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CENTRE DE MATHÉMATIQUES APPLIQUÉES
UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 41 50. Fax: 01 69 33 30 11

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K. Hamdache - I. Ngningone-Éya

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Abstract

We consider time harmonic solutions of a model to ferroelectric materials discussed recently by Greenberg, Coffeman and Mac Camy [5] in the time dependent case. We first discuss the full problem described by a system of coupled Maxwell's equations. We prove existence of solutions and the uniqueness is obtained for frequencies which are far from 0. The $\mathbb{H}^1$-regularity is proved for frequencies which are far from 0 and for the boundary condition $\mathbf{P} \times \mathbf{n} = 0$. Next, we discuss existence and uniqueness of the solution to the reduced model given by a couple of nonlinear Helmholtz equations satisfied by the electric field and the electric polarization of the material. We obtain uniform estimates with respect the frequency and deduce the behavior of the solution when the frequency tends to zero.

Key words. Ferroelectrics. Nonlinear Helmholtz equations. Low frequency behavior. Nonlinear Maxwell's equation.
1991 AMS subject classifications. 35L45, 35L10, 35K05, 73R05, 46E35

1 Introduction

In the present work we consider the model of ferroelectric materials introduced in [4] to describe the dynamics of elastic materials which exhibit phase changes. The model is derived from the classical Maxwell's system with the constitutive laws $D = \epsilon(E+P)$ and $B = \mu H$ where (H, E) is the electromagnetic field and P is the electric polarization field. This model, suggested by Greenberg in [4], have been studied in [5] where the long time behaviour of the solutions is given in the case of the T.E and T.M approximations. Let Ω be a bounded and regular domain of $\mathbb{R}^3$ with boundary $\partial\Omega$. We denote by $\mathbf{n}$ the outward unit normal to Ω. The electromagnetic field (H, E) and the electric polarization field P satisfy the set of equations

*Centre de Mathématiques Appliquées, CNRS UMR 7641 & Ecole Polytechnique, 91128 Palaiseau Cedex, France (Email: hamdache@cmapx.polytechnique.fr), (Email: ngningon@cmapx.polytechnique.fr).

$$
\begin{cases}
\partial_t(\mu H) - \text{curl}E = 0 \\
\partial_t(\epsilon(E+P)) + \text{curl}H + \sigma E = 0 \\
\partial_t^2 P + a\partial_t P + \lambda^2 \text{curl}^2 P + b\ P\phi'(|P|^2) = b\ E \\
E \times \mathbf{n} = \text{curl}P \times \mathbf{n} = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega
\end{cases}
\tag{1}
$$

where $a > 0$, $b > 0$ are fixed constants, $\theta = \dfrac{\sigma}{\mu\epsilon} > 0$ where $\sigma > 0$, $\mu > 0$ and $\epsilon > 0$ are respectively the electric conductivity of the material, the magnetic permeability and the electric permittivity of the vacuum.

We are dealing with time harmonic solutions of this model. Namely, we consider solutions of the type $E = e^{i\omega t}\mathbf{E}$, $H = e^{i\omega t}\mathbf{H}$ and $P = e^{i\omega t}\mathbf{P}$ where $\omega \geq 0$ is fixed. Then, the complex field $(\mathbf{E}, \mathbf{H}, \mathbf{P})$ satisfies in Ω the set of Maxwell's equations

$$
\begin{cases}
(-\omega^2 + i\omega\theta)\mathbf{E} + \lambda^2 \text{curl}\mathbf{E} = \omega^2 \mathbf{P} + i\omega \mathbf{F} \\
(-\omega^2 + ia\omega)\mathbf{P} + \lambda^2 \text{curl}^2 \mathbf{P} + b\ \mathbf{P}\phi'(|\mathbf{P}|^2) = b\ \mathbf{E} \\
\mathbf{E} \times \mathbf{n} = \text{curl}\mathbf{P} \times \mathbf{n} = 0 \text{ on } \partial\Omega
\end{cases}
\tag{2}
$$

where $\mathbf{F}$ is a given source term. We used the notation $|\mathbf{P}(x)|^2 = \mathbf{P}(x)\mathbf{P}(x)^*$ where $\mathbf{P}^*$ represents the complex conjugate of $\mathbf{P} \in \mathbb{C}^3$. Notice that the magnetic field $\mathbf{H}$ is recovert by the formula $\mathbf{H} = \frac{1}{i\ \omega\mu}\text{curl}\mathbf{E}$.

When $\Omega = \widehat{\Omega} \times (0, L)$ is a slender cylinder with $\widehat{\Omega} \subset \mathbb{R}^2$, then, the T.E or T.M approximation of the model reduces to the following set of Helmholtz equations satisfied in $\widehat{\Omega}$ by the electric and polarization functions $\mathbf{e}(x) \in \mathbb{C}$ and $\mathbf{p}(x) \in \mathbb{C}$

$$
\begin{cases}
(z_1(\omega) - \Delta)\mathbf{e} = \omega^2 \mathbf{p} + i\omega \mathbf{f} \text{ in } \widehat{\Omega} \\
(z_2(\omega) - \Delta)\mathbf{p} + b\ \mathbf{p}\phi'(|\mathbf{p}|^2) = b\ \mathbf{e} \text{ in } \widehat{\Omega} \\
\mathbf{e} = 0,\ \frac{\partial \mathbf{p}}{\partial n} = 0 \text{ on } \partial\widehat{\Omega}.
\end{cases}
\tag{3}
$$

where we setted $z_1(\omega) = -\omega^2 + i\omega\theta$ and $z_2(\omega) = -\omega^2 + i\omega a$.

The potential function $\phi : \mathbb{R}^+ \to \mathbb{R}$ is a C^2 convex function, satisfying the following hypotheses see [5]. There exist $0 < r_1 < r_0$ and $C_1' > 0$ such that

$$
\begin{cases}
\phi(0) = \phi(r_0) = 0,\ \phi'(r_1) = 0,\ C_1 := \phi'(0) < 0, \text{ and} \\
(s\phi'(s^2))' \leq C_1'\ \forall\ s \geq 0.
\end{cases}
\tag{4}
$$

Hence, for all $s \geq R > r_1$, there exists $C_R > 0$ such that $\phi'(s) \geq C_R$ and for all $s \geq 0$

$$
|\phi'(s)| \leq C_*, \text{ and } 0 \leq s^2\phi^{(2)}(s^2) \leq C_* \tag{5}
$$

where $C_* = \max(C_1, C_1')$. The following potentials defined on $\mathbb{R}^+$ satisfy our hypotheses $\phi_1(s) = b(1+s^2)^{1/2} - as - 1$, with $0 < a < 1$, $b > 0$, $\phi_2(s) = s/2 - \log(1+s)$, $\phi_3(s) = as + 1 - (1+s)^\alpha$ with $a > 0$ and $0 < \alpha < 1$ foe example. Let us introduce the vector valued complex function $\Psi(\mathbf{P}) = \mathbf{P}\phi'(|\mathbf{P}|^2)$ then, (4)-(5), imply

$$
|\Psi(\mathbf{P}) - \Psi(\mathbf{Q})| \leq C_*|\mathbf{P} - \mathbf{Q}| \tag{6}
$$

for all $\mathbf{P}, \mathbf{Q} \in \mathbb{C}^3$. Using this result and the continuity of ϕ' then, there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$\Psi(\mathbf{P}).\mathbf{P}^* \geq c_1|\mathbf{P}|^2 - c_2. \tag{7}$$

for all $\mathbf{P} \in \mathbb{C}^3$. In particular we have we have $\Psi(\mathbf{P}).\mathbf{P}^* \geq \phi'(R^2)|\mathbf{P}|^2$ for $|\mathbf{P}| \geq R > r_1$. The outline of the paper is the following. In section 2 we consider the full problem (2) and prove existence of solutions by using a contraction mapping argument. The existence proof holds when $\mathbf{P}$ satisfies either the boundary condition

$$\text{curl}\mathbf{P} \times \mathbf{n} + \beta \mathbf{n} \times (\mathbf{P} \times \mathbf{n}) = 0 \tag{8}$$

for $\beta \geq 0$ or,

$$\mathbf{P} \times \mathbf{n} = 0. \tag{9}$$

The boundary condition (9) will be used only in the proof of the $\mathbb{H}^1$-regularity of the solutions. The uniqueness of the solutions is proved for $\omega > \omega_* = bC_*/a$. Finally, the $\mathbb{H}^1(\Omega)$-regularity of the solutions holds for $\omega > \omega_* + 4C_\Omega$ and for $\mathbf{P}$ satisfying (9) where $C_\Omega > 0$ is a constant depending only of the domain Ω. In section 3, we consider the reduced model (3). We first prove uniform estimates of the solutions with respect to the parameter $\omega > 0$ and then we deduce the behavior of the solution when ω tends to zero. Finally, we prove the existence of the solutions for $\omega > 0$ and obtain their uniqueness for $\omega > \omega_*$. In the last section we give some remarks concerning the low frequency behavior of the solutions of the full model (2) assuming that they satisfy uniform bounds with respect ω.
Others questions related to this model has been discussed in [2] and [1]. In [2] the global existence of solutions of the time dependent model is studied and in [1] the behaviour of thin ferroelectric materials is discussed. In [1] the $\mathbb{H}^1$-regularity of the solutions is proved for the solutions of the problem with the boundary condition (8) where $\beta > 0$. In [6] a model for piezoelectric material is discussed.
We shall use the following functional spaces. We denote by $\mathbb{L}^2(\Omega) = (L^2(\Omega))^3$ the usual Hilbert space equipped with the norm $(U;V) = \int_\Omega U(x) \cdot (V(x))^* \, dx$. We set $\mathcal{X} = \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$ endowed with the scalar product $\langle \vec{U}; \vec{V} \rangle_{\mathcal{X}} = (U_1; V_1) + (U_2; V_2)$ where $\vec{U} = (U_1, U_2)$ and $\vec{V} = (V_1, V_2)$. Next, we set $\mathcal{H}(\text{curl}, \Omega) = \{U \in \mathbb{L}^2(\Omega), \text{curl}U \in \mathbb{L}^2(\Omega)\}$, $\mathcal{H}(\text{div}, \Omega) = \{U \in \mathbb{L}^2(\Omega), \text{div}U \in L^2(\Omega)\}$ and $\mathcal{H}_0(\text{curl}, \Omega) = \{U \in \mathcal{H}(\text{curl}, \Omega), U \times \mathbf{n} = 0 \text{ on } \partial\Omega\}$ equipped with their natural norms.

2 The full model

2.1 Existence theory in the case $\omega > \omega_*$.

We shall use the classical fixed point theorem to prove existence of solutions to problem (2)-(8) with $\beta \geq 0$ and for problem (2)-(9). For given $(\mathbf{F}, \mathbf{G}) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$ we consider the linear problem

$$\begin{cases} (z_1(\omega) + \lambda^2 \mathrm{curl}^2)\mathbf{E} - \omega^2 \mathbf{P} = i\omega \mathbf{F} \text{ in } \Omega \\ (z_2(\omega) + \lambda^2 \mathrm{curl}^2)\mathbf{P} - b\ \mathbf{E} = \mathbf{G} \text{ in } \Omega \\ \mathbf{E}\times\mathbf{n} = 0,\ \mathrm{curl}\mathbf{P}\times\mathbf{n} + \beta\ \mathbf{n}\times\mathbf{P}\times\mathbf{n} = 0 \text{ or } \mathbf{P}\times\mathbf{n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{10}$$

We introduce on $\mathcal{X} = \mathbb{L}^2(\Omega)\times\mathbb{L}^2(\Omega)$ the unbounded linear operator $\mathcal{A}$ with domain $D(\mathcal{A}) = \{(U,V)\in\mathcal{X},\ (\mathrm{curl}^2 U, \mathrm{curl}^2 V)\in\mathcal{X},\ U\times\mathbf{n} = 0, \text{ and } \mathrm{curl}V\times\mathbf{n} + \beta\mathbf{n}\times V\times\mathbf{n} = 0 \text{ or } V\times\mathbf{n} = 0 \text{ on } \partial\Omega\}$ with $\mathcal{A}(U,V) = (\mathrm{curl}^2 U, \mathrm{curl}^2 V))$ for $(U,V)\in D(\mathcal{A})$. We also introduce on $\mathcal{X}$ the bounded operators Λ and Γ defined by $\Lambda(U,V) = (z_1(\omega)U, z_2(\omega)V)$ and $\Gamma(U,V) = (\omega^2 V, bU)$. The following properties hold

$$\begin{cases} \langle\Lambda(U,V);(U,V)\rangle_{\mathcal{X}} = -\omega^2(|U|^2 + |V|^2) + i\omega(\theta|U|^2 + a|V|^2), \\ \Im\langle\Gamma(U,V);(bU,\omega^2 V)\rangle_{\mathcal{X}} = 0 \end{cases} \tag{11}$$

for $(U,V)\in D(\mathcal{A})$. Moreover, we have, see [3], [11] for example

$$\langle\mathcal{A}(U,V);(U,V)\rangle_{\mathcal{X}} = |\mathrm{curl}U|^2 + |\mathrm{curl}V|^2 + \beta|V\times\mathbf{n}|^2_{\mathbb{L}^2(\partial\Omega}. \tag{12}$$

if $\beta \geq 0$. In the case where we use the boundary condition (9) then (12) holds by setting $\beta = 0$. More generally, for all $(U_j, V_j)\in D(\mathcal{A})$ satisfying (8) with $j = 1,2$, we have $\langle\mathcal{A}(U_1,V_1);(U_2,V_2)\rangle_{\mathcal{X}} = (\mathrm{curl}U_1;\mathrm{curl}U_2) + (\mathrm{curl}V_1,\mathrm{curl}V_2) + \beta(V_1\times\mathbf{n};V_2\times\mathbf{n})^2_{\mathbb{L}^2(\partial\Omega)}$. Integrating by parts we obtain

$$\langle\mathcal{A}(U_1,V_1);(U_2,V_2)\rangle_{\mathcal{X}} = \langle(U_1,V_1);\mathcal{A}(U_2,V_2)\rangle_{\mathcal{X}} \tag{13}$$

which shows that operator $\mathcal{A}$ is a selfadjoint operator on $\mathcal{X}$. We have $\mathcal{A} = \mathcal{A}^*$ with $D(\mathcal{A}) = D(\mathcal{A}^*)$. Setting $U = \mathbf{E}$ and $V = \mathbf{P}$, $\vec{U} = (U,V)$ and $\vec{\mathbf{F}} = (i\omega\mathbf{F}, 0)$ then, the abstract formulation of problem (10) becomes

$$(\Lambda - \Gamma + \lambda^2\mathcal{A})(\vec{U}) = \vec{\mathbf{F}}. \tag{14}$$

We shall prove that operator $\mathcal{B} = (\Lambda - \Gamma + \lambda^2\mathcal{A})$ is invertible from $D(\mathcal{A})$ into $\mathcal{X}$ that is its kernel $N(\mathcal{B}) = \{\vec{0}\}$ and its range $R(\mathcal{B}) = \mathcal{X}$. Let $\vec{U} = (U,V)\in D(\mathcal{A})$ such that $(\Lambda - \Gamma + \lambda^2\mathcal{A})(\vec{U}) = \vec{0}$. Multiplying the first equation by bU^* and the second by $\omega^2 V^*$, integrating by parts and adding the results we get the Green formula $\langle\mathcal{B}(U,V);(bU,\omega^2 V)\rangle_{\mathcal{X}} = \lambda^2(b|\mathrm{curl}U|^2 + \omega^2|\mathrm{curl}V|^2 + \beta\omega^2|V\times\mathbf{n}|^2_{\mathbb{L}^2(\partial\Omega}) - \omega^2(b|U|^2 + \omega^2|V|^2) + i\omega(b\theta|U|^2 + a\omega^2|V|^2) = 0$. Taking the imaginary part of the Green formula we deduce that $U = V = 0$. Next, Let $\mathcal{B}^*$ be the adjoint operator of $\mathcal{B}$. We have $\mathcal{B}^* = \Lambda^* - \Gamma + \lambda^2\mathcal{A}$ with $\Lambda^*(U,V) = (z_1(\omega)^* U, z_2(\omega)^* V)$. Proceeding as previously, we get $N(\mathcal{B}^*) = \{\vec{0}\}$. Then $R(\mathcal{B})$ is dense in $\mathcal{X}$. The closedness of the range $R(\mathcal{B})$ in $\mathcal{X}$ is a consequence of the following coercivity property satisfied by $\mathcal{B}$. From the previous Green formula, we have $\Im\langle\mathcal{B}(U,V);(bU,\omega^2 V)\rangle_{\mathcal{X}} = \omega(b\theta|U|^2 + a\omega^2|V|^2)$ and $\Re\langle\mathcal{B}(U,V);(bU,\omega^2 V)\rangle_{\mathcal{X}} = \lambda^2(b|\mathrm{curl}U|^2 + \omega^2|\mathrm{curl}V|^2 + \beta\omega^2|V\times\mathbf{n}|^2_{\mathbb{L}^2(\partial\Omega}) - \omega^2(b|U|^2 + \omega^2|V|^2)$. We repalce in these equations (U,V) by $(\frac{U}{b^{1/2}}, \frac{V}{\omega})$ with $\omega > 0$, we get $\Re\langle(\mathcal{B} + \omega^2 I)(U,V);(U,V)\rangle_{\mathcal{X}} = \lambda^2(|\mathrm{curl}(U,V)|^2_{\mathcal{X}} + \beta|V\times\mathbf{n}|^2_{\mathbb{L}^2(\partial\Omega}) $ and $\Im\langle\mathcal{B}(U,V);(U,V)\rangle_{\mathcal{X}} \geq \omega C|(U,V)|^2_{\mathcal{X}}$ where $C = \min(\theta, a)$ and finally we arrive to

$$|\langle\mathcal{B}\vec{U};\vec{U})\rangle_{\mathcal{X}}| \geq C\omega|\vec{U}|^2_{\mathcal{X}}. \tag{15}$$

The coercivity property (15) ensures the closedness of the range $R(\mathcal{B})$. We proved

Lemma 2.1 *Let $\omega > 0$ and $\mathbf{F}, \mathbf{G} \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$ be fixed. Then, there exists a unique solution $(\mathbf{E}, \mathbf{P}) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$ of problems (2)-(8) and (2)-(9) such that $(\mathrm{curl}^2\mathbf{E}, \mathrm{curl}^2\mathbf{P}) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$ and $\mathbf{P} \times \mathbf{n} \in \mathbb{L}^2(\partial\Omega)$ if $\beta > 0$. Moreover, the solution satisfies the following bounds. There exists $C > 0$ which is independent of ω such the solutions of (2)-(8) satisfy*

$$\begin{aligned}
&|\mathbf{E}|^2 + |\omega \mathbf{P}|^2 \leq C(|\mathbf{F}|^2 + |\mathbf{G}^2) \\
&|\mathrm{curl}\mathbf{E}|^2 + |\mathrm{curl}(\omega\mathbf{P})|^2 + \beta|\omega\mathbf{P} \times \mathbf{n}|^2_{\mathbb{L}^2(\partial\Omega)} \leq C(\omega^2 + \omega)(|\mathbf{F}|^2 + |\mathbf{G}|^2) \\
&|\mathrm{curl}^2\mathbf{E}| + |\mathrm{curl}^2(\omega\mathbf{P})| \leq C(1+\omega)(|\mathbf{F}| + \omega|\mathbf{G}|.
\end{aligned} \tag{16}$$

The solutions of (2)-(9) satisfy the same estimates by setting $\beta = 0$ in the second inequality of (16).

Proof. Let us consider the second inequality of (**??**) with $\mathcal{B}(U, V) = (i\omega\mathbf{F}, \mathbf{G})$, $U = \mathbf{E}$ and $V = \mathbf{P}$. Its imaginary part gives $\omega(b\theta|\mathbf{E}|^2 + a|\omega\mathbf{P}|^2) \leq \omega b|\mathbf{F}||\mathbf{E}| + \omega|\mathbf{G}||\omega\mathbf{P}|$. Hence, we deduce the first estimate of (16). Next, the real part of the first equality of (**??**) gives $\lambda^2(b|\mathrm{curl}\mathbf{E}|^2 + |\mathrm{curl}(\omega\mathbf{P})|^2 + \beta|\omega\mathbf{P}\times\mathbf{n}|^2) \leq \omega^2(b|\mathbf{E}|^2 + |\omega\mathbf{P}|^2) + \omega b|\mathbf{F}||\mathbf{E}| + \omega|\mathbf{G}||\omega\mathbf{P}|$. Then we obtain the second estimate of (16). Finally, the last estimate follows by using the equations satisfied by $(\mathbf{E}, \mathbf{P})$.

For $\mathbf{Q} \in \mathbb{L}^2(\Omega)$ fixed we set $\mathbf{G}(\mathbf{Q}) = -\mathbf{Q}\phi'(|\mathbf{Q}|^2)$. Let $(\mathbf{E}, \mathbf{P})$ be the solution of the linear problem associated with the data $(\mathbf{F}, \mathbf{G}(\mathbf{Q}))$ given by Lemma 2.1. We introduce the operator $\mathcal{R} : \mathbb{L}^2(\Omega) \to \mathbb{L}^2(\Omega)$ defined by $\mathcal{R}(\mathbf{Q}) = \mathbf{P}$. We shall prove that $\mathcal{R}$ is a contraction mapping on $\mathbb{L}^2(\Omega)$ if $\omega > \omega_*$ where $\omega_* > 0$ is some constant which is defined later.

Lemma 2.2 *For all $\mathbf{Q}_1,\ \mathbf{Q}_2 \in \mathbb{L}^2(\Omega)$, we have*

$$|\mathcal{R}(\mathbf{Q}_1) - \mathcal{R}(\mathbf{Q}_2)| \leq \frac{bC_*}{\omega}|\mathbf{Q}_1 - \mathbf{Q}_2|. \tag{17}$$

Proof. For $j = 1, 2$ let $\mathbf{Q}_j \in \mathbb{L}^2(\Omega)$ be fixed and let $(\mathbf{E}_j, \mathbf{P}_j)$ be the solution of the problem. We set $\mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2$, $\mathbf{P} = \mathbf{P}_1 - \mathbf{P}_2$, $\mathbf{Q} = \mathbf{Q}_1 - \mathbf{Q}_2$ and $\Phi = \mathbf{G}(\mathbf{Q}_1) - \mathbf{G}(\mathbf{Q}_2)$ then $(\mathbf{E}, \mathbf{P})$ satisfies the problem

$$\begin{cases}
(z_2(\omega) + \lambda^2\mathrm{curl}^2)\mathbf{P} = b\mathbf{E} - b\Phi \\
(z_1(\omega) + \lambda^2\mathrm{curl}^2)\mathbf{E} = \omega^2\mathbf{P} \\
\mathbf{E}\times\mathbf{n} = 0,\ \mathrm{curl}\mathbf{P}\times\mathbf{n} + \beta\mathbf{n}\times\mathbf{P}\times\mathbf{n} = 0 \text{ or } \mathbf{P}\times\mathbf{n} = 0\ \partial\Omega
\end{cases} \tag{18}$$

Using (**??**) with $\mathcal{B}(U, V) = (0, -b\Phi)$ then the imaginary part gives

$$\omega(b|\mathbf{E}|^2 + a\omega^2|\mathbf{P}|^2) \leq \omega^2|\mathbf{P}||\Phi| \tag{19}$$

using hypothesis (4)-(5) we get (17).

We proved the following existence result

Theorem 2.3 *Let $\omega_* = bC_*$. We assume that $\mathbf{F} \in \mathbb{L}^2(\Omega)$ and $\omega > \omega_*$. Then under hypotheses (4) there exists a unique solution $(\mathbf{E}, \mathbf{P}) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$ of problems (2)-(8) and (2)-(9) such that $\mathbf{E} \in \mathcal{H}_0(\mathrm{curl}, \Omega)$, $\mathrm{curl}^2\mathbf{E} \in \mathbb{L}^2(\Omega)$, $\mathbf{P} \in \mathcal{H}(\mathrm{curl}, \Omega)$, $\mathrm{curl}^2\mathbf{P} \in \mathbb{L}^2(\Omega)$. Moreover, $\mathbf{P}\times\mathbf{n} \in \mathbb{L}^2(\partial\Omega)$ if $\beta > 0$. The solution satisfies estimates (16) with $\mathbf{G} = 0$.*

2.2 $\mathbb{H}^1(\Omega)$-regularity of the solutions

In this section we prove the $\mathbb{H}^1(\Omega)$-regularity of the solution satisfying the boundary condition (9) and for $\omega > \omega_{**}$ where $\omega_{**} > 0$ will be precised later. In order to simplify the presentation and without lost of generality we set $a = 1$, $b = 1$, $\lambda = 1$ and $\epsilon = 1$. In that case we have $z_1(\omega) = z_2(\omega)$. We assume that the source term $\mathbf{F}$ is such that $\mathbf{F} \in \mathcal{H}(\mathrm{div}, \Omega)$. To prove the $\mathbb{H}^1(\Omega)$- regularity of the solutions we first built a regularized sequence of solutions $(\mathbf{E}^\epsilon, \mathbf{P}^\epsilon)$ belonging to $\mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega)$ then, we prove an uniform estimate in $\mathbb{H}^1$-estimate for the regularized solutions for $\omega > \omega_{**}$. Finally, we pass to the limit when $\epsilon \to 0$.

Let $(\rho^\epsilon(x))$ be a regularising sequence of scalar functions such that $\rho^\epsilon(x) \geq 0$ in $\mathbb{R}^3$ and $\int_{\mathbb{R}^3} \rho^\epsilon(x)\, dx = 1$. We consider the regularized problem

$$\begin{cases} (z_1(\omega) + \mathrm{curl}^2)\mathbf{E}^\epsilon - \omega^2 \mathbf{P}^\epsilon = i\omega \mathbf{F} \text{ in } \Omega \\ (z_1(\omega) + \mathrm{curl}^2)\mathbf{P}^\epsilon - \mathbf{E}^\epsilon = -\mathbf{G}^\epsilon(\mathbf{P}^\epsilon) \text{ in } \Omega \\ \mathbf{E}^\epsilon \times \mathbf{n} = 0,\ \mathbf{P}^\epsilon \times \mathbf{n} = 0 \text{ on } \partial\Omega \end{cases} \tag{20}$$

where

$$\mathbf{G}^\epsilon(\mathbf{Q}) = (\rho^\epsilon \star \mathbf{Q})\phi'(|\rho^\epsilon \star \mathbf{Q}|^2) \tag{21}$$

for all $\mathbf{Q} \in \mathbb{L}^2(\Omega)$ with $(\rho^\epsilon \star \mathbf{Q})(x) = \int_\Omega \rho^\epsilon(x-y)\mathbf{Q}(y)\, dy$ a.e $x \in \Omega$. We have $\rho^\epsilon \star \mathbf{Q} \to \mathbf{Q}$ strongly in $\mathbb{L}^2(\Omega)$ and if $\mathbf{P}^\epsilon \to \mathbf{P}$ strongly in $\mathbb{L}^2(\Omega)$ then $\rho^\epsilon \star \mathbf{P}^\epsilon \to \mathbf{P}$ strongly in $\mathbb{L}^2(\Omega)$. Moreover, using hypotheses (4)-(5) we get

$$|\mathbf{G}^\epsilon(\mathbf{Q}_1) - \mathbf{G}^\epsilon(\mathbf{Q})_2| \leq C_*|\mathbf{Q}_1 - \mathbf{Q}_2|. \tag{22}$$

for all $\mathbf{Q}_1$ and $\mathbf{Q}_2$ in $\mathbb{L}^2(\Omega)$. Proceeding along the lines of the proof of Theorem 2.1, we obtain

Lemma 2.4 *Let $\mathbf{F} \in \mathbb{L}^2(\Omega)$ and $\epsilon > 0$ be fixed. Then, for $\omega > \omega_* = C_*$, there exist a unique solution $(\mathbf{E}^\epsilon, \mathbf{P}^\epsilon)$ of problem (20). Moreover, the solution belongs to $\mathcal{H}_0(\mathrm{curl}, \Omega) \times \mathcal{H}(\mathrm{curl}, \Omega)$ and satisfies $\mathrm{curl}^2\mathbf{E}^\epsilon, \mathrm{curl}^2\mathbf{P}^\epsilon \in \mathbb{L}^2(\Omega)$. The solution is uniformly bounded with respect to ϵ in these spaces .*

Proof. Since the operator $\mathbf{Q} \to \mathbf{G}^\epsilon(\mathbf{Q})$ is uniformly Lipschitzian then the solution exists. We have to show that the solution is uniformly bounded with respect to ϵ. The energy estimate gives

$$\begin{cases} z_1(\omega)|\mathbf{E}^\epsilon|^2 + |\mathrm{curl}\mathbf{E}^\epsilon|^2 - \omega^2(\mathbf{P}^\epsilon; (\mathbf{E}^\epsilon)^*) = i\omega(\mathbf{F}; (\mathbf{E}^\epsilon)^*) \\ z_1(\omega)|\mathbf{P}^\epsilon|^2 + |\mathrm{curl}\mathbf{P}^\epsilon|^2 - (\mathbf{E}^\epsilon; (\mathbf{P}^\epsilon)^*) = -(\mathbf{G}^\epsilon(\mathbf{P}^\epsilon); (\mathbf{P}^\epsilon)^*) \end{cases} \tag{23}$$

summing the imaginary parts of both equations we get $|\mathbf{E}^\epsilon|^2 + \omega^2|\mathbf{P}^\epsilon|^2 = \Re(\mathbf{F}; (\mathbf{E}^\epsilon)^*) - \omega\Im(\mathbf{G}^\epsilon(\mathbf{P}^\epsilon); (\mathbf{P}^\epsilon)^*$ which implies the estimate

$$|\mathbf{E}^\epsilon|^2 + \omega(\omega - C_*)|\mathbf{P}^\epsilon|^2 \leq |\mathbf{F}|^2. \tag{24}$$

This gives the uniform estimate of the solution in $\mathbb{L}^2(\Omega)$. Summing the real part of both equations we obtain

$$\left\{\begin{array}{l} |\mathrm{curl}\mathbf{E}^\epsilon|^2 \leq \omega^2|\mathbf{E}^\epsilon|^2 + \omega^2|\mathbf{P}^\epsilon||\mathbf{E}^\epsilon| + \omega|\mathbf{F}^\epsilon||\mathbf{E}^\epsilon| \\ \\ |\mathrm{curl}\mathbf{P}^\epsilon|^2 \leq \omega^2|\mathbf{P}^\epsilon|^2 + |\mathbf{P}^\epsilon||\mathbf{E}^\epsilon| + C_*|\mathbf{P}^\epsilon|^2. \end{array}\right. \tag{25}$$

The proof of the Lemma is then complete.

We shall show that under hypotheses $\mathbf{F} \in \mathcal{H}(\mathrm{div}, \Omega)$ and and $\omega > \omega_{**}$, the solution belongs in fact in $\mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega)$ where ω_{**} will be given below. First, writing $\mathrm{div}\mathbf{G}^\epsilon(\mathbf{P}^\epsilon) = \Big((\mathrm{grad}\rho^\epsilon)\star\mathbf{P}^\epsilon\Big)\phi'(|\rho^\epsilon\star\mathbf{P}^\epsilon|^2) + 2(\rho^\epsilon\star\mathbf{P}^\epsilon_j)(\partial_k\rho^\epsilon\star\mathbf{P}^\epsilon_j)(\rho^\epsilon\star\mathbf{P}^\epsilon_k)\phi^{(2)}(|\rho^\epsilon\star\mathbf{P}^\epsilon|^2)$ one observes that $\mathrm{div}\mathbf{G}^\epsilon(\mathbf{P}^\epsilon) \in L^2(\Omega)$ for $\mathbf{P}^\epsilon \in \mathbb{L}^2(\Omega)$. Then by using hypothesis (4), one deduces that there exists a constant $C_\epsilon > 0$ such that $|\mathrm{div}\mathbf{G}^\epsilon(\mathbf{P}^\epsilon)| \leq C_\epsilon|\mathbf{P}^\epsilon|$. We set $\mathbf{e}^\epsilon = \mathrm{div}\mathbf{E}^\epsilon$, $\mathbf{p}^\epsilon = \mathrm{div}\mathbf{P}^\epsilon$, $\mathbf{g}^\epsilon = \mathrm{div}\mathbf{G}^\epsilon(\mathbf{P}^\epsilon)$ and $\mathbf{f} = \mathrm{div}\mathbf{F}$. The compatibility conditions of the equations (20) take the form

$$\left\{\begin{array}{l} z_1(\omega)\mathbf{e}^\epsilon - \omega^2\mathbf{p}^\epsilon = i\omega\mathbf{f} \\ \\ z_1(\omega)\mathbf{p}^\epsilon - \mathbf{e}^\epsilon = -\mathbf{g}^\epsilon. \end{array}\right. \tag{26}$$

Multiplying the first equation by $(\mathbf{e}^\epsilon)^*$ and the second by $\omega^2(\mathbf{p}^\epsilon)^*$, integrating over Ω adding the results and taking the imaginary part of the sum we get

$$\omega^2|\mathbf{p}^\epsilon|^2 + |\mathbf{e}^\epsilon|^2 = \omega\Im\Big(\int_\Omega \mathbf{g}^\epsilon(\mathbf{p}^\epsilon)^*\ dx\Big) + \Re\Big(\int_\Omega \mathbf{f}(\mathbf{e}^\epsilon)^*\ dx\Big) \tag{27}$$

which implies the estimate

$$\omega^2|\mathbf{p}^\epsilon|^2 + |\mathbf{e}^\epsilon|^2 \leq C_\epsilon|\mathbf{P}^\epsilon|^2 + |\mathbf{f}|^2 \tag{28}$$

where $C_\epsilon > 0$ represents various constants depending of ϵ such that $C_\epsilon \to \infty$ when $\epsilon \to 0$.

Next, we use the following classical estimate see [3]. Any $\mathbf{W} \in \mathcal{H}_0(\mathrm{curl}, \Omega) \cap \mathcal{H}(\mathrm{div}, \Omega)$ belongs to $\mathcal{H}^1(\Omega)$ and satisfies the estimate, there exist $C_\Omega > 0$ depending only of Ω such that

$$|\mathrm{grad}\mathbf{W}|^2 \leq C_\Omega(|\mathrm{curl}\mathbf{W}|^2 + |\mathrm{div}\mathbf{W}|^2 + |\mathbf{W}|^2). \tag{29}$$

for all $\mathbf{W} \in \mathcal{H}_0(\mathrm{curl}, \Omega) \cap \mathcal{H}(\mathrm{div}, \Omega)$. We apply this estimate for $\mathbf{E}^\epsilon$ and $\mathbf{P}^\epsilon$ which satisfy the boundary condition $\mathbf{E}^\epsilon \times \mathbf{n} = 0$ and $\mathbf{P}^\epsilon \times \mathbf{n} = 0$. Using (27) and (28) we get the estimate

$$\left\{\begin{array}{l} |\mathrm{grad}\mathbf{E}^\epsilon|^2 \leq C_\Omega(|\mathbf{E}^\epsilon|^2 + |\mathrm{curl}\mathbf{E}^\epsilon|^2 + C_\epsilon|\mathbf{P}^\epsilon|^2 + |\mathbf{f}|^2) \\ \\ |\mathrm{grad}\mathbf{P}^\epsilon|^2 \leq C_\Omega(C_\epsilon|\mathbf{P}^\epsilon|^2 + |\mathrm{curl}\mathbf{P}^\epsilon|^2 + |\mathbf{f}|^2). \end{array}\right. \tag{30}$$

We proved the following result

Lemma 2.5 *Let $(\mathbf{E}^\epsilon, \mathbf{P}^\epsilon)$ be the solution of the regularized problem (20). Assume that $\mathbf{F} \in \mathcal{H}(\mathrm{div}, \Omega)$ and $\beta > 0$ then for all $\epsilon > 0$, $(\mathbf{E}^\epsilon, \mathbf{P}^\epsilon) \in \mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega)$.*

Now, we prove that $(\mathbf{E}^\epsilon, \mathbf{P}^\epsilon)$ is uniformly bounded in $\mathbb{H}^1(\Omega)\times\mathbb{H}^1(\Omega)$ with respect to ϵ. Since $(\mathbf{E}^\epsilon, \mathbf{P}^\epsilon) \in \mathbb{H}^1(\Omega)\times\mathbb{H}^1(\Omega)$ then we may write $\mathrm{div}\mathbf{G}^\epsilon(\mathbf{P}^\epsilon) = (\rho^\epsilon\star\mathrm{div}\mathbf{P}^\epsilon)\phi'(|\rho^\epsilon\star\mathbf{P}^\epsilon|^2)+2(\rho^\epsilon\star\mathbf{P}^\epsilon_j)(\rho^\epsilon\star\partial_k\mathbf{P}^\epsilon_j)(\rho^\epsilon\star\mathbf{P}^\epsilon_k)\phi^{(2)}(|\rho^\epsilon\star\mathbf{P}^\epsilon|^2)$ which gives, using hypotheses (4)-(5), the estimate

$$|\mathrm{div}\mathbf{G}^\epsilon(\mathbf{P}^\epsilon)| \leq C_*|\mathbf{p}^\epsilon| + 2C_*|\mathrm{grad}\mathbf{P}^\epsilon|. \tag{31}$$

The relations (27)-(31) imply the following estimate (recall that $\omega > \omega_* = C_*$)

$$(\omega^2 - C_*^2)|\mathbf{p}^\epsilon|^2 + |\mathbf{e}^\epsilon|^2 \leq 8C_*^2|\mathrm{grad}\mathbf{P}^\epsilon|^2 + |\mathbf{f}|^2. \tag{32}$$

We deduce the following regularity result

Theorem 2.6 *Let $\omega_{**} = \omega_* + 4C_\Omega$ with $\omega_* = C_*$. We assume that $\mathbf{F} \in \mathcal{H}(\mathrm{div},\Omega)$. Then if $\omega > \omega_{**}$ the unique solution $(\mathbf{E},\mathbf{P})$ of the full problem (2)-(9) belongs to $\mathbb{H}^1(\Omega)\times\mathbb{H}^1(\Omega)$ and satisfies the estimate*

$$\left\{\begin{array}{l} |\mathrm{grad}\mathbf{P}|^2 \leq C_\Omega(|\mathbf{P}|^2 + |\mathrm{curl}\mathbf{P}|^2) + \frac{C_\Omega}{\omega^2-\omega_{**}^2}|\mathrm{div}\mathbf{F}|^2 \\ |\mathrm{grad}\mathbf{E}|^2) \leq C_\Omega(|\mathbf{E}|^2 + |\mathrm{curl}\mathbf{E}|^2) + 8C_*^2|\mathrm{grad}\mathbf{P}|^2 + |\mathrm{div}\mathbf{F}|^2. \end{array}\right. \tag{33}$$

Proof. Let $(\mathbf{E}^\epsilon, \mathbf{P}^\epsilon)$ be the solution of the regularized problem (20). Since we have $\mathbf{E}^\epsilon\times\mathbf{n} = 0$ and $\mathbf{P}^\epsilon\times\mathbf{n} = 0$ on $\partial\Omega$ then, estimate (29) implies that

$$\left\{\begin{array}{l} |\mathrm{grad}\mathbf{P}^\epsilon|^2 \leq C_\Omega(|\mathbf{P}^\epsilon|^2 + |\mathrm{curl}\mathbf{P}^\epsilon|^2 + |\mathbf{p}^\epsilon|^2) \\ |\mathrm{grad}\mathbf{E}^\epsilon|^2 \leq C_\Omega(|\mathbf{E}^\epsilon|^2 + |\mathrm{curl}\mathbf{E}^\epsilon|^2 + |\mathbf{e}^\epsilon|^2). \end{array}\right. \tag{34}$$

Using (32) in the first equation of (33) one deduces that

$$|\mathrm{grad}\mathbf{P}^\epsilon|^2 \leq C_\Omega(|\mathbf{P}^\epsilon|^2 + |\mathrm{curl}\mathbf{P}^\epsilon|^2) + \frac{8C_*^2C_\Omega}{\omega^2-\omega_*^2}|\mathrm{grad}\mathbf{P}^\epsilon|^2 + \frac{C_\Omega}{\omega^2-\omega_*^2}|\mathbf{f}|^2. \tag{35}$$

This estimate implies the first inequality of (33) for $\mathbf{P}^\epsilon$. The second equation of (33) is also satisfied by $\mathbf{E}^\epsilon$. We get

$$\left\{\begin{array}{l} |\mathrm{grad}\mathbf{P}^\epsilon|^2 \leq C_\Omega(|\mathbf{P}^\epsilon|^2 + |\mathrm{curl}\mathbf{P}^\epsilon|^2) + \dfrac{C_\Omega}{\omega^2-\omega_{**}^2}|\mathrm{div}\mathbf{F}|^2 \\ |\mathrm{grad}\mathbf{E}^\epsilon|^2 \leq C_\Omega(|\mathbf{E}^\epsilon|^2 + |\mathrm{curl}\mathbf{E}^\epsilon|^2) + 8C_*^2|\mathrm{grad}\mathbf{P}^\epsilon|^2 + |\mathrm{div}\mathbf{F}|^2. \end{array}\right. \tag{36}$$

Extracting a subsequence still denoted $(\mathbf{E}^\epsilon, \mathbf{P}^\epsilon)$ we may pass to the limit in (36) by using the weak convergence in $\mathbb{H}^1(\Omega)\times\mathbb{H}^1(\Omega)$ as $\epsilon\to 0$. Let $(\mathbf{E},\mathbf{P})$ be the weak limit of this subsequence then we have $\mathbf{P}^\epsilon\phi'(|\mathbf{P}^\epsilon|^2)\to\mathbf{P}\phi'(|\mathbf{P}|^2)$ strongly in $\mathbb{L}^2(\Omega)$ and $(\mathbf{E},\mathbf{P})$ satisfies estimates (33). Moreover $(\mathbf{E},\mathbf{P})$ is a solution of the full problem (2)-(9). Finally since we have $\omega_{**} > \omega_*$ then this solution is unique.

Remark 2.7 *The $\mathbb{H}^1(\Omega)$-regularity of the solutions may be proved in the case of the boundary condition (8) with $\beta > 0$ by using a trace regularity result for Maxwell's equations proved in [1].*

3 The reduced model

Let Ω be a bounded and regular open set of $\mathbb{R}^2$ with boundary $\partial\Omega$ (denoted in section 1 by $\widehat{\Omega}$). In what follows we discuss some properties satisfied by the solutions $(\mathbf{e}(\omega), \mathbf{p}(\omega)) \in H_0^1(\Omega) \times H^1(\Omega)$ of problem (3). We first prove uniform estimates with respect to ω satisfied by the solutions then, we discuss their behavior when $\omega \to 0$. Finally, we prove the existence of the solutions.

3.1 Uniform estimates and convergences

Theorem 3.1 *There exists $C > 0$ depending only on the constants a, b, θ, c_1, c_2 and $|\Omega|$ such that for all $\omega > 0$, any solution $(\mathbf{e}(\omega), \mathbf{p}(\omega)) \in H_0^1(\Omega) \times H^1(\Omega)$ of problem (3) satisfies the estimates*

$$\left\{\begin{array}{l} |\mathbf{e}(\omega)| \leq C\ |\mathbf{f}| \\ \\ |\mathrm{grad}\mathbf{e}(\omega)| \leq C\ max(\omega, \sqrt{\omega})\ |\mathbf{f}|_{L^2(\Omega)} \\ \\ |\mathbf{p}(\omega)| + |\mathrm{grad}\mathbf{p}(\omega)| \leq C\ (1 + |\mathbf{f}|) \end{array}\right. \tag{37}$$

The proof is given in several steps. For simplicity, we omit to write the dependency of the solutions with respect the parameter ω. Multiplying the first equation of (3) by $\mathbf{e}^*$, the second one by $\mathbf{p}^*$ and integrating by parts we get

$$\left\{\begin{array}{l} z_1(\omega)|\mathbf{e}|^2_{L^2(\Omega)} + |\mathrm{grad}\mathbf{e}|^2_{L^2(\Omega)} = \omega^2\ \int_\Omega \mathbf{p}\mathbf{e}^*\ dx + i\ \omega\ \int_\Omega \mathbf{f}\mathbf{e}^*\ dx \\ z_2(\omega)|\mathbf{p}|^2 + |\mathrm{grad}\mathbf{p}|^2 + b\ \int_\Omega |\mathbf{p}|^2\phi'(|\mathbf{p}|^2)\ dx = b\ \int_\Omega \mathbf{e}\mathbf{p}^*\ dx. \end{array}\right. \tag{38}$$

Taking the real part then the imaginary part of the first equation of (38) we get

$$\left\{\begin{array}{l} -\omega^2\ |\mathbf{e}|^2 + |\mathrm{grad}\mathbf{e}|^2 = \omega^2\ \Re(\int_\Omega \mathbf{p}\mathbf{e}^*\ dx) - \omega\ \Im(\int_\Omega \mathbf{f}\mathbf{e}^*\ dx) \\ \theta\ \omega\ |\mathbf{e}|^2 = \omega^2\ \Im(\int_\Omega \mathbf{p}\mathbf{e}^*\ dx) + \omega\ \Re(\int_\Omega \mathbf{f}\mathbf{e}^*\ dx). \end{array}\right. \tag{39}$$

Next, the real and imaginary parts of the second equation of (38) can be written as follows

$$\left\{\begin{array}{l} -\omega^2\ |\mathbf{p}|^2 + |\mathrm{grad}\mathbf{p}|^2 + b\ \int_\Omega |\mathbf{p}|^2\phi'(|\mathbf{p}|^2)\ dx = b\ \Re(\int_\Omega \mathbf{e}\mathbf{p}^*\ dx) \\ \omega\ |\mathbf{p}|^2 = \frac{b}{a}\ \Im(\int_\Omega \mathbf{e}\mathbf{p}^*\ dx). \end{array}\right. \tag{40}$$

We multiply the second equation of (39) by $\frac{b}{a}$ and the second equation of (40) by ω^2 and add both equations we get

$$\omega^2\ |\mathbf{p}|^2_{L^2(\Omega)} + \frac{b\theta b}{a}|\mathbf{e}|^2 = \frac{b}{a}\ \Re(\int_\Omega \mathbf{f}\mathbf{e}^*\ dx). \tag{41}$$

In what follows $C > 0$ denotes various constants which are independent of ω and the solution $(\mathbf{e}, \mathbf{p})$. Equation (41) implies

Lemma 3.2 *Any solution $(\mathbf{e}, \mathbf{p})$ in $H_0^1(\Omega) \times H^1(\Omega)$ of problem (3) satisfies the uniform estimates*

$$|\mathbf{e}| \le C|\mathbf{f}|, \ \omega \ |\mathbf{p}| \le C|\mathbf{f}|. \tag{42}$$

The first equation of (39) and estimates (42) imply that

$$|\mathrm{grad}\mathbf{e}|^2 \le C \max(\omega, \omega^2)|\mathbf{f}|^2 \tag{43}$$

and the first equation of (40), estimates (42) and the property (6) give

$$|\mathrm{grad}\mathbf{p}|^2 + \frac{c_1}{2} \ |\mathbf{p}|^2 \le c_2 \ |\Omega| + C|\mathbf{f}|^2. \tag{44}$$

where $c_1 > 0$ and $c_2 > 0$ are the constants appearing in (7).

Remark 3.3 *By Poincaré's inequality, we get* $|\mathbf{e}| \le C \ max(\omega, \sqrt{\omega}) \ |\mathbf{f}|$.

3.2 Continuity for $\omega > 0$.

Let us consider the map $T : \mathbb{R}^+ \to \mathcal{V} = H_0^1(\Omega) \times H^1(\Omega)$ defined by $T(\omega) = (\mathbf{e}(\omega), \mathbf{p}(\omega))$ where $(\mathbf{e}(\omega), \mathbf{p}(\omega)) \in \mathcal{V}$ is a solution of the reduced problem (3). The Hilbert space $\mathcal{V}$ is equipped with its natural norm denoted by $||\cdot||_{\mathcal{V}}$. We also introduce the Hilbert space $\mathcal{H} = L^2(\Omega) \times L^2(\Omega)$ which is equipped with its natural norm denoted by $|\cdot|_{\mathcal{H}}$.

For $j = 1, 2$, let $T(\omega_j) = (\mathbf{e}(\omega_j), \mathbf{p}(\omega_j)) \in \mathcal{V}$ be a solution of (3) associated with $\omega_j > 0$ and the same source term $\mathbf{f} \in L^2(\Omega)$. We may suppose $\omega_1 \le \omega_2$. We set $\mathbf{E} = \mathbf{e}(\omega_1) - \mathbf{e}(\omega_2)$, $\mathbf{P} = \mathbf{p}(\omega_1) - \mathbf{p}(\omega_2)$ and $\Phi = \Psi(\omega_1) - \Psi(\omega_2)$ where Ψ is the function defined by (6). Moreover we set $\nu = \omega_1 - \omega_2$, $\mu = \omega_1^2 - \omega_2^2$ and $Z_j = z_j(\omega_1) - z_j(\omega_2)$ for $j = 1, 2$. We observe that $\mu = \nu S$, $Z_1 = \nu\xi_1$, $Z_2 = \nu\xi_2$ where $S = \omega_1 + \omega_2$, $\xi_1 = -S + i\theta$, $\xi_2 = -S + ia$. It follows that $(\mathbf{E}, \mathbf{P})$ satisfies the system of equations

$$\left\{ \begin{array}{l} (z_1(\omega_1) - \Delta)\mathbf{E} = \omega_1^2 \mathbf{P} + \nu \mathbf{F} \text{ in } \Omega \\ \\ (z_2(\omega_1) - \Delta)\mathbf{P} + b \ \Phi = b \ \mathbf{E} - \nu \mathbf{G} \text{ in } \Omega \\ \\ \mathbf{E} = 0, \ \dfrac{\partial \mathbf{P}}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega. \end{array} \right. \tag{45}$$

where we setted $\mathbf{F} = \Big(S\mathbf{p}(\omega_2) - \xi_1 \mathbf{e}(\omega_2) + i\mathbf{f}\Big)$ and $\mathbf{G} = \xi_2 \mathbf{p}(\omega_2)$. Proceeding as in the proof of Theorem 3.1 we get the estimate

$$\left\{ \begin{array}{l} -\omega_1^2 \ |\mathbf{E}|^2 + |\mathrm{grad}\mathbf{E}|^2 = \omega_1^2 \ \Re(\int_\Omega \mathbf{P}\mathbf{E}^* \ dx) + \nu \ \Im(\int_\Omega \mathbf{F}\mathbf{E}^* \ dx) \\ \\ \theta \ \omega_1 \ |\mathbf{E}|^2 = \omega_1^2 \ \Im(\int_\Omega \mathbf{P}\mathbf{E}^* \ dx) + \nu \ \Im(\int_\Omega \mathbf{F}\mathbf{E}^* \ dx). \end{array} \right. \tag{46}$$

Next, taking the real and imaginary parts of the second equation of (46) we obtain

$$\left\{ \begin{array}{l} -\omega_1^2 \ |\mathbf{P}|^2 + |\mathrm{grad}\mathbf{P}|^2 + b \ \int_\Omega \Phi\mathbf{P}^* \ dx = b \ \Re(\int_\Omega \mathbf{E}\mathbf{P}^* \ dx) + \nu\Re(\int_\Omega \mathbf{G}\mathbf{P}^* \ dx), \\ \\ \omega_1 a \ |\mathbf{P}|^2 = b \ \Im(\int_\Omega \mathbf{E}\mathbf{P}^* \ dx) + \nu\Im(\int_\Omega \mathbf{G}\mathbf{P}^* \ dx). \end{array} \right. \tag{47}$$

We multiply the second equation of (46) by b and the second equation of (47) by ω_1^2 and add both equations we get

$$\omega_1\Big(b\theta\ |\mathbf{E}|^2 + a\omega_1^2\ |\mathbf{P}|^2\Big) = \nu\Big(b\ \Im(\int_\Omega \mathbf{F}\mathbf{E}^*\ dx) + \omega_1^2\Im(\int_\Omega \mathbf{G}\mathbf{P}^*\ dx)\Big). \tag{48}$$

We deduce the estimate

Lemma 3.4 *There exists $C > 0$ which is independent of ω_1 and ω_2 such that the following holds*

$$\Big(|\mathbf{E}|^2 + |\mathbf{P}|^2\Big) \le C\ \frac{\max(\omega_1^2, \omega_2^2, 1)}{\min(\omega_1^2, \omega_2^2, 1)}|\nu||\mathbf{f}|. \tag{49}$$

Now let us consider the first equations of (47). By Cauchy-Schwartz inequality and by (49) we get $|\mathrm{grad}\mathbf{E}|^2 \le \omega_1^2(|\mathbf{E}|^2 + |\mathbf{E}||\mathbf{P}|) + |\nu||\mathbf{F}||\mathbf{E}|$ which implies the estimate

$$|\mathrm{grad}\mathbf{E}| \le A\ \sqrt{|\nu|} \tag{50}$$

where $A =$. Next, the first equation of (48) gives $|\mathrm{grad}\mathbf{P}|^2 \le b\ |\Phi||\mathbf{P}| + \omega_1^2|\mathbf{P}|^2 + b\ |\mathbf{E}||\mathbf{P}| + |\nu||\mathbf{G}||\mathbf{P}|$ and finally using (**??**) we obtain

$$|\mathrm{grad}\mathbf{P}| \le B\ \sqrt{|\nu|} \tag{51}$$

where $B =$. We proved the following result

Theorem 3.5 *There exists $C > 0$ such the map T satisfies for all $\omega_1 > 0$ and $\omega_2 > 0$ the estimate*

$$||T(\omega_1) - T(\omega_2)||_{\mathcal{V}} \le C\ \sqrt{|\omega_1 - \omega_2|} \tag{52}$$

the constant C depends of $\omega_1 > 0$, $\omega_2 > 0$, θ, a, b and the $L^2(\Omega)$-norm of $\mathbf{f}$.

3.3 The low frequency behavior.

We discuss the behavior of the solutions of problem (3) when ω tends to 0. Let ω_n be a sequence of $\mathbb{R}^+$ such that $\omega_n \to 0$ as $n \to +\infty$. We denote by $(\mathbf{e}_n, \mathbf{p}_n) = (\mathbf{e}(\omega_n), \mathbf{p}(\omega_n))$ the solution of the problem (3) associated with ω_n. Theorem 3.1 implies that there exist a constant $C > 0$ which is independent of n such that $||(\mathbf{e}_n, \mathbf{p}_n)||_{\mathcal{V}} \le C$. Hence, the sequence $(\mathbf{e}_n, \mathbf{p}_n)$ is uniformly bounded in $\mathcal{V}$. There exists a subsequence, still denoted $(\mathbf{e}_n, \mathbf{p}_n)$ such that

$$(\mathbf{e}_n, \mathbf{p}_n) \rightharpoonup (\mathbf{e}_\infty, \mathbf{p}_\infty) \text{ in } \mathcal{V} \text{ weak.} \tag{53}$$

Using the compactness of the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ we deduce for a subsequence still denoted $(\mathbf{e}_n, \mathbf{p}_n)$ we have $(\mathbf{e}_n, \mathbf{p}_n)$ converges to $(\mathbf{e}_\infty, \mathbf{p}_\infty)$ in $\mathcal{H}$ strong and $(\mathbf{e}_n(x), \mathbf{p}_n(x)) \to (\mathbf{e}_\infty(x), \mathbf{p}_\infty(x))$ a.e $x \in \Omega$. Now we are ready to prove the following theorem.

Theorem 3.6 *Let $(\mathbf{e}_\infty, \mathbf{p}_\infty) \in \mathcal{V}$ be the weak limit of the subsequence of $(\mathbf{e}_n, \mathbf{p}_n)$. Then it holds that $(\mathbf{e}_n, \mathbf{p}_n) \to (\mathbf{e}_\infty, \mathbf{p}_\infty)$ in $\mathcal{V}$ strongly and we have $\mathbf{e}_\infty = 0$ and*

$$-\Delta\mathbf{p}_\infty + b\ \mathbf{p}_\infty\phi'(|\mathbf{p}_\infty|^2) = 0 \text{ in } \Omega,\ \frac{\partial\mathbf{p}_\infty}{\partial n} = 0 \text{ on } \partial\Omega. \tag{54}$$

Proof. The estimates obtained in (37) show that all the sequence grad$\mathbf{e}_n \to 0$ strongly in $L^2(\Omega)$ and by Poincaré inequality we deduce that all the sequence $\mathbf{e}_n \to 0$ strongly in $L^2(\Omega)$. The sequence $\mathbf{p}_n$ satisfies the equation

$$-\Delta \mathbf{p}_n + b\ \mathbf{p}_n \phi'(|\mathbf{p}_n|^2) = \mathbf{F}_n \text{ in } \Omega,\ \frac{\partial \mathbf{p}_n}{\partial n} = 0 \text{ on } \partial\Omega \tag{55}$$

where $\mathbf{F}_n = -z_2(\omega_n)\mathbf{p}_n + b\ \mathbf{e}_n$. Clearly all the sequence $\mathbf{F}_n \to 0$ in $L^2(\Omega)$ strong. Thanks estimate to (**??**) we have $|\mathbf{p}_n\phi'(|\mathbf{p}_n|^2) - \mathbf{p}_\infty\phi'(|\mathbf{p}_\infty|^2)| \le C_*|\mathbf{p}_n - \mathbf{p}_\infty|$ and since the subsequence $\mathbf{p}_n$ converges strongly in $L^2(\Omega)$ then $|\mathbf{p}_n\phi'(|\mathbf{p}_n|^2) - \mathbf{p}_\infty\phi'(|\mathbf{p}_\infty|^2) \to 0$ strongly in $L^2(\Omega)$. Next, since we have $-\Delta(\mathbf{p}_n - \mathbf{p}_\infty) = \mathbf{F}_n - (\mathbf{p}_n\phi'(|\mathbf{p}_n|^2) - \mathbf{p}_\infty\phi'(|\mathbf{p}_\infty|^2)$ then multiplying this equation by $\mathbf{p}_n - \mathbf{p}_\infty$, integrating by parts and using the estimate $|\Big(\mathbf{p}_n\phi'(|\mathbf{p}_n|^2) - \mathbf{p}_\infty\phi'(|\mathbf{p}_\infty|^2)\Big)(\mathbf{p}_n - \mathbf{p}_\infty)| \le C_*|\mathbf{p}_n - \mathbf{p}_\infty|^2$ we get

$$|\text{grad}\mathbf{p}_n - \text{grad}\mathbf{p}_\infty|^2 \le C_*|\mathbf{p}_n - \mathbf{p}_\infty|^2 + |\mathbf{F}_n||\mathbf{p}_n - \mathbf{p}_\infty| \tag{56}$$

which implies that $\mathbf{p}_n \to \mathbf{p}_\infty$ in $H^1(\Omega)$ strong. Hence, the theorem is proved.

Remark 3.7 *Since the stationary equation (54) has multiple solutions see [5] then the low frequency behavior given by Theorem 2.3 is related only to the subsequence $\omega_{n'}$ of ω_n.*

3.4 Uniqueness for large frequencies

Theorem 3.8 *Let ω be such that $\omega > \omega_* = \dfrac{bC_*}{a}$. Then, any solution $(\mathbf{e},\mathbf{p}) \in H_0^1(\Omega) \times H^1(\Omega)$ associated with ω and $\mathbf{f} \in L^2(\Omega)$ is unique. Moreover, if $(\mathbf{e}_j,\mathbf{p}_j) \in H_0^1(\Omega) \times H^1(\Omega)$ is the solution of the problem (3) associated with ω as before and $\mathbf{f}_j \in L^2(\Omega)$ for $j = 1,2$ then, there exists $C > 0$ which independent of ω such that*

$$\begin{cases} |\mathbf{e}_1 - \mathbf{e}_2| \le C|\mathbf{f}_1 - \mathbf{f}_2|,\ (\omega - \omega_*)|\mathbf{p}_1 - \mathbf{p}_2| \le C|\mathbf{f}_1 - \mathbf{f}_2| \\ (\omega - \omega_*)^{1/2}|\text{grad}\mathbf{e}_1 - \text{grad}\mathbf{e}_2| \le C((\omega^2 + \omega)(\omega - \omega_*) + \omega^2)^{1/2}|\mathbf{f}_1 - \mathbf{f}_2| \\ (\omega - \omega_*)|\text{grad}\mathbf{p}_1 - \text{grad}\mathbf{p}_2| \le C(\omega^2 + \omega + 1)^{1/2}|\mathbf{f}_1 - \mathbf{f}_2|. \end{cases} \tag{57}$$

Proof. Let $(\mathbf{e}_j,\mathbf{p}_j) \in H_0^1(\Omega) \times H^1(\Omega)$ be the solution of problem (3) associated to the frequency $\omega > 0$ and the source term $\mathbf{f}_j$ for $j = 1,2$. We set $\mathbf{E} = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{P} = \mathbf{p}_1 - \mathbf{p}_2$ and $\mathbf{F} = \mathbf{f}_1 - \mathbf{f}_2$. Hence, the couple $(\mathbf{E},\mathbf{P})$ satisfies the set of equations

$$\begin{cases} (z_1(\omega) - \Delta)\mathbf{E} = \omega^2\mathbf{P} + i\omega\ \mathbf{F} \text{ in } \Omega \\ (z_2(\omega) - \Delta)\mathbf{P} + b\ \Phi = b\ \mathbf{E} \text{ in } \Omega \\ \mathbf{E} = 0,\ \dfrac{\partial \mathbf{P}}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{58}$$

where $\Phi = \mathbf{p}_1\phi'(|\mathbf{p}_1|^2) - \mathbf{p}_2\phi'(|\mathbf{p}_2|^2)$. We multiply the first equation by $\mathbf{E}^*$ and the second one by $\mathbf{P}^*$ and integrate by parts. We take the imaginary and real parts of each energy estimate. Then the imaginary parts write as

$$\begin{cases} \omega\theta|\mathbf{E}|^2 = \omega^2\Im(\int_\Omega \mathbf{P}\mathbf{E}^*\ dx) - \omega\Re(\int_\Omega \mathbf{F}\mathbf{E}^*\ dx) \\ \omega a|\mathbf{P}|^2 + b\ \Im(\int_\Omega \Phi\mathbf{P}^*\ dx) = b\ \Im(\int_\Omega \mathbf{E}\mathbf{P}^*\ dx) \end{cases} \tag{59}$$

which implies the inequality $b\theta|\mathbf{E}|^2 + a\omega(\omega - \omega_*)|\mathbf{P}|^2 \leq b|\mathbf{F}||\mathbf{E}|$. Adding the real parts we get

$$\begin{cases} |\mathrm{grad}\mathbf{E}|^2 = \omega^2|\mathbf{E}|^2 + \omega^2\Re(\int_\Omega \mathbf{P}\mathbf{E}^*\ dx) - \omega\Im(\int_\Omega \mathbf{F}\mathbf{E}^*\ dx) \\ |\mathrm{grad}\mathbf{P}|^2 = \omega^2|\mathbf{P}|^2 - b\Re(\int_\Omega \Phi\mathbf{P}^*\ dx) + b\Re(\int_\Omega \mathbf{E}\mathbf{P}^*\ dx). \end{cases} \tag{60}$$

It follows that

$$\begin{cases} |\mathrm{grad}\mathbf{E}|^2 \leq \omega^2|\mathbf{E}|^2 + \omega^2|\mathbf{P}||\mathbf{E}| + \omega|\mathbf{F}||\mathbf{E}| \\ |\mathrm{grad}\mathbf{P}|^2 \leq (\omega^2 + bC_*)|\mathbf{P}|^2 + b|\mathbf{P}||\mathbf{E}|. \end{cases} \tag{61}$$

This ends the proof of the theorem.

Remark 3.9 *Let* $(\mathbf{e}, \mathbf{p})$ *be a solution of the problem associated with* $\mathbf{f} = 0$. *Since* $\mathbf{p}\phi'(|\mathbf{p}|^2)\mathbf{p}^*$ *belongs to* $\mathbb{R}$ *then estimates (59) give* $\omega\theta|\mathbf{e}|^2 = \omega^2\Im(\int_\Omega \mathbf{p}\mathbf{e}^*\ dx)$ *and* $\omega a|\mathbf{p}|^2 = b\ \Im(\int_\Omega \mathbf{e}\mathbf{p}^*\ dx)$ *hence, we get* $b\omega\theta|\mathbf{e}|^2 + \omega^3 a|\mathbf{p}|^2 = 0$ *which implies* $\mathbf{e} = 0$ *and* $\mathbf{p} = 0$.

3.5 Existence of solutions

Let us introduce the map R defined on $L^2(\Omega)$ by $R(\mathbf{q}) = \mathbf{p}$ where $\mathbf{p}$ is the solution of the linear equation

$$(z_2(\omega) - \Delta)\mathbf{p} = -b\ \mathbf{q}\phi'(|\mathbf{q}|^2) + b\ \mathbf{e} \text{ in } \Omega,\ \frac{\partial \mathbf{p}}{\partial n} = 0 \text{ on } \partial\Omega. \tag{62}$$

and $\mathbf{e}$ is the solution of the linear equation

$$(z_1(\omega) - \Delta)\mathbf{e} = \omega^2\mathbf{q} + i\ \omega\mathbf{f} \text{ in } \Omega,\ \mathbf{e} = 0 \text{ on } \partial\Omega. \tag{63}$$

We introduce the unbounded operator $\mathcal{A}$ with domain $D(\mathcal{A}) = \{\mathbf{e} \in H_0^1(\Omega), \Delta\mathbf{e} \in L^2(\Omega)\}$ by setting $\mathcal{A}\mathbf{e} = (z_1(\omega) - \Delta)\mathbf{e}$ for $\mathbf{e} \in D(\mathcal{A})$. Operator $\mathcal{A}$ is densely defined on $L^2(\Omega)$. The following Green formula holds for $\mathbf{u}$ and $\mathbf{v}$ in $D(\mathcal{A})$

$$(\mathcal{A}\mathbf{u}; \mathbf{v}) = -\omega^2(\mathbf{u}; \mathbf{v}) + (\mathrm{grad}\mathbf{u}; \mathrm{grad}\mathbf{v}) + i\ \omega\theta(\mathbf{u}; \mathbf{v}) \tag{64}$$

where $(\mathbf{u}; \mathbf{v}) = \int_\Omega \mathbf{u}\mathbf{v}^*\ dx$ denotes the scalar product of $L^2(\Omega)$. We have the result

Lemma 3.10 *Operator* $\mathcal{A}$ *is an isomorphism from* $D(\mathcal{A})$ *into* $L^2(\Omega)$.

Proof. Let $N(\mathcal{A})$ be the kernel of $\mathcal{A}$. Let $\mathbf{u} \in N(\mathcal{A})$ then, by (64) we get $\Re(\mathcal{A}\mathbf{u}; \mathbf{u}) = -\omega^2|\mathbf{u}|^2 + |\mathrm{grad}\mathbf{u}|^2 = 0$ and $\Im(\mathcal{A}\mathbf{u}; \mathbf{u}) = \omega\theta|\mathbf{u}|^2 = 0$. We deduce that $\mathbf{u} = 0$ if $\omega > 0$. Hence we have $N(\mathcal{A}) = \{0\}$. Let $\mathcal{A}^*$ be the adjoint operator of $\mathcal{A}$ with domain $D(\mathcal{A}^*) = \{\mathbf{u} \in H_0^1(\Omega)\ \Delta\mathbf{u} \in L^2(\Omega)\}$ defined by $\mathcal{A}^*\mathbf{u} = (z_1^* - \Delta)\mathbf{u}$. Proceeding along the lines of the previous proof we deduce that $N(\mathcal{A}^*) = \{0\}$. Hence, $R(\mathcal{A})$, the range of $\mathcal{A}$, is densely defined in $L^2(\Omega)$. Since $\mathcal{A}$ satisfies the coercivity property

$|(\mathcal{A}\mathbf{u};\mathbf{u})| \geq \omega\theta|\mathbf{u}|^2$ for all $\mathbf{u} \in D(\mathcal{A})$ then it is easy to deduce that $R(\mathcal{A})$ is closed in $L^2(\Omega)$.

Let $\mathbf{e} \in D(\mathcal{A})$ be the solution of (63) associated with $\mathbf{f} \in L^2(\Omega)$ and $\mathbf{q} \in L^2(\Omega)$. Then, it holds

$$\begin{cases} |\mathbf{e}| \leq C(\omega|\mathbf{q}| + |\mathbf{f}|), \\ |\mathrm{grad}\mathbf{e}| \leq C(\omega^2+\omega)^{1/2}\Big(\omega|\mathbf{q}| + |\mathbf{f}|\Big) \\ |\Delta\mathbf{e}| \leq C(\omega^2+\omega+1)\Big(\omega|\mathbf{q}| + |\mathbf{f}|\Big) \end{cases} \tag{65}$$

where $C > 0$ is a constant which is independant of ω, $\mathbf{e}$ and $\mathbf{q}$. To solve problem (62) we proceed following the lines of the previous proof by introducing the unbounded operator $\mathcal{B}$ defined on $L^2(\Omega)$ with domain $D(\mathcal{B}) = \{\mathbf{u} \in H^1(\Omega), \dfrac{\partial \mathbf{u}}{\partial n} = 0 \text{ on } \partial\Omega,\ \Delta\mathbf{u} \in L^2(\Omega)\}$ and given by $\mathcal{B}\mathbf{u} = (z_2 - \Delta)\mathbf{u}$ for $\mathbf{u} \in D(\mathcal{B})$. As we have shown for lemma 3.1, the Green formula (64) holds for $\mathbf{u} \in D(\mathcal{B})$. Moreover, operator $\mathcal{B}$ is densely defined on $L^2(\Omega)$ and it is an isomorphism from $\mathcal{B}$ into $L^2(\Omega)$. Consequently, since $\mathbf{q}\phi'(|\mathbf{q}|^2) \in L^2(\Omega)$ and $\mathbf{e} \in L^2(\Omega)$ then there exists a unique solution $\mathbf{p} \in D(\mathcal{B})$ of problem (62). Moreover, by using (65), $\mathbf{p}$ satisfies the estimates

$$\begin{cases} \omega|\mathbf{p}| \leq C((1+\omega)|\mathbf{q}| + |\mathbf{f}|), \\ \omega^{1/2}|\mathrm{grad}\mathbf{p}| \leq C((1+\omega)|\mathbf{q}| + |\mathbf{f}|), \\ |\Delta\mathbf{p}| \leq C((1+\omega)|\mathbf{q}| + |\mathbf{f}|). \end{cases} \tag{66}$$

Let for $j = 1, 2$, $(\mathbf{e}_j, \mathbf{p}_j)$ the solution to problem (63) and (62) respectively associated with $\mathbf{q}_j$ and the source term $\mathbf{f}$. We set $\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2$, $\mathbf{e} = \mathbf{e}_1 - \mathbf{e}_2$ and $\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2$. Then $\mathbf{e}$ and $\mathbf{p}$ satify the equation

$$\begin{cases} z_1(\omega)\mathbf{e} - \Delta\mathbf{e} = \omega^2\mathbf{q} \\ z_2(\omega)\mathbf{p} - \Delta\mathbf{p} = -b\Phi + b\mathbf{e} \end{cases} \tag{67}$$

where $\Phi = \Psi(\mathbf{q}_1) - \Psi(\mathbf{q}_2)$. By using (4), we have $|\Phi| \leq C_*|\mathbf{q}|$ and using the standard energy estimates, as previously for both equations, we get the following bounds

$$\begin{cases} |\mathbf{e}_1 - \mathbf{e}_2| \leq C\omega|\mathbf{q}_1 - \mathbf{q}_2|,\ \omega|\mathbf{p}_1 - \mathbf{p}_2| \leq C(1+\omega)|\mathbf{q}_1 - \mathbf{q}_2|, \\ |\mathrm{grad}\mathbf{e}_1 - \mathrm{grad}\mathbf{e}_2| \leq C\omega(\omega^2+\omega)^{1/2}|\mathbf{q}_1 - \mathbf{q}_2|, \\ \omega^{1/2}|\mathrm{grad}\mathbf{p}_1 - \mathrm{grad}\mathbf{p}_2| \leq C(1+\omega)|\mathbf{q}_1 - \mathbf{q}_2|, \end{cases} \tag{68}$$

and

$$\begin{cases} |\Delta\mathbf{e}_1 - \Delta\mathbf{e}_2| \leq C(\omega^2+\omega+1)\omega|\mathbf{q}_1 - \mathbf{q}_2|, \\ |\Delta\mathbf{p}_1 - \Delta\mathbf{p}_2| \leq C(1+\omega)|\mathbf{q}_1 - \mathbf{q}_2|. \end{cases} \tag{69}$$

It follows that the operator R maps continuously $L^2(\Omega)$ into $H^1(\omega)$ and then it is compact from $L^2(\Omega)$ into $L^2(\Omega)$. Hence, by Schauder fixed point theorem, there exists a solution $\mathbf{p} \in H^1(\Omega)$ of the equation $R(\mathbf{p}) = \mathbf{p}$ which means that problem (1) admits a solution $(\mathbf{e}, \mathbf{p}) \in H_0^1(\Omega) \times H^1(\Omega)$. We proved the following existence and uniqueness result

Theorem 3.11 *Let $\omega > 0$ be fixed and let $\mathbf{f} \in L^2(\Omega)$. Then under hypothesis (3) there exists a solution $(\mathbf{e}, \mathbf{p}) \in H_0^1(\Omega) \times H^1(\Omega)$ to problem (1). Moreover the solution belongs to $H^2(\Omega) \times H^2(\Omega)$, satisfies estimates given in Theorem 2.1 and is unique for $\omega > \omega_*$.*

4 Concluding remarks

The results proved for the full model are not sufficient to discuss the low frequency behavior for such solutions. Nevertheless, it is easy, using the classical energy estimates, to see that any solution $(\mathbf{E}, \mathbf{P})$ of the full model (2) with the boundary condition (8) with $\beta = 0$ satisfies the following bounds $|\mathbf{E}|_{\mathbb{L}^2}^2 + |\mathrm{curl}\mathbf{E}|_{\mathbb{L}^2}^2 \leq C$ and $|\mathbf{P}|_{\mathbb{L}^2}^2 + |\mathrm{curl}\mathbf{P}|_{\mathbb{L}^2}^2 \leq C$ where $C > 0$ is independent of $\omega > 0$. Using (2) we deduce that $|\mathrm{curl}^2\mathbf{E}|_{\mathbb{L}^2}^2 \leq C\omega$ and $|\mathrm{curl}^2\mathbf{P}|_{\mathbb{L}^2}^2 \leq C$.

Let ω_n be a sequence of $\mathbb{R}^+$ such that $\omega_n \to 0$. We set $\mathbf{E}_n = \mathbf{E}(\omega_n)$ and $\mathbf{P}_n = \mathbf{P}(\omega_n)$ the solutions associated with ω_n. Hence, there exists a subsequences still denoted $(\mathbf{E}_n, \mathbf{P}_n)$ such that we have

$$\begin{cases} \mathbf{E}_n \rightharpoonup \mathbf{E}^\infty, \ \mathbf{P}_n \rightharpoonup \mathbf{P}^\infty \text{ weakly in } \mathcal{H}(\mathrm{curl}, \Omega) \\ \mathrm{curl}^2\mathbf{E}_n \to 0, \ \mathrm{curl}^2\mathbf{P}_n \rightharpoonup \mathrm{curl}^2\mathbf{P}^\infty \text{ weakly in } \mathbb{L}^2(\Omega), \ \mathrm{curl}^2\mathbf{E}^\infty = 0. \end{cases} \tag{70}$$

We set $\mathbf{U}_n = \mathrm{curl}\mathbf{P}_n$. We deduce that $\mathrm{div}\mathbf{U}_n = 0$, $\mathrm{curl}\mathbf{U}_n$ and $\mathbf{U}_n$ are uniformly bounded in $\mathbb{L}^2(\Omega)$ and $\mathbf{U}_n \times \mathbf{n} = 0$ on $\partial\Omega$. It follows, see [3], that $\mathbf{U}_n$ is uniformly bounded in $\mathbb{H}^1(\Omega)$ and consequently $\mathrm{curl}\mathbf{P}_n \to \mathrm{curl}\mathbf{P}^\infty$ strongly in $\mathbb{L}^2(\Omega)$.

First discuss the low fequency behavior of the solutions to Maxwell equations which are equivalent to the equation satisfied by $\mathbf{E}_n$. We have

$$\begin{cases} i\omega_n\mathbf{H}_n = \mathrm{curl}\mathbf{E}_n \\ i\omega_n\mathbf{E}_n + \mathrm{curl}\mathbf{H}_n + \sigma\mathbf{E}_n = -i\omega_n\mathbf{P}_n + \mathbf{F} \\ \mathbf{E}_n \times \mathbf{n} = 0 \textbf{ on } \partial\Omega. \end{cases} \tag{71}$$

As usually, we multiply the first equation by $\mathbf{H}_n^*$ and the second one by $\mathbf{E}_n^*$, integrating by parts we get the energy estimate $i\omega_n(|\mathbf{H}_n|^2 + |\mathbf{E}_n|^2) + \sigma|\mathbf{E}_n|^2 + 2i\Im(\mathrm{curl}\mathbf{H}_n; \mathbf{E}_n^*) = -i\omega_n(\mathbf{P}_n; \mathbf{E}_n^*) + (\mathbf{F}; \mathbf{E}_n^*)$. Taking the real part we get the estimate

$$\sigma|\mathbf{E}_n|_{\mathbb{L}^2} \leq \omega_n|\mathbf{P}_n|_{\mathbb{L}^2} + |\mathbf{F}|_{\mathbb{L}^2} \leq C \tag{72}$$

hence, the second equation of (81) gives $|\mathrm{curl}\mathbf{H}_n|_{\mathbb{L}^2} \leq (\sigma + \omega_n)|\mathbf{E}_n|_{\mathbb{L}^2} + C \leq C_1$ where $C_1 > 0$ is a constant which is independent of n. Taking the imaginary part of the energy estimate we get $\omega_n|\mathbf{H}_n|_{\mathbb{L}^2}^2 \leq CC_1 + C_1^2$. It follows that $\mathbf{E}_n \rightharpoonup \mathbf{E}^\infty$, $\mathbf{H}_n \rightharpoonup \mathbf{H}^\infty$ weakly and $\mathrm{curl}\mathbf{E}_n \to 0$ strongly in $\mathbb{L}^2(\Omega)$. We find

$$\mathrm{curl}\mathbf{E}^\infty = 0, \ \mathbf{E}^\infty \times \mathbf{n} = 0, \ \mathrm{curl}\mathbf{H}^\infty + \sigma\mathbf{E}^\infty = \mathbf{F}. \tag{73}$$

Thus, the electric field $\mathbf{E}^\infty$ satisfies the problem

$$\text{curl}\mathbf{E}^\infty = 0,\ \sigma\text{div}\mathbf{E}^\infty = \text{div}\mathbf{F},\ \mathbf{E}^\infty \times \mathbf{n} = 0 \tag{74}$$

which shows if one assumes div$\mathbf{F} = 0$ that $\mathbf{E}^\infty = 0$. Indeed, under this hypothesis on the souce terme we deduce that div$\mathbf{E}_n$ belongs to a compact set of $H^{-1}(\Omega)$ and div$\mathbf{E}_n \to 0$ strongly in $H^{-1}(\Omega)$.

The polarization field $\mathbf{P}_n$ satisfies the equation

$$\text{curl}^2\mathbf{P}_n + b\mathbf{P}_n\phi'(|\mathbf{P}_n|^2) = b\mathbf{E}_n - z_2(\omega_n)\mathbf{P}_n := \mathbf{F}_n \tag{75}$$

Let $(\nu_x^\infty(d\lambda))$ be the parametrized family of Young measures associated with the vector valued sequence $\mathbf{P}_n$. Passing to the limit in this equation we get, thanks the hypothesis div$\mathbf{F} = 0$, the limit equation

$$\text{curl}^2\mathbf{P}^\infty + b\left\langle \nu_x^\infty(d\lambda); \lambda\phi'(|\lambda|^2)\right\rangle = 0,\ \text{curl}\mathbf{P}^\infty \times \mathbf{n} = 0 \text{ on } \partial\Omega \tag{76}$$

Since we have div$(b\mathbf{E}_n - z_2(\omega_n)\mathbf{P}_n)$ converges strongly to 0 in $H^{-1}(\Omega)$ then it follows that div$(\mathbf{P}_n\phi'(|\mathbf{P}_n|^2))$ converges strongly to 0 in $H^{-1}(\Omega)$ and since curl$\mathbf{P}_n$ is uniformly bounded in $\mathbb{L}^2(\Omega)$ then by the classical compensated compactness result we get

$$\left\langle \nu_x^\infty(d\lambda), |\lambda|^2\phi'(|\lambda|^2)\right\rangle = \mathbf{P}^\infty \cdot \left\langle \nu_x^\infty(d\lambda), \lambda\phi'(|\lambda|^2)\right\rangle. \tag{77}$$

The main problem remaining to solve is the characterization of the measure Young $\nu_x(d\lambda)$.

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