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Dependence structure of multivariate Lévy processes with applications in risk management.

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Abstract

We describe a general method to model dependence between subordinators (positive Lévy processes) by introducing S-copulas. These objects have the same properties as copulas but are defined on a different domain. They can be used to separate dependence from the behavior of the components of a multivariate positive Lévy process. The method is then extended to model dependence of general Lévy processes. We construct parametric families of S-copulas and develop an algorithm for simulating multivariate Lévy processes using their S-copulas. Finally, we illustrate our method by showing how it can be used to build multivariate models with jumps for finance and insurance.

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1 Introduction

Almost all existing jump-diffusion models based on Lévy processes [1, 2, 3, 4, 5] are one-dimensional[1]. However, the domain of application of such models in finance is limited to pricing options on a single asset. All other applications including risk management, portfolio optimization, pricing of basket options and insurance loss modelling require a multivariate model.

In this paper we construct such models by first specifying any desirable one-dimensional process for each component and then defining the dependence structure to obtain a multivariate Lévy process. The multidimensional Lévy measure is thus split onto marginal one-dimensional Lévy measures and the copula function which parametrizes dependence. To do this, we generalize the notion of probabilistic copula to the case of σ-finite measures. It turns out that copulas cannot be directly extended to Lévy measures of general form because the latter have a non-integrable singularity at 0. Nevertheless, it is possible to define copulas for Lévy processes with jumps of constant sign (e.g. only positive jumps).

We call this new function S-copula (the letter S stands for subordinator, i.e., a positive Lévy process). It is an function $F : [0,\infty]^n \to [0,\infty]$ with the following properties:

1. F is grounded (that is, equal to zero if at least one of its arguments is equal to zero)
2. F is n-increasing, that is, dF is a positive measure on $[0,\infty]^n$
3. F has margins $F_k(x_k) = F(\infty,\cdots,x_k,\cdots,\infty)$, $k = 1,\cdots,n$ which satisfy $F_k(u) = u$ for all u in $[0,\infty]$.

S-copula allows to separate the dependence structure from the individual behavior of the components of a multivariate Lévy process with positive jumps. All kinds of dependence may be represented by an S-copula and for all S-copulas and all types of individual behavior a multivariate Lévy process may be constructed.

In insurance applications multivariate Lévy processes with positive jumps can be used directly. To treat processes of general form we observe that every Lévy process can be represented as sum of several independent Lévy

[1] [1] and [5] suggest to construct multivariate models by time changing a multivariate Brownian motion with a one-dimensional subordinator. Models of this type are not true multivariate Lévy models because they are conditionally Gaussian. In particular, they do not allow for complete independence between assets, because even if the components of the multivariate Brownian motion are uncorrelated, the absolute values of jumps in different assets are correlated.

processes with jumps of constant sign. Therefore, every multivariate Lévy process can be constructed by adding and subtracting several Lévy processes with positive jumps.

Our modelling approach is thus to specify separately the behaviour of components of a subordinator and its dependence structure. This method has the following advantages:

- One usually has less information about the dependence structure of several processes than about each one of them separately. Components may be described by complex parametric or non-parametric models whereas there may not be enough information to estimate many dependence parameters. Our approach allows to use any complicated model for individual behavior of each of the components, and at the same time pick a dependence S-copula from one of the simple parametric families that we provide.

- Using this approach, one can couple components of very different structure (e.g. infinite activity components and compound Poisson ones).

- The notion of S-copula and some of its properties allow us to develop an efficient method for simulating trajectories of a multivariate Lévy process using its S-copula.

Throughout the paper we will sometimes use the words "multidimensional subordinator" or "multidimensional positive Lévy process" to designate a Lévy process, every component of which is a subordinator, i.e. a positive Lévy process.

The paper is structured as follows. In section 2 we discuss multidimensional Lévy processes, show how various dependence concepts are expressed in terms of the Lévy measure and discuss methods to model dependence between components of a multivariate Lévy process. In section 3 we define S-copulas and state the main theorem about representation of multivariate Lévy processes. Section 4 discusses general methods for constructing S-copulas and suggests several parametric families. In section 5 we develop simulation methods and give examples of their use. Section 6 discusses possible applications of our methodology in finance and insurance.

2 Dependence concepts of Lévy processes

Levy processes are processes with independent stationary increments, continuous in probability. From independence and stationnarity of increments it follows that the law of the process $(X_t, t \geq 0)$ is completely determined by the distribution of X_t for any fixed t. The latter distribution can be parametrized by the characteristic triplet of the process through the Lévy-Khintchine formula: $E[e^{i\langle z, X_t\rangle}] = e^{t\phi(z)},\ z \in \mathbb{R}^d$ with

$$\phi(z) = -\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{|x|\leq 1})\nu(dx) \quad (2.1)$$

where A is a positive definite square matrix of size d, γ is a d-dimensional vector and ν is a positive measure on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$ (Lévy measure). The triplet (A, γ, ν) is called the characteristic triplet of the Lévy process. For more details on Lévy processes see [6].

In the sequel we restrict our attention to Lévy processes with $A = 0$ (processes with no Gaussian component) and we will also often omit the deterministic drift term γt.

Since the law of a Lévy process is completely determined by its distribution at time t, the dependence structure of components X_t and Y_t of a Lévy process could in principle be parametrized by the probabilistic copula C_t of X_t and Y_t at one fixed time t (see appendix A for details on copulas). However, this parametrization is not the most convenient one due to the following reasons.

- We will see later in this section (example 2.1) that this copula C_t is not constant in general, that is, $C_t \neq C_s$ if $t \neq s$.

- It is unclear which copulas correspond to multivariate infinitely divisible distributions. More precisely, if X_t and Y_t are distributions at time t of two Lévy processes, it is unclear, for which copulas C, the two-dimensional distribution with marginals X_t and Y_t and dependence structure given by C is infinitely divisible.

- Finally, all existing one-dimensional Lévy models are specified in terms of their Lévy measures and it would therefore be inconvenient to specify the dependence in terms of density at time t.

Thus, the solution is to parametrize dependence using the multidimensional Lévy measure. In the rest of this section we show how different dependence concepts can be expressed in terms of Lévy measure and in the next section we model dependence of Lévy processes by extending the notion of copula to Lévy measures.

The following proposition gives a method to compute marginal Lévy measures. The procedure turns out to be exactly the same as in the case of probability measures.

Proposition 2.1 (Margins of a Lévy measure). *Let (X_t, Y_t) be a Lévy process with Lévy measure ν. Then X_t has Lévy measure ν_x defined by*

$$\nu_x(A) = \nu(A \times (-\infty, \infty)),\ \forall A \in \mathcal{B}(\mathbb{R})$$

Proof. Using Lévy-Khintchine formula (2.1) for the two-dimensional process (X_t, Y_t) and Fubini's theorem, we find

$$\begin{aligned} E[e^{iuX_t}] &= \exp t \left\{ i\gamma_1 u + \int_{\mathbb{R}^2} (e^{iux} - 1 - iux 1_{x^2+y^2 \leq 1}) \nu(dx, dy) \right\} \\ &= \exp t \left\{ i\tilde{\gamma}_1 u + \int_{\mathbb{R}} (e^{iux} - 1 - iux 1_{x^2 \leq 1}) \nu(dx \times (-\infty, \infty)) \right\} \end{aligned}$$

This change of drift is possible because

$$\left| \int_{\mathbb{R}^2} x(1_{x^2+y^2 \leq 1} - 1_{x^2 \leq 1}) \nu(dx, dy) \right| \leq \left| \int_{\substack{x^2+y^2>0 \\ |x| \leq 1}} \nu(dx, dy) \right| < \infty$$

$\square$

This proposition can be easily generalized to more than two dimensions: to obtain m-dimensional margin of an n-dimensional Lévy measure, one has to integrate the latter with respect to $n - m$ variables which do not appear in the marginal measure.

The next proposition gives a useful criterion of independence of components of a Lévy process. For short, components of a Lévy process are independent if and only if they never jump together.

Proposition 2.2. *Let (X_t, Y_t) be a Lévy process without Gaussian component. Its components (X_t) and (Y_t) are independent if and only if the support of its Lévy measure is contained in the set $\{(x, y) : xy = 0\}$. Moreover, in this case the Lévy measure of (X_t, Y_t) has the following form:*

$$\nu(A) = \nu_x(A_x) 1_{0 \in A_y} + \nu_y(A_y) 1_{0 \in A_x} \tag{2.2}$$

where ν_x and ν_y are, respectively, Lévy measures of (X_t) and (Y_t), $A_x = \{x : (x, 0) \in A\}$ and $A_y = \{y : (0, y) \in A\}$.

Proof. Straightforward, by applying the Lévy-Khintchine formula to the process (X_t, Y_t). $\square$

This proposition also applies to the case of more than two dimensions.

Finally, we consider dependence structures of stable processes. Stable processes are Lévy processes that have the selfsimilarity property:

$$X_{at} \stackrel{\mathcal{L}}{=} a^{1/\alpha} X_t, \quad \forall a > 0 \tag{2.3}$$

where $\alpha \in (0, 2]$ is called the index of stability. The only stable process with $\alpha = 2$ is the Brownian motion, hence we only need to consider $\alpha \in (0, 2)$. In this case the Lévy measure of a stable process has the following form (see [6]):

$$\nu_\alpha(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{dr}{r^{1+\alpha}} \tag{2.4}$$

where $S = \{\mathbf{x} : |\mathbf{x}| = 1\}$ is the unit sphere. λ is a finite positive measure on S (called spherical part of the Lévy measure) which parametrizes dependence of the components of a stable process. It follows directly from the selfsimilarity property (2.3), that the probabilistic copula C_t of components a stable process at time t does not depend on t.

An example of stable process is given by the Cauchy process in 2 dimensions for which $\alpha = 1$ and $\lambda = \lambda_0$ is a uniform measure on the unit circle.

$$\nu^Z(B) = \int_S \lambda_0(d\xi) \int_0^\infty 1_B(r\xi) \frac{dr}{r^2}$$

In Cartesian coordinates this Lévy measure takes the form

$$\nu^Z(B) = \int_B \frac{dxdy}{(x^2 + y^2)^{\frac{3}{2}}}$$

Since this measure is not concentrated on the set $\{(x, y) : xy = 0\}$, we can conclude by proposition 2.2 that the components of the Cauchy process are dependent. This fact allows us to construct an example showing that the probabilistic copula of components of a Lévy process is in general time–dependent.

Example 2.1. Let (Z_t) be a 2-dimensional Cauchy process. Its probabilistic copula C^Z does not depend on time and is different from the independence copula $C^\perp(u, v) \equiv uv$. Let W_t be a standard planar Brownian motion. Since its components are independent, its probabilistic copula at all times is the independence copula $C^\perp(u, v)$. Consider the Lévy process $X_t = Z_t + W_t, t \geq 0$. Its characteristic triplet is $(Id_2, 0, \nu^Z)$. Let $X_t^{(1)} = \frac{X_t}{t}$ and $X_t^{(2)} = \frac{X_t}{\sqrt{t}}$. For each t, the Lévy triplet of the infinitely divisible random variable $X_t^{(1)}$

is $(\frac{1}{t}Id_2, 0, \nu^Z)$ and the Lévy triplet of $X_t^{(2)}$ is $(Id_2, 0, \sqrt{t}\nu^Z)$. It follows (see [7, thm. 13.14]) that $X_t^{(1)}$ converges in law to the standard bivariate Cauchy variable as t tends to infinity and $X_t^{(2)}$ converges in law to the standard bivariate normal variable as t tends to zero. Since the limit laws are continuous and therefore their copulas are unique, this means that the probabilistic copula C_t of the process X_t has the following properties

$$\forall u, v, \quad C_t(u,v) \to C^Z(u,v) \text{ as } t \to \infty$$
$$\forall u, v, \quad C_t(u,v) \to C^{\perp}(u,v) \text{ as } t \to 0$$

Since $C^Z(u,v)$ is different from $C^{\perp}(u,v)$, we conclude that the probabilistic copula C_t of the Lévy process X_t is not constant over time.

3 Modelling dependence using S-copulas

To separate the dependence structure of a Lévy measure from the behavior of its marginals, that is, to prove a statement, analogous to Sklar's theorem (see theorem A.2), two conditions must be satisfied. First, we must be able to compute the margins, integrating the measure with respect to variables that do not appear in the marginal measure (cf. proposition 2.1). Second, we must be able to represent the measure with an increasing grounded function F as $\nu(A) = V_F(A)$. This is not possible for general Lévy measure on $(-\infty, \infty)^d$ due to a non-integrable singularity at 0. More precisely, for a Lévy measure with positive and negative jumps of infinite activity, the function F given by

$$F = \int_{-\infty}^{x} \int_{-\infty}^{y} \nu(dx', dy')$$

is infinite for $x > 0$ and $y > 0$. This problem cannot be removed by multiplying the Lévy measure by some function to obtain a finite measure. Indeed, suppose that we have constructed a finite measure $\tilde{\nu}$ as follows:

$$\tilde{\nu}(A) = \int_A \phi(x,y)\nu(dx,dy)$$

One can for example take $\phi(x,y) = (x^2+y^2) \wedge 1$. Copulas cannot be applied to this new measure because for margins to have correct form, the following condition must be satisfied for all Lévy measures ν

$$\int_{-\infty}^{\infty} \phi(x,y)\nu(x,y)dy = \phi_1(x) \int_{-\infty}^{\infty} \nu(x,y)dy$$

Substituting $\nu(x,y) = \nu(x)\delta(y-y_0)$, which is a Lévy measure on $\mathbb{R}^2$ if $\nu(x)$ is a finite positive measure on $\mathbb{R}$, we see that $\phi(x,y_0) = \phi_1(x)$ for all y_0. This entails after applying the same reasoning to the first variable, that $\phi(x,y)$ is constant.

Therefore, copulas cannot be extended directly to Lévy measures on $(-\infty,+\infty)^d$ and one solution is to construct copulas for Lévy processes with jumps of constant sign. First, such processes have in independent interest, for example, in insurance. Second, every Lévy process may be represented as sum of several Lévy processes with jumps of constant sign. For example, a 2-dimensional process (X_t) with Lévy measure ν can be written as follows

$$X_t = X_t^{(11)} + X_t^{(12)} + X_t^{(21)} + X_t^{(22)}$$

where the processes $X^{(11)}$, $X^{(12)}$, $X^{(21)}$, $X^{(22)}$ are independent and their Lévy measures are given by

$$\begin{aligned}
\nu^{(11)}(dx,dy) &= \nu(dx,dy)1_{x>0,y>0} \qquad & \nu^{(12)}(dx,dy) &= \nu(dx,dy)1_{x>0,y<0} \\
\nu^{(21)}(dx,dy) &= \nu(dx,dy)1_{x<0,y>0} \qquad & \nu^{(22)}(dx,dy) &= \nu(dx,dy)1_{x<0,y<0}
\end{aligned}$$

Thus, to describe dependence stuctures of general Lévy processes it is sufficient to be able to model dependence of Lévy processes with positive jumps.

A further simplification can be achieved by supposing that the trajectories have finite variation (in finance this assumption seems reasonable, cf. Carr et al. [8]). Thus, in the sequel we will be working with processes that have the following Lévy-Khintchine representation:

$$E[e^{iuZ_t}] = \exp t\left\{\int_{\mathbb{R}^n_+}(e^{i\langle u,x\rangle}-1)\nu(dx)\right\}$$

with $\int_{\mathbb{R}^n_+}(|x|\wedge 1)\nu(dx) < \infty$, i.e. with processes, every component of which is a subordinator.

The key idea of our approach is to parametrize the dependence of Lévy processes in terms of the Lévy measure. For this we would like to extend the copula approach for probability distributions to Lévy measures. Cumulative distribution function will thus be replaced by the tail integral of the Lévy measure and margins of the probability distribution will be replaced by the margins of the Lévy measure. The tail integral of an n-dimensional Lévy process with positive jumps is defined in the following way:

$$U(x_1,\cdots,x_n) = \nu([x_1,\infty]\times\cdots\times[x_n,\infty])$$

$(-1)^n U$ is an n-increasing function (see appendix A for details on increasing functions). U has the following properties:

1. $\forall k \leq n, U(x_1, \cdots, x_{k-1}, \infty, x_{k+1}, \cdots, x_n) = 0$.

2. $\forall k \leq n, U(0, \cdots, 0, x_k, 0, \cdots, 0) = U_k(x_k)$ where U_k is the tail integral of the k-th component, or, in other words, the k-th margin.

Compound Poisson case If a Lévy process has infinite activity then $U(0, \cdots, 0) = \infty$. If it is of compound Poisson type (finite activity), we explicitly redefine the value of its tail integral at zero: $U(0, \cdots, 0) \stackrel{def}{=} \infty$, that is, we suppose that every compound Poisson process has an infinite atom at zero. Clearly, this definition does not alter the trajectory of the process in any way, but it allows us not to distinguish finite and infinite activity in the definition of an S-copula that follows.

Definition 3.1 (S-copula). A n-dimensional S-copula is a function F with domain $[0, \infty]^n$ such that

1. F is grounded and n-increasing.

2. F has margins F_k, $k = 1, \cdots, n$ which satisfy $F_k(u) = u$ for all u in $[0, \infty]$.

See appendix A for detailed definitions of increasing function, grounded function and margins of a function.

Remark 3.1. This new object is termed S-copula because it is used to specify dependence structure of subordinators. The only difference between this definition and the definition of a probabilistic copula (cf. definition A.3) is the domain of F. Probabilistic copulas are defined on $[0, 1]^n$ whereas S-copulas are definied on $\bar{\mathbb{R}}^n_+$. This difference appears because Lévy measures, unlike probability measures, can have infinite mass.

Before presenting various examples of S-copulas, it is worthwhile to state the main theorem which makes clear that S-copulas indeed allow to separate the dependence structure of a multidimensional Lévy process from the laws of its components.

Theorem 3.1 (Main theorem). *Let U be the tail integral of an n-dimensional Lévy process with positive components having tail integrals $U_1, \cdots, U_n$. Then there exists an n-dimensional S-copula F such that for all vectors $(x_1, \cdots, x_n)$ in $\mathbb{R}^n_+$,*

$$U(x_1, \cdots, x_n) = F(U_1(x_1), \cdots, U_n(x_n)). \tag{3.1}$$

If $U_1, \cdots, U_n$ are continuous (this is the case if the Lévy measures of the components have no atoms) then this S-copula is unique, otherwise it is unique on $\operatorname{Ran} U_1 \times \cdots \times \operatorname{Ran} U_n$ *(*$\operatorname{Ran} U$ *denotes the range of U).*

Conversely, if F is an n-dimensional S-copula and $U_1, \cdots, U_n$ are tail integrals of one-dimensional subordinators, then the function U defined above is the tail integral of an n-dimensional Lévy process with positive components having tail integrals $U_1, \cdots, U_n$.

To prove this theorem, we need to define the generalized inverse of a one-dimensional tail integral U. Let $t \in \operatorname{Ran} U$. In general, the inverse function $U^{-1}(t)$ is not unique but we can define a generalized inverse function with nice properties in the following way:

$$U^{(-1)}(t) \stackrel{def}{=} \sup\{x : U(x) = t\} \tag{3.2}$$

If $U(x)$ is continuous then the set in (3.2) is always non-empty and the generalized inverse is well defined for all x in $\bar{\mathbb{R}}^+$. It is always a strictly decreasing function with the following properties: $U(U^{(-1)}(t)) = t$ for all t in $\operatorname{Ran} U$, $U^{(-1)}(0) = \infty$ and $U^{(-1)}(\infty) = 0$.

Proof. To clarify the exposition, we give the proof for the simpler case when Lévy measures of the components have no atoms. The proof in the general case is available from the author on request.

To prove the existence part of the first statement of the theorem, we use generalized inverses $U_1^{(-1)}(x_1), \cdots, U_n^{(-1)}(x_n)$ of one-dimensional tail integrals and consider the object

$$\tilde{F}(x_1, \cdots, x_n) = U(U_1^{(-1)}(x_1), \cdots, U_n^{(-1)}(x_n))$$

defined on $[0, \infty]^n$ From the definition of the tail integral and property 1 in proposition A.1 it follows that $\tilde{F}$ is grounded, n-increasing and has required margins. We have thus constructed the required S-copula.

To prove the unicity suppose that there exist two functions with required properties, i.e. we have

$$F^1(U_1(x_1), \cdots, U_n(x_n)) = F^2(U_1(x_1), \cdots, U_n(x_n)), \forall x_1, \cdots, x_n$$

From the continuity of $U(x)$ it follows that for all $t_1, \cdots, t_n$ in $[0, \infty]^n$, there exist $x_1, \cdots, x_n$ in $\mathbb{R}^n_+$ such that $U_1(x_1) = t_1, , \cdots, U_n(x_n) = t_n$. This means that for all $t_1, \cdots, t_n$ we have $F^1(t_1, \cdots, t_n) = F^2(t_1, \cdots, t_n)$.

To show the converse statement consider the function U defined in (3.1). From the properties of S-copula it is clear that this expression defines a positive measure on $\mathbb{R}^n_+$ whose margins exist and have the required form. The only thing left to prove is that that this measure has the correct behavior in the neighborhood of zero (integrates $||\mathbf{x}||$). Denote by ν the positive measure on $\mathbb{R}_+^n$ that has tail integral U. Then there exists a constant C such that

$$\int_{[0,1]^n} ||\mathbf{x}||\nu(\mathbf{dx}) \leq C \int_{[0,1]^n} \sum_{i=1}^n x_i \nu(\mathbf{dx})$$
$$= C\sum_{i=1}^n \int_{[0,1]^n} x_i \nu(\mathbf{dx}) \leq C\sum_{i=1}^n \int_0^1 x_i \nu_i(dx_i) \quad (3.3)$$

Since the margins are Lévy measures with required property, this completes the proof. □

Remark 3.2 *(Lévy densities).* When the tail integral given by (3.1) is absolutely continuous, we can compute the corresponding Lévy density by differentiation

$$\nu(x_1, \cdots, x_n) = \frac{\partial^n F(y_1, \cdots, y_n)}{\partial y_1 \cdots \partial y_n}\Big|_{y_1=U_1(x_1),\cdots,y_n=U_n(x_n)} \nu_1(x_1)\cdots\nu_n(x_n) \quad (3.4)$$

where $\nu_1(x_1), \cdots, \nu_n(x_n)$ are marginal Lévy densities. Unlike the tail integral $U(x_1, \cdots, x_n)$ which is only singular at the origin, the Lévy density can be singular in any point. However it must be integrable everywhere except at the origin.

Now we give some examples that illustrate the notion of S-copula. More examples of parametric families that can be used in practice are given in section 4.

Example 3.1 *(independent processes).* It follows directly from proposition 2.2 that the components of a 2-dimensional positive Lévy process are independent if and only if its S-copula is

$$F_\perp(x_1, x_2) = x_1 1_{x_2=\infty} + x_2 1_{x_1=\infty} \quad (3.5)$$

Note the difference between this example and the probabilistic copula which corresponds to independent random variables $C_\perp(x_1, x_2) = x_1 x_2$. This probabilistic copula is not an S-copula because it does not have the correct margins.

Generalization of (3.5) to the case of n-dimensional process with independent components is straightforward:

$$F_\perp(x_1, \ldots, x_n) = x_1 1_{x_2=\infty,\ldots,x_n=\infty} + \ldots + x_n 1_{x_1=\infty,\ldots,x_{n-1}=\infty}$$

Example 3.2 *(complete dependence).* First consider a two-dimensional subordinator (X_t^1, X_t^2) with $X_t^1 = X_t^2$ for all t. In this case we have

$$\begin{aligned}
&\nu(x_1, x_2) = \nu(x_1)\delta(x_1 - x_2)\\
&U(x_1, x_2) = \int_{\max(x_1,x_2)}^{\infty} \nu(\xi)d\xi = \min(U(x_1), U(x_2))\\
&F_{\|}(x_1, x_2) = \min(x_1, x_2)
\end{aligned} \tag{3.6}$$

Here, unlike in previous example we find the exact analog of the classical copula of complete dependence (the upper Fréchet bound). Consider now two subordinators which have different marginal Lévy measures but the same S-copula of dependence (3.6). Then, conditionally on the fact that $\Delta X_t^1 = x$, if U_2 is strictly decreasing, the jump size ΔX_t^2 is non–random and we have $\Delta X_t^2 = U_2^{-1}(U_1(x))$. To see this observe that by theorem 5.1, the probability distribution of $U_2(\Delta X_2)$ conditionally on $\Delta X_t^1 = x$ is given by $\frac{\partial F_{\|}(x_1,x_2)}{\partial x_1} = 1_{x_2 \geq x_1}$ which means that $U_2(\Delta X_2) = U_1(x)$.

The generalization of 3.6 to arbitrary dimension is

$$F_{\|}(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)$$

Example 3.3 *(Compound Poisson processes).* One of the most important advantages of our approach is that it can be applied to infinite activity Lévy processes. It is more natural to apply it to infinite activity processes, because the dependence structure of a multivariate compound Poisson process can be described using only probabilistic copulas (see [9]). In spite of this, in this example we show how our method works for compound Poisson subordinators.

Let X_t and Y_t be two positive compound Poisson processes with jump intensities λ_x and λ_y and S-copula $F(x, y)$. This means that the tail integral of the two-dimensional process (X_t, Y_t) is $U(x, y) = F(U_x(x), U_y(y))$. Let λ be the intensity of this process. It is equal to the Lévy measure of $\mathbb{R}_+^2 \setminus \{0\}$ because in our notation all compound Poisson processes have an infinite atom at zero. We then have

$$\lambda = \nu(\mathbb{R}_+^2 \setminus \{0\}) = \lim_{x,y \to 0}(U_x(x) + U_y(y) - U(x, y)) = \lambda_x + \lambda_y - F(\lambda_x, \lambda_y)$$

Since for every S-copula $0 \leq F(x, y) \leq \min(x, y)$, we find that $\max(\lambda_x, \lambda_y) \leq \lambda \leq \lambda_x + \lambda_y$, that is, the jump intensity of the two-dimensional process lies between the sum of the intensities of its components (this is the case when components are independent) and the maximum of the two intensities (this is the case when both components are driven by the same Poisson shock).

4 Examples and construction of S-copulas

The following proposition shows how to construct S-copulas from probabilistic copulas.

Proposition 4.1. *Let C be an n-copula and $f(x)$ a function from $[0,1]$ to $[0,\infty]$ with positive derivatives of all orders up to n. Then*

$$F(x_1,\cdots,x_n) = f(C(f^{-1}(x_1),\cdots,f^{-1}(x_n))) \tag{4.1}$$

defines an n-dimensional S-copula.

Proof. This is a direct consequence of properties 1 and 5 in proposition A.1. □

Remark 4.1. This proposition can be used to construct S-copulas from probabilistic copulas but it does not work in the opposite direction. The converse result would require a function with positive derivatives of all orders up to n mapping $[0,\infty]$ into $[0,1]$ and such functions do not exist for $n \geq 2$.

Corollary 4.1. *If f is absolutely monotonic (has positive derivatives of all orders) then (4.1) defines an S-copula for all n.*

One example of absolutely monotonic function that maps $[0,1]$ into $[0,\infty]$ is $f(x) = \frac{x}{1-x}$.

The notion of Archimedean copulas (see [10]) can be extended to S-copulas as shown by the following proposition.

Proposition 4.2. *Let ϕ be a strictly decreasing function from $[0,\infty]$ to $[0,\infty]$ such that $\phi(0)=\infty$, $\phi(\infty)=0$ and ϕ^{-1} has derivatives of all orders up to n with alternating signs, that is, $(-1)^k \frac{d^k\phi^{-1}(t)}{dt^k} > 0$. Then*

$$F(x_1,\cdots,x_n) = \phi^{-1}(\phi(x_1)+\cdots+\phi(x_n)) \tag{4.2}$$

defines an n-dimensional S-copula.

By analogy to the probabilistic copulas, we call F Archimedean S-copula with generating function ϕ.

Proof. One can verify by direct substitution that $F(x_1,\cdots,x_n)$ is grounded and has correct margins. To prove that it is n-increasing, first consider the function

$$F^{**}(x_1,\cdots,x_n) = x_1+\cdots+x_n$$

which is clearly n-increasing. After variable change $x_i \mapsto -\phi(x_i)$ we obtain the function

$$F^*(x_1, \cdots, x_n) = -\phi(x_1) - \cdots - \phi(x_n)$$

which is n-increasing by property 1 in proposition A.1. Since the function $\phi^{-1}(-x)$ has positive derivatives of all orders up to n, we conclude by property 5 that $F(x_1, \cdots, x_n)$ is n-increasing. $\square$

Example 4.1 *(Clayton family of S-copulas).* Taking in previous proposition $n = 2$ and $\phi(t) = t^{-\theta}$ for $\theta > 0$, we obtain the following family of S-copulas:

$$F_\theta(u, v) = (u^{-\theta} + v^{-\theta})^{-1/\theta} \tag{4.3}$$

where $(u, v) \in \mathbb{R}_+^2$. Members of this family will be called Clayton S-copulas in the sequel because, as we will see later in this section, this family is closely related to the Clayton family of copulas

$$C_\theta(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$$

The parameter θ in (4.3) allows to adjust the dependence of two subordinators to the desired level from complete dependence to complete independence. It is easy to check, explicitly computing the limits $\theta \to 0$ and $\theta \to \infty$ in (4.3) that:

$$\lim_{\theta \to \infty} F_\theta(u, v) = F_{\|}(u, v) \quad \text{(perfect dependence)}$$
$$\lim_{\theta \to 0} F_\theta(u, v) = F_{\perp}(u, v) \quad \text{(independence)}$$

Let us compute the Lévy density of a subordinator which has α-stable margins and Clayton S-copula with parameter θ (we will see later that it is in fact a 2-dimensional stable process). The marginal Lévy densities are $\nu_1(x) = \nu_2(x) = \frac{1}{x^{\alpha+1}}$ for $x \geq 0$. Combining formulae (3.4) and (4.3) we obtain

$$\nu(x_1, x_2) = \alpha(1 + \theta)(x_1^{\alpha\theta} + x_2^{\alpha\theta})^{-2-1/\theta} x_1^{\alpha\theta-1} x_2^{\alpha\theta-1} \tag{4.4}$$

The first factor in (4.4) is only singular at the origin while the last two factors are singular on the coordinate axes if $\alpha\theta < 1$. This is easy to understand since for small values of θ the components are almost independent and hence the Lévy density is highly concentrated on the coordinate axes.

Next, we would like to study S-copulas of multidimensional positive stable processes. In the representation (2.4) dependence between components is defined by a finite measure λ on the unit sphere. However, not all types of dependence can be parametrized using this representation, because the class of subordinators with stable margins is larger than the class of stable subordinators. In particular, every linear comination of components of an α-stable subordinator is α-stable which is not the case for all subordinators with α-stable margins. Moreover, it is more convenient from the computational point of view to model dependence via an S-copula than via the spherical part of the Lévy measure. It is therefore desirable to characterize stable subordinators in terms of their S-copulas.

The following proposition gives a simple characterization of the class of S-copulas which can be obtained from α-stable subordinators.

Proposition 4.3. *Let (X_t) be an n-dimensional subordinator with α-stable margins and S-copula $F(x_1, \cdots, x_n)$. It is α-stable if and only if F is homogeneous of order one, that is*

$$\forall c > 0,\ \forall x_1, \cdots, x_n,\ F(cx_1, \cdots, cx_n) = cF(x_1, \cdots, x_n) \tag{4.5}$$

Proof. By theorem 14.3 in [6], (X_t) is α-stable if and only if its Lévy measure ν satisfies

$$\nu = c^{-\alpha} T_c \nu \quad \text{for every } c > 0$$

where T_c is a transformation of measures defined by $(T_c\nu)(B) \equiv \nu(c^{-1}B)$ for every $B \in \mathcal{B}(\mathbb{R}^n)$. Since there is a unique correspondence between Lévy measures and their tail integrals, (X_t) is α-stable if and only if the tail integral U of its Lévy measure satisfies

$$U(cx_1, \cdots, cx_n) = c^{-\alpha} U(x_1, \cdots, x_n)$$

Substituting one-dimensional α-stable margins into the formula (3.1), we conclude that (X_t) is α-stable if and only if the condition (4.5) is verified. $\square$

Example 4.2. To construct a subordinator with stable margins which is not stable, it suffices to take an Archimedean S-copula with generating function ϕ other than power. Consider $\phi(x) = \frac{e^{-x}}{1-e^{-x}}$. This function is strictly decreasing, maps $[0, \infty]$ into $[0, \infty]$ and its inverse $\phi^{-1}(y) = \log(1+y) - \log(y)$ has derivatives of alternating sign. Hence it is a valid generating function for an Archimedean S-copula. This S-copula (obtained using the formula (4.2)) is not homogeneous. A subordinator with stable margins and this S-copula is not stable and, in particular, linear combinations of its components are not stable in general.

We have seen earlier that probabilistic copulas of stable processes do not depend on time and thus provide an alternative parametrization of dependence. The following proposition shows how to compute the S-copula of a positive stable process from its probabilistic survival copula (which is related to the standard copula in a very simple way: see appendix A).

Proposition 4.4. *Let $\hat{C}$ be the probabilistic survival copula of a positive stable process. Then its S-copula F can be computed as follows*

- *If $x_1, \cdots, x_n$ are finite,*

$$F(x_1, \cdots, x_n) = \lim_{\tau \to \infty} \tau \hat{C}\left(\frac{x_1}{\tau}, \cdots, \frac{x_n}{\tau}\right) \tag{4.6}$$

- *If some of the variables $x_1, \cdots, x_n$ are equal to infinity, then the variables $\frac{x_k}{\tau}$ at the corresponding places in the right-hand side of (4.7) are replaced with 1. For example,*

$$F(\infty, x_2, \cdots, x_n) = \lim_{\tau \to \infty} \tau \hat{C}\left(1, \frac{x_2}{\tau}, \cdots, \frac{x_n}{\tau}\right)$$

To prove this proposition we need the following lemma.

Lemma 4.5. *Let X_t be an d-dimensional positive Lévy process with tail integral U continuous at the point $(x_1, \cdots, x_d)$. Then*

$$U(x_1, \cdots, x_d) = \lim_{t \to 0} t^{-1} P\{X_t^{(1)} > x_1, \cdots, X_t^{(d)} > x_d\}$$

Proof. Denote by $C_\sharp$ the set of bounded continuous functions $\mathbb{R}^d \to \mathbb{R}$, vanishing in the neighborhood of zero. By corollary 8.9 in [6], if ν is the Lévy measure of an infinitely divisible distribution μ then for every $f \in C_\sharp$ and for every sequence $t_n \downarrow 0$ we have

$$t_n^{-1} \int_{\mathbb{R}^d} f(x) \mu^{t_n}(dx) \to \int_{\mathbb{R}^d} f(x) \nu(dx) \tag{4.7}$$

Let $A = [x_1, \infty] \times \cdots \times [x_d, \infty]$. We can always find two sequences of functions f_m^+ and f_m^- in $C_\sharp$ such that $f_m^-(x) \le 1_A(x) \le f_m^+(x)$ for all x and m and such that $f_m^+ \to f_\infty^+$ and $f_m^- \to f_\infty^-$ where f_∞^- and f_∞^+ coincide with 1_A everywhere except on the boundary of A. For all m and n

$$t_n^{-1} \int_{\mathbb{R}^d} f_m^-(x) \mu^{t_n}(dx) \le t_n^{-1} \int_{\mathbb{R}^d} 1_A(x) \mu^{t_n}(dx) \le t_n^{-1} \int_{\mathbb{R}^d} f_m^+(x) \mu^{t_n}(dx)$$

Computing the limits on the right and on the left using (4.7) and the dominated convergence theorem, we find

$$\int_{\mathbb{R}^d} f_\infty^-(x)\nu(dx) \leq \lim_{n\to\infty} t_n^{-1}\int_{\mathbb{R}^d} 1_A(x)\mu^{t_n}(dx) \leq \int_{\mathbb{R}^d} f_\infty^+(x)\nu(dx)$$

The continuity assumption entails that

$$\int_{\mathbb{R}^d} f_\infty^-(x)\nu(dx) = \int_{\mathbb{R}^d} f_\infty^+(x)\nu(dx) = \int_{\mathbb{R}^d} 1_A(x)\nu(dx) = U(x_1,\cdots,x_d)$$

and the proof is completed. □

Proof of proposition 4.4. We can suppose without loss of generality that all variables $x_1,\cdots,x_n$ are finite, because if some of them is infinite, it suffices to consider the corresponding lower-dimensional margin. Denote by $\bar{F}$ the n-dimensional survival function of the subordinator and by $\bar{F}_i$ its one-dimensional survival margins. Since the margins of the subordinator are α-stable, property 3 in proposition A.1 entails that the tail integral $U(z_1,\cdots,z_n)$ is continuous as long as all variables $z_1,\cdots,z_n$ are nonzero. Marginal tail integrals are also continuous on $(0,\infty]$. Now we can write, using the selfsimilarity property and lemma 4.5:

$$U_i(z) = \lim_{t\downarrow 0} t^{-1}\bar{F}_i\left(\frac{z}{t^{1/\alpha}}\right), \quad i = 1\ldots n \tag{4.8}$$

$$U(z_1,\cdots,z_n) = \lim_{t\downarrow 0} t^{-1}\bar{F}\left(\frac{z_1}{t^{1/\alpha}},\cdots,\frac{z_n}{t^{1/\alpha}}\right) \tag{4.9}$$

Now we can estimate the quantity of interest as follows:

$$\begin{aligned}
&\left|t^{-1}\bar{F}(\bar{F}_1^{-1}(tx_1),\cdots,\bar{F}_n^{-1}(tx_n)) - U(U_1^{-1}(x_1),\cdots,U_n^{-1}(x_n))\right| \\
&\leq \left|t^{-1}\bar{F}(\bar{F}_1^{-1}(tx_1),\cdots,\bar{F}_n^{-1}(tx_n)) - t^{-1}F\left(\frac{U_1^{-1}(x_1)}{t^{1/\alpha}},\cdots,\frac{U_n^{-1}(x_n)}{t^{1/\alpha}}\right)\right| \\
&\quad + \left|t^{-1}F\left(\frac{U_1^{-1}(x_1)}{t^{1/\alpha}},\cdots,\frac{U_n^{-1}(x_n)}{t^{1/\alpha}}\right) - U(U_1^{-1}(x_1),\cdots,U_n^{-1}(x_n))\right|
\end{aligned}$$

The first term can be made smaller than every $\varepsilon > 0$ by taking t sufficiently big because by property 3 in proposition A.1

$$\begin{aligned}
&\left|t^{-1}\bar{F}(\bar{F}_1^{-1}(tx_1),\cdots,\bar{F}_n^{-1}(tx_n)) - t^{-1}F\left(\frac{U_1^{-1}(x_1)}{t^{1/\alpha}},\cdots,\frac{U_n^{-1}(x_n)}{t^{1/\alpha}}\right)\right| \\
&\qquad\qquad \leq \sum_{i=1}^n \left|x_i - t^{-1}F_i\left(\frac{U_i^{-1}(x_i)}{t^{1/\alpha}}\right)\right|
\end{aligned}$$

and (4.8) applies. Similarly, the second term can be made arbitrarily small because (4.9) holds. □

Example 4.3. The Clayton S-copula (4.3) satisfies the condition (4.5) and hence can describe the dependence structure of a stable subordinator. Moreover, if the probabilistic survival copula of a positive stable process is a Clayton copula with parameter θ, then by proposition 4.4 its S-copula is Clayton S-copula with the same parameter.

5 Probabilistic representation of S-copulas and simulation of multivariate dependent positive Lévy processes

Theorem 5.1. *Let F be an n-dimensional S-copula. Then for fixed x_k, the function*

$$F^{(k)}(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) = \frac{\partial}{\partial x_k} F(x_1, \cdots, x_n)$$

is an $n-1$-dimensional probability distribution function. It determines the distribution of jump sizes of $n-1$ components at a given time conditionally on the jump size of one of the components. More precisely, consider an n-dimensional positive Lévy process (X_t) with marginal tail integrals $U_i, i = 1 \ldots n$ and S-copula F. Suppose that (X_t) has a jump at time t. Let $\Delta X_t^{(1)}, \cdots, \Delta X_t^{(n)}$ be the sizes of jumps of different components at this time and let $Y_i = U_i(\Delta X_t^{(i)})$ for all i. Then, if U_k has no atom at x_k, $F^{(k)}$ is the distribution function of $Y_1, \cdots, Y_{k-1}, Y_{k+1}, \cdots, Y_n$ conditionally on $Y_k = U(x_k)$:

$$F^{(k)}(y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_n) = P\{Y_1 \le y_1, \cdots, Y_n \le y_n | Y_k = U(x_k)\} \quad (5.1)$$

Proof. We suppose without loss of generality that $k = 1$. First, let us compute the distribution of $\Delta X_t^{(2)}, \cdots, \Delta X_t^{(n)}$ conditionally on $a \le \Delta X_t^{(1)} < b$, with $0 < a < b$. To do this consider two independent subordinators: Y_t with Lévy measure $U_Y(A) = U(A \cap \{\mathbf{x} : x_1 \ge a\})$ and Z_t with Lévy measure $U_Z(A) = U(A \cap \{\mathbf{x} : x_1 < a\})$. In other words, the first component of all jumps of Y_t is greater or equal to a and the first component of all jumps of Z_t is smaller than a. Clearly, Y_t is a compound poisson process and $Y_t + Z_t$ has the same law as X_t. To compute the conditional distribution of interest it is enough to take into account Y_t only, jumps in Z_t are irrelevant. Since Y_t is compound Poisson, we know its jump size law and can compute the following conditional probability:

$$P(X_2 \geq x_2, \cdots, X_n \geq x_n | a \leq X_1 < b) = \frac{U(a, x_2, \cdots, x_n) - U(b, x_2, \cdots, x_n)}{U_1(a) - U_1(b)}$$

After change of variables we obtain that for all $\alpha \in \operatorname{Ran} U_1, \beta \in \operatorname{Ran} U_1, y_2 \in \operatorname{Ran} U_2, \cdots, y_n \in \operatorname{Ran} U_n$ such that $\alpha > \beta$,

$$P(Y_2 \leq y_2, \cdots, Y_n \leq y_n | \alpha \geq Y_1 > \beta) = \frac{F(\alpha, y_2, \cdots, y_n) - F(\beta, y_2, \cdots, y_n)}{\alpha - \beta}$$

To pass to the limit $\beta \to \alpha$ we need to find a sequence $\beta_n \to \alpha$ such that $\beta_n \in \operatorname{Ran} U_1$ for all n. This is always possible if U_1 has no atom at x_1. In this case we pass to the limit and obtain the desired result (5.1). $\square$

In the rest of this section $\{\Gamma_i\}$ denotes a sequence of jumping times of a standard Poisson process and $\{V_i\}$ denotes an independent sequence of independent random variables, uniformly distributed on $[0, 1]$. The next proposition recalls a well-known series representation for one-dimensional subordinators.

Proposition 5.2. *Let Z_s be a one-dimensional subordinator with tail integral $U(x)$. It can be represented in law on the time interval $[0, 1]$ as follows*

$$\{Z_s, 0 \leq s \leq 1\} \stackrel{\mathcal{L}}{=} \{\tilde{Z}_s, 0 \leq s \leq 1\}$$

where

$$\tilde{Z}_s = \sum_{i=1}^{\infty} U^{-1}(\Gamma_i) 1_{[0,s]}(V_i) \tag{5.2}$$

The series in (5.2) converges almost surely and uniformly on $s \in [0, 1]$.

For the proof of this result see Rosiński [11]. The next theorem generalizes this method to multidimensional subordinators with dependent components, using the notion of S-copula.

Theorem 5.3. *Let (Z_s) be an n-dimensional positive Lévy process with marginal tail integrals U_i, $i = 1, \cdots, n$ and S-copula $F(x_1, \cdots, x_n)$.*

We introduce n random sequences $\{\Gamma_i^{(1)}\}, \ldots, \{\Gamma_i^{(n)}\}$ as follows

- $\Gamma_i^{(1)} = \Gamma_i, \forall i$

- *Conditionally on $\{\Gamma_i\}$, all $n-1$-dimensional random vectors $\{\Gamma_i^{(2)}\}, \dots, \{\Gamma_i^{(n)}\}$ with different i's are independent and have the distribution function $\frac{\partial}{\partial x_1} F(x_1, \cdots, x_n)|_{x_1=\Gamma_i}$ (viewed as a function of $n-1$ variables $x_2, \cdots, x_n$).*

If F is continuous on $[0,\infty]^n$ then the process Z is representable in law, on the time interval $[0,1]$, as

$$\{Z_s, 0 \le s \le 1\} \stackrel{\mathcal{L}}{=} \{\tilde{Z}_s, 0 \le s \le 1\}$$

where

$$\tilde{Z}_s^{(k)} = \sum_{i=1}^{\infty} U_i^{-1}(\Gamma_i^{(k)}) 1_{[0,s]}(V_i), \quad k = 1, \cdots, n \tag{5.3}$$

All the series in (5.3) converge almost surely and uniformly on $s \in [0,1]$.

Remark 5.1. Here we enumerate the jumps of a multidimensional subordinator in such a way that the jump sizes of the first component are in descending order. This explains the special role of the first component in the theorem. Of course, we could have taken any other component.

Remark 5.2. F satisfies the Lipschitz condition on $[0,\infty)^n$ due to the special form of its margins (property 3 in proposition A.1). This means in particular that F is continuous on this set and that it is absolutely continuous with respect to any one variable when all others are fixed. Therefore the non-trivial continuity condition imposed in this theorem is "continuity at infinity" which means that the value of F in any point that lays at infinite distance from the origin is defined in such a way that it coincides with the limit of $F(\mathbf{x})$ as $\mathbf{x}$ tends to this point. For example, $F(x,\infty)$ must be equal to $\lim_{y\to\infty} F(x,y)$. This means, in particular, that expressions like $1_{x=\infty}$ are not allowed in the definition of F and that the independence S-copula (3.5) is not a valid S-copula for this theorem. In other words, the positive measure on $\bar{\mathbb{R}}_+^n$ corresponding to F must not have atoms at infinity.

To prove theorem 5.3 we need the following convergence result due to Rosiński [11].

Let $\{W_i\}_{i\ge 1}$ be an i.i.d. sequence of random elements in a measurable space S with common distribution F. Assume that $\{W_i\}_{i\ge1}$ is independent of $\{\Gamma_i\}_{i\ge1}$ and $\{V_i\}_{i\ge1}$. Let $H : (0,\infty) \times S \to \mathbb{R}^d$ be a measurable function.

We define the measure Q on $\mathbb{R}^d$ by

$$Q(A) = \int_0^\infty P(H(r, W_i) \in A) dr \tag{5.4}$$

Theorem 5.4. *If Q is a Lévy measure concentrated on $\mathbb{R}^d_+$, with*

$$\int_{\mathbb{R}^d_+} (x \wedge 1) Q(dx) < \infty$$

then the series $\sum_{i=1}^{\infty} H(\Gamma_i, W_i) 1_{V_i \le t}$ converges almost surely and uniformly on $t \in [0,1]$ to a multidimensional subordinator with Lévy measure Q.

Proof. This is a direct consequence of theorems 4.1 and 5.1 in [11] $\square$

Proof of theorem 5.3. Let $\{W_i\}_{i \ge 1}$ be a sequence of independent random variables, uniformly distributed on $[0,1]$. Then for each $\Gamma \in \mathbb{R}_+$ there exists a function $h_\Gamma(v) : [0,1] \to \mathbb{R}^{n-1}_+$ such that for each i, $h_\Gamma(W_i)$ has distribution function $\frac{\partial}{\partial x_1} F(x_1, \cdots, x_n)|_{x_1 = \Gamma}$ (see e.g. [7, lemma 2.22]). We can now define the function $H(r,v) : \mathbb{R}_+ \times [0,1] \to \mathbb{R}^n_+$ component by component as follows:

$$H^{(1)}(r,v) = U_1^{-1}(r) \tag{5.5}$$

$$H^{(k)}(r,v) = U_k^{-1}(h_\Gamma^{(k-1)}(v)), \quad k = 2, \cdots, n \tag{5.6}$$

It remains to check that $H(r,v)$, when integrated as in (5.4), yields the Lévy measure of the subordinator that we want to simulate. It is sufficient to consider the sets A of the form $A = A_1 \times A_2 \times \cdots \times A_n$ with $A_k = [x_k, \infty]$.

$$\begin{aligned} P[H(r,W_i) \in A] &= 1_{U_1^{-1}(r) \in A_1} P[U_2^{-1}(h_r^1(W_i)) \in A_2, \cdots, U_n^{-1}(h_r^{n-1}(W_i)) \in A_n] \\ &= 1_{r \in \tilde{A}_1} P[h_r^1(W_i) \in \tilde{A}_2, \cdots, h_r^{n-1}(W_i) \in \tilde{A}_n] \end{aligned}$$

where $\tilde{A}_k = [0, U_k(x_k)]$. The second factor in this expression is the probability distribution function of $h_r^1(W_i), \cdots, h_r^{n-1}(W_i)$, which is known, hence,

$$P[H(r,W_i) \in A] = 1_{r \in \tilde{A}_1} \frac{\partial}{\partial r} F(r, U_2(x_2), \cdots, U_n(x_n))$$

We now conclude that because F is absolutely continuous with respect to the first variable in all finite points and continuous on $[0,\infty]^n$,

$$\int_0^\infty P(H(r,W_i) \in A) dr = \int_0^{U_1(x_1)} \frac{\partial}{\partial r} F(r, U_2(x_2), \cdots, U_n(x_n)) dr = F(U_1(x_1), \cdots, U_n(x_n))$$

yields the tail integral of the Lévy measure of subordinator that we want to simulate and we can apply theorem 5.4. $\square$

To illustrate this method, we simulate, using the formula (5.3), a two-dimensional strictly $\frac{1}{2}$-stable subordinator with Clayton S-copula (4.3) for various values of θ. Marginal tail integrals of this process have a simple form: $U^{-1}(x) = \frac{1}{x^2}$, which speeds up the simulation. The conditional distribution function (5.1) of transformed jump sizes for Clayton S-copula is

$$F(v|u) = \frac{\partial F_\theta(u,v)}{\partial u} = \left\{1 + \left(\frac{u}{v}\right)^\theta\right\}^{-1-1/\theta}$$

and the inverse distribution function has an explicit form:

$$F^{-1}(y|u) = u\left(y^{-\frac{\theta}{1+\theta}} - 1\right)^{-1/\theta}$$

which allows efficient simulation of the jumps in the second component conditionally on the jumps in the first one. The formula (5.3) can be rewritten in the following way: let $\{\Gamma_i\}_{i\geq 1}$ be a sequence of jump times of a standard Poisson process, and $\{V_i\}_{i\geq 1}$ and $\{W_i\}_{i\geq 1}$ be two sequences of i.i.d. random variables, uniformly distributed on $[0,1]$. The first sequence determines the jump times and the second one will drive the jumps of the second component. Then we have the following representation for our two-dimensional subordinator on the time interval $[0,1]$:

$$X_s = \sum_{i=1}^{\infty} U^{-1}(\Gamma_i) 1_{[0,s]}(V_i)$$

$$Y_s = \sum_{i=1}^{\infty} U^{-1}(F^{-1}(W_i|\Gamma_i)) 1_{[0,s]}(V_i)$$

Figure 1 depicts the simulated trajectories of this subordinator for different modes of dependence (different values of θ). On the left graph the dependence is very weak; shocks in two components are almost completely independent. On the right graph the dependence is stronger; here big shocks in two components tend to arrive at the same time; however their exact magnitude may be very different. Finally, on the bottom graph we have almost perfect dependence: shocks arrive together and have approximately the same magnitude.

6 Building multivariate models for finance and insurance using S-copulas

In this section we show on several examples, how S-copulas can be used to build multivariate Lévy models for finance and insurance.

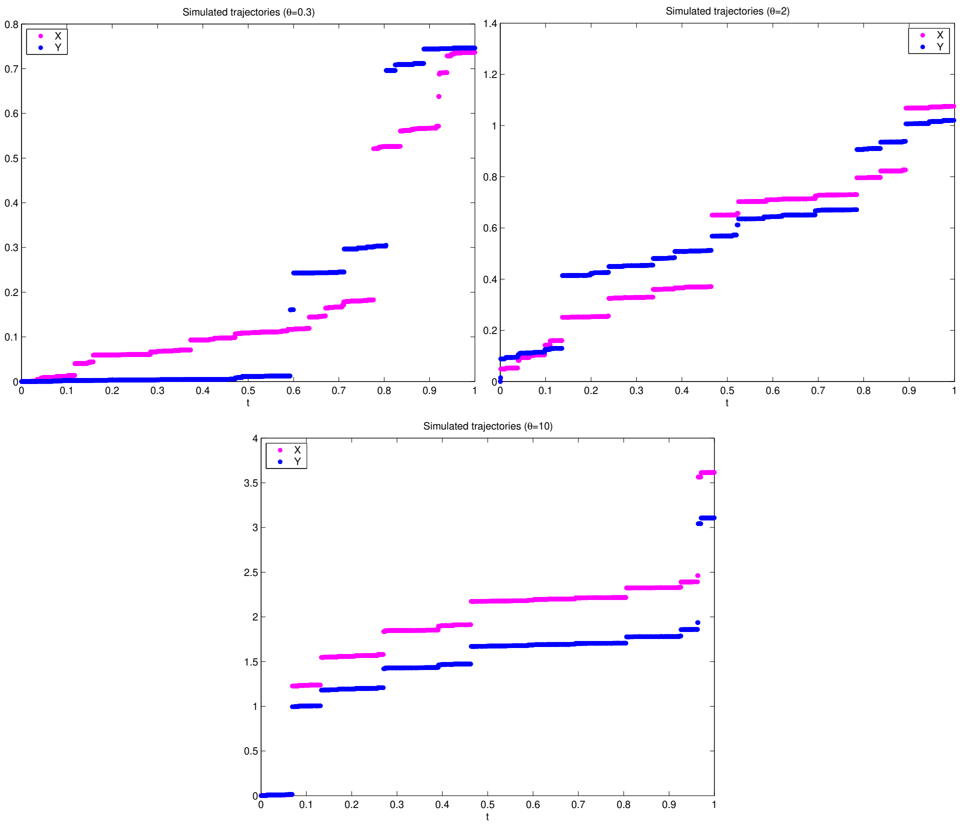

Figure 1: Simulated trajectories of a two-dimensional subordinator under weak (left), medium (right) and strong (bottom) dependence.

Example 6.1 *(Insurance model).* Consider an insurance company with subsidiaries in France and Germany. Suppose that the losses of French subsidiary are modelled by the process (X_t) which is a subordinator with Lévy measure ν_x, and the losses of German subsidiary are modelled by the process (Y_t), which is a subordinator with Lévy measure ν_y. (X_t) and (Y_t) may have different structure because both the subsidiaries and the risks differ. However, these processes are dependent because some risks (e.g. windstorms) are common.

In this setting, dependence of (X_t) and (Y_t) is best modelled using an S-copula $F_\theta(x, y)$. On one hand, any structure of (X_t) and (Y_t) can be taken into account and there are few additional parameters to calibrate (because

F_θ can be taken from some parametric family). On the other hand, this model gives an accurate vision of global risks because one can not only simulate joint scenarios but also compute the Lévy measure of $X_t + Y_t$ and assess overall ruin probabilities.

Example 6.2 *(Stock price model).* Consider two stock price processes (S_t^1) and (S_t^2) whose logarithms are modelled by variance gamma Lévy process. We remind that a variance gamma process (see [3]) can be obtained by subordinating a Brownian motion with drift with a gamma subordinator having Lévy measure $\nu(x) = \frac{\lambda e^{-x}}{x}$.

A standard method for introducing dependence between (S_t^1) and (S_t^2) consists in taking a 2-dimensional Brownian motion (W_t^1, W_t^2) with correlation ρ and one gamma subordinator Z_t. Then the 2-dimensional model takes the form:

$$\begin{cases} S_t^1 = \exp(X_t); & X_t = a_1 W_{Z_t}^1 + b_1 Z_t \\ S_t^2 = \exp(Y_t); & Y_t = a_2 W_{Z_t}^2 + b_2 Z_t \end{cases}$$

This model has the following drawbacks:

- (X_t) and (Y_t) always have the same jump structure parameter λ
- Even if $\rho = 0$, the two stocks are dependent because the absolute values of returns are correlated, in other words, big jumps in one stock tend to occur at the same time as big jumps in the other.
- Upward and downward jumps have the same correlation structure (because the Brownian motion is symmetric). This may not be very realistic since upward jumps usually correspond to events, favorable to a particular firm while downward jumps may more often correspond to global events like economic crises and hence exhibit a higher level of correlation.

A better model may be constructed using S-copulas as follows. Consider an alternative representation of variance gamma process: each VG process may be represented as a difference of two *independent* Gamma subordinators with the same intensity parameter. Our model has the following form

$$\begin{cases} X_t = & X_t^+ - X_t^- \\ Y_t = & Y_t^+ - Y_t^- \end{cases}$$

where X^+, X^-, Y^+, Y^- are gamma processes which model jumps of different signs. Here, (X_t^+) and (X_t^-) are independent and have the same intensity parameter; (Y_t^+) and (Y_t^-) are independent and have the same intensity parameter; X^+ and Y^+ are linked through S-copula $F^+(x,y)$ and finally X^- and Y^- are linked through S-copula $F^-(x,y)$. The modelling approach is as follows: first, specify arbitrary VG processes (X_t) and (Y_t) and then model the dependence between positive jumps of (X_t) and (Y_t) using S-copula $F^+(x,y)$ and between the negative jumps of (X_t) and negative jumps of (Y_t) using S-copula $F^-(x,y)$. This approach has the following advantages:

- Marginals are completely separated from dependence.
- Complete independence is included in this model: it suffices to take independence copulas.
- Modelling of upward and downward jumps can be done separately.

A Increasing functions and probabilistic copulas

In this appendix we define n-increasing functions and list some of their properties. For the sake of completeness we also define probabilistic copulas and formulate Sklar's theorem.

Definition A.1 (H-volume). Let $S_1,\cdots,S_n$ be nonempty subsets of $\bar{\mathbb{R}}$, let H be a real function of n variables such that $\operatorname{Dom} H = S_1\times\cdots\times S_n$ and for $\mathbf{a}\le\mathbf{b}$ ($a_k\le b_k$ for all k) let $B=[\mathbf{a},\mathbf{b}]$ be an n-box whose vertices are in $\operatorname{Dom} H$. Then the H-volume of B is defined by

$$V_H(B)=\sum \operatorname{sgn}(\mathbf{c})H(\mathbf{c}),$$

where the sum is taken over all vertices $\mathbf{c}$ of B, and $\operatorname{sgn}(\mathbf{c})$ is

$$\operatorname{sgn}(\mathbf{c})=\begin{cases} 1, & \text{if } c_k=a_k \text{ for an even number of } k\text{'s},\\ -1, & \text{if } c_k=a_k \text{ for an odd number of } k\text{'s},\end{cases}$$

H-volume is an additive quantity: for any two n-boxes A and B such that $V_H(A\cap B)=0$, $V_H(A\cup B)=V_H(A)+V_H(B)$. It is also linear in H: $V_{H+F}(A)=V_H(A)+V_F(A)$.

Definition A.2 (n-increasing function). A real function H of n variables is called n-increasing if $V_H(B)\ge 0$ for all n-boxes B whose vertices lie in $\operatorname{Dom} H$.

A n-increasing function H corresponds to a positive measure (denoted also by H) on $\mathrm{Dom}\, H$ (the measure of an n-box B is equal to its volume $V_H(B) \geq 0$). When H is absolutely continuous, this measure has a density given by

$$\nu(x_1, \cdots, x_n) = \frac{\partial^n H(x_1, \cdots, x_n)}{\partial x_1, \cdots, \partial x_n}$$

Suppose that the domain of a real function H of n variables is $S_1 \times \cdots \times S_n$ where each S_k has a smallest element a_k. We say that H is *grounded* if $H(\mathbf{t}) = 0$ for all $\mathbf{t}$ in $\mathrm{Dom}\, H$ such that $t_k = a_k$ for at least one k.

If each S_k is nonempty and has a greatest element b_k, then (one-dimensional) margins of H are functions H_k with $\mathrm{Dom}\, H_k = S_k$ defined by $H_k(x) = H(b_1, \cdots, b_{k-1}, x, b_{k+1}, \cdots, b_n)$ for all x in S_k.

The following elementary properties of n-increasing functions are useful throughout the paper.

Proposition A.1.

1. *The notion of n-increasing function is invariant under increasing variable changes: if H is n-increasing and $f_i, i = 1, \cdots, n$ are increasing, then $H(f_1(x_1), \cdots, f_n(x_n))$ is n-increasing.*

2. *Let H be an n-increasing grounded function and let $m < n$. Then as a function of any m of its arguments, the other ones being fixed, H is an m-increasing grounded function.*

3. *If H is n-increasing grounded function with margins $H_1, \cdots, H_n$ then*

$$|H(\mathbf{x}) - H(\mathbf{y})| \leq \sum_{k=1}^{n} |H_k(x_k) - H_k(y_k)|.$$

4. *Product of two n-increasing grounded functions is n-increasing and grounded.*

5. *Let H be an n-increasing grounded function with range $[0, a]$. If $f : [0, a] \to [0, b]$ has positive derivatives of all orders up to n then $f(H)$ is an n-increasing grounded function.*

Proofs of these properties are available on request from the author. Some of them can also be found in [12].

We close this appendix with the definition of classical copula and the famous Sklar's theorem. For more details on copulas see [10]. The proof of Sklar's theorem can be found in [13].

Definition A.3 (Copula). An n-dimensional copula is a function C with domain $[0,1]^n$ such that

1. C is grounded and n-increasing
2. C has margins $C_k, k = 1, 2, \cdots, n$, which satisfy $C_k(u) = u$ for all u in $[0,1]$.

We recall that an n-dimensional distribution function F is an n-increasing grounded function with domain $\bar{\mathbb{R}}^n$ such that $H(\infty, \cdots, \infty) = 1$, so copulas are distribution functions with uniform margins.

Theorem A.2 (Sklar). *Let F be an n-dimensional distribution function with margins $F_1, \cdots, F_n$. Then there exists an n-dimensional copula C such that for all $x \in \bar{\mathbb{R}}^n$,*

$$F(x_1, x_2, \cdots, x_n) = C(F_1(x_1), F_2(x_2), \cdots, F_n(x_n)) \tag{A.1}$$

if $F_1, \cdots, F_n$ are all continuous then C is unique; otherwise, C is uniquely determined on $\operatorname{Ran} F_1 \times \cdots \times \operatorname{Ran} F_n$. *Conversely, if C is an n-copula and $F_1, \cdots, F_n$ are distribution functions, then the function F defined by (A.1) is an n-dimensional distribution function with margins $F_1, \cdots, F_n$.*

The survival copula $\hat{C}$ of random variables $X_1, \cdots, X_n$ is defined as the copula of $f_1(X_1), \cdots, f_n(X_n)$ where $f_1, \cdots, f_n$ are strictly decreasing. The survival copula relates the joint survival function of n random variables to their marginal survival functions:

$$\bar{F}(x_1, \cdots, x_n) = \hat{C}(\bar{F}_1(x_1), \cdots, \bar{F}_n(x_n))$$

where $\bar{F}(x_1, \cdots, x_n) = P\{X_1 > x_1, \cdots, X_n > x_n\}$. The survival copula is related to the standard copula in a very simple way. For example, in two dimensions $\hat{C}(x, y) = x + y - 1 + C(1 - x, 1 - y)$.

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