ECOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES APPLIQUÉES
UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 41 50. Fax: 01 69 33 30 11

http://www.cmap.polytechnique.fr/

Global Existence and Regularity of Solutions to a System of Nonlinear Maxwell Equations

H. Ammari - K. Hamdache

R.I. N^0 494 *October 2002*

Global existence and regularity of solutions to a system of nonlinear Maxwell equations

Habib Ammari and Kamel Hamdache

Centre de Mathématiques Appliquées
CNRS UMR 7641 & Ecole Polytechnique
Palaiseau 91128 Cedex, France
ammari@cmapx.polytechnique.fr, hamdache@cmapx.polytechnique.fr

Abstract

We consider the model that has been suggested by Greenberg et al. [6] for the ferroelectric behavior of materials. In this model, the usual (linear) Maxwell's equations are supplemented with a constitutive relation in which the electric displacement equals a constant times the electric field plus an internal polarization variable which evolves according to an internal set of nonlinear Maxwell's equations. For such model we provide rigorous proofs of global existence, uniqueness, and regularity of solutions. We also provide some preliminary results on the long-time behavior of solutions. The main difficulties in this study are due to the loss of compactness in the system of Maxwell's equations. These results generalize those in [6], where only solutions with TM (transverse magnetic) symmetry were considered.

Key words: Maxwell's equations, Semilinear hyperbolic systems.

AMS(MOS) Subject classifications: 78A25, 35L45.

1 The model problem

In this paper we investigate the model that has been suggested by Greenberg et al. [6] for the ferroelectric behavior of (dielectric) materials. Other models of ferroelectrics has been discussed by Dàvi [4], [5]. In this model, the usual Maxwell's equations are supplemented with a constitutive relation in which the electric displacement equals a constant times the electric field plus an internal polarization variable. The latter evolves according to an internal set of (nonlinear) Maxwell's equations. More precisely, let Ω be a bounded, open subset of $\mathbb{R}^3$, with a smooth boundary. If the domain Ω is occupied by a ferroelectric (dielectric) material of constant magnetic permeability $\mu > 0$, constant conductivity $\sigma > 0$, and constant electric permittivity $\epsilon > 0$, then the time dependent Maxwell equations take the form:

$$
\begin{cases}
\mu\partial_t \mathbf{H} - \text{curl }\ \mathbf{E} = 0 \text{ in } \mathbb{R}^+ \times \Omega, \\
\epsilon\partial_t(\mathbf{E}+\mathbf{P}) + \sigma\mathbf{E} + \text{curl }\ \mathbf{H} = 0 \text{ in } \mathbb{R}^+ \times \Omega, \\
\mathbf{E}\times\mathbf{n} = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega, \\
\mathbf{H}_{|t=0} = \mathbf{H}_0 \text{ and } \mathbf{E}_{|t=0} = \mathbf{E}_0 \text{ in } \Omega.
\end{cases}
\tag{1.1}
$$

Here $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields and $\mathbf{P}$ the internal electric polarization. The field $\mathbf{P}$ satisfies the following set of (nonlinear) Maxwell's equations

$$
\begin{cases}
\partial_t^2\mathbf{P} + a\ \partial_t\mathbf{P} + \lambda^2 \text{curl}^{\,2}\ \mathbf{P} + b\ \mathbf{P}\phi'(|\mathbf{P}|^2) = b\ \mathbf{E} \text{ in } \mathbb{R}^+ \times \Omega, \\
\text{curl }\ \mathbf{P}\times\mathbf{n} = 0 \text{ or } \mathbf{P}\times\mathbf{n} = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega, \\
\mathbf{P}_{|t=0} = \mathbf{P}_0 \text{ and } \partial_t\mathbf{P}_{|t=0} = \mathbf{P}_1 \text{ in } \Omega,
\end{cases}
\tag{1.2}
$$

where $\lambda^2 = \dfrac{1}{\epsilon\mu}$ and a and b are positive constants. In the system of equations (1.2) we assume that $\phi : \mathbb{R}^+ \to \mathbb{R}$ is a C^2 convex function such that

$$
|\phi'(r)| \leq C_1,\ r\phi^{(2)}(r) \leq C_2 \text{ for } r \geq 0, \tag{1.3}
$$

where r_0^2 is the location of the unique minimum of the function $\phi(r^2)$. These assumptions guarantee that $|(r\phi'(r^2))'| \leq C_*$ for all $r \geq 0$, where $C_* = C_1 + 2C_2$. Therefore,

$$
|X\phi'(|X|^2) - Y\phi'(|Y|^2)| \leq C_*\ |X - Y|, \tag{1.4}
$$

for any $X, Y \in \mathbb{R}^3$.

In their important work [6], Greenberg, MacCamy, and Coffman considered particular solutions of ferroelectric system (1.1)-(1.2) with TM (transverse magnetic) symmetry. They supposed that the ferroelectric material occupies a cylinder with generators parallel to the x_3-axis and a uniform, simply connected cross section ω and considered only solutions which are independent of the variable x_3 and have the special form, $\mathbf{E} = r_0 e\mathbf{u}_3$, $\mathbf{H} = \beta(h_1\mathbf{u}_1 + h_2\mathbf{u}_2)$, and $\mathbf{P} = r_0 p\mathbf{u}_3$ in $\omega \subset \mathbb{R}^2$ where $\beta = r_0\sqrt{\frac{\epsilon}{\mu}}$ and $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is an orthonormal basis of $\mathbb{R}^3$. Reducing so the coupled full (complicated) systems of Maxwell's equations (1.1) and (1.2) to scalar wave equations, they were able to study the asymptotic behavior with respect to the time variable and prove that the reduced (scalar) ferroelectric system tends to a steady state in which the scalar polarization is governed by a nonlinear (scalar) equation that has multiple solutions.

Our aim in this paper is to study existence, uniqueness and space and time regularities of solutions to the full ferroelectric system (1.1)-(1.2). The main difficulties in this study are due to the loss of compactness in the system of Maxwell's equations which changes the spectral properties of the resolvent of its linear part. Further, one does not have any L^2-estimates on the divergences of the electric field and the internal electric polarization. Other questions related to the model proposed by Greenberg and all. are also discussed in [8] and [1]. In [8], the existence of time harmonic solutions is studied for the full model and the low frequency behaviour of the solutions for the reduced model is proved. Many examples of potential functions ϕ satisfying hypothesis (1.3) are also given. In [1], the behaviour of thin ferroelectric model is proved and in [7] it is proved a local existence result of solutions to the 1-D time dependent model proposed by [4].

The present paper is organized as follows. In section 2, we prove global existence and uniqueness of energy solutions of the full ferroelectric system (1.1)-(1.2). In section 3, we provide energy bounds and time regularity of solutions. In section 4, we study the space regularity of the solutions if the internal electric polarization is subject to the boundary condition $\mathbf{P} \times \mathbf{n} = 0$.

2 Global existence, uniqueness and time regularity

Throughout this paper, we shall use quite standard L^2- Sobolev spaces to measure regularity. We set $\mathbb{L}^2(\Omega) = (L^2(\Omega))^3$ and denote $|\cdot|_{\mathbb{L}^2}$ the $\mathbb{L}^2$-norm and $(\ ;\ )_{\mathbb{L}^2}$ its scalar product. We also consider the following classical spaces used in the theory of Maxwell equations $\mathcal{H}(\mathrm{curl}\,,\Omega) = \{\mathbf{u} \in \mathbb{L}^2(\Omega),\ \mathrm{curl}\,\mathbf{u} \in \mathbb{L}^2(\Omega)\}$ and $\mathcal{H}_0(\mathrm{curl}\,,\Omega) = \{\mathbf{u} \in \mathcal{H}(\mathrm{curl}\,,\Omega), \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Omega\}$ the closure of $(\mathcal{D}(\Omega))^3$ in $\mathcal{H}(\mathrm{curl}\,,\Omega)$ where $\mathbf{n}$ denotes the outward unit normal to Ω. These Hilbert spaces are equipped with the scalar product $(\mathbf{u};\mathbf{v})_{\mathcal{H}} = (\mathbf{u};\mathbf{v})_{\mathbb{L}^2} + (\mathrm{curl}\,\mathbf{u};\mathrm{curl}\,\mathbf{v})_{\mathbb{L}^2}$. We have that $\mathcal{H}(\mathrm{curl}\,,\Omega)$ and $\mathcal{H}_0(\mathrm{curl}\,,\Omega)$ are dense in $\mathbb{L}^2(\Omega)$. Moreover $(\mathcal{D}(\overline{\Omega}))^3$ is dense in $\mathcal{H}(\mathrm{curl}\,,\Omega)$. For any $\mathbf{u} \in \mathcal{H}(\mathrm{curl}\,,\Omega))$, the trace $\mathbf{u} \times \mathbf{n}$ is defined as an element of $\mathbb{H}^{-1/2}(\partial\Omega)$. Let us define the Hilbert space $\mathcal{H}_0^1(\mathrm{curl}\,,\Omega) = \{u \in \mathcal{H}(\mathrm{curl}\,,\Omega),\ \mathrm{curl}\,u \in \mathcal{H}_0(\mathrm{curl}\,,\Omega)\}$. Since we have $(\mathcal{D}(\overline{\Omega}))^3 \subset \mathcal{H}_0^1(\mathrm{curl}\,,\Omega) \subset \mathcal{H}(\mathrm{curl}\,,\Omega)$ then $\mathcal{H}_0^1(\mathrm{curl}\,,\Omega)$ is dense in $\mathcal{H}(\mathrm{curl}\,,\Omega)$, see [3] and [12] for example.

For $(\mathbf{H}_0, \mathbf{E}_0, \mathbf{P}_0, \mathbf{P}_1)$ satisfying the hypotheses

$$\mathbf{E}_0, \mathbf{H}_0,\ \mathbf{P}_0, \mathrm{curl}\,\mathbf{P}_0,\ \text{and } \mathbf{P}_1 \in \mathbb{L}^2(\Omega), \tag{2.5}$$

we say that $(\mathbf{H}, \mathbf{E}, \mathbf{P})$ is a weak (or mild) solution to our problem if (1.1)-(1.2) are satisfied in the sense of distributions and

$$\begin{cases} \mathbf{E}, \mathbf{H} \in L^\infty(\mathbb{R}^+, \mathbb{L}^2(\Omega)) \text{ and } \mathbf{E} \in L^2(\mathbb{R}^+, \mathbb{L}^2(\Omega)), \\ \\ \partial_t\ \mathbf{P}, \mathrm{curl}\ \ \mathbf{P} \in L^\infty(\mathbb{R}^+, \mathbb{L}^2(\Omega)) \text{ and } \partial_t \mathbf{P} \in L^2(\mathbb{R}^+, \mathbb{L}^2(\Omega)). \end{cases} \tag{2.6}$$

Moreover, for all $t \geq 0$, the following energy identity holds

$$\mathcal{E}(t) + 2\int_0^t (\frac{b}{\epsilon}\sigma|\mathbf{E}(s)|^2 + a\ |\partial_s \mathbf{P}(s)|^2) ds = \mathcal{E}_0, \tag{2.7}$$

where $\mathcal{E}(t)$ is the energy at the time $t > 0$ while $\mathcal{E}_0$ is the initial one. $\mathcal{E}(t)$ and $\mathcal{E}_0$ are defined by

$$\mathcal{E}(t) = |\partial_t \mathbf{P}|^2_{\mathbb{L}^2(\Omega)} + \lambda^2 |\mathrm{curl}\,\mathbf{P}|^2_{\mathbb{L}^2(\Omega)} + b\int_\Omega \phi(|\mathbf{P}(t)|^2))dx + b|\mathbf{E}|^2_{\mathbb{L}^2(\Omega)} + \frac{b\mu}{\epsilon}|\mathbf{H}|^2_{\mathbb{L}^2(\Omega)}, \tag{2.8}$$

and

$$\mathcal{E}_0 = |\mathbf{P}_1|^2_{\mathbb{L}^2(\Omega)} + \lambda^2 |\mathrm{curl}\,\mathbf{P}_0|^2_{\mathbb{L}^2(\Omega)} + b\int_\Omega \phi(|\mathbf{P}_0|^2))dx + b|\mathbf{E}_0|^2_{\mathbb{L}^2(\Omega)} + \frac{b\mu}{\epsilon}|\mathbf{H}_0|^2_{\mathbb{L}^2(\Omega)}. \tag{2.9}$$

We will say that $(\mathbf{H},\mathbf{E},\mathbf{P})$ is a classical solution to our problem, associated with the initial data $(\mathbf{H}_0,\mathbf{E}_0,\mathbf{P}_0,\mathbf{P}_1)$ satisfying hypotheses (2.5) and

$$\begin{cases} \operatorname{curl}\mathbf{E}_0, \operatorname{curl}\mathbf{H}_0,\ \operatorname{curl}^2\mathbf{P}_0,\ \text{and } \operatorname{curl}\mathbf{P}_1 \in \mathbb{L}^2(\Omega), \\ \mathbf{E}_0\times\mathbf{n} = 0 \text{ and } \operatorname{curl}\mathbf{P}_0\times\mathbf{n} = 0 \text{ or } \mathbf{P}_0\times\mathbf{n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.10}$$

if on one hand it satisfies (2.6) and on the other hand we have for all $T > 0$ and for all $t \in [0,T[$

$$\begin{cases} \operatorname{curl}\mathbf{E}(t), \operatorname{curl}\mathbf{H}(t),\ \operatorname{curl}^2\mathbf{P}(t),\ \text{and } \operatorname{curl}\partial_t\mathbf{P}(t) \in \mathbb{L}^2(\Omega), \\ \mathbf{E}(t)\times\mathbf{n} = 0 \text{ and } \operatorname{curl}\mathbf{P}(t)\times\mathbf{n} = 0 \text{ or } \mathbf{P}_0\times\mathbf{n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.11}$$

Moreover, the solution satisfies (1.1)-(1.2) in $L^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ and the following energy equality holds

$$\frac{d\mathcal{E}(t)}{dt} + \frac{b}{\epsilon}\sigma|\mathbf{E}(t)|^2 + a\ |\partial_t\mathbf{P}(t)|^2 = 0. \tag{2.12}$$

2.1 Existence and uniqueness

Our goal in this subsection is to prove existence and uniqueness of solutions to problem (1.1)-(1.2). We use classical results of the semigroups theory and its application to semilinear equations. We refer to the classical monographs [2], [9], [11], [10] for this subject. Here we confine our attention to the case where the polarization field $\mathbf{P}$ satisfies the boundary condition $\operatorname{curl}\mathbf{P}\times\mathbf{n} = 0$. The proofs of the results stated in this section in the case where we use the boundary condition $\mathbf{P}\times\mathbf{n} = 0$ can be obtained with only minor modifications.

We begin by giving an abstract formulation to our problem. We set $U = (U^1, U^2)$ with $U^1 = (U_1, U_2) \in \mathbb{R}^3\times\mathbb{R}^3$ and $U^2 = (U_3, U_4) \in \mathbb{R}^3\times\mathbb{R}^3$. Here $U_1 = \mathbf{H}$, $U_2 = \mathbf{E}$, $U_3 = \mathbf{P}$ and $U_4 = \partial_t\mathbf{P}$. It follows that U satisfies the abstract equation

$$\begin{cases} \Lambda\dfrac{dU}{dt} + \mathcal{J}(U) + \mathcal{K}(U) + \Psi(U) = 0, \\ U(0) = U_0, \end{cases} \tag{2.13}$$

with $U_0 = (U_0^1, U_0^2)$, $U_0^1 = (\mathbf{H}_0,\mathbf{E}_0)$ and $U_0^2 = (\mathbf{P}_0,\mathbf{P}_1)$. The entries λ_{ij} of the 12×12 diagonal matrix Λ are given by $\lambda_{ii} = \mu$ for $i = 1, \cdot, 3$, $\lambda_{ii} = \epsilon$ for $i = 4, \cdot, 6$, $\lambda_{ii} = 1$ for $i = 7, \cdots, 12$ and $\lambda_{ij} = 0$ for $i \neq j$. The matrix $\mathcal{K}$ is defined by $\mathcal{K}(U) = (0, \sigma U_2 + \epsilon U_4, 0, aU_4 - bU_2)$. The nonlinear function Ψ is given by $\Psi(U) = (0,0,0,-\alpha U_3 + bU_3\phi'(|U_3|^2))$ where $\alpha > 0$ is an arbitrary fixed constant. The unbounded operator $\mathcal{J}$ is formally given by $\mathcal{J}(U) = (\mathcal{A}(U^1), \mathcal{B}(U^2))$ with $\mathcal{A}(U^1) = (-\operatorname{curl}U_2, \operatorname{curl}U_1)$ and $\mathcal{B}(U^2) = (-U_4, \lambda^2\operatorname{curl}^2U_3 + \alpha U_3)$.

Let us define the Hilbert spaces $\mathcal{X} = \mathcal{H}_1\times\mathcal{H}_2$ with $\mathcal{H}_1 = \mathbb{L}^2(\Omega)\times\mathbb{L}^2(\Omega)$ and $\mathcal{H}_2 = \mathcal{H}(\operatorname{curl},\Omega)\times\mathbb{L}^2(\Omega)$ equipped with their natural scalar products but in the sequel $\mathcal{H}(\operatorname{curl},\Omega)$ is endowed with the weighted scalar product $(U;V)_{\mathcal{H}} = \lambda^2(\operatorname{curl}U;\operatorname{curl}U)_{\mathbb{L}^2} + \alpha(U;V)_{\mathbb{L}^2}$. The scalar product of $\mathcal{X}$ is denoted by $\langle U;V\rangle_{\mathcal{X}} = (U^1;V^1)_{\mathcal{H}_1} + (U^2;V^2)_{\mathcal{H}_2}$ with $(U^1;V^1)_{\mathcal{H}_1} = (U_1;V_1)_{\mathbb{L}^2} + (U_2;V_2)_{\mathbb{L}^2}$ and $(U^2;V^2)_{\mathcal{H}_2} = (U_3;V_3)_{\mathcal{H}} + (U_4;V_4)_{\mathbb{L}^2}$.

The unbounded operator $\mathcal{A}$ is defined on the Hilbert space $\mathcal{H}_1$ by setting $\mathcal{A}(U) = (-\operatorname{curl}U_1, \operatorname{curl}U_2)$ for all $U = (U_1,U_2) \in D(\mathcal{A})$ where the domain $D(\mathcal{A}) = \{U = (U_1,U_2) \in$

$\mathcal{H}_1,\ \mathcal{A}(U) \in \mathcal{H}_1,\ U_2 \times \mathbf{n} = 0$ on $\partial\Omega\}$. Hence, we have $D(\mathcal{A}) = \mathcal{H}(\mathrm{curl}\,,\Omega) \times \mathcal{H}_0(\mathrm{curl}\,,\Omega)$ which is dense in $\mathcal{H}_1$. Clearly, $\mathcal{A}$ is densely defined on $\mathcal{H}_1$. By using Green's formula we get

$$(\mathcal{A}(U);V)_{\mathcal{H}_1} = -(U;\mathcal{A}(V))_{\mathcal{H}_1} \text{ for all } U,V \in D(\mathcal{A}). \tag{2.14}$$

Hence, we have $(\mathcal{A}(U);U)_{\mathcal{H}_1} = 0$ for all $U \in D(\mathcal{A})$. Then $\mathcal{A}$ is dissipative on $\mathcal{H}_1$ and its adjoint operator is given by $\mathcal{A}^* = -\mathcal{A}$ with domain $D(\mathcal{A}^*) = D(\mathcal{A})$, see [10], [2] for example.

Next, the unbounded operator $\mathcal{B}$ is defined on $\mathcal{H}_2$ by $\mathcal{B}(U) = (-U_4, \lambda^2\mathrm{curl}^{\,2}U_3 + \alpha U_3)$ for all $U = (U_3,U_4) \in D(\mathcal{B})$ where its domain is given by $D(\mathcal{B}) = \{U = (U_3,U_4) \in \mathcal{H}_2,\ \mathcal{B}(U) \in \mathcal{H}_2,\ \mathrm{curl}\,U_3 \times \mathbf{n} = 0$ on $\partial\Omega\}$. It follows that $D(\mathcal{B}) = \mathcal{H}_0^1(\mathrm{curl}\,,\Omega) \times \mathcal{H}(\mathrm{curl}\,,\Omega)$ where $\mathcal{H}_0^1(\mathrm{curl}\,,\Omega) = \{U_3 \in \mathcal{H}(\mathrm{curl}\,,\Omega),\ \mathrm{curl}\,U_3 \in \mathcal{H}_0(\mathrm{curl}\,,\Omega)\}$. Thus $D(\mathcal{B})$ is dense in $\mathcal{H}_2$. By Green's formula (having in mind the new choice of the scalar product of the space $\mathcal{H}(\mathrm{curl}\,,\Omega)$) we get

$$(\mathcal{B}(U);V)_{\mathcal{H}_2} = -(U;\mathcal{B}(V))_{\mathcal{H}_2} \text{ for all } U,V \in D(\mathcal{B}), \tag{2.15}$$

which shows that the adjoint operator of $\mathcal{B}$ is given by $\mathcal{B}^* = -\mathcal{B}$ with $D(\mathcal{B}^*) = D(\mathcal{B})$. Moreover, we have $(\mathcal{B}(U);U)_{\mathcal{H}_2} = 0$ for all $U \in D(\mathcal{B})$ which means that $\mathcal{B}$ is dissipative on $\mathcal{H}_2$, see [10], [2] for example.

The unbounded operator $\mathcal{J}$ is defined on the Hilbert space $\mathcal{X}$ as follows. For all $U = (U^1,U^2) \in D(\mathcal{J}) = D(\mathcal{A}) \times D(\mathcal{B})$ we set $\mathcal{J}(U) = (\mathcal{A}(U^1), \mathcal{B}(U^2))$. Hence, $D(\mathcal{J})$ is dense in $\mathcal{X}$ and we have

$$\langle \mathcal{J}(U);V\rangle_{\mathcal{X}} = -\langle U;\mathcal{J}(V)\rangle_{\mathcal{X}} \text{ for all } U,V \in D(\mathcal{J}). \tag{2.16}$$

As previously, we deduce that the adjoint of $\mathcal{J}$ is given by $\mathcal{J}^* = -\mathcal{J}$ with $D(\mathcal{J}^*) = D(\mathcal{J})$ and that $\mathcal{J}$ is dissipative on $\mathcal{X}$ since it satisfies $\langle \mathcal{J}(U);U\rangle_{\mathcal{X}} = 0$ for all $U \in D(\mathcal{J})$. The following result holds.

Lemma 2.1 *Let $\nu > 0$ be fixed. Then, the operator $\mathcal{J} + \nu\Lambda$ is maximal dissipative on $\mathcal{X}$. Moreover, $\mathcal{J} + \nu\Lambda$ is the the infinitesimal generator of a C^0 semi-group on $\mathcal{X}$.*

Proof. For all $U \in D(\mathcal{J})$, we have $\langle \pm\mathcal{J}(U) + \nu\Lambda U;U\rangle_{\mathcal{X}} = \nu|\Lambda^{1/2}U|^2_{\mathcal{X}} \geq \beta|U|^2_{\mathbb{L}^2}$. Thus, it follows that $\mathcal{J} + \nu\Lambda$ and $\mathcal{J}^* + \nu\Lambda$ are one to one with closed range. We deduce that the operator $\mathcal{J} + \nu\Lambda$ is an isomorphism from $D(\mathcal{J})$ (equipped with the graph norm) into $\mathcal{X}$. Hence, the lemma is proved by using the classical results of the semi-groups theory, see [2] and [10] for example.

Let us consider problem (2.13). We introduce V by setting $U(t) = e^{\nu t}V(t)$. Then $V(t)$ satisfies the problem

$$\begin{cases} \Lambda \dfrac{dV}{dt} + \mathcal{J}(V) + \nu\Lambda V + \mathcal{K}(V) + \Psi(t,V) = 0 \\ V(0) = U_0, \end{cases} \tag{2.17}$$

where $\Psi(t,V) = e^{-\nu t}\Psi(e^{\nu t}V)$. By definition of Ψ and the hypothesis (1.4) satisfied by ϕ one deduces that for all $T > 0$ fixed the function $\Psi(t,V) : [0,T] \times \mathcal{X} \to \mathcal{X}$ is continuous in t and satisfies

$$|\Psi(t,V) - \Psi(t,W)| \leq (\alpha + C_*)|V - W|, \tag{2.18}$$

for all $V, W \in \mathbb{R}^6 \times \mathbb{R}^6$ and $t \geq 0$. Since the operator $\mathcal{K}$ is linear and bounded on $\mathcal{X}$ then the operator $\mathcal{K}(\cdot) + \Psi(t, \cdot)$ is continuous with respect t and is uniformly Lipschitz with respect to V with a different Lipschitz constant than the one involved in (2.17). Using Theorem 1.2, Chp.6 of [10] one immediately deduces that for $U_0 \in \mathcal{X}$ there exists a unique mild solution $V \in C^0([0, T); \mathcal{X})$ to problem (2.17). Moreover, since $\Psi(t, V) = \Psi(t, V_3)$ and ϕ belongs to C^2 then it is readily seen that $\Psi(t, V)$ is continuously differentiable from $[0.T] \times \mathcal{X}$ into $\mathcal{X}$. Therefore, for all $U_0 \in D(\mathcal{J})$ hypothesis of Theorem 1.5, Chp 6 of [10] are satisfied and then the mild solution is a classical solution of the initial value problem (1.1)-(1.2).

Combining the above results we obtain the following.

Theorem 2.1 (**Global existence and uniqueness**) *We assume that the potential function ϕ satisfies hypothesis (1.4). Then, if $(\mathbf{H}_0, \mathbf{E}_0, \mathbf{P}_0, \mathbf{P}_1) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega) \times \mathcal{H}(\mathrm{curl}\,, \Omega) \times \mathbb{L}^2(\Omega)$ there exists a unique mild solution $(\mathbf{H}, \mathbf{E}, \mathbf{P}) \in C^0([0, \infty[; \mathbb{L}^2(\Omega)) \times C^0([0, T]; \mathbb{L}^2(\Omega)) \times C^0([0, \infty[; \mathcal{H}(\mathrm{curl}\,, \Omega)$ to the problem (1.1)-(1.2) such that $\mathbf{P} \in C^1([0, \infty[; \mathcal{H}(\mathrm{curl}\,, \Omega))$. Moreover, the solution satisfies, for all $t > 0$, the energy identity*

$$\mathcal{E}(t) + \int_0^t \frac{b\sigma}{\epsilon} |\mathbf{E}(s)|^2_{\mathbb{L}^2} + a\ |\partial_s \mathbf{P}(s)|^2_{\mathbb{L}^2}\ ds = \mathcal{E}_0, \tag{2.19}$$

where

$$\mathcal{E}(t) = (\frac{b\mu}{\epsilon} |\mathbf{H}(t)|^2_{\mathbb{L}^2} + b|\mathbf{E}(t)|^2_{\mathbb{L}^2}) + |\partial_t \mathbf{P}(t)|^2_{\mathbb{L}^2} + \lambda^2 |\mathrm{curl}\,\mathbf{P}(t)|^2_{\mathbb{L}^2} + b \int_\Omega \phi(|\mathbf{P}(t)|^2)\ dx \tag{2.20}$$

and the initial energy is defined by

$$\mathcal{E}_0 = (\frac{b\mu}{\epsilon} |\mathbf{H}_0|^2_{\mathbb{L}^2} + b|\mathbf{E}_0|^2_{\mathbb{L}^2}) + |\mathbf{P}_1|^2_{\mathbb{L}^2} + \lambda^2 |\mathrm{curl}\,\mathbf{P}_0|^2_{\mathbb{L}^2} + b \int_\Omega \phi(|\mathbf{P}_0|^2)\ dx. \tag{2.21}$$

If moreover the initial data are such that $\mathrm{curl}\,\mathbf{H}_0, \mathrm{curl}\,\mathbf{E}_0, \mathrm{curl}^2\mathbf{P}_0, \mathrm{curl}\,\mathbf{P}_1 \in \mathbb{L}^2(\Omega)$ and $\mathbf{E}_0 \times \mathbf{n} = 0$, $\mathrm{curl}\,\mathbf{P}_0 \times \mathbf{n} = 0$ on $\partial\Omega$ then for all $t > 0$ we have $\mathrm{curl}\,\mathbf{H}(t), \mathrm{curl}\,\mathbf{E}(t), \mathrm{curl}^2\mathbf{P}_0 \in \mathbb{L}^2(\Omega)$ and $\partial_t \mathrm{curl}\,\mathbf{P}(t) \in \mathbb{L}^2(\Omega)$ and the following energy equality holds for all $t \geq 0$

$$\frac{d\mathcal{E}(t)}{dt} + (\frac{b\sigma}{\epsilon} |(\mathbf{E}(t)|^2_{\mathbb{L}^2} + a\ |\partial_t \mathbf{P}(t)|^2_{\mathbb{L}^2}) = 0. \tag{2.22}$$

Proof. Let V be the solution of problem (2.14) associated with the initial data $U_0 \in \mathcal{X}$. Hence $U(t) = e^{\nu t} V(t)$ is the unique solution of (2.10) satisfying $\partial_t U_3 = U_4$. It follows that $\mathbf{H} = U_1$, $\mathbf{E} = U_2$ and $\mathbf{P} = U_3$ is the unique solution on $[0, T]$ of problem (1.1)-(1.2) for all $T > 0$. The energy equality (2.18) follows directly and it holds for all $t \in [0, T]$. By a continuity argument, we obtain that the solution exists on $[0, \infty[$. Moreover, to obtain energy equality (2.19), for an initial data in $D(\mathcal{J})$, we multiply the first equation by $\frac{b}{\epsilon}\mathbf{H}$, the second by $\frac{b}{\epsilon}\mathbf{E}$ and the equation of $\mathbf{P}$ by $\partial_t \mathbf{P}$ and integrate by parts the resulting equations. Hence, the energy identity (2.16) follows by integrating with respect to the time variable. Next for $U_0 \in \mathcal{X}$ we use the density of $D(\mathcal{J})$ in $\mathcal{X}$ and the previous result to get (2.16).

2.2 Time-regularity of solutions

We have shown that $\partial_t \mathbf{P} \in C^0([0,\infty[;\mathbb{L}^2(\Omega)) \cap L^1(0,\infty;\mathbb{L}^2(\Omega))$. Then, by using

$$\partial_t(\mathbf{P}\phi'(|\mathbf{P}|^2)) = \phi'(|\mathbf{P}|^2))\partial_t(\mathbf{P}) + 2\phi^{(2)}(|\mathbf{P}|^2))\mathbf{P}\otimes\mathbf{P}(\partial_t\mathbf{P}), \tag{2.23}$$

and hypothesis (1.3), it follows that $\partial_t(\mathbf{P}\phi'(|\mathbf{P}|^2)) \in C^0([0,\infty[;\mathbb{L}^2(\Omega)) \cap L^1(0,\infty;\mathbb{L}^2(\Omega))$. We set

$$\mathbf{G}(t) = e^{-\nu t}V_3\phi'(e^{\nu t}|V_3|^2)), \tag{2.24}$$

where V is the classical solution of the problem (2.17). Since ϕ belongs to C^2 and satisfies hypothesis (1.4) then from

$$\begin{aligned} \partial_t\mathbf{G} &= -\nu e^{-\nu t}V_3\phi'(|e^{\nu t}V_3(t)|^2) + e^{-\nu t}(\partial_t V_3)\phi'(|e^{\nu t}V_3(t)|^2) \\ &+ 2\nu e^{-\nu t}V_3|e^{\nu t}V_3|^2\phi^{(2)}(|e^{\nu t}V_3|^2) + e^{-\nu t}(e^{\nu t}V_3\otimes e^{\nu t}V_3)(\partial_t V_3)\phi^{(2)}(|e^{\nu t}V_3|^2), \end{aligned} \tag{2.25}$$

we deduce that for all $t \geq 0$

$$|\partial_t\mathbf{G}(\mathbf{t})| \leq C_{**}e^{-\nu t}(|V_3(t)| + |\partial_t V_3(t)|), \tag{2.26}$$

where $C_{**} > 0$ depends only on the constants involved in (1.4). This shows that $\partial_t\mathbf{G} \in C^0([0,\infty[;\mathbb{L}^2(\Omega)) \cap L^1(0,\infty;\mathbb{L}^2(\Omega))$.

Next, we set $W = \partial_t V$. It is readily seen that W satisfies the problem

$$\begin{cases} \Lambda\dfrac{dW}{dt} + \mathcal{J}(W) + \nu\Lambda W + \mathcal{K}(W) = \mathcal{S}(t), \\ \Lambda W(0) = \Lambda W_0 := -\mathcal{J}(U_0) - \nu\Lambda U_0 - \mathcal{K}(U_0) - \Psi(U_0), \end{cases} \tag{2.27}$$

where $\mathcal{S}(t) = -\partial_t(\Psi(t,V(t))) = -(0,0,0,\partial_t\mathbf{G}(t))$. The source term $\mathcal{S} \in C^0(]0,T[;\mathcal{X}) \cap L^1(0,T;\mathcal{X})$ for all $T > 0$. Since we assume that $U_0 \in D(\mathcal{J})$ then $\Lambda W_0 \in \mathcal{X}$. It follows, by Theorem 1.2, Chp. 6 of [10], that W is the unique mild solution of problem (2.27). Hence we conclude that $W = \partial_t V \in C^0([0,T];\mathcal{X})$.

Let us now consider problem (1.1)-(1.2). We set $\mathbf{H}^1 = \partial_t\mathbf{H}$, $\mathbf{E}^1 = \partial_t\mathbf{E}$ and $\mathbf{P}_1 = \partial_t\mathbf{P}$. Then $(\mathbf{H}^1,\mathbf{E}^1,\mathbf{P}^1)$ satisfies, in the sense of distributions, the problem

$$\begin{cases} \mu\partial_t\mathbf{H}^1 - \text{curl }\mathbf{E}^1 = 0 \text{ in } \mathbb{R}^+\times\Omega, \\ \epsilon\partial_t(\mathbf{E}^1 + \mathbf{P}^1) + \sigma\mathbf{E}^1 + \text{curl }\mathbf{H}^1 = 0 \text{ in } \mathbb{R}^+\times\Omega, \\ \mathbf{E}^1\times\mathbf{n} = 0 \text{ on } \mathbb{R}^+\times\partial\Omega, \\ \mathbf{H}^1_{|t=0} = \mathbf{H}^1_0 \text{ and } \mathbf{E}^1_{|t=0} = \mathbf{E}^1_0 \text{ in } \Omega, \end{cases} \tag{2.28}$$

coupled to

$$\begin{cases} \partial_t^2 \mathbf{P}^1 + a\ \partial_t \mathbf{P}^1 + \lambda^2 \mathrm{curl}^{\,2}\ \mathbf{P}^1 + b\ \mathbf{P}^1 \phi'(|\mathbf{P}|^2)) = b\ \mathbf{E}^1 - 2b\phi^{(2)}(|\mathbf{P}|^2)\mathbf{P}\otimes\mathbf{P}(\mathbf{P}^1) \text{ in } \mathbb{R}^+ \times \Omega, \\ \\ \mathrm{curl}\ \ \mathbf{P}^1 \times \mathbf{n} = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega, \\ \\ \mathbf{P}^1_{|t=01} = \mathbf{P}^1_0 \\ \\ \text{and } \partial_t \mathbf{P}^1_{|t=0} = \mathbf{P}^1_1 \text{ in } \Omega, \end{cases} \tag{2.29}$$

where the initial data are given by

$$\begin{cases} \mathbf{H}^1_0 = \frac{1}{\mu}\mathrm{curl}\,\mathbf{E}_0 \in \mathbb{L}^2(\Omega),\ \ \mathbf{E}^1_0 = -\frac{1}{\epsilon}\mathrm{curl}\,\mathbf{H}_0 - \mathbf{P}_1 - \frac{\sigma}{\epsilon}\mathbf{E}_0 \in \mathbb{L}^2(\Omega), \\ \\ \mathbf{P}^1_0 = \mathbf{P}_1 \in \mathbb{L}^2(\Omega),\ \ \mathbf{P}^1_1 = -a\mathbf{P}_1 - \lambda^2 \mathrm{curl}^{\,2}\mathbf{P}_0 + b\mathbf{E}_0 - b\mathbf{P}_0\phi'(|\mathbf{P}_0|^2) \in \mathbb{L}^2(\Omega). \end{cases} \tag{2.30}$$

Since we have shown that $\partial_t V$ is the mild solution of the problem (2.27) then it is easy to see that $\partial_t U$ is also the mild solution of the equivalent problem where $\nu = 0$ and $\mathcal{S} = -b\partial_t(\mathbf{P}\phi'(|\mathbf{P}|^2))$. Then, in a similar manner as for (2.19), the following energy equality holds

$$\begin{cases} \mathcal{E}^1(t) + \int_0^t (\frac{b}{\epsilon}\sigma|\mathbf{E}^1(s)|^2_{\mathbb{L}^2(\Omega)} + a|\partial_s \mathbf{P}^1(s)|^2_{\mathbb{L}^2(\Omega)})ds = \\ \\ \qquad \mathcal{E}^1_0 - b\int_0^t (\partial_s(\mathbf{P}\phi'(|\mathbf{P}(s)|^2)); \partial_s \mathbf{P}^1(s)_{\mathbb{L}^2(\Omega)} ds \end{cases} \tag{2.31}$$

where

$$\mathcal{E}^1(t) = |\partial_t \mathbf{P}^1|^2_{\mathbb{L}^2(\Omega)} + \lambda^2|\mathrm{curl}\,\mathbf{P}^1|^2_{\mathbb{L}^2(\Omega)} + b|\mathbf{E}^1|^2_{\mathbb{L}^2(\Omega)} + \mu|\mathbf{H}^1|^2_{\mathbb{L}^2(\Omega)} \tag{2.32}$$

and

$$\mathcal{E}^1_0 = |\mathbf{P}^1_1|^2_{\mathbb{L}^2(\Omega)} + \lambda^2|\mathrm{curl}\,\mathbf{P}^1_0|^2_{\mathbb{L}^2(\Omega)} + b|\mathbf{E}^1_0|^2_{\mathbb{L}^2(\Omega)} + \mu|\mathbf{H}^1_0|^2_{\mathbb{L}^2(\Omega)}. \tag{2.33}$$

By Cauchy-Schwartz inequality and the derivation formula

$$\partial_t(\mathbf{P}\phi'(|\mathbf{P}|^2)) = \mathbf{P}^1\phi'(|\mathbf{P}|^2)) + 2\phi^{(2)}(|\mathbf{P}|^2)(\mathbf{P}\otimes\mathbf{P})(\mathbf{P}^1)$$

we deduce that $|\int_0^t (\partial_s(\mathbf{P}\phi'(|\mathbf{P}(s)|^2)); \partial_s \mathbf{P}^1(s)_{\mathbb{L}^2(\Omega)} ds|$ is bounded by $C\int_0^1 |\mathbf{P}^1||\partial_s \mathbf{P}^1| ds$ where the constant $C > 0$ depends only on the constant involved in hypothesis (1.4). The energy equality (2.31) becomes

$$\mathcal{E}^1(t) + \int_0^t (\frac{b}{\epsilon}\sigma|\mathbf{E}^1(s)|^2_{\mathbb{L}^2(\Omega)} + \frac{a}{2}|\partial_s \mathbf{P}^1(s)|^2_{\mathbb{L}^2(\Omega)})ds \le \mathcal{E}^1_0 + \frac{b^2C^2}{a}\int_0^t |\mathbf{P}^1(s)|^2_{\mathbb{L}^2(\Omega)} ds \tag{2.34}$$

and by (2.19) we get

$$\mathcal{E}^1(t) + \int_0^t (\frac{b}{\epsilon}\sigma|\mathbf{E}^1(s)|^2_{\mathbb{L}^2(\Omega)} + \frac{a}{2}|\partial_s \mathbf{P}^1(s)|^2_{\mathbb{L}^2(\Omega)})ds \le \mathcal{E}^1_0 + \frac{b^2C^2}{a^2}\mathcal{E}_0. \tag{2.35}$$

We have proved the following time regularity result.

Theorem 2.2 *Let* $(\mathbf{H}, \mathbf{E}, \mathbf{P})$ *be the classical solution of problem (1.1)-(1.2). Then* $\partial_t^2 \mathbf{P}$, $\partial_t \mathrm{curl}\, \mathbf{P}$, $\partial_t \mathbf{H}$, $\partial_t \mathbf{E}$ *belong to* $L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ *and* $\partial_t^2 \mathbf{P}$, $\partial_t \mathbf{E}$ *are in* $L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega))$*. Moreover, the energy inequality (2.35) holds.*

Using equations (1.1)-(1.2) we deduce also the following regularity results.

Lemma 2.2 *The classical solution* $(\mathbf{H}, \mathbf{E}, \mathbf{P})$ *to problem (1.1)-(1.2) is such that* $\mathrm{curl}\, \mathbf{H}$ *and* $\lambda^2 \mathrm{curl}^{\,2} \mathbf{P} + b\mathbf{P}\phi'(|\mathbf{P}|^2)$ *are uniformly bounded in* $L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega))$.

Remark 2.1 *As we have said in the introduction all the results performed in this section remain true in the case where* $\mathbf{P}$ *satisfies the boundary condition* $\mathbf{P} \times \mathbf{n} = 0$ *on* $\partial\Omega$.

3 Space-regularity of the solutions

Let $\mathbf{P}$ be a classical solution of the problem satisfying $\mathbf{P} \in L^\infty(\mathbb{R}^+; \mathcal{H}_0^1(\mathrm{curl}\,, \Omega))$. Setting $Z(t,x) = \mathrm{curl}\, \mathbf{P}(t,x)$ it follows that Z, $\mathrm{curl}\, Z \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ with $\mathrm{div}\, Z = 0$ and $Z(t,\cdot) \times \mathbf{n} = 0$. Using a classical estimate, see [3], we deduce that $\mathrm{curl}\, \mathbf{P} \in L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega))$ and satisfies

$$|\mathrm{curl}\, \mathbf{P}|^2_{L^\infty(\mathbb{R}^+;\mathbb{H}^1(\Omega))} \leq C |\mathrm{curl}\, \mathbf{P}|^2_{L^\infty(\mathbb{R}^+;\mathcal{H}_0(\mathrm{curl}\,,\Omega))}. \tag{3.36}$$

We will come back to this classical estimate since it will play a crucial role in the proof of our $\mathbb{H}^1$- estimate of the solutions of the problem.

Let $\mathbf{P}, \mathrm{curl}\, \mathbf{P}$, and $\mathrm{curl}^{\,2} \mathbf{P} \in \mathbb{L}^2(\Omega)$ be such that $\mathrm{curl}\, \mathbf{P} \times \mathbf{n} = 0$ on $\mathbb{R}^+ \times \partial\Omega$. It is easy to see that in general the $\mathrm{grad}\, \mathbf{P}$ does not belong to $\mathbb{H}^1(\Omega)$. To see that it suffices to consider $\mathbf{P} = \mathrm{grad}\, \varphi$ with $\varphi \in H^1(\Omega)$. Hence φ satisfies $\Delta\varphi \in H^{-1}(\Omega)$ without any boundary condition and so, $\mathrm{grad}\, \mathrm{grad}\, \varphi$ does not belong in general to $\mathbb{H}^1(\Omega)$.

Let $(\mathbf{E}, \mathbf{P})$ be a solution of the problem (1.1)-(1.2) satisfying the boundary condition

$$\mathbf{E} \times \mathbf{n} = 0,\ \ \mathbf{P} \times \mathbf{n} = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega. \tag{3.37}$$

We first remark that it was pointed out by Coffman, Greenberg and Mac Camy [6] that the right boundary condition satisfied by the polarization field $\mathbf{P}$ is $\mathrm{curl}\, \mathbf{P} \times \mathbf{n} = 0$ because the internal magnetic field $\mathbf{m}$ is subject to the boundary condition $\mathbf{m} \times \mathbf{n} = 0$. If $\mathbf{P}$ satisfies the boundary condition (3.37) then we deduce also that $\mathrm{curl}\,(\mathrm{curl}\, \mathbf{P}) \times \mathbf{n} = 0$. The main difficulty of the problem is its lack of compactness. Hence we would like to understand how to obtain the compactness of the solutions. This question is very important for example in the study of the stability of the model with respect to the oscillations of the initial data.

To perform the following computations we assume that the solutions to our problem with (3.37) belong to $L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega))$. The compatibility conditions associated with problem (1.1)-(1.2) subject to the boundary conditions (3.37) are given by the following set of equations that hold in the sense of distributions

$$\begin{cases} \mu\partial_t \mathbf{h} = 0,\ \epsilon\partial_t(\mathbf{e}+\mathbf{p}) + \sigma\mathbf{e} = 0, \\ \\ \partial_t^2\mathbf{p} + a\ \partial_t\mathbf{p} + b\phi'(|\mathbf{P}|^2)\ \mathbf{p} - b\ \mathbf{e} = -2b\ \phi^{(2)}(|\mathbf{P}|^2)\mathbf{P}_i\mathbf{P}_j\partial_{x_i}\mathbf{P}_j, \end{cases} \tag{3.38}$$

where we have used the notations $\mathbf{h} = \operatorname{div}\mathbf{H}$, $\mathbf{e} = \operatorname{div}\mathbf{E}$ and $\mathbf{p} = \operatorname{div}\mathbf{P}$ and the relation

$$\operatorname{div}(\mathbf{P}\phi'(|\mathbf{P}|^2)) = \mathbf{p}\phi'(|\mathbf{P}|^2) + 2\phi^{(2)}(|\mathbf{P}|^2)\mathbf{P}_i\mathbf{P}_j\partial_{x_i}\mathbf{P}_j.$$

Introducing the new variable $W = (\mathbf{e}, \mathbf{p}, \partial_t\mathbf{p})$ and $W_0 = (\mathbf{e}_0, \mathbf{p}_0, \mathbf{p}_1)$ we get the system of ordinary differential equations where the space variable $x \in \Omega$ appears as a parameter

$$\begin{cases} \dfrac{dW}{dt} + \mathcal{C}(\mathbf{P})W = \mathcal{S}(\operatorname{grad}\mathbf{P}), \\ \\ W(0) = W_0 \end{cases} \tag{3.39}$$

where we set

$$\mathcal{C}(\mathbf{P}) = \begin{pmatrix} \theta & 0 & 1 \\ \\ 0 & 0 & -1 \\ \\ -b & b\phi'(|\mathbf{P}|^2) & a \end{pmatrix},\ \mathcal{S}(\operatorname{grad}\mathbf{P}) = \begin{pmatrix} 0 \\ 0 \\ -2b\ \phi^{(2)}(|\mathbf{P}|^2)\mathbf{P}_i\mathbf{P}_j\partial_{x_i}\mathbf{P}_j \end{pmatrix}. \tag{3.40}$$

We shall assume that

$$\operatorname{div}\mathbf{H}_0,\ \operatorname{div}\mathbf{E}_0,\ \operatorname{div}\mathbf{P}_0,\ \operatorname{div}\mathbf{P}_1 \in L^2(\Omega), \tag{3.41}$$

which implies that $W_0 \in \mathbb{L}^2(\Omega)$. Since we have $|\mathcal{C}(\mathbf{P}(t,x))| \le C_1$ and $|\mathcal{S}(\operatorname{grad}\mathbf{P}(t,x))| \le C_2|\operatorname{grad}\mathbf{P}(t,x)|$ a.e (t,x) where $C_1 > 0$ and $C_2 > 0$ are independent of $\mathbf{P}$ but depend on ϕ, a, θ and b then, we deduce that $W(t,x)$ satisfies for all $T > 0$ and $0 \le t \le T$ the inequality

$$|W(t,x)|^2 \le \gamma e^{\beta t}(|W_0(x)|^2 + \int_0^t |\operatorname{grad}\mathbf{P}(s,x)|^2 ds), \tag{3.42}$$

where $\gamma > 0$ and $\beta > 0$ depend only on the constants C_1 and C_2.

We now use the following classical estimate, see [3] Chp. IX, p.345-346. There exists $C > 0$ depending only on Ω such that for all $\mathbf{Q} \in L^\infty(\mathbb{R}^+; \mathcal{H}_0(\operatorname{curl},\Omega) \cap \mathcal{H}(\operatorname{div},\Omega))$ we have

$$|\operatorname{grad}\mathbf{Q}(t,\cdot)|_{\mathbb{L}^2(\Omega)} \le C(|\operatorname{curl}\mathbf{Q}(t,\cdot)|^2_{\mathbb{L}^2(\Omega)} + |\operatorname{div}\mathbf{Q}(t,\cdot)|^2_{\mathbb{L}^2(\Omega)} + |\mathbf{Q}(t,\cdot)|^2_{\mathbb{L}^2(\partial\Omega)}), \tag{3.43}$$

for all $\mathbf{Q} \in L^\infty(\mathbb{R}^+; \mathcal{H}_0(\operatorname{curl},\Omega) \cap \mathcal{H}(\operatorname{div},\Omega))$. Hence, by using (3.42), we deduce that any mild solution of problem (1.1)-(1.2) satisfying the boundary condition $\mathbf{P} \times \mathbf{n} = 0$ verifies the estimate

$$\begin{cases} |\operatorname{grad}\mathbf{P}(t,\cdot)|_{\mathbb{L}^2(\Omega)} \le C(|\mathbf{P}(t,\cdot)|^2_{\mathbb{L}^2(\Omega)} + |\operatorname{curl}\mathbf{P}(t,\cdot)|^2_{\mathbb{L}^2(\Omega)}) \\ \\ \qquad +\gamma e^{\beta t}|W_0(\cdot)|^2_{L^2(\Omega)} + \gamma e^{\beta t}\int_0^t |\operatorname{grad}\mathbf{P}(s,\cdot)|^2_{\mathbb{L}^2(\Omega)} ds. \end{cases} \tag{3.44}$$

Since we have shown by Theorem 2.1 that $|\mathbf{P}(t,\cdot)|^2_{\mathbb{L}^2(\Omega)} + |\mathrm{curl}\,\mathbf{P}(t,\cdot)|^2_{\mathbb{L}^2(\Omega)} \leq C\mathcal{E}_0$ where $C > 0$ is independent of $\mathbf{P}$ estimate (3.44) becomes then

$$|\mathrm{grad}\,\mathbf{P}(t,\cdot)|_{\mathbb{L}^2(\Omega)} \leq C(\mathcal{E}_0 + \gamma e^{\beta t}|W_0(\cdot)|^2_{L^2(\Omega)}) + \gamma e^{\beta t}\int_0^t |\mathrm{grad}\,\mathbf{P}(s,\cdot)|^2_{\mathbb{L}^2(\Omega)}ds. \tag{3.45}$$

By using Gronwall's inequality one obtains that any mild solution $\mathbf{P}$ of problem (1.1)-(1.2)-(3.37) belongs to $L^\infty(0,T;\mathbb{H}^1(\Omega))$ and satisfies

$$|\mathrm{grad}\,\mathbf{P}(t,\cdot)|_{\mathbb{L}^2(\Omega)} \leq C_T(\mathcal{E}_0 + |W_0|^2_{L^2(\Omega)}), \tag{3.46}$$

for all $0 < T < \infty$ and for all $0 \leq t \leq T$. The constant $C_T > 0$ depends on T and is such that $C_T \to \infty$ as $T \to \infty$. Moreover the same estimate holds for the electric field $\mathbf{E}$ and we have

$$|\mathrm{grad}\,\mathbf{E}(t,\cdot)|_{\mathbb{L}^2(\Omega)} \leq C_T(\mathcal{E}_0 + |W_0|^2_{L^2(\Omega)}), \tag{3.47}$$

for all $0 < T < \infty$ and for all $0 \leq t \leq T$. We are now in position to prove the following result.

Theorem 3.1 ($\mathbb{H}^1$-regularity). *Let $(\mathbf{E},\mathbf{P})$ be a solution of problem (1.1)-(1.2)-(3.37) given by Theorem 2.1 and assume moreover that hypotheses (3.41) hold. Then for all $0 < T < \infty$, we have $(\mathbf{E},\mathbf{P}) \in L^\infty(0,T;\mathbb{H}^1(\Omega)\times\mathbb{H}^1(\Omega))$ and $|(\mathbf{E},\mathbf{P})|_{L^\infty(0,T;\mathbb{H}^1(\Omega)\times\mathbb{H}^1(\Omega))} \leq C_T$ where the constant $C_T > 0$ depends on the data and satisfies $C_T \to \infty$ as $T \to \infty$.*

To rigorously prove this result we proceed as in [8] by introducing the following regularized problem

$$\begin{cases} \mu\partial_t\mathbf{H}^\nu - \mathrm{curl}\;\mathbf{E}^\nu = 0 \text{ in } \mathbb{R}^+\times\Omega, \\ \epsilon\partial_t(\mathbf{E}^\nu + \mathbf{P}^\nu) + \sigma\mathbf{E}^\nu + \mathrm{curl}\;\mathbf{H}^\nu = 0 \text{ in } \mathbb{R}^+\times\Omega, \\ \mathbf{E}^\nu \times \mathbf{n} = 0 \text{ on } \mathbb{R}^+\times\partial\Omega, \\ \mathbf{H}^\nu_{|t=0} = \mathbf{H}_0 \text{ and } \mathbf{E}^\nu_{|t=0} = \mathbf{E}_0 \text{ in } \Omega, \end{cases} \tag{3.48}$$

and

$$\begin{cases} \partial_t^2\mathbf{P}^\nu + a\;\partial_t\mathbf{P}^\nu + \lambda^2\mathrm{curl}^2\;\mathbf{P}^\nu + b\;(\rho^\nu\star\mathbf{P}^\nu)\phi'(|\rho^\nu\star\mathbf{P}^\nu|^2) = b\;\mathbf{E}^\nu \text{ in } \mathbb{R}^+\times\Omega, \\ \mathbf{P}^\nu\times\mathbf{n} = 0 \text{ on } \mathbb{R}^+\times\partial\Omega, \\ \mathbf{P}^\nu_{|t=0} = \mathbf{P}_0 \text{ and } \partial_t\mathbf{P}^\nu_{|t=0} = \mathbf{P}_1 \text{ in } \Omega, \end{cases} \tag{3.49}$$

where ρ^ν is a regularizing sequence of scalar function in $\mathbb{R}^3$ satisfying $\rho^\nu(x) \geq 0$ and $\int_{\mathbb{R}^3}\rho^\nu(x)\,dx = 1$ and $(\rho^\nu\star\mathbf{P}^\nu)(x) = \int_\Omega \rho^\nu(x-y)\mathbf{P}^\nu(y)\,dy$. Setting $\Psi^\nu(\mathbf{Q}) = (\rho^\nu\star\mathbf{Q})\phi'(|\rho^\nu\star\mathbf{Q}|^2)$ for $\mathbf{Q}\in\mathbb{L}^2(\Omega)$ we get the estimate

$$|\Psi^\nu(\mathbf{Q}_1) - \Psi^\nu(\mathbf{Q}_2)|_{\mathbb{L}^2(\Omega)} \leq C_*|\mathbf{Q}_1 - \mathbf{Q}_2|_{\mathbb{L}^2(\Omega)}, \tag{3.50}$$

for all $\mathbf{Q}_1$ and $\mathbf{Q}_2$ in $\mathbb{L}^2(\Omega)$.

The existence, uniqueness and time-regularity results proved previously remain true for the regularized problem. Moreover the energy estimate changes as follows since we have

$$|\int_\Omega \Psi^\nu(\mathbf{P}^\nu)\partial_t\mathbf{P}^\nu\ dx| \le C_*|\mathbf{P}^\nu|_{\mathbb{L}^2(\Omega)}|\partial_t\mathbf{P}^\nu|_{\mathbb{L}^2(\Omega)}$$

and

$$|\mathbf{P}^\nu(t)|^2_{\mathbb{L}^2(\Omega)} \le |\mathbf{P}_0|^2_{\mathbb{L}^2(\Omega)} + \int_0^t |\partial_t\mathbf{P}^\nu(s)|^2_{\mathbb{L}^2(\Omega)}\ ds + \int_0^t |\mathbf{P}^\nu(s)|^2_{\mathbb{L}^2(\Omega)}\ ds.$$

We get

$$\begin{cases} \mathcal{E}_2(t) + \int_0^t \frac{b\sigma}{\epsilon}|\mathbf{E}^\nu(s)|^2_{\mathbb{L}^2(\Omega)} + a|\partial_t\mathbf{P}^\nu(s)|^2_{\mathbb{L}^2(\Omega)}\ ds \le \mathcal{E}_2(0) \\ +\ C_* \int_0^t |\mathbf{P}^\nu(s)|_{\mathbb{L}^2(\Omega)}|\partial_t\mathbf{P}^\nu(s)|_{\mathbb{L}^2(\Omega)}\ ds \text{ and} \\ |\mathbf{P}^\nu(s)|^2_{\mathbb{L}^2(\Omega)} \le |\mathbf{P}_0|^2_{\mathbb{L}^2(\Omega)} + 2\int_0^t |\mathbf{P}^\nu(s)|_{\mathbb{L}^2(\Omega)}|\partial_t\mathbf{P}^\nu(s)|_{\mathbb{L}^2(\Omega)}\ ds \end{cases} \tag{3.51}$$

where $\mathcal{E}_2(t) = \mathcal{E}(t) - \phi(|\mathbf{P}^\nu|^2)$. These estimates show that for all $T > 0$, there exists $C_T > 0$ which is independent of ν such that the regularized solution $(\mathbf{E}^\nu, \mathbf{P}^\nu)$ is uniformly bounded in $L^\infty(0,T;\mathcal{H}_0(\mathrm{curl}\,,\Omega) \times \mathcal{H}_0(\mathrm{curl}\,,\Omega))$. Consequently we have the following result.

Lemma 3.1 *Under hypotheses* (3.41)*, the regularized solution* $(\mathbf{E}^\nu, \mathbf{P}^\nu) \in L^\infty(0,T;\mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega))$ *for all* $0 < T < \infty$.

To show this assertion we use the fact that

$$|\mathrm{div}\,(\Psi^\nu(\mathbf{P}^\nu))|_{L^2(\Omega)} \le C_\nu|\mathbf{P}^\nu|_{\mathbb{L}^2(\Omega)},$$

where $C_\nu \to \infty$ as $\nu \to 0$. Then the compatibility divergence system satisfied by $(\mathbf{e}^\nu, \mathbf{p}^\nu)$ becomes

$$\begin{cases} \mu\partial_t\mathbf{h}^\nu = 0,\ \epsilon\partial_t(\mathbf{e}^\nu + \mathbf{p}^\nu) + \sigma\mathbf{e}^\nu = 0, \\ \partial_t^2\mathbf{p}^\nu + a\ \partial_t\mathbf{p}^\nu - b\ \mathbf{e}^\nu = -\mathrm{div}\,(\Psi^\nu(\mathbf{P}^\nu). \end{cases} \tag{3.52}$$

Introducing the new variable $W^\nu = (\mathbf{e}^\nu, \mathbf{p}^\nu, \partial_t\mathbf{p}^\nu)$ and $W_0 = (\mathbf{e}_0, \mathbf{p}_0, \mathbf{p}_1)$ we get the following system of ordinary differential equations where the space variable $x \in \Omega$ appears as a parameter

$$\begin{cases} \dfrac{dW^\nu}{dt} + \mathcal{C}W^\nu = \mathcal{S}(\mathbf{P}^\nu) \\ W^\nu(0) = W_0, \end{cases} \tag{3.53}$$

with

$$\mathcal{C} = \begin{pmatrix} \theta & 0 & 1 \\ 0 & 0 & -1 \\ -b & 0 & a \end{pmatrix},\ \mathcal{S}(\mathbf{P}^\nu) = \begin{pmatrix} 0 \\ 0 \\ -\mathrm{div}\,(\Psi^\nu(\mathbf{P}^\nu)) \end{pmatrix}. \tag{3.54}$$

It follows that the solution W^ν satisfies, for all $0 < T < \infty$ and for all $0 \le t \le T$, the estimate

$$|W^\nu(t)|^2_{\mathbb{L}^2(\Omega)} \le C_{T,\nu}(|W_0|^2_{\mathbb{L}^2(\Omega)} + \int_0^t |\mathbf{P}^\nu(s)|^2_{\mathbb{L}^2(\Omega} \, ds) \le C_{T,\nu}, \tag{3.55}$$

where $C_{T,\nu} > 0$ denotes a constant depending on T and ν. Finally, using the boundary condition $\mathbf{E}^\nu \times \mathbf{n} = 0$ and $\mathbf{P}^\nu \times \mathbf{n} = 0$ and the regularity estimate (3.44) we deduce that the solution $(\mathbf{E}^\nu, \mathbf{P}^\nu) \in L^\infty(0,T; \mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega))$. This proves the Lemma.

Since estimates (3.46) and (3.47) hold for $\mathbf{P}^\nu$ and $\mathbf{E}^\nu$ uniformly with respect to the parameter ν, they remain true as $\nu \to 0$. Denoting by $(\mathbf{E}, \mathbf{P})$ the weak-$\star$ limit in $L^\infty(0,T; \mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega))$ of the sequence $(\mathbf{E}^\nu, \mathbf{P}^\nu)$ we may pass to the limit in the regularized problem by using the strong convergence of the sequence $\mathbf{P}^\nu$ in $L^2(0,T; \mathbb{L}^2(\Omega))$ to get a solution of the full problem belonging in $L^\infty(0,T; \mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega))$ and is such that $\partial_t \mathbf{P} \in L^2(0,T; \mathbb{L}^2(\Omega))$ for $0 < T < \infty$ fixed. Theorem 3.1 is then proved.

References

[1] N. Aissa and K. Hamdache, Asymptotic behaviour of solutions to a model equations of thin ferroelectric materials, preprint, (2002).

[2] T. Cazenave and A. Haraux, Introduction aux problèmes d' évolutions semi-linéaires SMAI, Mathématiques et Applications, (1990).

[3] R. Dautray and J.L Lions, Analyse mathématique et calcul numérique pour les sciences et les techniques, Tome 2, Chp. IX, Masson.

[4] F. Dávi, On domain switching in deformable ferroelectrics, seen as a continua with microstructure, preprint, (2000).

[5] F. Dávi, Dynamics of non-linear thin piezoelectric rods with domain switching, preprint, (1999).

[6] J.M. Greenberg and R.C. MacCamy and C.V. Coffman, On the long-time behavior of ferroelectric systems, Physica D, 134, 362-383, (1999).

[7] D. Hamroun and K. Hamdache, Local solutions to a model of ferroelectric materials, preprint, (2002).

[8] K. Hamdache and I. Ngningone-Eya, Regularity and low frequency behavior of harmonic solutions to a model og ferroelectric material, preprint, (2002).

[9] S. Mizohata, The Theory of Partial Differential Equations, Cambridge University Press, New York, (1973).

[10] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Applied Mathematical Sciences, 44, Springer Verlag, New York, (1978).

[11] L. Tartar, Topics in Nonlinear Analysis, *Publications scientifiques d'Orsay 78, 13,* Université de Paris-sud, Département de Mathématiques, Orsay, (1978).

[12] L. Tartar, On the characterization of traces of a Sobolev space used Maxwell's equations, *Proceeding, Bordeaux November 6-7, 1997,* A.Y Leroux ed., (1988).