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CENTRE DE MATHÉMATIQUES APPLIQUÉES
UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 41 50. Fax: 01 69 33 30 11

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Homogenization of Layered Ferromagnetic Madia

Kamel Hamdache

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HOMOGENIZATION OF LAYERED FERROMAGNETIC MEDIA

KAMEL HAMDACHE*

Abstract. In this paper we adress the homogenization of some models of ferromagnetic materials. This question arise naturally when we consider a stack of composite constituted with two ferromagnets (F/F) or with ferromagnets and non magnetic spacers (F/nM/F), periodically distributed. In the second case, interlayer exchange coupling energy takes place to couple the adjacent interfaces of the ferromagnets. We use classical periodic homogenization techniques to obtain the effective Landau-Lifshitz-Gilbert (LLG) equations. For the F/F models the homogenized exchange anisotropy matrix is classical. For the F/nM/F models and under some assumptions on the strenghtness of the surface energy coefficient and the interlayer exchange coupling coefficient, we show that two effects are induced by the homogenization process . The first one is related to the surface energy and the second one to the small thickness parameter. Both effects are of the same type. The interlayer exchange coupling energy is too small compared to the lenght scale of the material and do not contribute in the limit problem.

Key words. Landau-Lifshitz equation, interlayer exchange coupling, surface energy, periodic homogenization, H-convergence

AMS subject classifications. 35D05, 78A25, 35Q60, 35B40, 82D40

1. Introduction. In recent years there has been a renaissance in the study of magnetic multilayers primarily because of the discoveries of giant magnetoresistance and oscillatory exchange coupling in periodic arrangements of thin sheets of ferromagnetic metal with spacers made of normal metal. These discoveries have lead to intense theoritical and experimental studies of extraordinary range of structures. To summarize, recent trends in magnetic thin film and multilayer research are concentrating on the interesting phenomena which occur (i) if the properties of magnetic layers are determined by surface and interface effects; (ii) if the layers are so thin and eventually additionally structured so that low-dimensional properties play a role, (iii) if successive magnetic layers are electronically coupled through nonmagnetic interlayers, or (iv) if the electronic transport through multilayer structures is considered, see for instance Hartmann [9]. Understand the origin of the indirect exchange coupling through the nonmagnetic layer is of great importance because the contact of the ferromagnetic layer to the nonmagnetic layer should change the electronic state of the nonmagnetic layer. In this work, we present a first attemp to study a part of the questions related to the giant magnetoresistance phenomena. We omit the electronic effects which are in fact the core in the modelisation of the problem. We only discuss the homogenization of the material. The full model equations for the giant magnetoresistance will be discussed in a forthcomming paper.

We assume that the behavior of the magnetization M of the ferromagnetic medium is governed by the Landau-Lifshitz-Gilbert equations coupled to the magneto-static equation satisfied by the potential φ, see [1], [4]. The interface effects are described by the Rado-Weertman boundary condition [12], which characterizes the surface anisotropy energy and the Hoffmann law ([10], [11]) to describe the interlayer exchange coupling energy which are expected to be important for very thin films.

In the sequel, $\Omega = \Omega_T \times I$ represents a cylinder of R^3 where $I = (0,1)$ and $\Omega_T \subset R^2$ is the cross section of the cylinder which is assumed to be an open, bounded and regular domain. A generic point of Ω is denoted by $x = (\hat{x}, z)$ where $\hat{x} \in \Omega_T$ and

*Centre de Mathématiques Appliquées, CNRS UMR 7641 & Ecole Polytechnique, 91128 Palaiseau Cedex, France (hamdache@cmapx.polytechnique.fr).

$z \in I$. We denote by $\partial\Omega$ the boundary of Ω which is assumed to be regular and $\mathbf{n}$ is the unit outward normal to Ω. We will use the following classical Hilbert spaces $L^2(\Omega) = (L^2(\Omega))^3$ and $H^1(\Omega) = (H^1(\Omega))^3$ equipped with their natural norms denoted respectively by $|\cdot|$ and $\|\cdot\|$. The canonical basis of R^3 is denoted by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The differential operators $\nabla\cdot_T$ and ∇_T represent respectively the divergence and the gradient operators with respect to the variable $\hat{x}$. The vector $\nabla_T g$ may be considered as well as 2-D vector or 3-D vector where the third component is 0. A vector $A \in R^3$ is written as $A = A_T + (A \cdot \mathbf{e}_3)\mathbf{e}_3$.
We shall discuss the homogenization of two models of ferromagnetic materials. The first one is a periodic stack composed by N identical samples ($N \in N$). The sample, with thickness $\frac{1}{N}$, is represented by two ferromagnetic media in contact with different characteristic parameters. This model is called a periodic Ferro/Ferro stack (F/F stack for shortness). The second model considered is a periodic stack of N identical samples. The thickness of the sample is $\frac{1}{N}$ which is composed by two ferromagnetic media separated by a nonmagnetic one. This model will be called a periodic Ferro/non Mag/Ferro stack, denoted shortly by F/nM/F stack.
Since the materials we consider are periodically layered in the z direction, we introduce some notations related to the periodicity. We denote by $Y = [0,1]$ the basic cell and $y \in Y$ represents the fast variable. An Y-periodic (vector valued) function with respect to the variable y is denoted by $f(t, \hat{x}, z, y)$ for $(t, \hat{x}, z)$ fixed. A periodically oscillating function is denoted by $f^N(t, \hat{x}, z) = f(t, \hat{x}, z, Nz)$ with $N \in N$.
Let $0 < \theta < 1/2$ be a fixed number. We set $h^{\pm} = (1 \pm \theta)/2$, $S = [h^-, h^+]$ and $Y^* = Y \setminus S$. For $N \in N$ we define the regular subdivision (z_k) of $[0,1]$ $z_k = \frac{k}{N}$ for $k = 0, \cdots, N$, the sets $Y_{k,N} = z_k + \frac{1}{N}Y$, $S_{k,N} = z_k + \frac{1}{N}S$ and $Y^*_{k,N} = z_k + \frac{1}{N}Y^*$ for $k = 0, \cdots, N-1$. Next we set $h^{\pm}_N = \frac{h^{\pm}}{N}$ and $z^{\pm}_k = z_k + h^{\pm}_N$ for $k = 0, \cdots, N-1$. The F/nM/F model is setted in the domain is $\Omega^N = \Omega_T \times I^N$ where $I^N = I \setminus S^N$ and $S^N = \cup_{k=0}^{N-1} S_{k,N}$ and the F/F model is setted in $\Omega_T \times I$.

1.1. The F/F layered stack. Let $A^N \in L^\infty(\Omega)^{3\times 3}$, $K^N_v \in L^\infty(\Omega)$ and $U^N \in L^\infty(\Omega)^3$ be respectively a periodically oscillating matrix function, a scalar function and a vector valued function defined respectively by $A^N(x) = A(x, Nz)$, $K^N_v(x) = K_v(Nz)$ and $U^N(x) = U(Nz)$ with $A(x,y) = A_1(x)$ if $0 \le y \le \theta$, $A(x,y) = A_2(x)$ if $\theta < y \le 1$, $K_v(y) = K^1_v$ if $0 \le y \le \theta$, $K_v(y) = K^2_v$ if $\theta < y \le 1$ and $U(y) = U^1$ if $0 \le y \le \theta$, $U(y) = U^2$ if $\theta < y \le 1$.
We assume that the 3×3 matrices A_k satisfy the ellipticity conditions $0 < a_1|\xi|^2 \le A_k(x)\xi \cdot \xi \le a_2|\xi|^2$ a.e $x \in \Omega$ for $\xi \ne 0$ and for $k = 1, 2$. The magnetization M^N and the potential φ^N satisfy in $Q = R^+ \times \Omega$ and $R^+ \times R^3$ respectively the problem

$$\begin{cases} \partial_t M^N - \alpha M^N \times \partial_t M^N = -(1+\alpha^2) M^N \times \mathcal{H}^N(M^N) \text{ in } Q \\ M^N(0,x) = M_0(x) \text{ in } \Omega, \ A^N \nabla M^N \cdot \mathbf{n} = 0 \text{ on } R^+ \times \partial\Omega \end{cases} \tag{1.1}$$

with

$$\mathcal{H}^N(M^N) = \nabla \cdot (A^N(x) \nabla M^N) + \Psi^N(M^N) + \nabla \varphi^N \tag{1.2}$$

where $\Psi^N(M^N) = K^N_v(M^N - (M^N \cdot U^N)U^N)$ and φ^N satisfies the magneto-static equation

$$\nabla \cdot (\nabla \varphi^N + \chi(\Omega) M^N) = 0 \text{ in } R^+ \times R^3. \tag{1.3}$$

In fact we shall discuss the more general problem of F/F composite without periodicity hypothesis.

1.2. The F/nM/F layered stack. In $Q^N = R^+ \times \Omega^N$, the magnetization M^N satisfies the Landau-Lifshitz-Gilbert (LLG) equations

$$\left\{ \begin{array}{l} \partial_t M^N - \alpha M^N \times \partial_t M^N = -(1+\alpha^2) M^N \times \mathcal{H}^N(M^N) \text{ in } Q^N \\ M^N(0,x) = M_0(x) \text{ in } \Omega^N, \; A\nabla M^N \cdot \mathbf{n} = 0 \text{ on } R^+ \times (\partial\Omega^N \setminus \Gamma_k^{\pm}). \end{array} \right. \tag{1.4}$$

for $k = 0, \cdots, N-1$, coupled to the potential φ^N satisfying in $R^+ \times R^3$ the magnetostatic equation

$$\nabla \cdot (\nabla \varphi^N + \chi(\Omega^N) M^N) = 0. \tag{1.5}$$

The characteristic function $\chi(\Omega^N)$ of the domain Ω^N may be written $\chi(\Omega^N) = \chi(\Omega)\chi(I^N)$. Since the holes S_k are periodically distributed in I then $\chi(I^N) = \chi(Nz)$ where $\chi(y)$ is the Y-periodic characteristic function of the basic cell Y^* in Y. The effective magnetic field is

$$\mathcal{H}(M^N) = A\Delta M^N + \Psi(M^N) + \nabla\varphi^N \tag{1.6}$$

where $A > 0$ is the bulk exchange stiffness constant and $\Psi(M)$ is given by $\Psi(M) = \nabla_M \psi(M)$ where the scalar function $\psi(M)$ is assumed to be convex, nonnegative and such that $D_M^2 \Psi(U)$ is bounded. On the interfaces $\Gamma_k^{\pm} = \Omega_T \times \{z = z_k^{\pm}\}$ the Rado-Weerteman- Hoffman boundary condition setted om $R^+ \times \Gamma_k^{\pm}$ is

$$M_{\pm}^N \times (\mp(A\frac{\partial M^N}{\partial z})_{|\pm} + \frac{s}{N}\ (M_{\pm}^N - (M_{\pm}^N \cdot \mathbf{e}_3)\mathbf{e}_3) + i\ (M_{\pm}^N - M_{\mp}^N) = 0 \tag{1.7}$$

for $k = 0. \cdots, N-1$ where we setted $M_{\pm}^N = M^N(z_k^{\pm})$. We assume the surface anisotropy coefficient $s^N = \frac{s}{N}$ with $s > 0$ and the interlayer exchange coupling coefficient $i^N = i > 0$. This boundary condition is deduced from the surface anisotropy energy on each interface $\Gamma_k^{\pm}$ given by $\frac{s}{N}\int_{\Omega_T} |M^N(z_k^{\pm}) \times \mathbf{e}_3|^2 d\hat{x}$ and from the interlayer exchange coupling energy between the interfaces Γ_k^+ and Γ_k^- given by $i\int_{\Omega_T}(1 - M^N(z_k^-) \cdot M^N(z_k^+))d\hat{x}$. The interlayer exchange coupling energy also contains the so called biquadratic one given by $j^N \int_{\Omega_T}(1 - (M^N(z_k^+) \cdot M^N(z_k^-))^2)\ d\hat{x}$ which is neglected in this work, see [10], [20], [12] for more details and explanations.

Before describing the content of the paper we recall some useful results from the two-scale convergence introduced in Allaire [2] see also Nguetseng [18], which will be used in the sequel. Let M^N be a bounded sequence of $L^\infty(R^+; L^2(\Omega))$ and let M be the weak-$\star$ limit of a subsequence. We say that M^N two-scale converge to the Y-periodic profile $\mathcal{M}(t,x,y)$ if

$$\int_Q M^N \cdot F^N dxdt \to \int_{Q\times Y} \mathcal{M} \cdot F dxdtdy \tag{1.8}$$

for all Y-periodic test functions $F(t,x,y)$. Moreover we have the relation $M(t,x) = |Y|^{-1}\int_Y \mathcal{M}(t,x,y)dy$. We set $\langle f \rangle = |Y|^{-1}\int_Y f(y)dy$. If $M^N \to M$ a.e then $\mathcal{M}$ is independent of the fast variable y and we have $\mathcal{M} = M$. If the sequence M^N is bounded in $L^\infty(R^+; H^1(\Omega))$ then we define the Y-corrector profile $\mathcal{M}^1(t,x,y)$ by the convergence

$$
\begin{aligned}
&\textstyle\int_Q \partial_z M^N \cdot F^N dxdt \to \int_{Q\times Y} (\partial_z \mathcal{M} + \partial_y \mathcal{M}^1) \cdot F dxdtdy \\
&\textstyle\int_Q \nabla_{\hat{x}} M^N \cdot F^N dxdt \to \int_{Q\times Y} \nabla_{\hat{x}} \mathcal{M} \cdot F dxdtdy
\end{aligned} \tag{1.9}
$$

for all periodically oscillating test functions $F^N(t,x) = F(t,\hat{x},z,Nz)$.
Now, let M^N be a bounded sequence of $L^\infty(R^+; H^1(\Omega^N))$. If $V^N = P_N M^N$ is a uniformly bounded extention in $L^\infty(R^+; H^1(\Omega))$ of M^N to Ω then we have $V^N \rightharpoonup M$ weakly-$\star$ in $L^\infty(R^+; H^1(\Omega))$. As previously, the Y-periodic profile $\mathcal{M}$ and the Y-corrector profile $\mathcal{M}^1$ are defined by

$$
\begin{aligned}
&\textstyle\int_{Q^N} V^N \cdot F^N dxdt \to \int_{Q\times Y} \chi(y)\mathcal{M} \cdot F dxdtdy \\
&\textstyle\int_{Q^N} \partial_z V^N \cdot F^N dxdt \to \int_{Q\times Y} \chi(y)(\partial_z \mathcal{M} + \partial_y \mathcal{M}^1) \cdot F dxdtdy.
\end{aligned} \tag{1.10}
$$

Moreover, we have

$$
\int_{Q^N} \nabla_{\hat{x}} V^N \cdot F^N dxdt \to \int_{Q\times Y} \chi(y) \nabla_{\hat{x}} \mathcal{M} \cdot F dxdtdy. \tag{1.11}
$$

If $V^N \to M$ a.e then we have $\mathcal{M} = M$ in Q.
The content of the paper is the following. In section 2, we discuss the homogenization of F/F layered stacks. We consider in fact a more general setting without any periodicity condition. Namely, we assume that the sequences A^N, K_v^N and U^N satisfy the hypotheses $0 < a_1|\xi|^2 \le A^N(x)\xi\cdot\xi \le a_2|\xi|^2$, $|K_v^N(x)| \le k_v$, $|U^N(x)| \le u$ a.e $x \in \Omega$ and for all $\xi \in R^3$, $\xi \ne 0$. We deduce from the energy equality uniform bounds for the solution (M^N, φ^N) and show that the H-convergence method [21], [15], [17] allows to deduce the homogenized problem where the homogenized effective magnetic field $\mathcal{H}^{hom}(M) = \nabla\cdot(A^{hom}\nabla M)) + \mathcal{B}^{hom}(M) + \nabla\varphi$. Here A^{hom} is the classical homogenized matrix associated with the sequence A^N and the homogenized volume anisotropy $\mathcal{B}^{hom}$ is obtained as the weak-$\star$ limit of a subsequence of $K_v^N U^N \otimes U^N$. In section 3 we discuss the F/nM/F layered stack. We show that only the surface anisotropy energy induces a homogenized volume anisotropy effect of the form $\frac{2s}{1-\theta}(M - (M\cdot\mathbf{e}_3)\mathbf{e}_3)$. Moreover, the demagnitizing field induces a Gioia-James type effect $\theta(M - (M\cdot\mathbf{e}_3)\mathbf{e}_3)$ due to the small thickness in the z direction. The interlayer exchange energy is too small in comparaison to the size $\frac{1}{N}$ and disappear in the homogenization process. In the last section, we give a new global existence result for the F/nM/F model based on a hyperbolic regularization of the LLG equations combined with a particular choice of the penalization operator for the saturation condition.

2. Homogenization of F/F stacks. We consider the homogenization of materials of the type F/F. We assume that the parameters $A^N(x)$, $K_v^N(x)$ and $U^N(x)$ of the mixture, namely the exchange anisotropy coefficient, the volume anisotropy coefficient, and the easy axis respectively of the mixture satisfy the following hypotheses

$$
\begin{aligned}
&0 < a_1|\xi|^2 \le A^N(x)\xi\cdot\xi \le a_2|\xi|^2,\ |K_v^N(x)| \le k_v,\ |U^N(x)| \le u \\
&A^N \rightharpoonup A,\ K_v^N \rightharpoonup K_v,\ U^N \rightharpoonup U \text{ in } L^\infty(\Omega) \text{ weak} - \star
\end{aligned} \tag{2.1}
$$

for all $\xi \in R^3$, $\xi \ne 0$ and a.e $x \in \Omega$.
Let $M^N(t,x)$ be the magnetization field defined in $R^+ \times \Omega$ satisfying the Landau-Lifshitz-Gilbert (LLG) equations (1.1) with the Neumann boundary condition on

$R^+ \times \partial\Omega$ and with the effective magnetic field (1.2) where φ^N satisfies in $R^+ \times R^3$ the equation

$$\nabla \cdot (\nabla \varphi^N + \chi(\Omega) M^N) = 0. \tag{2.2}$$

We assume that the initial data M_0^N satisfy

$$M_0^N \in H^1(\Omega),\ |M_0^N(x)|^2 = 1 \text{ a.e } x \in \Omega,\ M_0^N \to M_0 \text{ in } H^1(\Omega) \text{ weak}. \tag{2.3}$$

Notice that we have $|M_0(x)|^2 = 1$ a.e.
The solutions (M^N, φ^N) of the problem exist globaly see Visintin [23], [3] for example. We have $M^N \in L^\infty(R^+, H^1(\Omega))$, $\partial_t M^N \in L^2(R^+, L^2(\Omega))$ and $|M^N(t,x)|^2 = |M_0(x)|^2$ a.e. Furthermore the following energy inequality holds

$$\mathcal{E}^N(t) + \frac{2\alpha}{1+\alpha^2} \int_0^t \int_\Omega |\partial_t M^N|^2 dx \le \mathcal{E}_0^N \tag{2.4}$$

for all $t \ge 0$, where the total energy is defined by

$$\mathcal{E}^N(t) = \int_\Omega (A^N |\nabla M^N|^2 + K_v^N |M^N.U^N|^2) dx + \int_{R^3} |\nabla \varphi^N|^2 dx. \tag{2.5}$$

We define the volume anisotropy matrix $\mathcal{B}_v^N$ by

$$\mathcal{B}_v^N(\Phi) = K_v^N (U^N \otimes U^N)(\Phi). \tag{2.6}$$

We may write $K_N^v |M^N.U^N|^2 = (\mathcal{B}_v^N(M^N); M^N)$. Our main result is the following:

THEOREM 2.1. *Assume hypotheses* (2.1) *and* (2.3) *hold. Let* (M^N, φ^N) *be a global solution of problem* (1.1)- (1.2)-(1.3). *Let* (M, φ) *be the weak-$\star$ limit of a subsequence of* (M^N, φ^N). *Then* φ *satisfies the magneto-static equation* (1.3) *and* M *satisfies the homogenized LLG equations with the homogenized effective magnetic field*

$$\mathcal{H}_{hom}(M) = \nabla \cdot (A^{hom} \nabla M) + \mathcal{B}_v^{hom}(M) + \nabla \varphi. \tag{2.7}$$

The matrix A^{hom} is the classical homogenized matrix given by the H-convergence of a subsequence of A^N and the symmetric volume anisotropy matrix $\mathcal{B}_v^{hom}$ is defined as the weak-$\star$ limit in $L^\infty(\Omega)^{3\times 3}$ of a subsequence of $\mathcal{B}_v^N$.

Remark: (The layered case). Assume that the coefficient A^N, K_v^N and U^N are periodically oscillating with respect to the variable z and satisfy hypotheses (2.1). Then the associated homogenized coefficient are given by

$$\mathcal{A}_{hom}(x) = \begin{pmatrix} \Big\langle A(x,\cdot) \Big\rangle & 0 & 0 \\ 0 & \Big\langle A(x,\cdot) \Big\rangle & 0 \\ 0 & 0 & \Big\langle A(x,\cdot)(1 + \partial_y \psi(\cdot)) \Big\rangle \end{pmatrix} \tag{2.8}$$

and

$$\mathcal{B}_v^{hom} = \langle K_v(\cdot) U(\cdot) \otimes U(\cdot) \rangle. \tag{2.9}$$

where $\psi(x,y)$ is the solution of the local problem $\partial_y(A(x,y)\partial_y \psi) = -\partial_y A(x,y)$ which is Y-periodic with respect to the variable y and $\langle\cdot\rangle$ denotes the mean value over the

cell Y.
The proof of our main theorem will be given in several steps. Let (M^N, φ^N) be a global solution of the problem satisfying the energy estimate (2.4). We have the following results:

LEMMA 2.2. *There exists $C > 0$, which is independent of N, such that the solutions of problem* (1.1)-(1.2)-(1.3) *satisfy the following uniform bounds*

$$\begin{cases} |M^N|_{L^\infty(R^+,H^1(\Omega))} + |\partial_t M^N|_{L^2(R^+,L^2(\Omega))} \leq C, \ |M^N(t,x)|^2 = 1 \text{ a.e} \\ |\nabla\varphi^N|_{L^\infty(R^+,L^2(R^3))} \leq C. \end{cases} \tag{2.10}$$

For a subsequence still denoted by (M^N, φ^N) it holds

$$\begin{aligned} &M^N \rightharpoonup M \text{ in } L^\infty(R^+, H^1(\Omega)) \text{ weak} - \star, \\ &\partial_t M^N \rightharpoonup \partial_t M \text{ in } L^2(R^+, L^2(\Omega)) \text{ weak}, \\ &\nabla\varphi^N \rightharpoonup \nabla\varphi, \text{ in } L^\infty(R^+, L^2(R^3)) \text{ weak} - \star. \end{aligned} \tag{2.11}$$

By using Aubin's compactness results, we get the convergences

$$M^N(t,x) \to M(t,x) \text{ a.e} \tag{2.12}$$

and in $L^2_{loc}(R^+; L^2(\Omega))$ strongly. The associated weak formulation of LLG has the form

$$\begin{cases} \dfrac{1}{1+\alpha^2}\displaystyle\int_Q (\partial_t M^N - \alpha M^N \times \partial_t M^N)\cdot\Phi^N dxdt = \\ \int_Q (M^N \times A^N \nabla M^N)\cdot\nabla\Phi^N + \\ K_v^N (U^N \cdot M^N) M^N \times U^N \cdot \Phi^N + M^N \times \nabla\varphi^N \cdot \Phi^N) dxdt \end{cases} \tag{2.13}$$

where Φ^N is an oscillating test function (which will be precised later), see Tartar [21]. The test functions Φ^N are defined as follows. Let $F \in H^{-1}(\Omega)$ be fixed with components f_k for $k = 1,2,3$. Let $\Psi^N \in H^1_0(\Omega)$ with components $\psi_k^N \in H^1_0(\Omega)$ defined as the unique solution of the homogeneous Dirichlet problem

$$-\nabla\cdot(A^N \nabla\psi_k^N) = f_k \text{ in } \Omega. \tag{2.14}$$

Let $W \in R^3$ be a fixed vector with components w_k and let $\Theta \in \mathcal{D}(R^+)^3$ with components θ_k. We set

$$\Phi^N(t,x) = w_k \psi_k^N(x)\Theta(t) = (W \cdot \Psi^N(x))\Theta(t). \tag{2.15}$$

Let A^{hom} be the homogenized matrix associated with problem (2.14) defined, see Tartar [21] and [15], [17], by

$$\begin{cases} \psi_k^N \rightharpoonup \psi_k \text{ in } H^1_0(\Omega) \text{ weak} \\ A^N \nabla\psi_k^N \rightharpoonup A^{hom}\nabla\psi_k \text{ in } L^2(\Omega) \text{ weak}. \end{cases} \tag{2.16}$$

Further, ψ_k satisfies the equation $-\nabla\cdot(A^{hom}\nabla\psi_k) = f_k$ in Ω. It follows that $\Phi^N \to \Phi = (\Psi \cdot W)\Theta$ in $L^2(Q)$ strongly and $\partial_j \phi_k^N \rightharpoonup \partial_j \phi_k = \partial_j(\Psi \cdot W)\theta_k$ in $L^2(Q)$ weak. Here ∂_j denotes the partial derivative with respect to the component x_j.

Using the strong convergence of M^N togheter with the weak convergences given by (2.11) we easily pass to the limit in the left-hand side of (2.13) and in the second and third integrals of the right-hand side of (2.13) to get

$$\begin{aligned} &\int_Q (\partial_t M^N - \alpha M^N \times \partial_t M^N) \cdot \Phi^N dxdt \to \int_Q (\partial_t M - \alpha M \times \partial_t M) \cdot \Phi dxdt \\ &\int_Q M^N \times \nabla \varphi^N \cdot \Phi^N dxdt \to \int_Q M \times \nabla \varphi \cdot \Phi dxdt \end{aligned} \tag{2.17}$$

and

$$\int_Q K_v^N (U^N \cdot M^N) M^N \times U^N \cdot \Phi^N dxdt \to \int_Q \mathcal{B}_v^{hom}(M) \cdot M \times \Phi dxdt \tag{2.18}$$

where the symmetric 3×3 matrix $\mathcal{B}_v^{hom}$ is defined by

$$\mathcal{B}_v^N \rightharpoonup \mathcal{B}_v^{hom} \text{ in } (L^\infty(\Omega))^{3\times 3} \text{ weak} - \star. \tag{2.19}$$

We obtain this result by writing $K_v^N (U^N \cdot M^N) M^N \times U^N \cdot \Phi^N = K_v^N U^N \otimes U^N (M^N) \cdot M^N \times \Phi^N$ and pass to the limit by using the strong concergence of M^N and Φ^N. Notice that, passing to the limit in the magneto-static equation, one deduces that φ satisfies in $R^+ \times R^3$ the equation $\nabla \cdot (\nabla \varphi + \chi(\Omega) M) = 0$. The next step is to prove the following result.

PROPOSITION 2.3. *It holds that*

$$\int_Q M^N \times A^N \nabla M^N \cdot \nabla \Phi^N dxdt \to \int_Q M \times A^{hom} \nabla M \cdot \nabla \Phi dxdt \tag{2.20}$$

Proof. We set $\mathcal{J}^N(\Phi^N) = \int_Q M^N \times A^N \nabla M^N \cdot \nabla \Phi^N dxdt$. We have $\mathcal{J}^N(\Phi^N) = \mathcal{J}_1^N(\Phi^N) + \mathcal{J}_2^N(\Phi^N)$ with $\mathcal{J}_1^N(\Phi^N) = \int_Q M \times A^N \nabla M^N \nabla \Phi^N dxdt$ and $\mathcal{J}_2^N(\Phi^N) = \int_Q (M^N - M) \times A^N \nabla M^N \nabla \Phi^N dxdt$. We first pass to the limit in $\mathcal{J}_1^N(\Phi^N) = \sum_{k=1}^3 w_k G_k$ where G_k is defined by

$$G_k = \int_\Omega (M \times \Theta(t)) \cdot \partial_j M^N A^N \partial_j \psi_k^N dxdt. \tag{2.21}$$

Since ψ_k is independent of the variable t we may write G_k in the following form

$$G_k = \int_\Omega \nabla Z^N \cdot A^N \nabla \psi_k^N dx - \int_Q (\partial_j M \times \Theta(t)) \cdot M^N A^N \partial_j \psi_k^N dxdt \tag{2.22}$$

where we set $Z^N(x) = \int_0^\infty M \times \Theta(t) \cdot M^N dt$. Using Lemma 2.1 we deduce that the sequence (Z^N) is uniformly bounded in $H^1(\Omega)$ since we have $|Z^N|_{L^2}^2 \le |\Omega||\Theta|_{L^1}^2$ and $|\nabla Z^N|_{L^2}^2 \le (|\nabla M|_{L^\infty(L^2)} + |\nabla M^N|_{L^\infty(L^2)})|\Theta|_{L^2}$. Moreover, for a subsequence, still denoted by Z^N, we have $Z^N \rightharpoonup Z$ in $H^1(\Omega)$ weak and, simple computations show that $Z = \int_0^\infty M \times \Theta(t) \cdot M dt = 0$. To pass to the limit in the first integral in the right hand side of (2.22), we use the standard compensated compactness results see, [15], [17], applied to the sequences ∇Z^N and $A^N \nabla \psi_k^N$ which verify $\nabla \times (\nabla Z^N) = 0$ and $\nabla \cdot (A^N \nabla \psi_k^N) = f_k$. Hence, since $Z = 0$, we get the result

$$\int_\Omega \nabla Z^N \cdot A^N \nabla \psi_N^k dx \to 0. \tag{2.23}$$

Now, we pass to the limit in the second integral of the right-hand side of (2.22). Let V^N be the sequence defined by $V^N(t,x) = \partial_j M \times \Theta(t) \cdot M^N$. By lemma 2.1, this sequence is uniformly bounded in $L^\infty(R^+, L^2(\Omega))$. Let V defined by $V(t,x) = \partial_j M \times \Theta(t) \cdot M$. We shall prove that $V^N \to V$ strongly in $L^2(Q)$. We have $|V^N - V|_{L^1(Q)} \leq \int_Q |\partial_j M \times \Theta(t)||M^N - M| dxdt \leq |\partial_j M \times \Theta|_{L^2(Q)} |M^N - M|_{L^2(Q)}$ where we assumed support$(\Theta) \subset (0,T)$. This shows that $V^N \to V$ in $L^1(Q)$ strongly. Next for a subsequence, $V^N \rightharpoonup V$ weakly in $L^2(Q)$. Moreover, we have the estimate $|V^N(t,x)|^2 \leq |\partial_j M(t,x) \times \theta(t)|^2$ and since the right-hand side belongs to $L^1(Q)$ and $|V^N(t,x)|^2 \to |V(t,x)|^2$ a.e (t,x) then, by using the Lebesgue dominated convergence theorem, we deduce that $|V^N|^2 \to |V|^2$ strongly in $L^1(Q)$ and finally $V^N \to V$ strongly in $L^2(Q)$. Using this strong convergence togheter with the weak convergence in $L^2(\Omega)$ of $A^N \nabla \psi_N^k$, we deduce that the second integral in the right hand side of (2.22) satisfies the convergence $\int_Q V^N \cdot A^N \partial_j \psi_k^N dxdt \to \int_Q (\partial_j M \times \Theta(t)) M \cdot A^{hom} \partial_j \psi_k dxdt$. Combining these above results we get

$$\mathcal{J}_1^N(\Phi^N) \to \sum_{k=1}^{3} w_k \int_Q (M \times \Theta(t)) \cdot \partial_j M A^{hom} \partial_j \psi_k dxdt. \tag{2.24}$$

Next, we shall prove that $\mathcal{J}_2^N(\Phi^N) \to 0$. In the sequel $C > 0$ denotes various constant which are independent of N. Since $\mathcal{D}(\Omega)$ is densely embedded in $H^{-1}(\Omega)$ we may assume that $f_k \in W^{-1,p}(\Omega)$ for some $p \geq 2$. By Meyers regularity result, see [14], there exists $p^* > 2$ such that the solution ψ_k^N of problem (2.14) belongs to $W^{1,p}(\Omega)$ for all $2 \leq p < p^*$ and satisfies the estimate $|\partial_j \psi_k^N|_{L^p(\Omega)} \leq C |f_k|_{W^{-1,p}(\Omega)}$. Using these results and Hölder's inequality, we deduce that

$$|\mathcal{J}_2^N(\Phi^N)| \leq C |M^N - M|_{L^q_{loc}(Q)} |\partial_j M^N \Theta|_{L^2(Q)} |\partial_j \psi_k^N|_{L^p(\Omega)} \tag{2.25}$$

for all $q \geq 1$ such that $1/q + 1/p = 1/2$. By using the strong convergence of the sequence M^N and its uniform bound in $L^\infty(Q)$ we deduce that $|M^N - M|_{L^q_{loc}(Q)} \to 0$ for all $q \geq 1$. Next, the uniform bound of M^N in $L^\infty(R^+; H^1(\Omega))$ implies that $|\partial_j M^N \cdot \Theta|_{L^2(Q)} \leq C$. Finally, using Meyers estimate, we have $|\partial_j \psi_k^N|_{L^p(\Omega)} \leq C$. We conclude that

$$\mathcal{J}_2^N(\Phi^N) \to 0. \tag{2.26}$$

for all $f_k \in \mathcal{D}(\Omega)$. Using a density argument, we deduce that the convergence (2.26) holds for all $f^k \in H^{-1}(\Omega)$.
□

2.1. The F/F periodically layered case. We assume that the coefficients are periodically oscillating in the z-direction and are of the form $G(x, Nz)$ and take two values $G_1(x)$ and $G_2(x)$ with the proportion $0 < \theta < 1$. This means that the mean value of $G(y)$ over the basic cell $[0,1]$ is given by $\langle G(\cdot) \rangle = \theta G_1 + (1-\theta) G_2$. To be precise we set $A^N(x) = A(\hat{x}, Nz)$, $K_v^N(x) = K_v(\hat{x}, Nz)$, $U^N(x) = U(\hat{x}, Nz)$ where $A(\hat{x}, y) = A_1(\hat{x})$ if $0 \leq y < \theta$ and $A(\hat{x}, y) = A_2(\hat{x})$ if $\theta \leq y < 1$, $K_v(y) = K_v^1$ if $0 \leq y < \theta$ and $K_v(y) = K_v^2$ if $\theta \leq y < 1$, $U(y) = U^1$ if $0 \leq y < \theta$ and $U(y) = U^2$ if $\theta \leq y < 1$. We assume for simplicity $A(x,y) \in R^+$. Clearly, the homogenized matrix A^{hom} is diagonal. Since the material in periodically layered in the z-direction, the homogenization effect takes place only in this direction for the third component of the homogenized exchange coefficient while for the two first ones we get the mean value of the function. The homogenized volume anisotropy matrix $\mathcal{B}_v^{hom}$ is defined as the mean value over the basic cell Y of the matrix $K_v(y)U(y) \otimes U(y)$.

The weak formulation (2.13) with periodically oscillating test functions allows to give the different orders ,with respect to the parameter $\frac{1}{N}$, to build the homogenized problem. At the order N we get the local problem setted in Y for (t,x) fixed satisfied by the Y-periodic profile $\mathcal{M}^1$. Namely, passing to the limit in the weak formulation we get

$$\partial_y(A(x,y)\partial_y(M \times \mathcal{M}^1)) = -(\partial_y A(x,y))M \times \partial_z M. \tag{2.27}$$

Here we used the result $\mathcal{M} = M$. Solving this equation we get

$$M \times \mathcal{M}^1 = \psi(x,y)M \times \partial_z M \tag{2.28}$$

where $\psi(y)$ is the Y-periodic solution of the classical problem

$$\partial_y(A(x,y)\partial_y\psi) = -\partial_y A(x,y). \tag{2.29}$$

Next, we choose in the weak formulation (2.13) test functions Φ which are independent of the fast variable y. Passing to the limit in the main integral we deduce that $\int_Q M^N \times A^N \partial_z M^N \cdot \partial_z \Phi dxd$ converges to $\int_Q M \times \left\langle A(\cdot)(\partial_z M + \partial_y \mathcal{M}^1(\cdot))\right\rangle \cdot \partial_z \Phi dxdt$. The remainder of the proof follows the lines of the periodic homogenization proofs.

3. Homogenization of F/nM/F stacks. Let us consider in this section the homogenization of the problem (1.4)-(1.5)-(1.6)-(1.7) where the homogeneous Neumann boundary condition is setted on $R^+ \times (\partial\Omega \setminus \Gamma_k^{\pm})$ for $k = 0, \cdots, N-1$. Let (M^N, φ^N) be a global weak solution of the problem. In section 4 we give a proof for the global existence of such solutions satisfying the saturation condition $|M^N(t,x)|^2 = |M_0^N(x)|^2$ in $R^+ \times \Omega^N$ and the energy inequality

$$\mathcal{E}^N(t) + \frac{2\alpha}{1+\alpha^2}\int_0^t \int_{\Omega^N} |\partial_t M^N|^2 dx \leq \mathcal{E}_0^N \tag{3.1}$$

where the total energy $\mathcal{E}^N(t)$, at the time $t \geq 0$, is expressed by

$$\mathcal{E}^N(t) = \mathcal{E}_e^N(t) + \mathcal{E}_v^N(t) + \mathcal{E}_s^N(t) + \mathcal{E}_i^N(t) + \mathcal{E}_d^N(t) \tag{3.2}$$

where the exchange energy is given by $\mathcal{E}_e^N(t) = \int_{\Omega^N} A|\nabla M^N|^2 dx$ while the volume anisotropy energy takes the form $\mathcal{E}_v^N(t) = \int_{\Omega^N} \Psi(M^N)dx$. Moreover the stray energy is $\mathcal{E}_d^N(t) = \int_{R^3} |\nabla\varphi^N|^2 dx$. The function Ψ is assumed to be nonnegative convex and to satisfy $|D_U^2\Psi(U)| \leq C$. The surface energy and the interlayer exchange coupling energy are given by

$$\begin{aligned} \mathcal{E}_s^N(t) &= \frac{s}{N}\sum_{k=0}^{N-1}\int_{\widehat{\Omega}}(|M^N(z_k^-)\cdot \mathbf{e}_3| + |M^N(z_k^+)\cdot \mathbf{e}_3|^2)d\hat{x} \\ \mathcal{E}_i^N(t) &= \frac{i}{2}\sum_{k=0}^{N-1}\int_{\widehat{\Omega}}|M^N(z_k^+) - M^N(z_k^-)|^2 d\hat{x}. \end{aligned} \tag{3.3}$$

The initial energy $\mathcal{E}_0^N$ is defined by $\mathcal{E}_0^N = \mathcal{E}^N(0)$.
The above estimates provide uniform bounds for the solutions (M^N, φ^N). In what follows, $C > 0$ stands for various constants which are independent of N. We assume that the initial data M_0^N satisfy the hypothesis

$$M_0^N = M_{0|\Omega^N} \text{ with } M_0 \in H^1(\Omega),\ |M_0(x)|^2 = 1 \text{ a.e. in } \Omega. \tag{3.4}$$

Under this hypothesis we have $0 \le \mathcal{E}_e^N(0) \le C$ and $0 \le \mathcal{E}_v^N(0) \le C$. Next the demagnetizing initial energy satisfies $\mathcal{E}_d^N(0) = -\int_{R^3} \chi^N M^N \cdot grad\varphi^N\, dx$ which implies that $0 \le \mathcal{E}_d^N(0) \le C$. The initial surface anisotropy energy and the interlayer echange coupling energy satisfy the bounds

$$0 \le \mathcal{E}_s^N(0) \le 2\ s|\Omega_T|, \text{ and, } 0 \le \mathcal{E}_i^N(0) \le \theta \frac{i}{N} |\partial_z M_0|^2_{L^2(\Omega)} \le C \frac{i}{N}. \tag{3.5}$$

To bound uniformly the interlayer exchange coupling initial energy $\mathcal{E}_i^N(0)$ we use the fact that $|M_0^N(z_k^+) - M_0^N(z_k^-)|^2 \le |z_k^+ - z_k^-| \int_{z_k^-}^{z_k^+} |\partial_z M_0|^2 dz$ and $|z_k^+ - z_k^-| = \frac{\theta}{N}$. This shows that $0 \le \mathcal{E}_s^N(0) \le C$, $0 \le \mathcal{E}_i^N(0) \le \frac{C}{N}$ and finally $0 \le \mathcal{E}^N(0) \le C$. Using the global existence result given in Section 4 we get the estimates

LEMMA 3.1. *Under hypotheses* (3.4), *there exists $C > 0$ which is independent of N, such that any global solution (M^N, φ^N) of problem* (1.4)-(1.5)-(1.6)-(1.7) *satisfies the estimates*

$$\begin{aligned} &|M^N|_{L^\infty(R^+, H^1(\Omega^N))} + |\partial_t M^N|_{L^2(R^+, L^2(\Omega^N))} \le C \quad a.e \\ &|\nabla \varphi^N|_{L^\infty(R^+, L^2(R^3))} \le C,\ |M^N(t,x)|^2 = 1 \text{ a.e } (t,x) \end{aligned} \tag{3.6}$$

and for a.e t,

$$\begin{aligned} &\frac{s}{N} \sum_{k=1}^N \int_{\widehat{\Omega}} |M^N(z_k^\pm) \cdot \mathbf{e}_3|^2 d\hat{x} \le C \\ &\frac{i}{2} \sum_{k=1}^N \int_{\widehat{\Omega}} |M^N(z_k^+) - M^N(z_k^-)|^2 d\hat{x} \le C. \end{aligned} \tag{3.7}$$

As usually, the homogenization in a domain with holes, needs to use an extension operator from $H^1(\Omega^N)$ into $H^1(\Omega)$. We proceed along the lines of the proof given by [6], see also [21], to get the result

LEMMA 3.2. *There exists a linear continuous extension operator P_N from $H^1(I^N)$ into $H^1(I)$ such that $P_N u = u$ in I^N and for all $u \in H^1(I^N)$ we have*

$$\|P_N u\|_{H^1(I)} \le C \|u\|_{H^1(I^N)}, |\partial_z P_N u|_{L^2(I)} \le C |\partial_z u|_{L^2(I^N)}. \tag{3.8}$$

The operator P_N maps also continuously $H^1(\Omega^N)$ into $H^1(\Omega)$ and satisfies $P_N U = U$ in Ω^N. Moreover, for all $U \in H^1(\Omega^N)$ we have

$$\|P_N U\|_{H^1(\Omega)} \le C \|\ U\|_{H^1(\Omega^N)}, |\partial_z P_N U|_{L^2(\Omega)} \le C |\partial_z U|_{L^2(\Omega^N)} \tag{3.9}$$

where the constant $C > 0$ is independent of N.

We shall use the method of oscillating test functions [21] to pass to the limit in the weak formulation of the problem. Using the periodicity of the domain we consider test functions of the type $F^N(t,x) = F(t,\hat{x},z)\sigma(Nz)$ where $F(t,\hat{x},z)$ is a regular test function and $\sigma(y)$ is a Y-periodic function with respect to the fast variable $y \in Y$. Recall that the characteristic function χ^N of the domain I^N depends only of the variable z and is the form $\chi^N(z) = \chi(Nz)$ where $\chi(y)$ is the Y-periodic characteristic function of Y^*.

We set $V^N = P_N M^N$. The weak formulation associated with the problem (1.4)-(1.5)-(1.6)-(1.7) is the following

$$
\left\{
\begin{array}{l}
\dfrac{1}{1+\alpha^2}\displaystyle\int_Q \chi^N(\partial_t V^N - \alpha V^N \times \partial_t V^N)\cdot F^N dxdt = \\
\int_Q \chi^N V^N \times A\nabla_{\hat{x}} V^N \cdot \nabla_{\hat{x}} F^N dxdt + \\
\int_Q \chi^N V^N \times A\partial_z V^N \cdot (\sigma^N \partial_z F + NF(\partial_y \sigma)^N)dxdt - \\
\int_Q \chi^N V^N \times (\Psi(V^N) - \nabla\varphi^N)\cdot F^N dxdt + \\
\mathcal{S}_+^N(F^N) + \mathcal{S}_-^N(F^N) - \mathcal{I}^N(F^N)
\end{array}
\right. \tag{3.10}
$$

where we have set

$$
\begin{aligned}
\mathcal{S}_\pm^N(F^N) &= \frac{s}{N}\sum_{k=0}^{N-1}\int_{Q_T} V^N(z_k^\pm)\cdot \mathbf{e}_3 V^N(z_k^\pm)\times \mathbf{e}_3 \cdot F^N(z_k^\pm)d\hat{x}dt \\
\mathcal{I}^N(F^N) &= i\sum_{k=0}^{N-1}\int_{Q_T} V^N(z_k^+)\times V^N(z_k^+)(F^N(z_k^+) - F^N(z_k^-))d\hat{x}dt
\end{aligned} \tag{3.11}
$$

and put $Q_T = R^+ \times \Omega_T$. Before to pass to the limit in each integral terms of the weak formulation, let us consider the surface boundary integral and the interlayer boundary integral defined by (3.11). The surface integral term $\mathcal{S}^N(F^N) = \mathcal{S}_+^N(F^N) + \mathcal{S}_-^N(F^N)$ satisfies the estimate

$$
|\mathcal{S}_\pm^N(F^N)| \leq Cs|\sigma|_\infty \|F\|_{L^2(R^+,H^1(\Omega))}. \tag{3.12}
$$

We discuss the interlayer coupling boundary integral $\mathcal{I}^N(F^N)$. By the Y-periodicity of σ we have $\sigma(Nz_k^\pm) = \sigma(h^\pm)$ where $h^\pm = (1\pm\theta)/2$. Hence we may write $F^N(z_k^+) - F^N(z_k^-) = (\sigma(h^+)+\sigma(h^-))(F(z_k^+) - F(z_k^-))/2 + (F(z_k^+)+F(z_k^-))(\sigma(h^+)-\sigma(h^-))/2$. Moreover, using the relation $V^N(z_k^+)\times V^N(z_k^-) = (V^N(z_k^+) - V^N(z_k^-))\times V^N(z_k^-)$ and $V^N(z_k^+)\times V^N(z_k^-) = (\int_{z_k^-}^{z_k^+}\partial_z V^N(z)dz)\times V^N(z_k^-)$ we see that $\mathcal{I}^N(F\sigma)$ can be decomposed into two parts $\mathcal{I}^N(F\sigma) = \mathcal{I}_1^N(F\sigma) + \mathcal{I}_2^N(F\sigma)$ with $\mathcal{I}_1^N(F\sigma) = i(\sigma(h^+) + \sigma(h^-))/2\sum_{k=0}^{N-1}\int_{Q_T}(\int_{z_k^-}^{z_k^+}\partial_z V^N(z)dz)\times(V^N(z_k^+)+V^N(z_k^-))/2\cdot(\int_{z_k^-}^{z_k^+}\partial_z F(z)dz)d\hat{x}dt$ and $\mathcal{I}_2^N(F\sigma) = i(\sigma(h^+)+\sigma(h^-))\sum_{k=0}^{N-1}\int_{Q_T} V^N(z_k^+)\times V^N(z_k^-)\cdot(F(z_k^+)+F(z_k^-))/2d\hat{x}dt$. Using Cauchy-Schwartz inequality and the uniform bound of $\partial_z V^N$ in $L^\infty(R^+,L^2(\Omega))$ then we get

LEMMA 3.3. *The following estimates hold*

$$
\begin{aligned}
|\mathcal{I}_1^N(F\sigma)| &\leq \frac{i}{N}C(1-\theta)|\sigma|_\infty|\partial_z F|_{L^2(\Omega)} \\
|\mathcal{I}_2^N(F\sigma)| &\leq i\ C\ (1-\theta)^{1/2}|\sigma(h^+)-\sigma(h^-)|\|F\|_{H^1(\Omega)}.
\end{aligned} \tag{3.13}
$$

Moreover, if $\sigma = 1$, we have $\mathcal{I}_2^N(F) = 0$ and $\mathcal{I}^N(F)\to 0$ as $N\to+\infty$. This result shows that the interlayer exchange coupling betwen the interfaces is too small to contribute to the limit process.

3.1. Passing to the limit. First step. Wet pass to the limit in the weak formulation (3.10). Multiplying (3.10) by N and using Lemmas 2 and 3 we deduce that $\int_Q \chi(Nz)V^N \times A\partial_z V^N \cdot (\partial_z F)_N dxdt = \mathcal{O}(\frac{1}{N})$ and

$$\int_Q \chi(Nz)V^N \times A\partial_z V^N \cdot (\partial_z F)_N dxdt \to 0. \tag{3.14}$$

We introduce, by using the two-scale convergence results, see [18], [2], the Y-periodic profile $\mathcal{M}(t,x,y)$ and the Y-periodic corrector profile $\mathcal{M}^1(t,x,y)$ associated with a bounded sequence V^N in $L^\infty(R^+; H^1(\Omega))$. The definition of these profiles are recalled at the end of Section 1. By using Lemma 1, we deduce that there exist subsequences of V^N and φ^N such that

$$\begin{cases} V^N \rightharpoonup M \text{ weakly-}\star \text{ in } L^\infty(R^+, H^1(\Omega)) \\ \partial_t V^N \to \partial_t M \text{ weakly in } L^2(R^+, L^2(\Omega)) \\ V^N \to M \text{ strongly in } L^2_{\text{loc}}(R^+; L^2(\Omega)) \\ \nabla\varphi^N \rightharpoonup \nabla\phi \text{ weakly-}\star \text{ in } L^\infty(R^+; L^2(R^3)). \end{cases} \tag{3.15}$$

Using the weak convergences of Lemma 1 we get

$$\begin{aligned} &\textstyle\int_Q \chi^N \partial_t V^N \cdot F^N dQ \to \int_{Q\times Y} \chi(y)\partial_t \mathcal{M} \cdot F dxdtdy \\ &\textstyle\int_Q \chi^N \nabla_{\hat{x}} V^N \cdot F^N dxdt \to \int_{Q\times Y} \chi(y)\nabla_{\hat{x}} \mathcal{M} \cdot F dxdtdy \\ &\textstyle\int_Q \chi^N \partial_z V^N \cdot F^N dxdt \to \int_{Q\times Y} \chi(y)(\partial_z \mathcal{M} + \partial_y \mathcal{M}^1) \cdot F dxdtdy \end{aligned} \tag{3.16}$$

where $\mathcal{M}^1$ is the Y-periodic corrector profile associated with the bound of $\partial_z V^N$. By the strong convergence of V^N, see Lemma 3.1, we deduce that $\mathcal{M}$ is independent of y. Thus, we have $\mathcal{M} = M$ a.e $(t,x) \in Q$. Since M^N satisfies the saturation condition $|M^N(t,x)|^2 = 1$ a.e $(t,x) \in Q^N$ (but not in Q) we have

LEMMA 3.4. *The magnetization field M satisfies the saturation condition*

$$|M(t,x)|^2 = 1 \quad a.e \ in \ Q. \tag{3.17}$$

Proof. Let $g(t,x)$ be a test function defined in Q. We have $\int_Q \chi^N g(t,x)dxdt = \int_Q |V^N(t,x)|^2 \chi^N g(t,x)dxdt$. Using the strong convergence of the sequence V^N and the two-scale convergence we deduce the following equality $\int_Y \int_Q g(t,x)\chi(y)\, dtdxdy = \int_Y \int_Q |M(t,x)|^2 g(t,x)\chi(y)\, dtdxdy$ which implies (3.17).

Passing to the limit in the left-hand side of (60) and using test functions of the type $F^N(t,x) = F(t,x)\sigma(Nz)$ where σ is Y-periodic, we get

$$\int_{Y\times Q} M \times A(\partial_z M + \partial_y \mathcal{M}^1)F(t,x)\partial_y\sigma(y)\chi(y)dydxdt = 0 \tag{3.18}$$

which implies the

LEMMA 3.5. *The Y-corrector profile $\mathcal{M}^1$ satisfies the equation*

$$\partial_z M(t,x) + \partial_y \mathcal{M}^1(t,x,y) = 0 \quad a.e. \ (t,x,y) \in Q \times Y^*. \tag{3.19}$$

Proof. Using (3.18) and integrating by parts with respect to the fast variable $y \in Y^*$ we deduce that $\mathcal{M}^1$ satisfies, in the sense of distributions, the local problem

$$\partial_y(M \times A\partial_y\mathcal{M}^1) = 0 \text{ in } Y^*, \; M \times A(\partial_z M + \partial_y\mathcal{M}^1)(h^{\pm}) = 0 \tag{3.20}$$

for a.e (t,x) fixed. It follows that $M \times (\partial_z M + \partial_y\mathcal{M}^1) = 0$ a.e in $Q \times Y^*$. The saturation condition gives $M^N(t,x) \cdot \partial_z M^N(t,x) = 0$ then passing to the limit in the sense of the two-scale convergence we get $M \cdot (\partial_z M + \partial_y\mathcal{M}^1) = 0$ a.e in $Q \times Y^*$. Hence the lemma is proved.
Since the limit M is independent of the fast variable y then, in the sequel the test function used in the weak formulation (3.10) are chosen to be independent of y (in our case we set $\sigma = 1$). The following convergences hold

LEMMA 3.6. *We have*

$$\begin{aligned} &\textstyle\int_Q V^N \times \partial_t V^N \cdot F\chi^N dxdt \to \overline{\chi} \int_Q M \; times\partial_t M \cdot F dtdx \\ &\textstyle\int_Q V^N \times \partial_z V^N \cdot (\partial_z F)\chi^N dtdx \to 0 \\ &\textstyle\int_Q V^N \times \nabla_{\hat{x}} V^N \cdot (\nabla_{\hat{x}} F)\chi^N dtdx \to \overline{\chi} \int_Q M \times \nabla_{\hat{x}} M \cdot \nabla_{\hat{x}} F dtdx. \end{aligned} \tag{3.21}$$

Here we have set $\langle\chi\rangle = \overline{\chi}$.

Proof. The strong convergence of the subsequence V^N combined with the weak convergence of $\partial_t V^N$ in $L^2(R^+; L^2(\Omega)$ and the weak- $\star$ convergence of ∇V^N in $L^\infty(R^+; L^2(\Omega)$ imply the results.
Now we deal with the magneto-static equation. Let ϕ be defined by the convergence $\nabla\varphi^N \rightharpoonup \nabla\phi$ weak-$\star$ in $L^\infty(R^+; L^2(R^3))$. We have,

PROPOSITION 3.7. *The contribution of the stray field satisfies*

$$\int_Q V^N \times \nabla\varphi^N \cdot F\chi^N dtdx \to \overline{\chi} \int_Q M \times (\nabla\phi - (1-\overline{\chi})(M \cdot \mathbf{e}_3)\mathbf{e}_3) \cdot F dtdx \tag{3.22}$$

where $\overline{\chi} = \langle\chi(\cdot)\rangle = (1-\theta)$ *and* ϕ*, the weak-$\star$ limit in* $L^\infty(R^+; H^1_{loc}(R^3))$ *of a subsequence of* φ^N*, satisfies the equation*

$$\nabla \cdot (\nabla\phi + \chi(\Omega)\overline{\chi}M) = 0 \text{ in } R^+ \times R^3. \tag{3.23}$$

Proof. Since $\nabla\varphi^N$ is uniformly bounded in $L^\infty(R^+; L^2(R^3))$ it follows that φ^N so is in $L^\infty(R^+; L^2_{loc}(R^3))$. Let ϕ be the weak-$\star$ limit in $L^\infty(R^+; L^2_{loc}(R^3)$ of a subsequence of φ^N such that $\nabla\varphi^N$ converges to $\nabla\phi$ weakly-$\star$ in $L^\infty(R^+; L^2(R^3)$. We first characterize the limit equation satisfied by ϕ. Let $\Phi(t,x,y)$ be the Y-periodic profile and $\Phi^1(t,x,y)$ be the Y-periodic corrector profile associated with subsequences of φ^N and $\partial_z\varphi^N$ see [2], [18]. We have $\langle\Phi(t,x,\cdot)\rangle = \phi(t,x)$. We shall prove that Φ is independent of the fast variable y and thus we deduce that $\Phi = \phi$. This is a consequence of the strong convergence in $L^2_{loc}(R^+ \times R^3)$ of φ^N to ϕ which can be proved as follows. Since $\partial_t M^N$ is uniformly bounded in $L^2(R^+; L^2(\Omega^N))$ then using the magneto-static equation satisfied by φ^N, we deduce that $\partial_t\varphi^N$ satisfies equation $\nabla \cdot (\nabla\partial_t\varphi^N + \chi(\Omega)\chi(Nz)\partial_t V^N) = 0$. Hence, we get the estimate $|\nabla\partial_t\varphi^N|^2_{L^2(R^+;L^2(R^3))} \leq |\partial_t V^N|_{L^2(R^+;L^2(\Omega))}||\nabla\partial_t\varphi^N|_{L^2(R^+;L^2(R^3))}$. This implies that $\nabla\partial_t\varphi^N$ is uniformly bounded in $L^2(R^+; L^2(R^3))$. Consequently, $\partial_t\varphi^N$ is uniformly bounded in $L^2_{loc}(R^+; L^2(R^3))$ and then the full gradient of φ^N with respect to the variables t and x so is. Thus, a subsequence of φ^N converges strongly to ϕ in $L^2_{loc}(R^+ \times R^3))$. Hence we get that $\Phi = \phi$. Finally, passing to the limit in the magneto-static equation satisfied by φ^N and using the strong convergence of V^N to M we get the homogenized equation

$$\nabla \cdot (\nabla \phi + \chi(\Omega)\overline{\chi} M) = 0 \text{ in } R^+ \times R^3. \tag{3.24}$$

The next step in the proof concerns the convergence of $\int_Q V^N \times \nabla \varphi^N \cdot F \chi^N dxdt$ where $F(t,x)$ is a test function. Using the two-scale convergence we deduce that

$$\begin{aligned} &\int_Q V^N \times \nabla_{\hat{x}} \varphi^N \cdot F \chi^N dxdt \to \overline{\chi} \int_Q M \times \nabla_{\hat{x}} \phi \cdot F dxdt \\ &\int_Q V^N \times \mathbf{e}_3 \partial_z \varphi^N \cdot F \chi^N dxdt \to \int_{Q\times Y} \chi(y) M \times \mathbf{e}_3 (\partial_z \phi + \partial_y \Phi^1) \cdot F dxdtdy. \end{aligned} \tag{3.25}$$

To characterize the corrector profile Φ^1 we consider the weak formulation of the magneto-static equation. We use periodically oscillating test function of the type $F^N = F(t,x)\sigma(Nz)$. We get

$$\begin{aligned} &\int_{R^+\times R^3} \nabla_{\hat{x}} \varphi^N \nabla_{\hat{x}} F \sigma_N \; dxdt + \int_{R^+\times R^3} \partial_z \varphi^N (\partial_z F \sigma_N + NF(\partial_y \sigma)_N) dxdt = \\ &-\int_Q \chi^N (V^N)_{\hat{x}} \nabla_{\hat{x}} F \sigma_N dxdt - \int_Q \chi^N (V^N) \cdot \mathbf{e}_3 (\partial_z F \sigma_N + NF(\partial_y \sigma)_N) dxdt. \end{aligned} \tag{3.26}$$

Multiplying the equation by N and passing to the limit, we obtain

$$\int_{R^+\times R^3} F(\int_Y (\partial_z \phi + \partial_y \Phi^1) \partial_y \sigma dy) = -\int_{R^+\times R^3} F(M \cdot \mathbf{e}_3)(\int_Y \chi \partial_y \sigma dy) \tag{3.27}$$

which gives the local problem satisfied by Φ^1

$$\partial_y (\partial_z \phi + \partial_y \Phi^1 + \chi(\Omega)\chi(y)(M \cdot \mathbf{e}_3)) = 0 \text{ in } Y. \tag{3.28}$$

a.e (t,x). Solving this equation we obtain $\partial_z \phi + \partial_y \Phi^1 = -\chi(\Omega)\chi(y)(M \cdot \mathbf{e}_3) + C(t,x)$ where $C(t,x)$ is an unknown function which is independent of y. Hence since Φ^1 is Y-periodic we get

$$\partial_y \Phi^1 = -\chi(\Omega)(\chi(y) - \overline{\chi})(M \cdot \mathbf{e}_3) \tag{3.29}$$

Using the decomposition $\nabla \varphi^N = \nabla_{\hat{x}} \varphi^N + \partial_z \varphi^N \mathbf{e}_3$ then $\int_Q \chi^N V^N \times \nabla \varphi^N \cdot F dxdt = \int_Q \chi^N V^N \times \nabla_{\hat{x}} \varphi^N \cdot F dxdt + \int_Q \chi^N V^N \times \mathbf{e}_3 \cdot F \partial_z \varphi^N \; dxdt$. Passing to the limit, in the sense of the two-scale convergence, we get

$$\begin{aligned} &\int_Q \chi^N V^N \times \nabla \varphi^N \cdot F \; dxdt \to \overline{\chi} \int_Q M \times \nabla_{\hat{x}} \varphi \cdot F \; dxdt \\ &+ \; \int_Y \chi(y) \int_Q M \times \mathbf{e}_3 (\partial_z \phi + \partial_y \Phi^1) \cdot F \; dxdtdy. \end{aligned} \tag{3.30}$$

Using the expression of $\partial_y \Phi^1$ we arrive at the desired result

$$\begin{aligned} &\int_Q \chi^N V^N \times \nabla \varphi^N \cdot F \; dxdt \to \overline{\chi} \int_Q M \times \nabla \phi \cdot F \; dxdt \\ &- \overline{\chi}(1 - \overline{\chi})(\int_Q M \times \mathbf{e}_3 (M \cdot \mathbf{e}_3)) \cdot F \; dxdt). \end{aligned} \tag{3.31}$$

3.2. Passing to the limit. Second step. We shall pass to the limit in the surface boundary integral term $\mathcal{S}_N(F) = \mathcal{S}_N^+(F) + \mathcal{S}_N^-(F)$, see (61). We have shown that $|\mathcal{S}_N^{\pm}(F)| \leq sC_T|\Omega_T||F|_\infty$. To characterize the limit of $\mathcal{S}_N^{\pm}$, we proceed as in [6] and we follow the lines of the proof given in [6]. Let $\beta(y)$ be the Y-periodic solution of the local problem

$$\begin{cases} -\partial_y(A\partial_y\beta) = -\kappa \text{ in } Y^*, \text{ with } \kappa = -\dfrac{2s}{1-\theta} \\ A\partial_y\beta(h^-) = -s, A\partial_y\beta(h^+) = s. \end{cases} \tag{3.32}$$

Simple computations give $\langle\chi(\cdot)A\partial_y\beta(\cdot)\rangle = 0$. We extend β to R by periodicity. We define $\beta^N(z) = \beta(Nz)$ for $z \in I^N$. Then β^N is the solution of the problem

$$\begin{cases} -\partial_z(A\partial_z\beta^N) = -N^2\kappa \text{ in } I_N \\ A\partial_z\beta^N(z_k^-) = -Ns, A\partial_z\beta^N(z_k^+) = Ns, \text{ for } k = 0, \cdots, N-1 \end{cases} \tag{3.33}$$

Notice that $\beta^N(0) = \beta(0) = \beta(1)$ and $\beta^N(1) = \beta(N) = \beta(1)$ by periodicity and $\partial_z\beta^N = N(\partial_y\beta)(Nz)$. For any $v \in H^1(I_N)$ we have

$$\begin{aligned} &\frac{s}{N}\sum_{k=0}^{N-1}(v(z_k^-) + v(z_k^+)) = -\kappa\int_{I_N} vdz - \frac{1}{N}\int_{I_N} A(\partial_y\beta)(Nz)\partial_z vdz \\ &+\frac{1}{N}(A(\partial_y\beta)(1)v(1) - A(\partial_y\beta)(0)v(0)). \end{aligned} \tag{3.34}$$

Now, we may prove the following convergence result.

Proposition 3.8. *The surface boundary integral term $\mathcal{S}^N(F)$ satisfies*

$$\mathcal{S}^N(F) \to 2s\int_Q M\cdot\mathbf{e}_3 M\times\mathbf{e}_3\cdot F dxdt. \tag{3.35}$$

Proof. Let F be a test function. We set, in (3.34), $v = M^N\cdot\mathbf{e}_3 M^N\times\mathbf{e}_3\cdot F$. We have $v \in L^\infty(R^+, H^1(\Omega^N))$. We integrate (91) over $R^+\times\Omega_T$ to get

$$\begin{aligned} &\mathcal{S}^N(F) = -\kappa\textstyle\int_Q V^N\cdot\mathbf{e}_3 V^N\times\mathbf{e}_3\cdot F\chi^N dxdt \\ &-\tfrac{1}{N}\textstyle\int_Q A(\partial_y y\beta)^N\partial_z(V^N\cdot\mathbf{e}_3 V^N\times\mathbf{e}_3\cdot F)\chi^N dxdt \\ &+\tfrac{1}{N}\displaystyle\int_{Q_T}(A(\partial_y\beta)(1)(V^N\cdot\mathbf{e}_3 V^N\times\mathbf{e}_3\cdot F)(1) \\ &-A(\partial_y\beta)(0)(V^N\cdot\mathbf{e}_3 V^N\times\mathbf{e}_3\cdot F)(0))\, d\hat{x}dt. \end{aligned} \tag{3.36}$$

First, we have the result

$$\int_Q V^N\cdot\mathbf{e}_3 V^N\times\mathbf{e}_3\cdot F\chi(Nz)dxdt \to \overline{\chi}\int_Q M\cdot\mathbf{e}_3 M\times\mathbf{e}_3 F dxdt. \tag{3.37}$$

Next, since M^N is uniformly bounded in $L^\infty(R^+, H^1(\Omega^N))$ then the last term in (3.36) satisfies $\frac{1}{N}\int_{\widehat{Q}}(A(\partial_y\beta)(1)(V^N\cdot\mathbf{e}_3 V^N\times\mathbf{e}_3\cdot F)(1) - A(\partial_y\beta)(1)(V^N\cdot\mathbf{e}_3 V^N\times\mathbf{e}_3\cdot F)(0))d\hat{x}dt \to 0$. Then using the uniform bound of M^N in $L^\infty(R^+, H^1(\Omega^N))$ we deduce that

$$\frac{1}{N}\int_Q A(\partial_y\beta)^N\chi^N\partial_z(V^N\times\mathbf{e}_3 V^N\cdot\mathbf{e}_3\cdot F)dxdt \to 0. \tag{3.38}$$

Hence the proposition is proved.
Combining the above convergence results we get the homogenization result

THEOREM 3.9. *Let (M^N, φ^N) be a global solution of the LLG equations under the previous hypotheses. Let (M, φ) be the weak-$\star$ limit of a subsequence. Then (M, ϕ) satisfies the homogenized LLG problem*

$$
(3.39)\quad \begin{cases} \dfrac{1}{1+\alpha^2}\Big(\partial_t M - \alpha M \times \partial_t M\Big) = -M \times \mathcal{H}_{\mathrm{hom}}(M) \ \ in \ \ R^+ \times \widehat{\Omega} \\ M(0) = M_0, \dfrac{\partial M}{\partial n} = 0 \ \ on \ \ R^+ \times \partial\Omega \\ \mathcal{H}_{\mathrm{hom}}(M) = \nabla \cdot_{\hat{x}} (A \nabla_{\hat{x}} M) + (\dfrac{2s}{1-\theta} + \theta) M \cdot \mathbf{e}_3 \mathbf{e}_3 + \Psi(M) + \nabla\phi \end{cases}
$$

where φ satisfies the equation

$$
(3.40)\qquad \nabla \cdot (\nabla\phi + \chi(\Omega)\overline{\chi} M) = 0 \ \ in \ \ R^+ \times R^3
$$

and $\overline{\chi} M_0$ is the weak limit in $H^1(\Omega)$ of the subsequence $\chi(Nz) P_N M_0$.

Proof. We pass to the l limit in the weak formulation and use the convergence results proved previously we get

$$
(3.41)\quad \begin{cases} \dfrac{1}{1+\alpha^2} \displaystyle\int_{Q\times Y} (\partial_t M - \alpha M \times \partial_t M) \cdot F\chi(y)\, d\hat{x}dzdtdy = \\ \displaystyle\int_{Q\times Y} M \times A \nabla_{\hat{x}} M \cdot \nabla_{\hat{x}} F\chi(y)\, d\hat{x}dzdtdy + \\ \dfrac{2s}{1-\theta} \displaystyle\int_{Q\times Y} M \cdot \mathbf{e}_3 M \times \mathbf{e}_3 \cdot F\chi(y)\, d\hat{x}dzdtdy + \\ \displaystyle\int_{Q\times Y} M \times \Psi(M) \cdot F\chi(y)\, d\hat{x}dzdtdy + \\ \displaystyle\int_{R^+\times R^3\times Y} \chi(\Omega) M \times \nabla\phi \cdot F\chi(y)\, d\hat{x}dzdtdy. \end{cases}
$$

Let us prove the last result stated in the theorem. We pass to the limit in the integral $\int_{R^+\times R^3} \chi(\Omega) V^N \times \nabla\varphi^N \cdot F\chi^N dxdt$. The potential φ^N satisfies the magnetostatic equation in $R^+ \times R^3$. Let $\phi(t,x,y)$ be the profile associated to the sequence φ^N and $\phi^1(t,x,y)$ be the corrector profile associated to the sequence $\nabla\varphi^N$. Passing to the limit in the weak formulation associated to this equation for the test function $h_N(t,x) = h(t,x,Nz)$ with $h(t,x,y)$ is 1-periodic with respect to the fast variable y. We get $\partial_y(\partial_z \Phi + \partial_y \Phi^1) = 0$ in the sense of the distributions. Next, the local strong convergence of the sequence φ^N implies that Φ is independent of the fast variable y and we have $\Phi = \phi$. The previous equation implies that Φ^1 is constant with respect to the variable y and we choose Φ^1 with mean value equal to 0 in Y. Finally we obtain the convergence

$$
(3.42)\int_{R^+\times R^3} \chi(\Omega) V^N \times \nabla\varphi^N \cdot F\chi(Nz) \to \int_{R^+\times R^3\times Y} \chi(\Omega) M \times \nabla\phi \cdot F\chi(y) dy.
$$

Hence the theorem is proved.

4. Global existence results for LLG equations. The purpose of this Section is to prove the global existence of weak solutions to the problem (1.4)-(1.5)-(1.6)-(1.7)

under the assumption that $s^N = \frac{s}{N}$ and $i^N = i$ given in Section 3. We also assume that the convex function ψ satisfies the hypothesis $\Psi(U) \geq 0$ and $D_U\Psi(U)| \leq C$. The general framework, in the case $i^N = 0$ and $s^N = 0$ has been established in earlier papers (see for instance [23],[3]) using the Galerkin method and the penalization of the saturation constraint. This method gives an approximated sequence of solutions converging to a global solution of the problem and has been used in [8] in the case of the interlayer exchange coupling with $N = 1$ and $i > 0$. In the following, we give an alternative proof by using an hyperbolic regularization of a linear problem.
We will say that (M, φ) is a weak solution to the LLG equations if

$$(4.1)\quad \begin{cases} M \in L^\infty(R^+; H^1(\Omega^N)),\ \partial_t M \in L^2(R^+; L^2(\Omega^N)),\ |M(t,x)|^2 = 1 \text{ a.e} \\ \nabla\varphi \in L^\infty(R^+; L^2(R^3)) \end{cases}$$

and satisfies the LLG equations with the boundary conditions and the magnetostatic equation in the sense of distributions. A weak solution of the problem satisfies $M \times \partial_t M \in L^2(R^+; L^2(\Omega^N))$, $M \times (\nabla_M \Psi)(M) \in L^\infty(R^+; L^2(\Omega^N))$ and $M \times \nabla\varphi \in L^2(R^+; L^2(R^3))$. Moreover we have $M \times \Delta M = \nabla\cdot(M \times \nabla M)$. Indeed, more generally we have $M \times \partial_{x_j}(a_{ij}\partial_{x_j} M) = \partial_{x_j}(M \times a_{ij}\partial_{x_j} M)$ for any symmetric matrix (a_{ij}) with entries in $L^\infty(\Omega)$. We use the usual summation notation with repeted indices. In the sequel, for simplicity, we denote by Ω the domain $\Omega^N = \Omega_T \times I^N$.
Let us explain the method we shall use. The first remark is that if one uses the saturation condition $|M|^2 = 1$ and the vector decomposition $\partial_t M = (\partial_t \cdot M)M - M \times (M \times \partial_t M)$ then we can rewrite the LLG equation as following

$$M \times (-\mathcal{H}(M) + \frac{\alpha}{1+\alpha^2}\partial_t M + \frac{1}{1+\alpha^2} M \times \partial_t M) = 0.$$

The second remark concerns the possible interpretation of the energy equality associated with LLG equations

$$\frac{d\mathcal{E}(t)}{dt} + \frac{2\alpha}{1+\alpha^2}|\partial_t M|^2_{L^2} = 0$$

which may be understood as the energy equality of a wave equation with damping if one adds to $\mathcal{E}(t)$ the potential energy $\varepsilon^2|\partial_t M|^2_{L^2}$. This suggests to regularize the LLG equation (recall that $M(t,x)|^2 = 1$ in $R^+ \times \Omega$) as the following

$$(4.2)\ M \times (\varepsilon^2\partial_t^2 M - \mathcal{H} + \frac{\alpha}{1+\alpha^2}(\partial_t M + \frac{2M}{1+|M|} \times \partial_t M) + 2\nu\beta'(|M|^2)M) = 0$$

where $\varepsilon > 0$ is the regularization parameter and β is any regular scalar function. The regularized LLG equation takes the form

$$\partial_t M - \alpha M \times \partial_t M = -(1+\alpha^2) M \times (-\varepsilon^2\partial_t^2 M + \mathcal{H} - 2\nu\beta'(|M|^2)M).$$

These remarks suggest to introduce the Regularized Intermediary Penalyzed (RIP) problem

$$(4.3)\qquad \begin{cases} \varepsilon^2\partial_t^2 U - \mathcal{H}(U) + \frac{\alpha}{1+\alpha^2}\partial_t U = F_1^\varepsilon(U, \partial_t U) - 2\nu\beta'(|U|^2)U, \\ F_1^\varepsilon(U, \partial_t U) = -\frac{1}{1+\alpha^2}\frac{2U}{1+|U|} \times \frac{\partial_t U}{1+\varepsilon^2|\partial_t U|_{L^2(\Omega)}}. \end{cases}$$

Of course, the function β should penalize the saturation condition $|U|^2 = 1$ which is not satisfied by the solutions of the RIP problem and then $\nu > 0$ is the penalization parameter. We choose the function β in regard to the energy which should be conserved. The properties satisfied by $\beta(|U|^2)$ are the following: $\beta(|U|^2) \sim |U|^2$ for $|U|$ large and $\beta(|U|^2) = 0$ if and only if $|U|^2 = 1$. An example of such functions is given by $\beta(|U|^2) = 2(1 - (\frac{|U|^2+1}{2})^{1/2})^2$. In the sequel the penalization operator will be denoted by P_ν.

The RIP problem we shall discuss is the following

$$\left\{ \begin{array}{l} \varepsilon^2 \partial_t^2 U_\nu^\varepsilon - A\Delta U_\nu^\varepsilon + \dfrac{\alpha}{1+\alpha^2}\partial_t U_\nu^\varepsilon = F^\varepsilon(U_\nu^\varepsilon, \partial_t U_\nu^\varepsilon) \text{ in } R^+ \times \Omega \\ F^\varepsilon(U_\nu^\varepsilon, \partial_t U_\nu^\varepsilon) = F_1^\varepsilon(U_\nu^\varepsilon, \partial_t U_\nu^\varepsilon) - (\nabla_M \psi)(U_\nu^\varepsilon) + \nabla \varphi_\nu^\varepsilon + P_\nu(U_\nu^\varepsilon) \\ U_\nu^\varepsilon(0,x) = M_0(x), \ \varepsilon \partial_t U_\nu^\varepsilon(0,x) = 0, \text{ in } \Omega \end{array} \right. \tag{4.4}$$

with the boundary conditions

$$\left\{ \begin{array}{l} A\frac{\partial U_\nu^\varepsilon}{\partial n} = 0 \text{ on } R^+ \times (\partial\Omega \setminus \Gamma_k^\pm), \\ \mp A(\frac{\partial U_\nu^\varepsilon}{\partial z})^\pm = -\dfrac{s}{N}((U_\nu^\varepsilon)^\pm - ((U_\nu^\varepsilon)^\pm \cdot \mathbf{e}_3)\mathbf{e}_3) - i((U_\nu^\varepsilon)^\pm - (U_\nu^\varepsilon)^\mp) \end{array} \right. \tag{4.5}$$

on $R^+ \times \Gamma_k^\pm$ for $k = 1, \cdots, N-1$ and where φ_ν satisfies the stray equation. Notice that the initial data M_0 is assumed to verify the saturation condition $|M_0(x)|^2 = 1$ a.e. The energy equality takes the form

$$\frac{2\alpha}{1+\alpha^2}|\partial_t U_\nu^\varepsilon|^2 + \frac{d}{dt}(\varepsilon^2 |\partial_t U_\nu^\varepsilon|^2 + \mathcal{E}_\nu(U_\nu^\varepsilon(t))) = 0. \tag{4.6}$$

where the local penalization operator (which penalizes the saturation condition) is given by

$$P_\nu(U) = \nu \frac{U}{(1+|U|^2)^{1/2}}(2^{1/2} - (1+|U|^2)^{1/2}) \tag{4.7}$$

and satisfies the properties

$$\left\{ \begin{array}{l} U \times P_\nu(U) = 0, \ P_\nu(U) = -\nabla_U(p_\nu(U)) \\ p_\nu(U) = \dfrac{\nu}{2}(2^{\frac{1}{2}} - (1+|U|^2)^{\frac{1}{2}})^2, \\ p_\nu(U) = 0 \text{ if and only if } |U| = 1. \end{array} \right. \tag{4.8}$$

Moreover, there exists $C > 0$ which is independent of ν such that for all $U_1, U_2 \in R^3$ we have

$$|P_\nu(U_1) - P_\nu(U_2)| \leq \nu C |U_1 - U_2|. \tag{4.9}$$

Computing explicitely $\nabla_U(P_\nu(U)$ we deduce that $|(\nabla_U(P_\nu(U))\xi;\xi)| \leq \nu(2+\sqrt{2})|\xi|^2$ for all $\xi \in R^3$ and then the result follows.

The energy equality is given formally by (102) where

$$\begin{array}{l} \mathcal{E}_\nu(U_\nu^\varepsilon(t)) = A|\nabla U_\nu^\varepsilon|^2_{L^2(\Omega)} + |\nabla \varphi_\nu^\varepsilon|^2_{L^2(R^3)} + \\ + \ \int_\Omega (2\psi(U_\nu^\varepsilon) + p_\nu(U_\nu^\varepsilon))d\hat{x} + \mathcal{E}_s(U_\nu^\varepsilon) + \mathcal{E}_i(U_\nu^\varepsilon) \end{array} \tag{4.10}$$

with

$$\begin{aligned}
\mathcal{E}_s(U_\nu^\varepsilon) &= \frac{s}{N}\sum_{k=1}^{N-1}\int_{\Omega_T}(|U_\nu^\varepsilon(z_k^+)\times \mathbf{e}_3|^2 + |U_\nu^\varepsilon(z_k^-)\times \mathbf{e}_3|^2)d\hat{x} \\
\mathcal{E}_i(U_\nu^\varepsilon) &= i\sum_{k=1}^{N-1}\int_{\Omega_T}|U_\nu^\varepsilon(z_k^+) - U_\nu^\varepsilon(z_k^-)|^2 d\hat{x}.
\end{aligned} \tag{4.11}$$

Usually, see [23], [3] for example, we use the penalization operator $Q_\nu(U) = \nabla_U q_\nu(U)$, $\nabla_U q_\nu(U) = \nu U(1-|U|^2$ with $q_\nu(U) = -\frac{\nu}{4}(1-|U|^2)^2$. It is easy to see that Q_ν maps continuously $H^1(\Omega)$ into $H^1(\Omega)$ and $q_\nu(U) \sim |U|^4$. Hence, in regard of the semigroup perturbation theory see [19], this operator can not be considered as a bounded perturbation in $L^2(\Omega)$ of the operator $-A\Delta$. As we have said in the motivation of our method the penalization operator the penelization operator is chosen in order it is a local Lipchitz perturbation in $L^2(\Omega)$ of $-A\Delta$.

To prove the global existence of solutions to LLG equations we first solve the RIP problem then, we pass to the limit when $\varepsilon \to 0$ to get global solutions of the Intermediary Penalized (IP) problem. Finally we pass to the limit when $\nu \to \infty$ in the weak formulation of the IP problem by using test functions (depending of the solution) of the type $v \times U_\nu$ where v is an arbitrary regular test function to get global solutions of LLG problem. We shall prove the following results

PROPOSITION 4.1. *Let $M_0 \in H^1(\Omega)$ be such that $|M_0(x)|^2 = 1$ a.e in Ω and $\|M_0\|_{H^1(\Omega)} \leq C$ uniformly with respect N. Then, there exists a unique solution weak $U_\nu^\varepsilon \in L^\infty(R^+; H^1(\Omega))$ of the RIP problem. Moreover it satisfies the energy equality*

$$\mathcal{E}_\nu(U_\nu^\varepsilon(t)) + \varepsilon^2|\partial_t U_\nu^\varepsilon(t)|^2_{L^2(\Omega)} + \frac{2\alpha}{1+\alpha^2}\int_0^t |\partial_t U_\nu^\varepsilon(s)|^2_{L^2(\Omega)}ds = \mathcal{E}_0(M_0) \tag{4.12}$$

where

$$\begin{aligned}
&\mathcal{E}_0(M_0) = A|\nabla M_0|^2_{L^2(\Omega)} + 2\textstyle\int_\Omega \Psi(M_0)dx \\
&+ |\nabla\varphi_0|^2_{L^2(R^3)} + \mathcal{E}_s(M_0) + \mathcal{E}_i(M_0)
\end{aligned} \tag{4.13}$$

and φ_0 solves the equation $\nabla\cdot(\nabla\varphi_0 + \chi(\Omega)M_0) = 0$ in R^3.

PROPOSITION 4.2. *Let (U_ν^ε) be the sequence of solutions of the RIP problem given by the Theorem 3. There exists a subsequence still denoted by U_ν^ε such that*

$$\begin{aligned}
&U_\nu^\varepsilon \to U_\nu \text{ in } L^2_{loc}(R^+; L^2(\Omega)) \text{ strong} \\
&U_\nu^\varepsilon \rightharpoonup U_\nu \text{ in } L^\infty(R^+; H^1(\Omega)) \text{ weak} - \star \\
&\partial_t U_\nu^\varepsilon \rightharpoonup \partial_t U_\nu \text{ in } L^2(R^+; L^2(\Omega)) \text{ weak}
\end{aligned} \tag{4.14}$$

and U_ν is a weak solution of the Intermediary Penalyzed problem

$$\begin{cases}
\dfrac{\alpha}{1+\alpha^2}\partial_t U_\nu - A\Delta U_\nu = F(U_\nu, \partial_t U_\nu) \text{ in } R^+\times\Omega \\
F(U_\nu, \partial_t U_\nu) == F_1(U_\nu, \partial_t U_\nu) - (\nabla_M\psi)(U_\nu) + \nabla\varphi_\nu + P_\nu(U_\nu) \\
F_1(U_\nu, \partial_t U_\nu) = -\dfrac{1}{1+\alpha^2}(\dfrac{2U_\nu}{1+|U_\nu|}\times\partial_t U_\nu)
\end{cases} \tag{4.15}$$

with the boundary conditions

$$\left\{\begin{array}{l} A\frac{\partial U_\nu}{\partial n}=0 \text{ on } R^+\times(\partial\Omega\setminus\Gamma_k^\pm) \\ \mp A(\frac{\partial U_\nu}{\partial z})^\pm=-\dfrac{s}{N}((U_\nu)^\pm-((U_\nu)^\pm\cdot\mathbf{e}_3)\mathbf{e}_3)-i((U_\nu)^\pm-(U_\nu)^\mp)\end{array}\right. \tag{4.16}$$

on $R^+\times\Gamma_k^\pm$ *for* $k=1,\cdots,N-1$ *and where* φ_ν *satisfies the stray equation* $\nabla\cdot(\nabla\varphi_\nu+\chi(\Omega)U_\nu)=0$ *in* $R+^\times R^3$. *The energy equality takes the form*

$$\mathcal{E}_\nu(U_\nu(t))+\frac{2\alpha}{1+\alpha^2}\int_0^t|\partial_tU_\nu(s)|^2_{L^2(\Omega^N)}ds=\mathcal{E}_0(M_0). \tag{4.17}$$

Finally we have the main result where the domain is denoted by Ω^N to express the dependency of the solutions with respect to the parameter N

THEOREM 4.3. *Let* $M_0\in H^1(\Omega^N)$ *be such that* $|M_0(x)|^2=1$ *a.e in* Ω^N *satisfying hypothesis* (53). *Then, there exists a global weak solution* M^N *to LLG equations satisfying for all* $t\geq 0$ *the enegy inequality*

$$\mathcal{E}^N(M^N(t))+\frac{2\alpha}{1+\alpha^2}\int_0^t|\partial_tM^N(s)|^2ds\leq\mathcal{E}_0^N(M_0) \tag{4.18}$$

where

$$\begin{aligned}\mathcal{E}^N(M^N(t))&=A|\nabla M^N(t)|^2_{L^2(\Omega^N)}+\textstyle\int_{\Omega^N}\Psi(M^N(t,x))dx+\\ &+|\nabla\varphi^N(t)|^2_{L^2(R^3)}+\mathcal{E}_s(M^N(t))+\mathcal{E}_i(M^N(t))\end{aligned} \tag{4.19}$$

$\mathcal{E}_0^N(M_0^N)=\mathcal{E}^N(M^N(0))$ *and* φ^N *satisfie the stray equation* $\nabla\cdot(\nabla\varphi^N+\chi(\Omega^N)M^N)=0$ *in* $R^+\times R^3$. *These solutions are obtained as the limit, as* $\nu\to\infty$, *of a subsequence of solutions* (U_ν) *to the IP problem.*

Proof. We first prove Theorem 3 by using Proposition 5. Let (U_ν) be the sequence of solutions of the IP problem given by Theorem 4. As a consequence of the energy equality satisfied by U_ν we have the following corollaries

COROLLARY 4.4. *There exists* $C>0$ *which is independent of* ν *such that the solution* U_ν *of the IP problem satisfies the estimates*

$$\begin{aligned}&\|U_\nu\|_{L^\infty(R^+;H^1(\Omega^N))}+\|\partial_tU_\nu\|_{L^2(R^+;L^2(\Omega^N))}\leq C\\ &\|\sqrt{2}-\sqrt{1+|U_\nu|^2}\|_{L^\infty(R^+;L^2(\Omega^N))}\leq\tfrac{C}{\sqrt{\nu}},\\ &\|\nabla\varphi_\nu\|_{L^\infty(R^+;L^2(R^3))}\leq C\end{aligned} \tag{4.20}$$

COROLLARY 4.5. *There exists a subsequence still denoted* U_ν *such that when* $\nu\to+\infty$ *we have the convergences*

$$\begin{aligned}&U_\nu\to M^N \text{ in } L^2_{loc}(R^+;L^2(\Omega^N)) \text{ strong and a.e}\\ &\tfrac{2U_\nu}{1+|U_\nu|}\to M^N \text{ in } L^2_{loc}(R^+;L^2(\Omega^N)) \text{ strong and a.e,}\\ &|M^N(t,x)|^2=1 \text{ a.e } (t,x)\\ &U_\nu\rightharpoonup M^N \text{ in } L^\infty(R^+;H^1(\Omega^N)) \text{ weak}-\star\\ &\partial_tU_\nu\rightharpoonup\partial_tM^N \text{ in } L^2(R^+;L^2(\Omega^N)) \text{ weak}\\ &\nabla\varphi_\nu\rightharpoonup\nabla\varphi^N \text{ in } L^\infty(R^+;L^2(R^3)) \text{ weak}-\star\end{aligned} \tag{4.21}$$

We pass to the limit in the weak formulation of the IP problem when $\nu \to \infty$. Let $v \in \mathcal{D}(Q)$ be a regular test function. We set $Q = R^+ \times \Omega^N$ and $Q_T = R^+ \times \Omega_T$. We multiply the IP problem satisfied by U_ν by $v \times U_\nu$ and integrate by parts. We get

$$\begin{aligned}
&\frac{1}{1+\alpha^2}\int_Q (\alpha\partial_t U_\nu + \frac{2U_\nu}{1+|U_\nu|} \times \partial_t U_\nu) \cdot v \times U_\nu dxdt = \\
&-A\int_Q \nabla U_\nu \cdot \nabla(v \times U_\nu)dxdt - \int_Q \nabla_U \Psi(U_\nu) \cdot v \times U_\nu dxdt \\
&+\int_Q \nabla\varphi_\nu \cdot v \times U_\nu dxdt - \mathcal{S}^+(v) - \mathcal{S}^-(v) - \mathcal{I}(v)
\end{aligned} \tag{4.22}$$

where

$$\begin{aligned}
\mathcal{S}^\pm(v) &= \tfrac{s}{N}\sum_{k=1}^{N-1}\int_{Q_T} (U_\nu(z_k^\pm) - (U_\nu(z_k^\pm)\cdot \mathbf{e}_3)\mathbf{e}_3) \cdot (v \times U_\nu)(z_k^\pm) d\hat{x}dt \\
\mathcal{I}(v) &= i\sum_{k=1}^{N-1}\int_{Q_T} (U_\nu(z_k^+) - U_\nu(z_k^-)) \cdot ((v \times U_\nu)(z_k^+) - (v \times U_\nu)(z_k^-))d\hat{x}dt.
\end{aligned} \tag{4.23}$$

Notice that we have $\nabla U_\nu \cdot \nabla(v \times U_\nu) = \nabla U_\nu \cdot \nabla v \times U_\nu$. Combining convergences of $\nabla U_\nu \rightharpoonup \nabla M^N$ in $L^\infty(R^+; L^2(\Omega^N))$ weak-$\star$, $\nabla v \times U_\nu \to \nabla v \times M^N$ in $L^2(R^+; L^2(\Omega^N)$ strong we deduce that the first integral in the right-hand side of the weak formulation converges to$-A\int_Q \nabla M^N \cdot \nabla v \times M^N dxdt$. Next, the other volume integrals appearing in the right hand side of the weak formulation converge by using the same convergence arguments. This is also the case for the left hand side of the weak formulation by using also $\partial_t U_\nu \rightharpoonup \partial_t M^N$ in $L^2(R^+; L^2(\Omega^N))$ weak.
It remains to pass to the limit in the boundary integral terms $\mathcal{S}^\pm(v)$ and $\mathcal{I}(v)$. Since the solution U_ν is bounded in $L^\infty(R^+; H^1(\Omega^N))$ and $\partial_t U_\nu$ so is in $L^2(R^+; L^2(\Omega^N))$ then the sequence U_ν is compact in $L^2_{loc}(R^+; H^s(\Omega^N))$ for all $0 \le s < 1$. In particular, its traces $U_\nu(z_k^\pm)$ are compact in $L^2_{loc}(R^+; L^2(\Omega_T))$. Finally, the strong convergence of these traces allows to pass to the limit in the boundary integral terms. This prove that M^N satifies the LLG equation in the sense of distribution and then it is a global weak solution of LLG equations. The convergences stated in the previous Corollaries show that $M^N \in L^\infty(R^+; H^1(\Omega^N))$, $\partial_t M^N \in L^2(R^+; L^2(\Omega^N)$, $|M^N(t,x)|^2 = 1$ a.e and $\nabla\varphi^N \in L^\infty(R^+; L^2(R^3))$. Moreover the energy inequality (4.17) is satisfied by using a convexity argument.
We now prove Proposition 5 by using Proposition 4. The parameter ν is fixed and we shall pass to the limit when $\varepsilon \to 0$. Let (U_ν^ε) be the sequence of solutions of RIP problem associated to the initial data M_0. The energy equality gives uniform bounds for ∇U_ν^ε in $L^\infty(R^+; L^2(\Omega))$ and for $\partial_t U_\nu^\varepsilon$ in $L^2(R^+; L^2(\Omega))$. Moreover $\varepsilon\partial_t U_\nu^\varepsilon$ is uniformly bounded in $L^\infty(R^+; L^2(\Omega))$ and $\nabla\varphi_\nu^\varepsilon$ is bounded uniformly in $L^\infty(R^+; H^1(R^3))$ where φ_ν^ε is the solution of the magneto-static equation. Next, since $p_\nu(U_\nu^\varepsilon)$ is bounded in $L^1(R^+; L^1(\Omega)$ and $\nu|U_\nu^\varepsilon| \le p_\nu(U_\nu^\varepsilon) + \nu\sqrt{2}$ then we deduce that the solution U_ν^ε is uniformly bounded in $L^\infty(R^+; L^2(\Omega))$ and then it is uniformly bounded in $L^\infty(R^+; H^1(\Omega))$. Using the compactness of the embedding of $H^1(\Omega)$ in $L^2(\Omega)$ and Aubin's compactness Lemma we deduce that the sequence U_ν^ε is compact in $L^2(R^+; L^2(\Omega))$. We easily get the convergences stated in Theorem 4. Passing to the limit in the RIP problem when $\varepsilon \to 0$, we obtain that U_ν satisfies the IP problem in the sense of distributions. Using the convexity of the energy $\mathcal{E}_\nu$ we get an energy inequality instead of the equality stated in Theorem 4. To prove the equality we first observe that $\Delta U_\nu \in L^2(R^+; L^2(\Omega))$. Hence we may multiply the equation by $\partial_t U_\nu$ lying in $L^2(R^+; L^2(\Omega))$ and integrate by parts with respect to the space variable x.

Hence, we get the energy equality stated in the Theorem.
Let us finally prove Propsition 4. We consider the abstract formulation of the RIP problem which is a semilinear wave equation with damping

$$\begin{cases} \Lambda^\varepsilon \dfrac{d}{dt}W + \mathcal{A}W + \mathcal{C}W = \mathcal{F}(W) \\ W(0) = W_0 = (M_0, 0) \end{cases} \tag{4.24}$$

where $W = (W_1, W_2) = (U, \partial_t U)$ and the 6×6 matrices Λ^ε and $\mathcal{C}$ are given by

$$\Lambda^\varepsilon = \begin{pmatrix} I_3 & 0 \\ 0 & \varepsilon^2 I_3 \end{pmatrix}, \ \mathcal{C} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha/(1+\alpha^2). \end{pmatrix}, \tag{4.25}$$

The nonlinear source term $\mathcal{F}(W) = (\mathcal{F}_1(W), \mathcal{F}_2(W))$ is such that $\mathcal{F}_1(W) = 0$ and

$$\mathcal{F}_2(W) = F_1^\varepsilon(W) - \nabla_M \Psi(W_1) + R(W_1) + P_\nu(W_1) + \beta W_1 \tag{4.26}$$

where $F_1^\varepsilon(W) = -\frac{2}{1+\alpha^2}\frac{W_1}{1+|W_1|} \times \frac{W_2}{1+\varepsilon^2 |W_2|_{L^2(\Omega)}}$ and $\beta > 0$ is an arbitrary fixed constant. The nonlocal operator R is defined by $R(W_1) = \nabla\varphi_1$ where φ_1 solves the magnetostatic equation

$$\nabla \cdot (\nabla\varphi_1 + \chi(\Omega) W_1) = 0 \text{ in } R^+ \times R^3. \tag{4.27}$$

Finally the unbounded operator $\mathcal{A}$ is defined by

$$\mathcal{A} = \begin{pmatrix} 0 & -I_3 \\ \mathcal{L} & 0 \end{pmatrix}, \text{ with } \mathcal{L} = -A\Delta + \beta. \tag{4.28}$$

We introduce the following Hilbert space $\mathcal{G} = H^1(\Omega) \times L^2(\Omega)$ equipped with the scalar product $(\cdot;\cdot)_\mathcal{G}$ defined by $(U;V)_\mathcal{G} = (U_1;V_1)_{H^1} + (U_2;V_2)_{L^2}$ where the scalar product of $H^1(\Omega)$ is given by $(U_1;V_1)_{H^1} = A(\nabla U_1; \nabla V_1)_{L^2} + \beta(U_1;V_1)_{L^2}$. We denote by $|\cdot|_\mathcal{G}$ the norm of $\mathcal{G}$ and by $\|\cdot\|_{H^1}$ the norm of $H^1(\Omega)$.
The domain on $L^2(\Omega)$ of the unbounded operator $\mathcal{L}$ is given by $D(\mathcal{L}) = \{U \in L^2(\Omega),\ \mathcal{L}U \in L^2(\Omega), A\frac{\partial U}{\partial \mathbf{n}} = 0$ on $(\partial\Omega \setminus \Gamma_k^\pm),\ \mp A(\frac{\partial U}{\partial z})^\pm = -\frac{s}{N}(U^\pm - (U^\pm \cdot \mathbf{e}_3)\mathbf{e}_3) - i(U^\pm - U^\mp))$ on $\Gamma_k^\pm,\ k = 1, \cdots, N-1\}$ and for $U \in \mathcal{D}(\Omega)$, we have $\mathcal{L}U = (-A\Delta + \beta)U$. Notice that if $U \in D(\mathcal{L})$ then $U \in H^1(\Omega)$, $U^\pm \in H^{1/2}(\Gamma_k^\pm,)$ and $A(\frac{\partial U}{\partial z})^\pm \in H^{1/2}(\Gamma_k^\pm)$. Moreover for all $U \in D(\mathcal{L})$ the Green formula gives

$$\begin{cases} (\mathcal{L}U;U)_{L^2} = A|\nabla U|^2 + \beta|U|^2 + \\ \dfrac{s}{N}\displaystyle\sum_{k=1}^{N-1}(|U_k^+ \times \mathbf{e}_3|^2_{L^2(\Omega_T)} + |U_k^- \times \mathbf{e}_3|^2_{L^2(\Omega_T)}) + i\sum_{k=1}^{N-1}|U_k^+ - U_k^-|^2_{L^2(\Omega_T)}, \end{cases} \tag{4.29}$$

which implies the coercivity property satisfied by $\mathcal{L}$, $(\mathcal{L}U;U)_{L^2} \geq \|U\|^2_{H^1(\Omega)}$.
We introduce on $H^1(\Omega) \times H^1(\Omega)$ the bilinear form $L(U,V)$ defined by

$$\begin{cases} L(U,V) = A\int_\Omega \nabla U \cdot \nabla V dx + \beta\int_\Omega U \cdot V dx + \\ \dfrac{s}{N}\displaystyle\sum_{k=1}^{N-1}\int_{\Omega_T}(U_k^+ \times \mathbf{e}_3 \cdot V_k^+ \times \mathbf{e}_3 + U_k^- \times \mathbf{e}_3 \cdot V_k^- \times \mathbf{e}_3) d\hat{x} + \\ + i\displaystyle\sum_{k=1}^{N-1}\int_{\Omega_T}(U_k^+ - U_k^-)\cdot(V_k^+ - V_k^-) d\hat{x}. \end{cases} \tag{4.30}$$

As we have seen earlier the bilinear form L satisfies the coercivity property $L(U,U) \geq \|U\|^2_{H^1(\Omega)}$ for all $U \in H^1(\Omega)$. Using the identity $|U(z_k^\pm)|^2 = |U(z)|^2 - 2\int_{z_k^\pm}^{z} \partial_z U(z) \cdot U(z)dz$ we get, for $k = 1, \cdots, N-1$, the estimate

$$|U(z_k^\pm)|^2_{L^2(\Omega_T)} \leq \frac{N\varepsilon + 1}{\varepsilon}|U|^2_{L^2(\Omega_T \times Y_k^*)} + \varepsilon|\partial_z U|^2_{L^2(\Omega_T \times Y_k^*)}. \tag{4.31}$$

Simple computations show that there exists a constant $C_N > 0$ such that $U, V \in H^1(\Omega)$ we have the estimate

$$|L(U,V)| \leq C_N \|U\|_{H^1(\Omega)} \|V\|_{H^1(\Omega)}. \tag{4.32}$$

By the Lax-Milgram Lemma, we deduce that $\mathcal{L}$ is an isomorphism from $D(\mathcal{L})$ into $L^2(\Omega)$. The domain $D(\mathcal{A})$ in the Hilbert space $\mathcal{G}$ of the unbounded operator $\mathcal{A}$ is given by $D(\mathcal{A}) = \{U = (U_1,U_2) \in \mathcal{G}, \mathcal{A}(U) \in \mathcal{G}\} = D(\mathcal{L}) \times H^1(\Omega)$. For $U = (U_1,U_2) \in D(\mathcal{A})$ we have $\mathcal{A}(U) = (-U_2, \mathcal{L}U_1)$. The operator $\mathcal{A}$ satisfies for all $U, V \in D(\mathcal{A})$ the properties

$$\begin{cases} (\mathcal{A}U;V)_\mathcal{G} = -(U_2;V_1)_{H^1(\Omega)} + (\mathcal{L}U_1;V_2)_{L^2(\Omega)} = -(U;\mathcal{A}V)_\mathcal{G} \\ (\mathcal{A}U;U)_\mathcal{G} = 0,\ ((\mathcal{A}+\mathcal{C})U;U)_\mathcal{G} = \frac{\alpha}{1+\alpha^2}|U_2|^2_{L^2(\Omega)} \end{cases} \tag{4.33}$$

These properties show, see Pazy [19], that $\mathcal{A}+\mathcal{C}$ is the infinitesimal generator of a C^0 semi-group on $\mathcal{G}$. Now we shall prove that the source term $\mathcal{F}(W)$. Let us consider the source term $\mathcal{F}(W)$. Recalling that $\mathcal{F}_1(W) = 0$ then we obtain

LEMMA 4.6. *There exists $C > 0$ depending of ν and ε such that for all $W, Z \in L^2(\Omega)$*

$$|\mathcal{F}_2(W) - \mathcal{F}_2(Z)|_{L^2(\Omega)} \leq C(|W_1 - Z_1|_{L^2(\Omega)} + |W_2 - Z_2|_{L^2(\Omega)}) \tag{4.34}$$

Proof. Since we have $\mathcal{F}_2(W) = F_1^\varepsilon(W) - \nabla_M \Psi(W_1) + R(W_1) + P_\nu(W_1) + \beta W_1$ then we shall prove this property for each trem. We have shown previously that the penalized operator P_ν satisfies $|(\nabla_W P_\nu)(W)| \leq \nu(2+\sqrt{2})$ for all $W \in R^3$ then the result for P_ν follows. Next, by using the hypotheses satisfied by Ψ, we have $|D^2_W \Psi(W)| \leq C$. Then the result holds for $\nabla_W \Psi(W)$. Let us consider operator $R(W) = \nabla\varphi$ where φ satisfies the magneto-static equation $\nabla \cdot (\nabla\varphi + \chi(\Omega)W) = 0$ in R^3. Since the problem is linear then by an integration by parts we get $|R(W) - R(Z)|_{L^2(R^3)} \leq |W - Z|_{L^2(\Omega)}$. It remains to prove the result for $F_1^\varepsilon(W_1,W_2)$. We have

$$\begin{aligned} &(1+\alpha^2)|F_1^\varepsilon(W_1,W_2) - F_1^\varepsilon(Z_1,Z_2)| \leq |\tfrac{W_2}{1+\varepsilon^2|W_2|_{L^2}} - \tfrac{Z_2}{1\varepsilon^2|Z_2|_{L^2}}|+ \\ &+ |\tfrac{Z_2}{1+\varepsilon^2|Z_2|_{L^2}}||\tfrac{W_1}{1+|W_1|} - \tfrac{Z_1}{1+|Z_1|}|. \end{aligned}$$

We get after simple computations the inequality

$$\begin{aligned} &(1+\alpha^2)|F_1^\varepsilon(W_1,W_2) - F_1^\varepsilon(Z_1,Z_2)| \leq 2|W_2 - Z_2| + \tfrac{\varepsilon^2|Z_2|}{(1+\varepsilon^2|W_2|_{L^2})(1+\varepsilon^2|Z_2|_{L^2})}\cdot \\ &||W_2|_{L^2} - |Z_2|_{L^2}| + 2|W_1 - Z_1| + \tfrac{|Z_2|}{1+\varepsilon^2|Z_2|_{L^2}}||W_1| - |Z_1||. \end{aligned}$$

Integrating on Ω, we get that F_1^ε is Lipchitzian with Lipchitz constant of order ε^{-2}.

Since $\mathcal{F}(W)$ is a Lipchitz perturbation on $\mathcal{G}$ of the operator $\mathcal{A}+\mathcal{C}$ then, see [19], for all $T>0$ fixed and for all $W_0 \in \mathcal{G}$ there exists a unique mild solution $W \in C^0([0,T];\mathcal{G})$ satisfying the integral form of the problem

$$\begin{cases} \Lambda^\varepsilon \dfrac{d}{dt}W + \mathcal{A}W + \mathcal{C}W = \mathcal{F}(W) \\ W(0) = W_0. \end{cases} \tag{4.35}$$

Moreover, if $W_0 \in D(\mathcal{A})$ then $W \in C^0([0,T];D(\mathcal{A})) \cap C^1(]0,T[;\mathcal{G}))$. The following energy estimate holds for $W_0 \in \mathcal{G}$ and for all $t \in [0,T]$

$$\|W(t)\|^2_{\mathcal{G}} + \frac{2\alpha}{1+\alpha^2}\int_0^t |W_2|^2_{L^2(\Omega)} = \int_0^t (\mathcal{F}_2(W(s)); W_2(s))_{L^2(\Omega} ds. \tag{4.36}$$

Since we have $W_2 = \partial_t W_1$ then using the notation $U = W_1$ we deduce that U satisfies in the sense of distributions the RIP problem and the energy equality

$$\mathcal{E}^\varepsilon_\nu(U^\varepsilon_\nu(t)) + \frac{2\alpha}{1+\alpha^2}\int_0^t |\partial_t U^\varepsilon_\nu(s)|^2_{L^2(\Omega)} ds = \mathcal{E}_0(M_0) \tag{4.37}$$

for all $t \geq 0$ where the total energy is defined by

$$\mathcal{E}^\varepsilon_\nu(U^\varepsilon_\nu(t)) = \varepsilon^2 |\partial_t U^\varepsilon_\nu(t)|^2_{L^2(\Omega)} + \mathcal{E}_\nu(U^\varepsilon_\nu) \tag{4.38}$$

and the initial energy is $\mathcal{E}_0 = \mathcal{E}(M_0)$ with

$$\mathcal{E}_0(M_0) = A|\nabla M_0|^2_{L^2(\Omega)} + \int_\Omega \Psi(M_0)dx + |\nabla\varphi_0|^2_{L^2(R^3)} + \mathcal{E}_s(M_0) + \mathcal{E}_i(M_0) \tag{4.39}$$

which is independent of ν and ε. We have used the fact that $|M_0(x)|^2 = 1$ a.e and $\varepsilon^2 \partial_t U(0) = 0$. Hence Theorem 3 is then proved.

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