \\
&= \Big| \int_s^t \psi^n(u, v^n(u))du - \int_s^t \psi^m(u, v^m(u))du \Big| \\
&\leq \int_s^t R^n(u)du + \int_s^t R^m(u)du + \int_s^t \Big|\psi^\infty(u, v^n(u)) - \psi^\infty(u, v^m(u))\Big| du, \qquad (45)
\end{aligned}$$

where for any $u \in [0,t]$,

$$\begin{aligned}
R^n(u) &= \Big|\psi^n(u, v^n(u)) - \psi^\infty(u, v^n(u))\Big| \\
&\leq e^{gs+\delta_s^n}\Big|\psi_0\big(e^{-gs-\delta_s^n}v^n(u)\big) - \psi_0\big(e^{-gs-\delta_s}v^n(u)\big)\Big| + e^{gs}\psi_0\big(e^{-gs-\delta_s}v^n(u)\big)\Big|e^{\delta_s^n} - e^{\delta_s}\Big|.
\end{aligned}$$

Moreover, from (39), (40), and (41), we obtain

$$\begin{aligned}
R^n(u) &\leq SC\lambda\Big|e^{-\delta_s^n} - e^{-\delta_s}\Big| + e^{gt}\psi_0(\lambda S)\Big|e^{\delta_s^n} - e^{\delta_s}\Big| \\
&\leq \Big(SC\lambda + e^{gt}\psi_0(\lambda S)\Big) \sup_{s\in[0,t]} \Big\{\Big|e^{-\delta_s^n} - e^{-\delta_s}\Big|, \Big|e^{\delta_s^n} - e^{\delta_s}\Big|\Big\} := s_n,
\end{aligned}$$

which implies

$$\sup\Big\{\int_s^t R^n(u)du, s \in [0,t]\Big\} \leq ts_n \underset{n\to\infty}{\to} 0.$$

Using similar arguments as above, we get from (41),

$$\Big|\psi^\infty(u, v^n(u)) - \psi^\infty(u, v^m(u))\Big| \leq CS^2\Big|v^n(u) - v^m(u)\Big|.$$

From (45), we use Gronwall's Lemma (see Lemma 3.2 in [Dyn91] or Lemma 4.6 in [BS47]) with

$$R_{m,n}(s) = \int_s^t R^n(u)du + \int_s^t R^m(u)du,$$

to deduce that for all $0 \leq s \leq t$,

$$|v^n(s) - v^m(s)| \leq R_{m,n}(s) + CS^2 e^{CS^2(t-s)} \int_s^t R_{m,n}(s)ds.$$

Hence for every $n_0 \in \mathbb{N}^*$,

$$\sup_{m,n\geq n_0, s\in[0,t]} |v^n(s) - v^m(s)| \leq t\Big[1 + CS^2e^{CS^2t}t\Big] \sup_{m,n\geq n_0}(s_n + s_m).$$

Thus $(v^n(s), s \in [0,t])_{n\geq 0}$ is a Cauchy sequence under the uniform norm. Then there exists v a continuous function on $[0,t]$ such that $v^n \to v$, as n goes to ∞. Now we prove that v is solution of the equation (8). By continuity, v satisfies (39) and (41). Then for any $s \in [0,t]$ and $n \in \mathbb{N}^*$:

$$\begin{aligned}
&\Big|v(s) - \int_s^t \psi^\infty(s,v(s))ds - \lambda\Big| \\
&\quad \le \Big|v(s) - v^n(s)\Big| + \int_s^t \Big|\psi^\infty(s,v(s)) - \psi^n(s,v(s))\Big|ds + \int_s^t \Big|\psi^n(s,v(s)) - \psi^n(s,v^n(s))\Big|ds \\
&\quad \le ts_n + (1+CS^2)\sup\Big\{\Big|v(s)-v^n(s)\Big|, s\in[0,t]\Big\}.
\end{aligned}$$

so that letting $n \to \infty$ yields

$$\Big|v(s) - \int_s^t \psi^\infty(s,v(s))ds - \lambda\Big| = 0.$$

It proves that v is solution of (8), whereas the uniqueness follows from Gronwall's lemma. □

6.3 An upper bound for ψ_0

The study of the Laplace exponent of $\widetilde{Z}$ in Corollary 2 requires a fine control of the branching mechanism ψ_0.

Lemma 18. *Assume that the process $(gs + \Delta_s, s \ge 0)$ drifts to $+\infty$ a.s. There exists a non negative increasing function h on $\mathbb{R}^+$ such that for every $\lambda \ge 0$*

$$\psi_0(\lambda) \le \lambda h(\lambda) \qquad \textit{and} \qquad \int_0^\infty h\Big(e^{-(gt+\Delta_t)}\Big)dt < \infty.$$

Proof. Inequality (44) implies that for every $\lambda \ge 0$,

$$\begin{aligned}
\psi_0(\lambda) &\le \sigma^2\lambda^2 + \int_0^\infty \left(\lambda^2x^2\mathbf{1}_{\{\lambda x\le 1\}} + \lambda x\mathbf{1}_{\{x\lambda>1\}}\right)\mu(dx) \\
&\le \left(\sigma^2 + \int_0^1 x^2\mu(dx)\right)\lambda^2 + \lambda^2\mathbf{1}_{\{\lambda<1\}}\int_1^{1/\lambda} x^2\mu(dx) + \lambda\int_{1/\lambda}^\infty x\mu(dx).
\end{aligned}$$

Now, using condition (13) we obtain the existence of a positive constant c such that

$$\lambda\int_{1/\lambda}^\infty x\mu(dx) \le \lambda\log(1+1/\lambda)^{-(1+\epsilon)}\int_{1/\lambda}^\infty x\log(1+x)^{1+\epsilon}\mu(dx) \le c\lambda\log(1+1/\lambda)^{-(1+\epsilon)}.$$

Next, let us introduce the following function f:

$$f(x) = \frac{\log(1+x)^{1+\varepsilon}}{x}, \quad \text{for } x\in[1,\infty).$$

By differentiation, we check that there exists a positive real number A such that f is decreasing on $[A,\infty)$. Therefore, for every $\lambda < 1/A$,

$$\begin{aligned}
\int_1^{1/\lambda}\lambda^2x^2\mu(dx) &\le \lambda^2\int_A^{1/\lambda}x^2\mu(dx) \\
&\le \lambda\log(1+1/\lambda)^{-(1+\varepsilon)}f(1/\lambda)\int_A^{1/\lambda}\frac{x\log(1+x)^{1+\varepsilon}}{f(x)}\mu(dx) \\
&\le \lambda\log(1+1/\lambda)^{-(1+\varepsilon)}\int_A^{1/\lambda}x\log(1+x)^{1+\varepsilon}\mu(dx)
\end{aligned}$$

Adding that for $\lambda \in [1, 1/A]$, $\int_1^{1/\lambda} \lambda^2 x^2 \mu(dx) \le \lambda^2 \int_1^A x^2 \mu(dx) \le \lambda^2 A \int_1^\infty x\mu(dx)$ and using again condition (13), we deduce that there exists a positive constant c' such that for every $\lambda \geq 0$,

$$\psi_0(\lambda) \le c'\Big(\lambda^2 + \lambda \log(1+1/\lambda)^{-(1+\varepsilon)}\Big).$$

Since λ^2 is negligible with respect to $\lambda \log(1+1/\lambda)^{-(1+\varepsilon)}$ when λ is near 0 or infinity, we conclude that there exists a positive constant c'' such that

$$\psi_0(\lambda) \le c''\lambda \log(1+1/\lambda)^{-(1+\varepsilon)}.$$

Defining the function $h(x) = c''x\log(1+1/x)^{-(1+\varepsilon)}$, for $x > 0$, we get:

$$0 \le \int_0^\infty h\Big(e^{-(gt+\Delta_t)}\Big)dt \le c'' \int_0^\infty (gt+\Delta_t)^{-(1+\varepsilon)}dt,$$

which is finite since the process $(gs + \Delta_s, s \geq 0)$ goes linearly to $+\infty$ a.s. More precisely, one can find $\epsilon > 0$ such that $(gs+\Delta_s - \epsilon s : s \geq 0)$ has positive expectation for $s = 1$, which ensures that it goes to ∞ a.s. and there exists $L > -\infty$ a.s. such that $gs + \Delta s \geq L + \epsilon s$ a.s. This ends the proof. $\square$

6.4 Extinction versus explosion

Let us here check that Y_t can be properly renormalized as $t \to \infty$ on the non-extinction event.

Lemma 19. *Let Y be a non negative Markov process which satisfies the branching property.*

Assume also that there exists a positive function a_t such that for every $x_0 > 0$, there exists a non negative finite random variable W such that

$$a_t Y_t \xrightarrow[t\to\infty]{} W \quad a.s, \qquad \mathbb{P}_{x_0}(W > 0) > 0, \qquad a_t \overset{t\to\infty}{\longrightarrow} 0.$$

Then

$$\{W = 0\} = \Big\{Y_t \xrightarrow[t\to\infty]{} 0\Big\} \qquad \mathbb{P}_{x_0} \quad a.s.$$

Proof. First, we prove that

$$\mathbb{P}_{x_0}(\limsup_{t\to\infty} Y_t = \infty \mid \limsup_{t\to\infty} Y_t > 0) = 1. \tag{46}$$

Let $0 < x \le x_0 \le A$ be fixed. As $a_t \to 0$ and $\mathbb{P}_x(W > 0) > 0$, there exists $t_0 > 0$ such that $\alpha := \mathbb{P}_x(Y_{t_0} \geq A) > 0$. By the branching property, the process is stochastically monotone as a function of its initial value. Thus, for every $y \geq x$ (including $y = x_0$),

$$\mathbb{P}_y(Y_{t_0} \ge A) \ge \alpha > 0.$$

Let us define the following stopping times

$$T_0 := 0, \qquad T_{i+1} = \inf\{t \ge T_i + t_0 : Y_t \ge x\} \qquad (i \ge 0)$$

For any $i \in \mathbb{N}^*$, by strong Markov property

$$\mathbb{P}_{x_0}(Y_{T_i+t_0} \ge A \mid (Y_t : t \le T_i),\ T_i < \infty) \ge \alpha.$$

Conditionally on $\{\limsup_{t\to\infty} Y_t > x\}$, the stopping times T_i are finite a.s. and for all $0 < x \le x_0 \le A$,

$$\mathbb{P}_{x_0}(\forall i \ge 0 : Y_{T_i+t_0} < A,\ \limsup_{t\to\infty} Y_t > x) = 0.$$

Then, $\mathbb{P}_{x_0}(\limsup_{t\to\infty} Y_t < \infty,\ \limsup_{t\to\infty} Y_t > x) = 0$. Adding that $\{\limsup_{t\to\infty} Y_t > 0\} = \cup_{x\in(0,x_0]}\{\limsup_{t\to\infty} Y_t > x\}$ yields (46).

Let us now consider the stopping times $T_n = \inf\{t \ge 0 : Y_t \ge n\}$. We get by strong Markov property and branching property,

$$\mathbb{P}_{x_0}(W = 0; T_n < \infty) = \mathbb{E}_{x_0}\Big(\mathbf{1}_{T_n<\infty}\mathbb{P}_{Y_{T_n}}(W = 0)\Big) \le \mathbb{P}_n(a_tY_t \underset{t\to\infty}{\longrightarrow} 0) = \mathbb{P}_1(a_tY_t \underset{t\to\infty}{\longrightarrow} 0)^n,$$

which goes to zero as $n \to \infty$, since $\mathbb{P}_1(a_tY_t \overset{t\to\infty}{\longrightarrow} 0) = \mathbb{P}_1(W = 0) < 1$. Then,

$$0 = \mathbb{P}_{x_0}(W = 0; \forall n : T_n < \infty) = \mathbb{P}_{x_0}(W = 0, \limsup_{t\to\infty} Y_t = \infty) = \mathbb{P}_{x_0}(W = 0, \limsup_{t\to\infty} Y_t > 0),$$

where the last identity comes from (46). It completes the proof. $\square$

6.5 A Central limit theorem

Finally, we need the following central limit theorem of Lévy processes in Corollary 3.

Lemma 20. *Under the assumption (14) we have*

$$\frac{gt + \Delta_t - \boldsymbol{m}t}{\rho\sqrt{t}} \xrightarrow[t\to\infty]{d} N(0, 1).$$

Proof. For simplicity, let η be the image measure of ν under the mapping $x \mapsto e^x$. The assumption (14) is equivalent to $\int_{|x|\ge1} x^2\eta(\mathrm{d}x) < \infty$, or $\mathbb{E}[\Delta_1^2] < \infty$.

Next, we define $T(x) = \eta\big((-\infty, -x)\big) + \eta\big((x, \infty)\big)$ and $U(x) = 2\int_0^x yT(y)\mathrm{d}y$, and assume that $T(x) > 0$ for all $x > 0$. According to Theorem 3.5 in Doney and Maller [DM02] there exist two functions $a(t), b(t) > 0$ such that

$$\frac{gt + \Delta_t - a(t)}{b(t)} \xrightarrow[t\to\infty]{d} N(0, 1), \qquad \text{if and only if} \quad \frac{U(x)}{x^2T(x)} \xrightarrow[x\to\infty]{} \infty.$$

If the above condition is satisfied, then b is regularly varying with index $1/2$ and it may be chosen to be strictly increasing to ∞ as $t \to \infty$. Moreover $b^2(t) = tU(b(t))$ and $a(t) = tA(b(t))$, where

$$A(x) = g + \int_{\{|z|<1\}} z\eta(\mathrm{d}z) + \eta\big((1, \infty)\big) - \eta\big((-\infty, -1)\big) + \int_1^x \Big(\eta\big((y, \infty)\big) - \eta\big((-\infty, -y)\big)\Big)\mathrm{d}y.$$

Note that under our assumption $x^2T(x) \to 0$, as $x \to \infty$. Moreover, note

$$U(x) = x^2T(x) + \int_{(-x,0)} z^2\eta(\mathrm{d}x) + \int_{(0,x)} z^2\eta(\mathrm{d}x),$$

and

$$A(x) = g + \int_{\{|z|<x\}} z\eta(\mathrm{d}z) + x\Big(\eta\big((x, \infty)\big) - \eta\big((-\infty, -x)\big)\Big).$$

Hence assumption (14) implies that

$$U(x) \xrightarrow[x\to\infty]{} \int_{(-\infty,\infty)} z^2\eta(\mathrm{d}z) = \rho^2, \quad A(x) \xrightarrow[x\to\infty]{} g + \int_{\mathbb{R}} z\eta(\mathrm{d}z) = \mathbf{m},$$

Therefore, we deduce $U(x)/(x^2T(x)) \to \infty$ as $x \to \infty$, $b(t) \sim \rho\sqrt{t}$ and $a(t) \sim \mathbf{m}t$, as $t \to \infty$.

Now assume that $T(x) = 0$, for x large enough. Define

$$\Psi(\lambda, t) = -\log \mathbb{E}\left[\exp\left\{i\lambda\left(\frac{gt + \Delta_t - a(t)}{b(t)}\right)\right\}\right],$$

where the functions $a(t)$ and $b(t)$ are defined as above. Hence, we can write

$$\begin{aligned}\Psi(\lambda,t) = t\int_{\{|x|<b(t)\}}\Big(1 - e^{\frac{i\lambda}{b(t)}x} + \frac{i\lambda}{b(t)}x + \frac{(i\lambda)^2}{2b^2(t)}x^2\Big)\eta(\mathrm{d}x) + t\int_{\{|x|\geq b(t)\}}\Big(1 - e^{\frac{i\lambda}{b(t)}x}\Big)\eta(\mathrm{d}x)\\ - \frac{t(i\lambda)^2}{2b^2(t)}\int_{\{|x|<b(t)\}} x^2\eta(\mathrm{d}x) + i\lambda t\Big(\eta(b(t),\infty) - \eta(-\infty,-b(t))\Big).\end{aligned}$$

Since $T(x) = 0$ for all x large, $b(t) \to \infty$ and $t^{-1}b^2(t) \to \rho$, as $t \to \infty$, therefore

$$\Psi(\lambda, t) \xrightarrow[t\to\infty]{} \frac{\lambda^2}{2},$$

which implies the result thanks to Lévy's Theorem. □

6.6 Two technical Lemma

We now give two technical lemmas, useful in the proofs of Section 5.

Lemma 21. *Assume that F is defined as in (26), then there exist two positive constants η and M such that for all (x,y) in $\mathbb{R}_+^2$ and ε in $[0,\eta]$,*

$$\Big|F(x) - C_F x^{-1/\beta}\Big| \leq Mx^{-(1+\varepsilon)/\beta}, \tag{47}$$

$$0 < F(x) \leq M(x+1)^{-1/\beta}, \tag{48}$$

$$\Big|F(x) - F(y)\Big| \leq M|x-y|, \tag{49}$$

$$\Big|F(x) - F(y)\Big| \leq M\Big|x^{-1/\beta} - y^{-1/\beta}\Big|. \tag{50}$$

Proof. We only prove (50), since the others inequalities follows from straightforward computations. We define the function $\tilde{h} : x \in \mathbb{R}^+ \mapsto (1+x)^{1-\varsigma}h(x)$ and let $0 \leq x \leq y$. Then,

$$\begin{aligned}0 \leq (F(x) - F(y))/C_F \ \leq\ & \Big((x+1)^{-1/\beta} - (y+1)^{-1/\beta}\Big) + (1+y)^{-1/\beta-1}\Big|\tilde{h}(x) - \tilde{h}(y)\Big|\\ & + \Big|\tilde{h}(x)\Big|\Big((1+x)^{-1/\beta-1} - (1+y)^{-1/\beta-1}\Big).\end{aligned} \tag{51}$$

As $\beta \in (0,1]$, we have the following inequalities :

$$\begin{aligned}(1+x)^{-1/\beta-1} - (1+y)^{-1/\beta-1} &\leq (1+y)^{-1/\beta-1}\Big(\Big(\frac{1+y}{1+x}\Big)^{1/\beta} - 1\Big)\Big(\frac{1+y}{1+x} + 1\Big)\\ &\leq (1+y)^{-1/\beta}\Big(\Big(\frac{y}{x}\Big)^{1/\beta} - 1\Big)\frac{1}{1+y}.2\frac{1+y}{1+x}\\ &\leq 2\Big(x^{-1/\beta} - y^{-1/\beta}\Big).\end{aligned}$$

Moreover, the Mean Value Theorem applied to the function $z \in \mathbb{R}_+ \mapsto (z+1)^{-1/\beta}$ on $[x, y]$ ensures that

$$\frac{1}{\beta}(y+1)^{-1/\beta-1}(y-x) \le (x+1)^{-1/\beta} - (y+1)^{-1/\beta}.$$

Now, we denote by k the Lipschitz constant of h. The equation (51) finally gives

$$0 \le F(x) - F(y) \quad \le \quad C_F(1 + 2\|h\|_\infty + k\beta)\Big(x^{-1/\beta} - y^{-1/\beta}\Big).$$

which ends up the proof. $\square$

Lemma 22. *Assume that the positive sequences $(a_{n,q})_{(n,q)\in\mathbb{N}^2}$, $(a'_{n,q})_{(n,q)\in\mathbb{N}^2}$ and $(b_n)_{n\in\mathbb{N}}$ satisfy for every $(n,q) \in \mathbb{N}^2$:*

$$a_{n,q} \le b_n \le a'_{n,q},$$

and that there exist two sequences $(c^-(q))_{q\in\mathbb{N}}$ and $(c^+(q)_{q\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} a_{n,q} = c^-(q)a(q), \quad \lim_{n\to\infty} a'_{n,q} = c^+(q)a(q), \quad \textit{and} \quad \lim_{q\to\infty} c^-(q) = \lim_{q\to\infty} c^+(q) = 1$$

Then there exists a positive finite constant a such that

$$\lim_{q\to\infty} a(q) = \lim_{q\to\infty} a'(q) = \lim_{n\to\infty} b_n = a$$

Proof. Letting n go to infinity, we have for every $q \in \mathbb{N}$

$$\limsup b_n \le c^+(q)a(q) \quad \text{and} \quad c^-(q)a(q) \le \liminf b_n.$$

Then letting q go to infinity, we obtain

$$\limsup b_n \le \liminf a(q) \quad \text{and} \quad \limsup a(q) \le \liminf b_n,$$

which ends the proof. $\square$

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