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Adaptation of regular grid filterings to irregular grids

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Abstract

This note is about filtering and subsampling of signals sampled on irregular sampling points. The number of filters and wavelets available in the litterature on regular grids is far higher than on irregular grids, because the filtering scheme is uniform and can be efficiently examined by means of a Fourier transform or a z-transform. What is described here is how to adapt a filtering scheme designed for regular grids to irregular grids. The filtering and subsampling process is decomposed into lifting steps, and then each of the steps are adapted to the irregular grid structure.

1 Introduction

The lifting scheme is regarded as a way to devise filtering and subsampling schemes adapted to irregular grids. Using the lifting scheme, it is relatively easy to build interpolating subdivisions: a Lagrange interpolation (as used in the construction of Deslauriers-Dubuc wavelets) can easily be implemented on an irregular grid in 1D.

It then possible to slightly modify such schemes so that the resulting wavelet basis is well conditioned in L_2 (which is not the case of Deslauriers-Dubuc wavelets whose dual wavelets are Radon measures, and do live in $L_2(\mathbb{R})$). In a recent paper "Commutation for irregular subdivisions", Daubechies, Guskov and Sweldens provide a technique to enrich the set of wavelets available on irregular grids. They extend Lemarié's commutation idea to irregular grids. With commutation, it is possible to derive a new wavelet system from an older one by derivating the wavelets and integrating the dual wavelets, thus transferring some of the wavelet smoothness to the dual wavelets. In this way, it is possible to transform interpolating schemes into other schemes that are better conditioned in $L_2(\mathbb{R})$. Daubechies et al. support their description with extensive theoretical analysis and caracterize under which conditions the new wavelet system obtained with commutation behaves well. However, it is not possible, using commutation ideas and starting with interpolating subdivisions, to reach the full spectrum of all possible wavelet transforms as available on regular grids. Especially, it is not possible to construct a lifting based transform that coincides with a 7-9 biorthogonal wavelet transform on regular grids.

In this technical note, we introduce another approach devised to adapt various existing wavelet systems for regular grids to irregular grids, and to preserve their number of vanishing moments. This is done by modifying each lifting step in order to preserve polynomial reproduction properties in the irregular grid.

The note is thus organized as follows: a short presentation of the lifting scheme, as well as a main result by Daubechies and Sweldens is presented in Sec. 2. The limits of the spectrum covered by commutation are described in Sec. 3. Our adaptation method of regular grid wavelets to irregular grids is described in Sec. 4. Sec. 5 concludes the technical note.

2 Decomposing a wavelet transform into lifting steps

A lifting transform is a sequence of operations applied to a sequence of samples (s_k), to compute two sequences of low-pass and high-pass coefficients (l_k) and (h_k).

The first step, called splitting step, consists in splitting the (s_k) sequence into even and odd sample sequences:

$$\begin{aligned} e_k^{(0)} &= s_{2k} \\ o_k^{(0)} &= s_{2k+1} \end{aligned}$$

Then follow a variable number of predict and update steps, each computing from sequences $(e_k^{(n)})_{k\in\mathbb{Z}}$ and $(o_k^{(n)})$ two new sequences $(e_k^{(n+1)})_{k\in\mathbb{Z}}$ and $(o_k^{(n+1)})$. In a predict step, the even sample sequence is unchanged, and the odd sample sequence is computed by addition of a function of the even sample sequence, i.e.:

$$\begin{aligned} e_k^{(n+1)} &= e_k^{(n)} \\ o_k^{(n+1)} &= o_k^{(n)} + F_k(\{e_{k'}^{(n)}\}_{k'\in\mathbb{Z}}) \end{aligned}$$

while an update step has the reverse structure:

$$\begin{aligned} e_k^{(n+1)} &= e_k^{(n)} + G_k(\{o_{k'}^{(n)}\}_{k'\in\mathbb{Z}}) \\ o_k^{(n+1)} &= o_k^{(n)} \end{aligned}$$

Traditionally, the operators F_k and G_k perform a filtering on the input coefficients, so that for example:

$$F_k(\{e_{k'}^{(n)}\}_{k'\in\mathbb{Z}}) = \sum_{k'\in\mathbb{Z}} F[k-k']e_{k'}^{(n)}$$

and

$$G_k(\{o_{k'}^{(n)}\}_{k'\in\mathbb{Z}}) = \sum_{k'\in\mathbb{Z}} G[k-k']o_{k'}^{(n)}.$$

Finally, a last rescaling step is performed to compute the output coefficients:

$$\begin{aligned} l_k &= \zeta e_k^{(N)} \\ h_k &= 1/\zeta\, o_k^{(N)} \end{aligned}$$

for some nonzero ζ.

Obviously, two successive update steps can be merged into a single one, and two successive predict steps can also be merged into a single one. For this reason, we can always assume that in the sequence of predict and update steps, an update step is always followed by a predict step, and conversely a predict step is always followed by an update step.

A lifting transform is trivially invertible, because each step composing it is separately invertible. This even holds if the functions F_k and G_k are nonlinear, and lifting was for this reason considered as a way to build nonlinear wavelet transform schemes.

In their paper of 1996, "Factoring wavelet transform into lifting steps", Daubechies and Sweldens provide an extensive description on how to decompose any filtering and subsampling step into a sequence of lifting steps.

3 Existing wavelets on irregular grids

In this section, we introduce some notations on wavelets and filters used in filter design, and then indicate using these notations what kinds of filters can be implemented using commutation on irregular grids.

3.1 Notations for filter design

Constructing a wavelet basis on a regular grid in $\mathbb{R}$ consists in choosing a low-pass filter m_0 and a high-pass filter m_1, which define a scaling function ϕ and a wavelet ψ with the following formulas:

$$\hat{\phi}(\omega) = \prod_{j=1}^{+\infty} m_0(2^{-j}\omega) \quad (1)$$

$$\hat{\psi}(\omega) = m_1(\omega/2)\hat{\phi}(\omega/2) \quad (2)$$

In this the coefficients of the filter m_0 are written $m_0[k]$ for $k \in \mathbb{Z}$, and $m_0(\omega)$ denotes its Fourier transform:

$$m_0(\omega) = \sum_{k\in\mathbb{Z}} m_0[k]e^{-i\omega}$$

and similarly for m_1. The hat denotes the Fourier transform, so $\hat{f}$ is the Fourier transform of f.

A necessary condition for the resulting wavelet family to be a basis of L_2 is that the pair of filters m_0, m_1 form a perfect reconstruction pair of filters. This requires that the transfer matrix

$$\begin{bmatrix} m_0(\omega) & m_1(\omega) \\ m_0(\omega+\pi) & m_1(\omega+\pi) \end{bmatrix}$$

is invertible for all ω. We can then define dual filters $\tilde{m}_0$ and $\tilde{m}_1$ such that

$$\begin{bmatrix} \tilde{m}_0(\omega) & \tilde{m}_1(\omega) \\ \tilde{m}_0(\omega+\pi) & \tilde{m}_1(\omega+\pi) \end{bmatrix}^T \times \begin{bmatrix} m_0(\omega) & m_1(\omega) \\ m_0(\omega+\pi) & m_1(\omega+\pi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\tilde{m}_0$ and $\tilde{m}_1$ define dual wavelets with formulas similar to (1)–(2).

As a consequence of the above formula, we have:

$$\overline{\tilde{m}_0(\omega)}m_0(\omega) + \overline{\tilde{m}_1(\omega)}m_1(\omega) = 1 \qquad \text{for all } \omega.$$

The filter M_0 defined by $M_0(\omega) = \overline{\tilde{m}_0(\omega)}m_0(\omega)$ is an interpolating filter. The design of a compactly supported biorthogonal wavelet system then proceeds as follows:

1. An interpolating filter M_0 is chosen (for example a Deslauriers-Dubuc).
2. M_0 is a trigonometric polynomial. It is factored as a product $\overline{\tilde{m}_0(\omega)}m_0(\omega)$, to obtain expressions of m_0 and $\tilde{m}_0$.
3. From this, the inversion formula leaves little choice for m_1 and $\tilde{m}_1$, and usually m_1 is defined by

   $$m_1(\omega) = e^{-i\omega}\tilde{m}_0(\omega+\pi)$$

   and similarly for $\tilde{m}_1$.

Owing to this, the two mains choices done in a wavelet design consists in choosing the interpolating filter M_0 that is factored, and then choosing how the filter is factored. Usually the filter M_0 is factored into a product of irreducible terms, putting aside the irreducible terms vanishing at π:

$$M_0(\omega) = \left(\frac{e^{i\omega}+1}{2}\right)^d R(e^{i\omega})$$

where $R(e^{i\omega})$ is a trigonometric polynomial that does not vanish on π. The factoring of M_0 into a product $\overline{\tilde{m}_0(\omega)} m_0(\omega)$ consists in choosing $d' \in [0, d]$, factoring $R(e^{i\omega})$ into $R_1(e^{i\omega})R_2(e^{i\omega})$, and in writing:

$$\begin{aligned} m_0(\omega) &= \left(\frac{e^{i\omega}+1}{2}\right)^{d'} R_1(e^{i\omega}) \\ \tilde{m}_0(\omega) &= \overline{\left(\frac{e^{i\omega}+1}{2}\right)^{d-d'} R_2(e^{i\omega})} \end{aligned}$$

The design of 5-3 filters starts with a Deslauriers-Dubuc filter M_0 of order 4:

$$\begin{aligned} M_0(\omega) &= \frac{-e^{-3i\omega} + 9e^{-i\omega} + 16 + 9e^{i\omega} - e^{3i\omega}}{32} \\ &= \left(\frac{e^{i\omega}+1}{2}\right)^2 \left(\frac{e^{-i\omega}+1}{2}\right)^2 \left(\frac{-e^{-i\omega}+4-e^{i\omega}}{2}\right) \end{aligned}$$

The choice for m_0 and $\tilde{m}_0$ is then:

$$\begin{aligned} m_0(\omega) &= \frac{e^{-i\omega}+2+e^{i\omega}}{4} && \text{(reconstruction filter)} \\ \tilde{m}_0(\omega) &= \frac{(e^{-i\omega}+2+e^{i\omega})(-e^{-i\omega}+4-e^{i\omega})}{8} && \text{(analysis filter)} \end{aligned}$$

Hence in the distribution of the irreducible factors of $M_0(\omega)$ between m_0 and $\tilde{m}_0$, only $\tilde{m}_0$ actually receives factors of the polynomial R: R_1 is a constant polynomial.

The design of the 9-7 wavelets starts with the Deslauriers-Dubuc filter M_0 of order 8:

$$M_0(\omega) = \left(\frac{e^{i\omega}+1}{2}\right)^4 \left(\frac{e^{-i\omega}+1}{2}\right)^4 R(e^{i\omega})$$

where

$$R(\omega) = \left(\frac{-5e^{-3i\omega} + 40e^{-2i\omega} - 131e^{-i\omega} + 208 - 131e^{i\omega} + 40e^{2i\omega} - 5e^{3i\omega}}{8}\right)$$

The polynomial $R(\omega)$ has 6 zeros: $\{r_1, 1/r_1, r_2, \overline{r_2}, 1/r_2, 1/\overline{r_2}\}$, where

$$\begin{aligned} r_1 &\simeq 3.04066 \\ r_2 &\simeq 2.03114 + 1.73895i \end{aligned}$$

In this case, the polynomial R_1 inherits the zeros r_1 and $1/r_1$, while R_2 inherits all 4 other zeros.

Note that interpolating filters are obtained with the trivial factorization:

$$\begin{aligned} m_0(\omega) &= M_0(\omega) \\ \tilde{m}_0(\omega) &= 1 \end{aligned}$$

3.2 Filters covered by commutation

Commutation on a regular grid amounts to replacing the low-pass reconstruction and analysis filters m_0 and $\tilde{m}_0$ with

$$\mu_0(\omega) = \frac{m_0(\omega)}{\left(\frac{e^{i\omega}+1}{2}\right)}$$
$$\tilde{\mu}_0(\omega) = \tilde{m}_0(\omega) \times \left(\frac{e^{-i\omega}+1}{2}\right)$$

i.e. transferring in the factorization of M_0 into the product of m_0 and $\tilde{m}_0$ one term $(e^{i\omega}+1)/2$ from m_0 to $\tilde{m}_0$.

When used on B-spline subdivision schemes, commutation simply transforms a B-spline subdivision scheme into a B-spline subdivision scheme of a different order, so in this case commutation does provide new subdivisions.

When used on a Lagrange subdivision scheme, corresponding to Deslauriers-Dubuc interpolating wavelets, r steps of commutation transfer a factor $(e^{i\omega}+1)^r/2^r$ from m_0 to $\tilde{m}_0$. So if we start with any interpolating filter like a Deslauriers-Dubuc filter, of dual filter equal to 1:

$$m_0(\omega) = \left(\frac{e^{i\omega+1}}{2}\right)^d R(e^{i\omega})$$
$$\tilde{m}_0(\omega) = 1$$

the filters corresponding to some commuted subdivision scheme always has the form:

$$m_0(\omega) = \left(\frac{e^{i\omega+1}}{2}\right)^{d-r} R(e^{i\omega})$$
$$\tilde{m}_0(\omega) = \left(\frac{e^{-i\omega+1}}{2}\right)^{r}$$

up a multiplication with some power of $e^{i\omega}$. However, no zero of R is transferred from m_0 to $\tilde{m}_0$. The dual analysis scaling function is always a B-spline function. The 5-3 dual scaling function is a B-spline function. However the 9-7 dual scaling function is not, so commutation cannot be used to implement a 9-7 equivalent on an irregular grid.

4 Porting regular grid wavelets to irregular grids

In their paper “Factoring wavelet transforms into lifting steps”, Daubechies and Sweldens proved that any wavelet transform can be factored into a sequence of lifting steps.

4.1 Example of the biorthogonal wavelets

For example, defining the predict operator of parameter α as:

$$P_\alpha : \{e_k\}, \{o_k\} \mapsto \{e'_k\}, \{o'_k\}$$

by

$$\begin{aligned} e'_k &= e_k \\ o'_k &= o_k + \alpha(e_k + e_{k+1}) \end{aligned}$$

and the update operator of parameter β as:

$$U_\beta : \{e_k\}, \{o_k\} \mapsto \{e'_k\}, \{o'_k\}$$

where

$$\begin{aligned} e'_k &= e_k + \beta(o_{k-1} + o_k) \\ o'_k &= o_k \end{aligned}$$

and scaling operator of parameter ζ as:

$$S_\zeta : \{e_k\}, \{o_k\} \mapsto \{e'_k\}, \{o'_k\}$$

where

$$\begin{aligned} e'_k &= \zeta e_k \\ o'_k &= o_k/\zeta \end{aligned}$$

and denoting Split the splitting operator that separates the even and odd samples in a sample sequence, one filtering and subsampling of a 5-3 transform is implemented as

$$S_{\sqrt{2}} \circ U_{1/4} \circ P_{-1/2} \circ \mathrm{Split}$$

One filtering and subsampling of a 9-7 transform is implemented as

$$S_\zeta \circ U_\delta \circ P_\gamma \circ U_\beta \circ P_\alpha \circ \mathrm{Split}$$

where

$$\begin{aligned} \alpha &\simeq -1.5861 & \beta &\simeq -0.0530 \\ \gamma &\simeq 0.8829 & \delta &\simeq 0.4435 \\ \zeta &\simeq 1.1496 \end{aligned}$$

In the same way, one filtering and subsampling of a 13-11 transform is implemented with 3 update and 3 predict steps of the same kind.

4.2 Adapting the lifting steps to irregular sampling

The notations we use for irregular grid subdivision are the following: each sample value s_k is placed at location x_k, where the x_k sequence is increasing.

The 5-3, 9-7 and 13-11 wavelet transforms have respectively 2, 4 and 6 vanishing moments. This means that if the input samples are polynomial values of degree 1, 3, or 5, i.e. $s_k = Q(x_k)$ where Q is a polynomial of degree 1, 3 or 5, the output low-pass coefficients are polynomial values of the same degree, and the output high-pass coefficients are 0.

Also, the intermediate lifting coefficients $e_k^{(n)}$ and $o_k^{(n)}$ also take polynomial values.

The basic principle underlying the design of an irregular grid filtering and subsampling after any of the above regular grid filtering and subsampling schemes consists in writing that coefficients $e^{(n)}$ and $o^{(n)}$ through all lifting steps obtained with an input polynomial Q should correspond to the same polynomials as when Q is transformed with a regular grid lifting step: the irregular grid filtering and subsampling must reproduce polynomials in all intermediate computation steps in the same way as some reference regular grid filtering and subsampling where the sampling points x'_k in the reference regular grid have a grid step $x'_{k+1} - x'_k$ that depends on the irregular grid structure (like the minimum grid step, or some average grid step of $(x_k)_{k\in\mathbb{Z}}$).

For this, each predict or update step must be replaced with a modified predict or update step. When we want the irregular grid transform to inherit $2p$ vanishing moments from the original regular grid transform, we must modify the predict operator P_α and replace it with:

$$P_\alpha : \{e_k\}, \{o_k\} \mapsto \{e'_k\}, \{o'_k\}$$

with

$$o'_k = o_k + \sum_{l=-p+1}^{p} \alpha_{k,l} e_{k-l}$$

That for each k, the $2p$ coefficients $(\alpha_{k,l})_{l=-p+1,\dots,p}$ are uniquely defined with the polynomial reproduction constraints is a direct consequence of the following proposition: if

$$\mathbb{P}_d(x) = (1, x, x^2, \dots, x^d)$$

then $\mathbb{P}_{2p-1}(x_{2k+1})$ can be expanded in a unique way in the basis of $2p$ vectors:

$$(\mathbb{P}_{2p-1}(x_{2k-2p+2}), \mathbb{P}_{2p-1}(x_{2k-2p+4}), \dots, \mathbb{P}_{2p-1}(x_{2k+2p}))$$

The system matrix solved to find the expansion is an invertible Van der Monde matrix.

4.3 Example: keeping 2 vanishing moments

When adapting a lifting transform corresponding to the 5-3, 9-7 or 13-11 wavelet transform to an irregular grid, the resulting formulas are particularly simple. The transform is again written as a sequence of the same number of lifting steps P_α and U_β with various parameters α and β, where the irregular grid implementation of the lifting steps P_α and U_α are respectively

$$P_\alpha : \{e_k\}, \{o_k\} \mapsto \{e'_k\}, \{o'_k\}$$

where

$$\begin{aligned} e'_k &= e_k \\ o'_k &= o_k + 2\alpha \left(\frac{(x_{2k+2} - x_{2k+1})e_k + (x_{2k+1} - x_{2k})e_{k+1}}{x_{2k+2} - x_{2k}} \right) \end{aligned}$$

and the update operator of parameter β as:

$$U_\beta : \{e_k\}, \{o_k\} \mapsto \{e'_k\}, \{o'_k\}$$

where

$$\begin{aligned} e'_k &= e_k + 2\beta \left(\frac{(x_{2k+1} - x_{2k})o_{k-1} + (x_{2k} - x_{2k-1})o_k}{x_{2k+1} - x_{2k-1}} \right) \\ o'_k &= o_k \end{aligned}$$

This can be interpreted very simply: in a regular grid predict step, an odd rank coefficient o_k is modified by addition or subtraction of some multiple of the average $(e_k + e_{k+1})/2$ of the neighboring even rank coefficients. In an irregular grid predict step, the same modification is done, but the average is replaced with an average use weighting factors that depend on the respective locations of the sampling points corresponding to o_k, e_k and e_{k+1}. The same holds for the update step.

5 Conclusion

In this technical note, we showed how to implement on an irregular grid a symmetric wavelet transform designed on a regular grid in order to keep an even number of its vanishing moments. Since the modified transform is implemented with a lifting scheme, it is clear that the modified transform is perfectly invertible. Preliminary numerical experimentations show that as long as the irregular grid steps $x_{k+1} - x_k$ do not vary too fast, i.e. if $(x_{k+1} - x_k)/(x_k - x_{k-1})$ is bounded, then the irregular grid transform does not become ill-conditioned.

References

[1] I. Daubechies, I. Guskov, P. Schröder, and W. Sweldens. Wavelets on irregular point sets. *Phil. Trans. R. Soc. Lond. A*, 357(1760):2397–2413, 1999.

[2] I. Daubechies, I. Guskov, and W. Sweldens. Commutation for irregular subdivision. *Constr. Approx.*, 15(3):381–426, 2001.

[3] I. Daubechies and W. Sweldens. Factoring wavelet transforms into lifting steps. *J. Fourier Anal. Appl.*, 4(3):245–267, 1998.

[4] S.G. Mallat. A theory for multiresolution signal decomposition. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 11(7):674–693, 1989.

[5] S.G. Mallat. *A Wavelet Tour of Signal Processing*. Academic Press, 1997.