Local solutions to a model of piezoelectric materials

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Abstract

A local existence theorem is proved for a nonlinear coupled system modeling the electromechanical motion of a one-dimensional piezoelectric body with domain switching. The system is composed by a heat equation describing the behaviour of the number of electric dipoles and by a wave equation governing the dynamic of the electric displacement. The main coupling in the system appears in the time dependent velocity of the waves depending on the number of electric dipoles. The proof of the result relies on a time decay estimate satisfied by the number of electric dipoles and an uniform estimate of the solution of the regularized wave equation.

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1 Introduction and main result

We are dealing with a one-dimensional model to describe the dynamic of piezoelectric bodies with switchable domain introduced by Daví in [6] and [7]. The model equations proposed is the following. In $(0,1)$, we have $\partial_x(C(f) + H\partial_x\varphi) = \rho\partial_t^2 u$, $K\partial_x\varphi - H\partial_x u = 0$ and $\sigma^2\partial_x^2 f + Mf - D\partial_x u = \gamma^2\partial_t f$ where $K > 0$ is the permittivity constant, $H > 0$ the piezoelectric stress, and $C(f) = C + Df$ with $C > 0$ is the elasticity constant and $D > 0$, $M > 0$, σ^2 and $\rho > 0$ are material parameters. The number of aligned dipoles $f(t,x)$ is defined as the projection of the dipole moments onto the direction of the electric field. The displacement field is denoted by $u(t,x)$ and the electric displacement is

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given by $\partial_x u$. The above equations are equivalent to the following ones where we assumed $M = 0$ and we equated the other constants to one

$$\begin{cases} \partial_t f - \partial_x^2 f = -\partial_x u \ \text{ in } (0,T) \times \mathbf{R} \\ f(0,x) = f_0(x) \ \text{ in } \mathbf{R} \\ \partial_t^2 u - \partial_x[(1+f)\partial_x u] = 0 \ \text{ in } (0,T) \times \mathbf{R} \\ u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x) \ \text{ in } \mathbf{R} \end{cases} \tag{1}$$

where $t \geq 0$ is the time variable, $x \in \mathbf{R}$ is the space one. The main observation is that the coupling of both equations makes the velocity of the wave equation time dependent and depends of the solution of the heat equation see Remark below. Moreover, the model equation do not have any natural conserved quantities as for example the energy of the system. This difficulty together with the fact that the wave operator is nonlinearly time dependent do not allow to give a general result.

Theorem 1 *Under the following hypotheses,*

$$\begin{cases} f_0 \in W^{1,\infty}(\mathbf{R}) \,,\; f_0^{(2)}(x) \, \textit{exists a.e. in} \, \mathbf{R} \\ u_0 \in H^1(\mathbf{R}), u_1 \in L^2(\mathbf{R}), f_1 \equiv f_0^{(2)} - u_0' \in L^\infty(\mathbf{R}), \\ f_0(x) \geq F_0 > 0 \;,\; f_1(x) \leq -F_0 \ \textit{ a.e. in} \, \mathbf{R} \end{cases} \tag{2}$$

there exists $T > 0$ depending only on the data f_0, u_0, u_1, F_0 such that (1) *has a solution (f,u) such that $f \in W^{1,\infty}((0,T) \times \mathbf{R})$ and $u \in L^\infty(0,T; H^1(\mathbf{R})) \cap W^{1,\infty}(0,T; L^2(\mathbf{R}))$ satisfying (1) in the sense of distributions and*

$$f \geq 0 \,,\; \partial_t f \leq 0 \ \textit{a.e. in} \ (0,T) \times \mathbf{R}.$$

Remark 1 Let us give an example of initial data satisfying our hypotheses. We set $u_0 = a\exp(-x^2)$, then $u_0 \in H^1(\mathbf{R})$ and let f_0 be a $2w$-periodic function with $0 < w < 1$ fixed, defined by $f_0(x) = 1 - x^2$ for $-w \leq x \leq w$. We have $f_0 \in W^{1,\infty}(\mathbf{R}), f_0(x) \geq 1 - w^2 > 0, f_0^{(2)}(x) - u_0'(x) = -2 + 2ax\exp(-x^2)$ *a.e.* so that we have $f_0^{(2)}(x) - u_0'(x) \leq 2(-1 + \frac{1}{\sqrt{2}}\exp(-\frac{1}{2})) < 0$ *a.e* if we have choosen a to be such that $0 < a < \sqrt{2}\exp(\frac{1}{2})$.

Remark 2 Solving explicitly the heat equation it follows that u satisfies the following nonlocal nonlinear wave equation with memory

$$\begin{aligned} &\partial_t^2 u - \partial_x((1 + (H(t,\cdot) \star f_0)(x))\partial_x u)+ \\ &\qquad + \partial_x((\textstyle\int_0^t H(t-s, x-\cdot) \star \partial_x u(s,\cdot)ds)\partial_x u) = 0. \end{aligned} \tag{3}$$

Remark 3 Let us explain formally the method we use to obtain existence of local solutions. The elastic energy $E(t) = (|\partial_t u|_2^2 + |(1+f)^{1/2}\partial_x u|_2^2)^{1/2}$ deduced from the wave equation satisfies

$$\frac{dE(t)^2}{dt} = \int_{\mathbf{R}} \partial_t f |\partial_x u|^2 dx$$

then $E(t)^2$ decreases if we have $\partial_t f(t,x) \le 0$ a.e. The second condition is $f(t,x) \ge 0$ a.e to insure the nonnegativity of the energy. Since the sign of $\partial_x u$ is not fixed then the heat equation implies that $f(t,x) \ge 0$ for a short time depending on $E(0)^2$ assuming the initial data to be such that $f_0 \ge F_0 > 0$. The estimate $\partial_t f(t,x) \le 0$ is satisfied also for a short time depending on $E(0)$ by assuming that $\partial_t f(0,x) \le -F_0$ a.e. If we are not dealing with the dissipation of the energy then we can use the estimate $|\partial_t f(t,x)| \le |\partial_t f(0)|_\infty + c\int_0^t (t-s)^{-1/4}E(s)ds$ to get the inequality $\frac{dE(t)^2}{dt} \le (|\partial_t f(0)|_\infty + c(\int_0^t (t-s)^{-1/4}E(s)ds)E(t)^2$. We deduce that $\frac{dE(t)^2}{dt} \le E(t)^2(A + Bt^{1/4}(\int_0^t E(s)ds))$ where $A > 0$ and $B > 0$ are two fixed constants. Clearly this estimate do not allow to obtain a better result than the one presented in this work. Finally, to solve the wave equation satisfied by u it is convenient to have $\partial_{tx}^2 u$ in L^2. We have added to $E(t)^2$ the artificial viscosity energy $\varepsilon \int_{(0,T)\times\mathbf{R}} |\partial_{tx}^2 u|^2 dxdt$ which allows to control the regularity of u.

Remark 4 Let us discuss the stability of the system for the weak-$\star$ convergence. Let (f^n, u^n) be a sequence of solutions of the problem such that $0 \le f^n(t,x) \le C$ and $0 \le E(t)^n \le C$ uniformly. Let f be the weak-$\star$ limit in $L^\infty((0,T)\times\mathbf{R})$ of (f^n) and u such that $\partial_t u^n \rightharpoonup \partial_t u$, $\partial_x u^n \rightharpoonup \partial_x u$ weakly-$\star$ in $L^\infty(0,T; L^2(\mathbf{R}))$. Using the integral representation of the solutions f^n and the regularizing effect of the heat equation we easily deduce that $f^n(t,x) \to f(t,x)$ a.e (t,x) and then (f,u) is a solution of the problem. If (f^n) is just assumed to be a nonnegative uniformly bounded sequence then, in general, the stability of the wave equation fail without assuming an uniformly bound for $\partial_t f^n$ in $L^\infty((0,T)\times\mathbf{R})$.

The content of the paper is the following. In section 2, we recall the L^∞-estimates satisfied by the solution of the heat equation and its first partial derivatives. In section 3, we consider the wave equation with given time dependent velocity. To solve it, we introduce an artificial viscosity energy in the model namely $\epsilon \int_0^T \int_{\mathbf{R}} |\partial_{tx}^2 u|^2 \, dxds$ where $\epsilon > 0$ and consider the associated approximated problem. We establish existence and uniqueness of the solution and the decay of its energy. Then, in section 4, we introduce the approximate nonlinear problem associated with (1). The proof of the existence of a solution for this problem is based on an iteration method and on uniform estimates which allow to obtain the convergence of the iterative scheme. The last paragraph is devoted to the proof of Theorem 1.

In the sequel, we use the following notations. We denote by $|.|_p$ the norms of the Lebesgues spaces either $L^p(\mathbf{R})$ or $L^p((0,T) \times \mathbf{R})$ and by $\|.\|$ the norm of the space $L^\infty(0,T; L^2(\mathbf{R}))$. We set ∂_t, ∂_x for the partial derivatives with respect to t and x, $\partial_t^2, \partial_{tx}^2, \partial_x^2$ the second partial derivatives. Next, for a function $f(x)$ with $x \in \mathbf{R}$, we set f', $f^{(2)}$ for the first and the second derivative of the function.

2 The heat equation

In this section we discuss the heat equation in order to show that for some initial data the associated solution f satisfies $f(t,x) \geq 0$ and $\partial_t f(t,x) \leq 0$. The first result shows that the velocity of the wave equation verifies $1 + f(t,x) \geq 1$ and the second one implies that the dissipated energy satisfies $\int_{\mathbf{R}} \partial_t f |\partial_x u|^2 \, dx \leq 0$. This inequality gives the decay of the energy. We shall prove a maximum principle for f and its derivatives (see [1] for example). Let $T > 0$ be fixed, we consider the heat equation

$$\begin{cases} \partial_t f - \partial_x^2 f = -\partial_x v \ \text{ in } (0,T) \times \mathbf{R} \\ \\ f(0,x) = f_0(x) \ \text{ in } \mathbf{R} \end{cases} \tag{4}$$

where v is a fixed function. The following result holds

Proposition 2 *Let v be such that $v \in L^\infty(0,T; H^1(\mathbf{R})) \cap C([0,T]; L^2(\mathbf{R}))$, $\partial_{tx}^2 v \in L^2((0,T) \times \mathbf{R})$ and $v(0,.) = u_0$. Then, under the hypotheses of Theorem 1, there exists a unique solution $f \in W^{1,\infty}((0,T) \times \mathbf{R})$ of problem (4). Moreover, there exists $C_1, C_2 > 0$ which are independent of the data such that the following estimates hold in $(0,T) \times \mathbf{R}$*

$$f(t,x) \geq F_0 - C_1 T^{\frac{3}{4}} \|\partial_x v\|, \ \ |f|_\infty \leq |f_0|_\infty + C_1 T^{\frac{3}{4}} \|\partial_x v\| \tag{5}$$

$$\partial_t f(t,x) \le -F_0 + C_2 T^{\frac{1}{4}} \|\partial_t v\|, \quad |\partial_t f|_\infty \le |f_1|_\infty + C_2 T^{\frac{1}{4}} \|\partial_t v\| \tag{6}$$

$$|\partial_x f|_\infty \le |f_0'|_\infty + C_2 T^{\frac{1}{4}} \|\partial_x v\| \tag{7}$$

Proof. The solution of (4) is given by

$$f(t,x) = H(t,\cdot) * f_0(\cdot)(x) - \int_0^t H(t-s,\cdot) * \partial_x v(s,\cdot)(x) ds. \tag{8}$$

where $H(t,x) = \frac{1}{\sqrt{4\pi t}} \exp\Big(-\frac{x^2}{4t}\Big)$ is the heat kernel. Using Cauchy- Schwartz inequality, we easily deduce that

$$\begin{aligned} &F_0 - \textstyle\int_0^t (8\pi(t-s))^{-1/4} |\partial_x v(s,\cdot)| ds \le f(t,x) \text{ and,} \\ &f(t,x) \le |f_0|_\infty + \textstyle\int_0^t (8\pi(t-s))^{-1/4} |\partial_x v(s,\cdot)| ds \end{aligned} \tag{9}$$

which implies (5). Taking in (8) the derivative of f with respect to the x variable, we get similarly (7). To prove (6) we proceed as follows. The time derivative $\partial_t f$ is a solution of the problem $\partial_t(\partial_t f) - \partial_x^2(\partial_t f) = -\partial_{tx}^2 v$ in $(0,T) \times \mathbf{R}$ with $\partial_t f(0,x) = f_1(x)$ in $\mathbf{R}$ which is given by $\partial_t f(t,x) = H(t) * f_1(x) - \int_0^t \partial_x H(t-s,\cdot) * \partial_t v(s,\cdot)(x) ds$. Proceeding as previously we get the stated result.

3 The wave equation with artificial viscosity

Let $g(t,x)$ be a fixed function. We consider the Cauchy problem for the wave equation with artificial viscosity

$$\begin{cases} \partial_t^2 u - \partial_x[(1+g)\partial_x u] - \varepsilon \partial_x^2[\partial_t u] = 0 \\ u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x). \end{cases} \tag{10}$$

For $T > 0$ and $\varepsilon > 0$ fixed, we shall prove the following result

Proposition 3 *Let u_0, u_1, f_0, f_1 as in Theorem 1. and let g be such that $g \in W^{1,\infty}(0,T; L^\infty(\mathbf{R}))$, $g \ge 0$, $\partial_t g \le 0$ a.e. in $(0,T) \times \mathbf{R}$ a nd $g(0,x) = f_0(x)$. Then, problem* (10) *has a unique solution $u \in C([0,T]; H^1(\mathbf{R}))$ such* *that* $u \in C^1([0,T]; L^2(\mathbf{R})) \cap H^1(0,T; H^1(\mathbf{R}))$ *and* $\partial_t^2 u \in L^2(0,T; H^{-1}(\mathbf{R}))$. *Finally,for every* $t \in [0,T]$, *the following energy inequality holds*

$$E(t)^2 + 2\varepsilon \int_0^t |\partial_{tx}^2 u(s)|_2^2 \, ds \leq E_0^2 \tag{11}$$

where $E(t)^2 = |\partial_t u(t)|_2^2 + |\sqrt{1+g}\partial_x u(t)|_2^2$ *and* $E_0 = E(0)$.

Our existence proof is based on a theorem of Lions [9] (Chap. III) that we recall hereafter. For a detailed study of a wave equation with time dependent velocity, we refer the reader to [8] for example in the case of regular coefficients and to [3] if not. We get in [2] other results related to the time dependent velocity of the wave equation. The wave equation with such an artificial viscosity had been studied in [13]. Note that (10) is equivalent to a Cauchy problem associated to a first order evolution differential equation of the form $\frac{du(t)}{dt} = A(t)u(t)$ and we can solve it using the method developed in [12], [14] and [4], [5].

Theorem 4 *(see [9] Chap. III) Let* $\mathcal{H}$ *be a Hilbert space provided with a norm* $|\cdot|$ *and an inner product* $(\cdot,\cdot)$. *Let* $\mathcal{V}$ *be a subspace of* $\mathcal{H}$ *provided with a prehilbertian norm* $\|\cdot\|$ *such that the injection* $\mathcal{V} \subset \mathcal{H}$ *is continuous. We consider a bilinear form* B

$$B : \mathcal{H} \times \mathcal{V} \ni (U, \phi) \to B(U, \phi) \in \mathbf{R}$$

such that for any fixed $\phi \in \mathcal{V}$, $B(\cdot, \phi)$ *is continuous on* $\mathcal{H}$ *and satisfies*

$$|B(\phi,\phi)| \geq \alpha \|\phi\|^2, \forall \phi \in \mathcal{V}, \text{ with } \alpha > 0.$$

Then, given a linear form L *in* $\mathcal{V}'$, *there exists a solution* U *in* $\mathcal{V}$ *of the problem*

$$B(U, \phi) = L(\phi) \ , \forall \phi \in \mathcal{V}.$$

The following result will be also useful in the sequel

Lemma 5 *For* $k > 0$ *fixed and under hypotheses of Proposition 3, the operator* $(-\partial_x((1+g)\partial_x) + kI)$ *is an isomorphism from* $L^2(0,T,H^1(\mathbf{R}))$ *into* $L^2(0,T,H^{-1}(\mathbf{R}))$ *and from* $H^1(0,T,H^1(\mathbf{R}))$ *into* $H^1(0,T,H^{-1}(\mathbf{R}))$.

Proof. Let $V = L^2(0,T,H^1(\mathbf{R}))$ and V' its dual. Thanks to the properties of g, the bilinear form $b(u,v) = \int_0^T \int_{\mathbf{R}} ((1+g)\partial_x u \partial_x v + kuv) dxdt$ is continuous and coercive on $V \times V$. So, for all $f \in V'$, there exists a unique solution to the variational equation $b(u,v) = \langle f, v \rangle_{V',V}$ for all $v \in V'$. Hence, u is the unique solution in V of the equation

$$-\partial_x[(1+g)\partial_x u] + ku = f. \tag{12}$$

Now, if $f \in H^1(0,T,H^{-1}(\mathbf{R}))$, then since $\partial_t g$ belongs to $L^\infty((0,T)\times\mathbf{R})$ we deduce that $\partial_t u \in V$.

Now we come back to the proof of Proposition 3.

Proof. Let u be a solution of (10). We set $\partial_t u = v, (\tilde{u},\tilde{v}) = \exp(-kt)(u,v)$ where $k>0$ is a constant. Hence (10) is equivalent to the system

$$\begin{cases} \partial_t\tilde{u} + k\tilde{u} - \tilde{v} = 0 \ \text{ in } (0,T)\times\mathbf{R} \\ \partial_t\tilde{v} - \partial_x[(1+g)\partial_x\tilde{u}] - \varepsilon\partial_x^2\tilde{v} + k\tilde{v} = 0 \text{ in } (0,T)\times\mathbf{R} \\ \tilde{u}(0,x) = u_0(x), \tilde{v}(0,x) = u_1(x) \ \text{ in } \mathbf{R} \end{cases} \tag{13}$$

Let $\mathcal{H}$ be the Hilbert space $\mathcal{H} = L^2(0,T;H^1(\mathbf{R}))\times L^2(0,T;H^1(\mathbf{R}))$ provided with the norm $||.||_{\mathcal{H}}$ defined for all $U=(u,v)\in\mathcal{H}$ by

$$\|U\|_{\mathcal{H}}^2 = \int_0^T (\||u(t)\||^2_{H^1(\mathbf{R})} + \|v(t)\|^2_{H^1(\mathbf{R})})dt \tag{14}$$

where $\|.\|_{H^1(\mathbf{R})}$ is the usual norm in $H^1(\mathbf{R})$ and

$$\||u(t)\||^2_{H^1(\mathbf{R})} = \int_{\mathbf{R}} [(1+g)|\partial_x u|^2 + k|u|^2]dx. \tag{15}$$

As $g\in L^\infty((0,T)\times\mathbf{R})$, $g\geq 0$ a.e., this norm is equivalent to the natural one. Next, we define the space of infinitely differentiable functions with compact support in $[0,T[\times\mathbf{R}$, $\mathcal{V} = (\mathcal{D}([0,T[\times\mathbf{R}))^2$ provided with the norm

$$|\phi|^2_{\mathcal{V}} = \|\phi\|^2_{\mathcal{H}} + \frac{1}{2}(\||\varphi(0)\||^2_{H^1(\mathbf{R})} + |\psi(0)|^2),\ \forall\phi = (\varphi,\psi)\in\mathcal{V} \tag{16}$$

which makes the injection $\mathcal{V} \subset \mathcal{H}$ continuous. Now we define on $\mathcal{H} \times \mathcal{V}$ the bilinear form B and $\mathcal{V}$ the linear form by setting by

$$\begin{aligned} B(U,\phi) &= \int_0^T \int_{\mathbf{R}} \Big(-(1+g)\partial_x u \partial_{tx}^2 \varphi - ku\partial_t\varphi - (\partial_t g)(\partial_x u)(\partial_x \varphi)\Big) dxdt \\ &+ \int_0^T \int_{\mathbf{R}} \Big((1+g)(\partial_x(ku-v))(\partial_x\varphi) + k(ku-v)\varphi\Big) dxdt \\ &+ \int_0^T \int_{\mathbf{R}} \Big(-v\partial_t\psi + (1+g)(\partial_x u)(\partial_x\psi) + \varepsilon(\partial_x v)(\partial_x \psi) + kv\psi\Big) dxdt \end{aligned} \tag{17}$$

and

$$L(\phi) = \int_{\mathbf{R}} [(1+g(0))\partial_x u_0(x)\partial_x\varphi(0) + ku_0(x)\varphi(0) + u_1(x)\psi(0)]dx \tag{18}$$

where $U=(u,v), \phi=(\varphi,\psi)$. Then clearly $B(\cdot,\phi)$ is continuous on $\mathcal{H}$, for any fixed $\phi \in \mathcal{V}$ and L is continuous on $\mathcal{V}$. Let us prove the corciveness of B on the space $\mathcal{V}$. First for $(U,\phi) \in \mathcal{H}\times\mathcal{V}$, we set $B_1(U,\phi) = \int_0^T \int_{\mathbf{R}}(-(1+g)\partial_x u\partial_{tx}^2\varphi - ku\partial_t\varphi - \partial_t g\partial_x u\partial_x\varphi - v\partial_t\psi)dxdt$ then for all $(U,\phi)\in\mathcal{V}\times\mathcal{V}$, we have $B_1(U,\phi) = -B_1(\phi,U) - \int_0^T\int_{\mathbf{R}}\partial_t g\partial_x u\partial_x\varphi)dxdt + \int_{\mathbf{R}}[(1+g(0,x))\partial_x u(0)\partial_x\varphi(0) + ku(0)\varphi(0) + v(0)\psi(0)]dx$ so for $\phi\in\mathcal{V}$, we get $B_1(\phi,\phi) = -\frac{1}{2}\int_0^T\int_{\mathbf{R}}\partial_t g|\partial_x\varphi|^2dxdt + \frac{1}{2}\int_{\mathbf{R}}[(1+g(0))|\partial_x\varphi(0)|^2 + k|\varphi(0)|^2 + |\psi(0)|^2]dx$ and, since we have $\partial_t g \leq 0$ a.e then we get

$$B_1(\phi,\phi) \geq \frac{1}{2}(\||\varphi(0)\||^2_{H^1(\mathbf{R})} + |\psi(0)|^2) \tag{19}$$

Now, we set $B_2(U,\phi) = B(U,\phi) - B_1(U,\phi)$. We have $B_2(\phi,\phi) = k\int_0^T\int_{\mathbf{R}}((1+g)|\partial_x\varphi|^2 + k|\varphi|^2)dxdt - k\int_0^T\int_{\mathbf{R}}\varphi\psi dxdt + \varepsilon\int_0^T\int_{\mathbf{R}}|\partial_x\psi|^2dxdt + k\int_0^T\int_{\mathbf{R}}|\psi|^2dxdt$ then we get for all $\phi\in\mathcal{V}$

$$\begin{aligned} B_2(\phi,\phi) \geq \frac{k}{2}\int_0^T\int_{\mathbf{R}}(1+g)|\partial_x\varphi|^2dxdt + (k^2 - \frac{k}{2})\int_0^T\int_{\mathbf{R}}|\varphi|^2dxdt \\ + \varepsilon\int_0^T\int_{\mathbf{R}}|\partial_x\psi|^2dxdt + \frac{k}{2}\int_0^T\int_{\mathbf{R}}|\psi|^2dxdt. \end{aligned} \tag{20}$$

Adding (19) and (20) we get the coerciveness estimate

$$B(\phi,\phi) \geq \alpha\|\phi\|^2_{\mathcal{V}}, \forall\,\phi\in\mathcal{V} \tag{21}$$

where $\alpha = min(\varepsilon, \frac{k}{2}, 1) > 0$. Thus thanks to the Theorem 4, then there exists a solution $\tilde{U}$ in $\mathcal{H}$ to the variational equation

$$B(\tilde{U}, \phi) = L(\phi),\ \forall \phi \in \mathcal{V}. \tag{22}$$

Set $\tilde{U} = (\tilde{u}, \tilde{v})$, then using (22), we get

$$-\partial_x[(1+g)\partial_x(\partial_t\tilde{u} + k\tilde{u} - \tilde{v})] + k(\partial_t\tilde{u} + k\tilde{u} - \tilde{v}) = 0 \tag{23}$$

and

$$\partial_t\tilde{v} - \partial_x[(1+g)\partial_x\tilde{u}] - \varepsilon\partial_x^2\tilde{v} + k\tilde{v} = 0 \tag{24}$$

in the sense of distributions. In particular (23) gives $\partial_t\Big[-\partial_x[(1+g)\partial_x\tilde{u}]+k\tilde{u}\Big] = \partial_x[(1+g)\partial_x(k\tilde{u}-\tilde{v})] - k(k\tilde{u}-\tilde{v}) - \partial_x[(\partial_t g)\partial_x\tilde{u}]$ so $\partial_t\Big[-\partial_x[(1+g)\partial_x\tilde{u}]+k\tilde{u}\Big] \in L^2(0,T,H^{-1}(\mathbf{R}))$, then $-\partial_x[(1+g)\partial_x u] + ku \in H^1(0,T,H^{-1}(\mathbf{R}))$. Therefore using lemma 1, we get $\tilde{u} \in H^1(0,T,H^1(\mathbf{R}))$. Coming back to (23), we conclude that

$$\partial_t\tilde{u} + k\tilde{u} - \tilde{v} = 0 \text{ in } L^2(0,T,H^1(\mathbf{R})). \tag{25}$$

In the other hand (24) leads to $\partial_t v \in L^2(0,T,H^{-1}(\mathbf{R}))$. Thus $\partial_t(\tilde{u},\tilde{v})$ belongs to $L^2(0,T,H^1(\mathbf{R})) \times L^2(0,T,H^{-1}(\mathbf{R}))$ and $(\tilde{u},\tilde{v}) \in \mathcal{C}([0,T];H^1(\mathbf{R}) \times L^2(\mathbf{R}))$, so the traces $\tilde{u}(0,.), \tilde{v}(0,.)$ are well defined in $H^1(\mathbf{R})$ and $L^2(\mathbf{R})$ respectively (see [10]). Taking the duality product of (23) by a function $\varphi \in \mathcal{D}([0,T[\times\mathbf{R})$, the duality product of (24) by a function $\psi \in \mathcal{D}([0,T[\times\mathbf{R})$, adding the obtained terms and using (22) we show that $\tilde{u}(0,x) = u_0(x), \tilde{v}(0,x) = u_1(x)$ a.e in $\mathbf{R}$ so, $(\tilde{u},\tilde{v})$ solves the problem (13) and $u = \exp(kt)\tilde{u}$ is a solution of (10) satisfying the properties of Proposition 2.

Now, to prove the energy estimate (11), we multiply the equation (10) by $\partial_t u$ and integrate by parts with respect to x , we get

$$\frac{1}{2}\frac{d}{dt}|\partial_t u|_2^2 + \frac{1}{2}\int_{\mathbf{R}} (1+g)\,\partial_t\left[(\partial_x u)^2\right]\,dx + \varepsilon|\partial_{tx}^2 u(t)|_2^2 = 0$$

which implies

$$\frac{d}{dt}\left[|\partial_t u|_2^2 + |\sqrt{1+g}\,\partial_x u|_2^2\right] + 2\varepsilon|\partial_{tx}^2 u(t)|_2^2 = \int_{\mathbf{R}} (\partial_t g)|\partial_x u|^2\,dx \le 0 \tag{26}$$

and leads to (11). For the uniqueness, since (10) is linear, it is enough to show that if $u_0 = 0$, $u_1 = 0$, then $u = 0$ and this is an immediate consequence of the energy inequality (11).

4 The coupling problem with artificial viscosity

We consider the following approximation of the problem (1)

$$\begin{cases} \partial_t f - \partial_x^2 f = -\partial_x u \ \text{ in } (0,T) \times \mathbf{R} \\ f(0,x) = f_0(x) \ \text{ in } \mathbf{R} \\ \partial_t^2 u - \partial_x[(1+f)\partial_x u] - \varepsilon \partial_x^2[\partial_t u] = 0 \ \text{ in } (0,T) \times \mathbf{R} \\ u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x) \ \text{ in } \mathbf{R} \end{cases} \tag{27}$$

where $\varepsilon > 0$ is fixed. Then we have

Theorem 6 *Under hypotheses of Theorem 1, there exist $T > 0$, which is independent of ε, satisfying*

$$F_0 - C_1 T^{\frac{3}{4}} E_0 \geq 0, \ F_0 - C_2 T^{\frac{1}{4}} E_0 \geq 0 \tag{28}$$

where $E_0 = \left(|u_1|_2^2 + |\sqrt{1+f_0}\, u_0'|_2^2\right)^{\frac{1}{2}}$ and C_1, C_2 are the constants given in Proposition 1 such that problem (27) admits a unique weak solution $(f^\varepsilon, u^\varepsilon)$ defined on a time interval [0,T]. This solution satisfies

$$\begin{aligned} & u^\varepsilon \in C([0,T]; H^1(\mathbf{R})) \cap C^1([0,T]; L^2(\mathbf{R}))) \cap H^1(0,T; H^1(\mathbf{R}) \\ & f^\varepsilon \in W^{1,\infty}((0,T) \times \mathbf{R}), \\ & f^\varepsilon \geq 0, \ \partial_t f^\varepsilon \leq 0, \quad a.e. \ in \ (0,T) \times \mathbf{R}. \end{aligned} \tag{29}$$

Moreover, the following uniform bounds with respect to ε hold

$$|f^\varepsilon|_\infty \leq |f_0|_\infty + F_0, \ |\partial_x f^\varepsilon|_\infty \leq |f_0'|_\infty + F_0, \ |\partial_t f^\varepsilon|_\infty \leq |f_1|_\infty + F_0 \tag{30}$$

and

$$\|u^\varepsilon\|^2 \le \exp T\Big(|u_0|_2^2 + E_0^2 T\Big) \tag{31}$$

$$\|\partial_t u^\varepsilon\|^2 + \|\partial_x u^\varepsilon\|^2 \le |u_1|_2^2 + |\sqrt{1+f_0}\, u_0'|_2^2. \tag{32}$$

The proof will be done in several steps (see [14] and [11] for example). We first prove the existence of solution by using the following iterative scheme.

4.1 The iterative scheme

We set $u^{(0)} = 0$ and $f^{(0)} = 0$ and define for all $n \in \mathbf{N}$, f^{n+1} as the solution of the linear heat equation

$$\partial_t f^{n+1} - \partial_x^2 f^{n+1} = -\partial_x u^n \ \text{ in } (0,T)\times\mathbf{R},\ f^{n+1}(0,x) = f_0(x) \ \text{ in } \mathbf{R} \tag{33}$$

then we define u^{n+1} as the solution of the linear wave equation

$$\begin{cases} \partial_t^2 u^{n+1} - \partial_x[(1+f^{n+1})\partial_x u^{n+1}] - \varepsilon\partial_x^2[\partial_t u^{n+1}] = 0 \ \text{ in } (0,T)\times\mathbf{R} \\ \\ u^{n+1}(0,x) = u_0(x), \partial_t u(0,x) = u_1(x) \ \text{ in } \mathbf{R}. \end{cases} \tag{34}$$

The following results give uniform estimates for the sequence $(f^n, u^n)_n$ of solutions.

Proposition 7 *Under hypotheses of Theorem 1, let $T > 0$ be the time existence defined by* (28)*. Then the sequence of solutions $(f^n, u^n)_n$ is well defined on [0,T], $f^n(t,x) \ge 0$, $\partial_t f^n(t,x) \le 0$ a.e. in $(0,T)\times\mathbf{R}$ for all $n \in \mathbf{N}$ and the uniform bounds with respect to n and ε are satisfied*

$$|f^n|_\infty \le |f_0|_\infty + F_0,\ |\partial_x f^n|_\infty \le |f_0'|_\infty + F_0,\ |\partial_t f^n|_\infty \le |f_1|_\infty + F_0 \tag{35}$$

$$\begin{aligned} &\|u^n\|^2 \le \exp T\Big(|u_0|_2^2 + E_0^2 T\Big), \\ &\|\partial_t u^n\|^2 + \|\partial_x u^n\|^2 + 2\varepsilon \textstyle\int_0^T |\partial_{tx}^2 u^n(s)|^2\, ds \le E_0^2. \end{aligned} \tag{36}$$

Proof. The proof is done by induction on n. Suppose the results of Proposition 3 hold for (f^n, u^n). Using the results of Proposition 1, we deduce that f^{n+1} is well defined and thanks to (36) at order n, we obtain $f^{n+1}(t,x) \ge$

$F_0 - C_1 T^{\frac{3}{4}} \|\partial_x u^n\| \geq F_0 - C_1 T^{\frac{3}{4}} E_0 \geq 0$ and $\partial_t f^{n+1}(t,x) \leq -F_0 + C_2 T^{\frac{1}{4}} \|\partial_t u^n\| \leq -F_0 + C_2 T^{\frac{1}{4}} E_0 \leq 0$ a.e. in $(0,T)\times\mathbf{R}$ then we have $f^{n+1}(t,x) \geq 0,\ \partial_t f^{n+1}(t,x) \leq 0$ *a.e.* in $(0,T)\times\mathbf{R}$.
Using again Proposition 1, thanks to (28) we get $|f^{n+1}|_\infty \leq |f_0|_\infty + C_1 T^{\frac{3}{4}} \|\partial_x u^n\|$ and then $|f^{n+1}|_\infty \leq |f_0|_\infty + C_1 T^{\frac{3}{4}} E_0 \leq |f_0|_\infty + F_0$. Similarly, $|\partial_x f^{n+1}|_\infty \leq |f_0'|_\infty + C_2 T^{\frac{1}{4}} \|\partial_x u^n\|$ and thus we have $|\partial_x f^{n+1}|_\infty \leq |f_0'|_\infty + C_2 T^{\frac{1}{4}} E_0 \leq |f_0'|_\infty + F_0$ and $|\partial_t f^{n+1}|_\infty \leq |f_1|_\infty + C_2 T^{\frac{1}{4}} \|\partial_t u^n\| \leq |f_1|_\infty + C_2 T^{\frac{1}{4}} E_0 \leq |f_1|_\infty + F_0$ and finally we get (35) at order $(n+1)$.
From Proposition 2 it follows that u^{n+1} is well defined and we get easily (36) at order $(n+1)$ by using (11). Since we have $(u^{n+1})^2(t,x) = u_0^2(x) + 2\int_0^t u^{n+1}(s,x)\partial_t u^{n+1}(s,x)\,ds$, we deduce the estimate

$$|u^{n+1}(t)|_2^2 \leq |u_0|_2^2 + \int_0^t |u^{n+1}(s)|_2^2\,ds + \int_0^t |\partial_t u^{n+1}(s)|_2^2\,ds \tag{37}$$

so by using (36) and Gronwall's inequality we obtain (31).

4.2 Convergence of the iterative scheme

We introduce the Banach space $\mathcal{E} = \mathcal{C}([0,T];H^1(\mathbf{R})) \cap \mathcal{C}^1([0,T];L^2(\mathbf{R})) \cap H^1(0,T;H^1(\mathbf{R}))$. We shall show that $(f^n)_n$ and $(u^n)_n$ are Cauchy sequences in $W^{1,\infty}((0,T)\times\mathbf{R})$ and $\mathcal{E}$ respectively. For this purpose, we set $g^n = f^{n+1} - f^n$, $v^n = u^{n+1} - u^n$ for all $n \in \mathbf{N}$ then g^n satisfies the homogeneous heat equation

$$\partial_t g^n - \partial_x^2 g^n = -\partial_x v^{n-1} \text{ in } (0,T)\times\mathbf{R},\ g^n(0,x) = 0 \text{ in } \mathbf{R} \tag{38}$$

and v^n the homogeneous wave equation with source term

$$\begin{cases} \partial_t^2 v^n - \partial_x[(1+f^{n+1})\partial_x v^n] - \varepsilon\partial_x^2[\partial_t v^n] = \partial_x[g^n\partial_x u^n] \text{ in } (0,T)\times\mathbf{R} \\ \\ v^n(0,x) = 0, \partial_t v^n(0,x) = 0 \text{ in } \mathbf{R}. \end{cases} \tag{39}$$

Multiplying (39) by $\partial_t v^n$ and integrating by parts with respect to space variable then, simple computations show that

$$\frac{1}{2}\frac{d}{dt}[|\partial_t v^n(t)|_2^2 + |\sqrt{1+f^{n+1}}\,\partial_x v^n(t)|_2^2] + \varepsilon|\partial_{tx}^2 v^n(t)|_2^2$$

$$=\frac{1}{2}\int_{\mathbf{R}}\partial_t f^{n+1}|\partial_x v^n|^2\,dx-\int_{\mathbf{R}}g^n(\partial_x u^n)(\partial^2_{tx}v^n)\,dx.$$

Using the property satisfied by $\partial_t f^n$ and Young's inequality we get the estimate $|\partial_t v^n(t)|_2^2+|\sqrt{1+f^{n+1}}\,\partial_x v^n(t)|_2^2+2\varepsilon\int_0^t|\partial^2_{tx}v^n(s)|_2^2\,ds\le\frac{1}{\varepsilon}\int_0^t|g^n\partial_x u^n(s)|_2^2\,ds+\varepsilon\int_0^t|\partial^2_{tx}v^n(s)|_2^2\,ds$. Hence, by (36), it holds that

$$|\partial_t v^n(t)|_2^2+|\sqrt{1+f^{n+1}}\,\partial_x v^n(t)|_2^2+\varepsilon\int_0^t|\partial^2_{tx}v^n(s)|_2^2\,ds\le\frac{1}{\varepsilon}E_0^2\int_0^t|g^n(s)|_\infty^2\,ds.$$

Let us consider the heat equation(38). Proceeding as in the first section we get the inequality $|g^n(t)|_\infty\le\Big[\int_0^t|H(t-s)|_2^2\,ds\Big]^{\frac{1}{2}}\Big[\int_0^t|\partial_x v^{n-1}(s)|_2^2\,ds\Big]^{\frac{1}{2}}$ which shows that

$$|g^n(t)|_\infty\le C_3t^{\frac{1}{4}}\Big[\int_0^t|\partial_x v^{n-1}(s)|_2^2\,ds\Big]^{\frac{1}{2}}\tag{40}$$

where $C_3>0$ is a constant which is independent of the data, ε and n. Since we have $f^{n+1}\ge 0$, then we get the estimate

$$|\partial_t v^n(t)|_2^2+|\partial_x v^n(t)|_2^2+\varepsilon\int_0^t|\partial^2_{tx}v^n(s)|_2^2ds\le\frac{C_3^2T^{\frac{3}{2}}E_0^2}{\varepsilon}\int_0^t|\partial_x v^{n-1}(s)|_2^2ds\tag{41}$$

for all $n\in\mathbf{N}^*$. In particular we have

$$|\partial_x v^n(t)|_2^2\le\frac{C_3^2T^{\frac{3}{2}}}{\varepsilon}E_0^2\int_0^t|\partial_x v^{n-1}(s)|_2^2\,ds.\tag{42}$$

We set $V^n(t)=|\partial_x v^n(t)|_2^2$ and $C_\varepsilon=C_3^2T^{\frac{3}{2}}E_0^2/\varepsilon$. Then (42) takes the form

$$V^n(t)\le C_\varepsilon\int_0^t V^{n-1}(s)\,ds\ \ \forall t\in[0,T]\ ,\forall n\in\mathbf{N}^*\tag{43}$$

which implies that $V^n(t)\le C_\varepsilon^n\frac{t^{n-1}}{(n-1)!}\int_0^t V^0(s)\,ds$ for all $t\in[0,T]$ and for all $n\in\mathbf{N}^*$. Thus we obtain

$$\|\partial_x u^{n+1}-\partial_x u^n\|^2\le 2\frac{C_\varepsilon^nT^n}{(n-1)!}E_0^2\ ,\ \forall n\in\mathbf{N}.$$

We deduce that $(\partial_x u^n)_n$ is a Cauchy sequence in $\mathcal{C}([0,T];L^2(\mathbf{R}))$, then coming back to (41), we see that $(\partial_t u^n)_n$ and $(\partial^2_{tx} u^n)_n$ are also Cauchy sequences in $\mathcal{C}([0,T];L^2(\mathbf{R}))$ and $L^2((0,T)\times\mathbf{R})$ respectively. Proceeding as in the proof of (36), we can show that

$$\|v^n\|^2 \leq \exp T\|\partial_t v^n\|^2. \tag{44}$$

We conclude that $(u^n)_n$ is a Cauchy sequence in $\mathcal{E}$. Moreover, thanks to (40), we see that $(f^n)_n$ is a Cauchy sequence in $L^\infty((0,T)\times\mathbf{R})$. Then, proceeding along the lines of the proof of Proposition 1, we obtain

$$|\partial_x g^n|_\infty \leq C_2 T^{\frac{1}{4}}\|\partial_x v^{n-1}\|,\ |\partial_t g^n|_\infty \leq C_2 T^{\frac{1}{4}}\|\partial_t v^{n-1}\|$$

and therefore$(f^n)_n$ is a Cauchy sequence in $W^{1,\infty}((0,T)\times\mathbf{R})$. It follows that the sequences $(u^n)_n$ and $(f^n)_n$ converge strongly respectively in $\mathcal{E}$ and $W^{1,\infty}((0,T)\times\mathbf{R})$ towards u^ε and f^ε. Letting $n\to+\infty$ in (33) and (34), we conclude that $(f^\varepsilon,u^\varepsilon)$ is a solution of the coupling problem with viscosity (27) and satisfies the inequalities (29) to (32).

To prove the uniqueness of the solution we proceed as follows. Let $(f^\varepsilon,u^\varepsilon)$ and $(h^\varepsilon,w^{\varepsilon)}$ be two solutions of (27), we set $g^\varepsilon = f^\varepsilon - h^\varepsilon$ and $v^\varepsilon = u^\varepsilon - w^\varepsilon$. Then, proceeding as in the proof of (42), (44) and (40), we obtain the following inequalities

$$|\partial_x v^\varepsilon(t)|_2^2 \leq \frac{C_3^2 T^{\frac{3}{2}}}{\varepsilon} E_0^2 \int_0^t |\partial_x v^\varepsilon(s)|_2^2\, ds \tag{45}$$

$$\|v^\varepsilon\|^2 \leq \exp T\|\partial_x v^\varepsilon\|^2 \tag{46}$$

$$|g^\varepsilon(t)|_\infty^2 \leq C_3^2 T^{\frac{1}{2}} \int_0^t |\partial_x v^\varepsilon(s)|_2^2\, ds. \tag{47}$$

So, by Gronwall's inequality, (45) gives $|\partial_x v^\varepsilon(t)|_2^2 = 0$ a.e. in $(0,T)$ then using (46) and (47), we obtain $v^\varepsilon(t,x)=0$, $g^\varepsilon(t,x)=0$ a.e. in $(0,T)\times\mathbf{R}$ which ends the proof of Theorem 3.

5 End of the proof of Theorem 1

Let $(f^\varepsilon, u^\varepsilon)$ be the solution of (27) provided by Theorem 3. Using the estimates therein, we deduce that $(f^\varepsilon, u^\varepsilon)_\varepsilon$ is uniformly bounded in the Banach space

$$\mathcal{F} = W^{1,\infty}((0,T)\times\mathbf{R}) \times \Big(L^\infty(0,T;H^1(\mathbf{R})) \cap W^{1,\infty}(0,T;L^2(\mathbf{R}))\Big).$$

Therefore we can extract a subsequence, still denoted $(f^\varepsilon, u^\varepsilon)_\varepsilon$ such that when ε goes to 0, $(f^\varepsilon, u^\varepsilon)_\varepsilon$ converges weakly$\star$ in $\mathcal{F}$ towards a limit (f,u). Thanks to the compactness of the imbedding of $W^{1,\infty}(0,T)\times\mathbf{R})$ into $L^\infty_{loc}((0,T)\times\mathbf{R})$, we can pass to the limit in (27) and we obtain

$$\begin{cases} \partial_t f - \partial_x^2 f = -\partial_x u \ \text{ in } \mathcal{D}'((0,T)\times\mathbf{R}) \\ \\ \partial_t^2 u - \partial_x[(1+f)\partial_x u] = 0 \ \text{ in } \mathcal{D}'((0,T)\times\mathbf{R}) \end{cases} \tag{48}$$

Moreover as $W^{1,\infty}((0,T)\times\mathbf{R})\times W^{1,\infty}(0,T;L^2(\mathbf{R}))$ is continuously embedded in $C([0,T];L^\infty(\mathbf{R}))\times C([0,T];L^2(\mathbf{R}))$, we deduce that

$$f(0,x) = f_0(x) \text{ in } L^\infty(\mathbf{R}),\ u(0,x) = u_0(x) \text{ in } L^2(\mathbf{R}). \tag{49}$$

In the other hand, since $f^\varepsilon \to f$ a.e.in $(0,T)\times\mathbf{R}$ then thanks to (29), we get

$$f \geq 0 \text{ a.e.in } (0,T)\times\mathbf{R}. \tag{50}$$

Now let $\varphi \in \mathcal{D}((0,T)\times\mathbf{R}), \varphi \geq 0$. Since we have

$$0 \geq \int_0^T\!\!\int_{\mathbf{R}} \partial_t f^\varepsilon \varphi(t,x)\,dxdt \to \int_0^T\!\!\int_{\mathbf{R}} \partial_t f \varphi(t,x)\,dxdt,$$

then we deduce that $\partial_t f \leq 0$ in $\mathcal{D}'((0,T)\times\mathbf{R})$ and since $\partial_t f \in L^\infty((0,T)\times\mathbf{R})$, we conclude that

$$\partial_t f \leq 0 \text{ a.e. in } (0,T)\times\mathbf{R}. \tag{51}$$

Finally, (48) implies $\partial_t^2 u \in L^2(0,T;H^{-1}(\mathbf{R}))$ and then $\partial_t u \in H^1(0,T;H^{-1}(\mathbf{R}))$ and $\partial_t u(0,.)$ is well defined in $L^2(\mathbf{R})$. Thus multiplying the second pde of (27) by $\phi \in \mathcal{D}([0,T[\times\mathbf{R})$ and integrating by parts, we get

$$\int_0^T \int_{\mathbf{R}} \left[u^\varepsilon \partial_t^2 \phi + (1+f^\varepsilon)(\partial_x u^\varepsilon)(\partial_x \phi) + \varepsilon(\partial_{tx}^2 u^\varepsilon)(\partial_x \phi)\right] dxdt =$$

$$\int_{\mathbf{R}} u_1(x)\phi(0,x)\,dx - \int_{\mathbf{R}} u_0(x)\partial_t\phi(0,x)\,dx,$$

which gives when ε goes to 0,

$$\int_0^T \int_{\mathbf{R}} \left[u \partial_t^2 \phi + (1+f)(\partial_x u)(\partial_x \phi)\right] dxdt =$$

$$\int_{\mathbf{R}} u_1(x)\phi(0,x)\,dx - \int_{\mathbf{R}} u_0(x)\partial_t\phi(0,x)\,dx.$$

Integrating by parts this last equality and making use of (48) we are led to

$$\int_{\mathbf{R}} [\partial_t u(0,x) - u_1(x)]\phi(0,x)\,dx - \int_{\mathbf{R}} [u(0,x) - u_0(x)]\partial_t\phi(0,x)\,dx = 0$$

which gives thanks to (49)

$$\int_{\mathbf{R}} [\partial_t u(0,x) - u_1(x)]\phi(0,x)\,dx = 0 \quad, \forall \phi \in \mathcal{D}([0,T[\times\mathbf{R}).$$

Therefore, we have $\partial_t u(0,x) = u_1(x)$ in $\mathcal{D}'(\mathbf{R})$ and then in $L^2(\mathbf{R})$. This ends the proof of Theorem 1.

Let us discuss the regularity of the solutions. We have the result

Lemma 8 *We assume that hypotheses of Theorem 1 hold and $\partial_x^2 u_0, \partial_x u_1 \in L^2(\mathbf{R})$. Then the solutions u of the wave equation satisfy the regulary result $\partial_x^2 u, \partial_{xt}^2 u, \partial_t^2 u \in L^\infty(0,T,L^2(\mathbf{R}))$.*

Proof. From the equation satisfied by u we get $\partial_t(\partial_{xt}^2 u) - \partial_x((1+f)\partial_x^2 u) = \partial_x(\partial_x f \partial_x u)$. Multiplying by $\partial_{xt}^2 u$ and integrating by parts we get

$$\frac{d}{dt} E(\partial_x u(t))^2 = \int_{\mathbf{R}} \partial_t f |\partial_x^2 u|^2 dx + 2\int_{\mathbf{R}} \partial_x(\partial_x f \partial_x u)\partial_{xt}^2 u dx \qquad (52)$$

where $E(w(t))^2 = |\partial_t w(t)|^2 + |(1+f)^{1/2}\partial_x u(t)|^2$ for some function w. Using the heat equation we get $\int_{\mathbf{R}} \partial_x(\partial_x f \partial_x u)\partial_{xt}^2 u dx = \int_{\mathbf{R}}(\partial_t f + \partial_x u)\partial_x u \partial_{xt}^2 u dx -$

$\int_{\mathbf{R}} \partial_x f \partial_x^2 u \partial_{xt}^2 u dx$. It follows that we have

$$\frac{d}{dt} E(\partial_x u(t))^2 - \frac{2}{3} \int_{\mathbf{R}} (\partial_x u)^3 dx) \leq |\nabla_{xt} f|_\infty (E_0^2 + E(\partial_x u(t))^2). \tag{53}$$

Next, one observes that $| \int_R (\partial_x u)^3 dx| \leq |\partial_x u(t)|_\infty E_0^2$ which implies the estimate $| \int_{\mathbf{R}} (\partial_x u)^3 dx| \leq c \|\partial_x u(t)\|_{\mathbf{H}^1} E_0^2 \leq c E_0^2 (E_0 + E(\partial_x u(t)))$ where $c > 0$ is independent of the data. Integrating the inequation we obtain

$$\begin{aligned} E(\partial_x u(t))^2 + E_0^2 \leq E_0^2 + E(\partial_x u_0)^2 - \tfrac{2}{3} \int_R (\partial_x u_0)^3 + \tfrac{2}{3} \int_{\mathbf{R}} (\partial_x u(t))^3 + \\ + |\nabla_{xt} f|_\infty \int_0^t (E_0^2 + E(\partial_x u(s))^2) ds. \end{aligned} \tag{54}$$

Finally, we have the estimate $E(\partial_x u(t))^2 + E_0^2 \leq E_0^2 + E(\partial_x u_0)^2 - \frac{2}{3} \int_{\mathbf{R}} (\partial_x u_0)^3 + \frac{2c}{3} E_0^2 (E_0^2 + E(\partial_x u(t))^2)^{1/2} + |\nabla_{xt} f|_\infty \int_0^t (E_0^2 + E(\partial_x u(s))^2) ds$ which implies the result

$$\frac{1}{2}(E(\partial_x u(t))^2 + E_0^2) \leq A_0 + |\nabla_{xt} f|_\infty \int_0^t (E_0^2 + E(\partial_x u(s))^2) ds \tag{55}$$

where$A_0 = E_0^2 + E(\partial_x u_0)^2 - \frac{2}{3} \int_{\mathbf{R}} (\partial_x u_0)^3$. By Gronwall inequality it holds that for all $t \in [0, T]$,

$$E(\partial_x u(t))^2 \leq C e^{T |\nabla_{xt} f|_\infty} |A_0|. \tag{56}$$

where $C > 0$ is independent of$T$ and the data. This result shows that $\partial_{xt}^2 u, \partial_x^2 u \in L^\infty(0, T; L^2(\mathbf{R}))$ and using the wave equation we get $\partial_t^2 u \in L^\infty(0, T; L^2(\mathbf{R}))$.

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