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SYMMETRIZED EULER SCHEME FOR AN EFFICIENT APPROXIMATION OF REFLECTED DIFFUSIONS

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Abstract

In this note, we analyze the error involved by using an Euler scheme with a symmetry procedure near the boundary for the simulation of diffusion processes with an oblique reflection on a smooth boundary. This procedure is straightforward to implement, and further accurate: indeed, we prove that it yields a rate of convergence for the weak error of order 1 w.r.t. the time discretization step. Results were previously announced in a weaker form in [7].

Keywords: reflected diffusion, discretization schemes, weak approximation, Neumann boundary condition for parabolic PDE.

AMS 2000 Subject Classification: Primary 65CXX
Secondary 35K20;60-08

1. Introduction

The numerical resolution by deterministic methods of second order Partial Differential Equations (PDEs in short) becomes unefficient in high dimension. An alternative approach consists in developing Monte Carlo methods from the probabilistic representations of the solutions as expectations of functionals of diffusion processes $X = (X_t)_{t\geq 0}$. Usually, exact simulations of X are impossible and time discretization procedures are needed.

A lot of attention has been paid for PDEs in the whole space. In that case, the processes to simulate is the solution in the whole space of

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s,$$

where W is a standard multidimensional Brownian motion (BM in short). Optimal convergence rates are now well established. For example, consider the Euler scheme with time step $h = T/N$ ($t_i = ih$ being the discretization times of $[0, T]$):

$$X^N_{t_{i+1}} = X^N_{t_i} + b(X^N_{t_i})h + \sigma(X^N_{t_i})(W_{t_{i+1}} - W_{t_i}).$$

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The weak error $\mathbf{E}(f(X_T)) - \mathbf{E}(f(X_T^N))$ can be expanded in terms of powers of h, provided some regularity conditions on f (see [22]), or some non degeneracy condition on the process X (i.e. hypo-ellipticity, see [1]).

If the PDE has a Dirichlet condition on the boundary ∂D of a domain D, then the diffusion process X needs to be killed or stopped when it hits ∂D. In that situation, if one naively kills or stops the Euler scheme, then the weak convergence rate is of order $\frac{1}{2}$. However, one can develop an efficient killing [6, 8] or stopping [16] procedure leading to a convergence rate of order 1.

For PDEs with Neuman boundary condition on ∂D, X needs to be a diffusion process with reflection on ∂D in some oblique direction γ, i.e. solution of

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \gamma(X_s)\, dk_s$$

where the so-called local time k_t is increasing only ∂D. In this work, we focus in the evaluation of quantities like $\mathbf{E}(f(X_T))$ for a fixed time T. This issue may be of interest if one seeks to reconstruct (see [4]) the three dimensional brain activity via the resolution of some PDE with Neumann boundary conditions. It is also pertinent in some approximation models for open queuing networks in heavy traffic (see [23] and references therein). At last, reflected processes have been recently introduced to solve one-dimensional viscous scalar conservation law in an interval (see [2]).

From the numerical point of view, starting from a regular mesh of the interval $[0, T]$ with time step h, one can use an Euler scheme with projection, for which the weak error is of order $\frac{1}{2}$ as established in [3] for normal reflections $\gamma = n$ (see also [21], [17]). One can also use a penalty method: the convergence has been studied only in $\mathcal{L}_p$-sense (see e.g. [15], [18], [9]). For a more complete presentation of these methods, see [8]. More recently, in [8] the second author combined the Lépingle's procedure [11, 12] (which is exact when D is a half-space and the coefficients are constant) and some local half-space approximation to construct implementable procedures, which are of order 1 under the condition that γ lies in the co-normal direction: $\gamma(s)$ is parallel to $\sigma\sigma^*(s)n(s)$ for any $s \in \partial D$.

Hence, so far, the question of getting an easy implementable procedure providing a first order convergence is still open for general oblique reflection problems. Our so-called *symmetrized Euler scheme* below solves this issue. Results in this paper have been presented at the conference *Monte Carlo and probabilistic methods for partial differential equations* which held at Monte Carlo (Monaco) in July 2000 and announced in [7]. This symmetrized scheme has been recently studied in [14] where the convergence is not analyzed in details.

Outline of the paper.
In the section 2, we set some preliminary geometry notations, state our assumptions and define the symmetrized Euler scheme. Then, we state our main result of convergence. The section 3 is devoted to the proof. In the section 4, a numerical example is considered, which illustrates the efficiency of our algorithm.

Notations.
We adopt the following usual convention on the gradients: if $\psi : \mathbf{R}^{p_2} \mapsto \mathbf{R}^{p_1}$ is a differentiable function, its gradient $\nabla\psi(x) = (\partial_{x_1}\psi(x), \cdots, \partial_{x_{p_2}}\psi(x))$ takes values in $\mathbf{R}^{p_1} \otimes \mathbf{R}^{p_2}$. In particular, the gradient of a linear function ψ is a *row* vector. Its Hessian

matrix is denoted by H^{ψ}. Usually, the gradient is computed w.r.t. the space variables only.
We use the generic notation $K(T)$ for all finite, non-negative and non-decreasing functions: they neither depend on x, nor the function f, nor the discretization step h, but they may depend on the coefficients b, σ, γ and on the domain D.
A quantity R is equal to $O_{\exp}(h)$ if $|R| \leq K(T)\exp(-c/h)$ for some constants $K(T)$ and $c > 0$.
The conditional expectation $\mathbf{E}(Z|\mathcal{F}_{t_i})$ is denoted by $\mathbf{E}^{\mathcal{F}_{t_i}}(Z)$.

2. Assumptions and main result

2.1. Assumptions

In the sequel, we consider a domain $D \subset \mathbf{R}^d$, with the following smoothness property.

(D) The boundary ∂D is bounded and of class C^5.

The set of points in the ϵ-neighborhood of ∂D is denoted by $V_{\partial D}(\epsilon) = \{x : d(x, \partial D) \leq \epsilon\}$. The vector $n(s)$ denotes the unit inward normal vector at $s \in \partial D$. In addition, the vector field defining the reflection direction is uniformly non tangent to the boundary.

(C) γ is a unit vector field of class C^4 and there exists $\rho_0 > 0$ such that $\gamma(s).n(s) \geq \rho_0$ for all $s \in \partial D$.

We remind some classical results concerning the distance to the boundary along γ (see Appendix in [8]).

Proposition 1. *Assume* **(D)** *and* **(C)**. *There exists a constant $R > 0$ such that:*

i) for any $x \in V_{\partial D}(R)$, there are unique $s = \pi^{\gamma}_{\partial D}(x) \in \partial D$ and $F^{\gamma}(x) \in \mathbf{R}$ such that $x = \pi^{\gamma}_{\partial D}(x) + F^{\gamma}(x)\gamma(\pi^{\gamma}_{\partial D}(x))$.

ii) The projection of x on ∂D parallel to γ, that is, the function $x \longmapsto \pi^{\gamma}_{\partial D}(x)$, is of class C^4 on $V_{\partial D}(R)$.

iii) The algebraic distance of x to ∂D parallel to γ, that is, the function $x \longmapsto F^{\gamma}(x)$, is of class C^4 on $V_{\partial D}(R)$. One has $F^{\gamma} > 0$ on $V_{\partial D}(R) \cap D$, $F^{\gamma} < 0$ on $V_{\partial D}(R) \cap \overline{D}^c$, $F^{\gamma} = 0$ on ∂D: we can extend F^{γ} into a $C^4_b(\mathbf{R}^d, \mathbf{R})$ function, with the conditions $F^{\gamma} > 0$ on D and $F^{\gamma} < 0$ on $\overline{D}^c$.

iv) The above extensions for F^{γ} and F^n can be performed in a way such that the functions F^{γ} and F^n are equivalent in the sense that

$$\frac{1}{c_1}\,|F^n(x)| \leq |F^{\gamma}(x)| \leq c_1\,|F^n(x)|, \qquad \forall x \in \mathbf{R}^d,$$

for some constant $c_1 > 1$.

v) For $x \in \partial D$, one has $\nabla F^{\gamma}(x) = \dfrac{n^*}{n.\gamma}(x)$.

We sometimes use the notation $n(x)$ or $\gamma(x)$ even if $x \notin \partial D$: for $x \in V_{\partial D}(R)$, we set $n(x) = n(\pi^{\gamma}_{\partial D}(x))$ and $\gamma(x) = \gamma(\pi^{\gamma}_{\partial D}(x))$ and, for $x \notin V_{\partial D}(R)$, arbitrary values are given.

The coefficients defining (1) below are supposed to satisfy

(S) b and σ are $C_b^4(\overline{D}, \mathbf{R}^d)$ and $C_b^4(\overline{D}, \mathbf{R}^d \otimes \mathbf{R}^d)$ functions.

Given a d-dimensional BM $(W_t)_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ (satisfying the usual conditions), one knows (see [20], [13]) that under **(C)**, **(D)** and **(S)**, there is an unique strong solution to

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \gamma(X_s)\, dk_s \tag{1}$$

where k_t is a process increasing only on ∂D: $k_t = \int_0^t \mathbf{1}_{X_s \in \partial D}\, dk_s$. The initial value $x \in D$ is fixed in all the sequel.

We also need the following non degeneracy condition.

(E) The matrix $\sigma\sigma^*$ is uniformly elliptic: $\forall\, x \in \overline{D},\ \ \sigma\sigma^*(x) \geq \sigma_0^2\ I_{\mathbf{R}^d \otimes \mathbf{R}^d}$ for some $\sigma_0 > 0$.

2.2. The algorithm

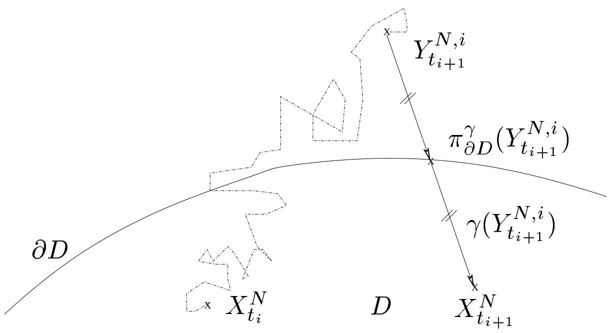

FIGURE 1: Description of the algorithm when $Y^{N,i}_{t_{i+1}}$ is outside D.

We start with $X_0^N = x$ and assume that we have obtained $X^N_{t_i} \in \overline{D}$.

a) For $t \in [t_i, t_{i+1}]$, we set $Y_t^{N,i} := X^N_{t_i} + b(X^N_{t_i})(t - t_i) + \sigma(X^N_{t_i})(W_t - W_{t_i})$. Observe that $Y^{N,i}_{t_{i+1}}$ is simulated by simply drawing d independent Gaussian variables.

b) Then,

i) If $Y^{N,i}_{t_{i+1}} \notin \overline{D}$ (i.e. $F^\gamma(Y^{N,i}_{t_{i+1}}) < 0$), we set $X^N_{t_{i+1}} = \pi^\gamma_{\partial D}(Y^{N,i}_{t_{i+1}}) - F^\gamma(Y^{N,i}_{t_{i+1}})\gamma(Y^{N,i}_{t_{i+1}})$ which coincides with the the symmetric point of $Y^{N,i}_{t_{i+1}}$ w.r.t. $\pi^\gamma_{\partial D}(Y^{N,i}_{t_{i+1}})$ (see Figure 1).

ii) If $Y^{N,i}_{t_{i+1}} \in \overline{D}$ (i.e $F^\gamma(Y^{N,i}_{t_{i+1}}) \geq 0$), we simply set $X^N_{t_{i+1}} = Y^{N,i}_{t_{i+1}}$.

To sum up, we have $X^N_{t_{i+1}} = Y^{N,i}_{t_{i+1}} + 2[F^\gamma(Y^{N,i}_{t_{i+1}})]^- \gamma(Y^{N,i}_{t_{i+1}})$.

c) It is possible that $Y^{N,i}_{t_{i+1}} \notin D \cup V_{\partial D}(R)$, that is a huge increment has occurred: this event has a probability exponentially small w.r.t. h (see below) and, in that case, we suggest to restart the simulation of $Y^{N,i}_{t_{i+1}}$.

This way to proceed using a symmetry is actually very natural: indeed, in dimension 1, we know by the Lévy's identity (see Section VI.2 in [19]) that the BM reflected on the positive axis has the same law as the absolute value of the standard BM. In more general situations, an analoguous procedure is used by Freidlin [5] to prove the existence of a solution to (1).

2.3. Rate of convergence

We denote by L is the infinitesimal generator of $(X_t)_{t\geq 0}$, that is,

$$Lu = \nabla u\ b + \frac{1}{2}\mathrm{Tr}(H^u a)$$

(with $a = \sigma\sigma^*$).

We suppose

(F) The function f is of class $C_b^5(\overline{D}, \mathbf{R})$ and satisfies the compatibility condition on ∂D: $\forall z \in \partial D,\ \ [\nabla f\ \gamma](z) = [\nabla (Lf)\ \gamma](z) = 0$.

For $f \in C_b^5(\overline{D}, \mathbf{R})$, we set $\|f\|^{(5)} = \sum_{\alpha:|\alpha|\leq 5} \|\partial_x^\alpha f\|_\infty$. Our main result is the following

Theorem 1. *Under* **(C)**, **(D)**, **(F)**, **(S)**, *we have*

$$|\mathbf{E}(f(X_T^N)) - \mathbf{E}(f(X_T))| \leq K(T)\|f\|^{(5)}h$$

for some constant $K(T)$ uniform in x and f.

The rest of the paper is devoted to its proof.

3. Proofs

We follow the usual trick consisting in decomposing the error in a sum of local errors using an appropriate partial differential equation (PDE in short). For this, we set $u(t,x) = \mathbf{E}\left[f(X_{T-t})|X_0 = x\right]$, which is a smooth solution of the following PDE (see [10] Theorem 5.3 p.320), with Neumann boundary condition:

$$\left\{\begin{array}{ll} (\partial_t u + L\,u)(t,x) = 0 & \text{for } (t,x) \in [0,T]\times\overline{D}, \\ \nabla u(t,x)\gamma(x) = 0 & \text{for } (t,x) \in [0,T]\times\partial D, \\ u(T,x) = f(x) & \text{for } x \in D. \end{array}\right. \tag{2}$$

Under the assumptions of Theorem 1, the solution u is at least of class $C^{2,4}([0,T]\times\overline{D})$ with uniformly bounded derivatives (the compatibility conditions in **(F)** are crucial for this): namely, for $2p + |\alpha| \leq 4$, one has

$$\forall (t,x)| \in [0,T]\times\overline{D}\ \ |\partial_t^p\partial_x^\alpha u(t,x)| \leq K(T)\|f\|^{(5)}. \tag{3}$$

We extend u in a $C^{2,4}([0,T]\times\mathbf{R}^d)$-function (see [10]) which still satisfies the estimates (3).

We introduce a continuous-time version of the symetrized Euler scheme by setting $X_t^N = Y_t^{N,i} + 2[F^\gamma(Y_t^{N,i})]^-\gamma(Y_t^{N,i})$ for $t \in [t_i, t_{i+1}[$. Define $\tau = \inf\{t \geq 0 : Y_t^{N,i} \notin D \cup V_{\partial D}(R)$ with $t_i \leq t \leq t_{i+1}\}$. On the event $\{\tau > T\}$, $(X_t^N)_{0\leq t\leq T}$ lives in D. In addition, one has

$$\mathbf{P}[\tau \leq T] = O_{\exp}(h) \tag{4}$$

which is a straightforward consequence of the following standard lemma.

Lemma 1. *Consider an Itô process with uniformly bounded coefficients:* $dU_t = b_t\, dt + \sigma_t\, dW_t$. *There exist some constants* $c > 0$ *and* $K(T)$ *(depending on* $p \geq 1$*) such that, for any stopping times* S *and* S' *(*$0 \leq S \leq S' \leq \delta \leq T$*) and any* $\eta \geq 0$,

$$\mathbf{P}\big(\sup_{t\in[S,S']} \|U_t - U_S\| \geq \eta\big) \leq K(T) \exp\big(-c\frac{\eta^2}{\delta}\big), \tag{5}$$

$$\mathbf{E}[\sup_{t\in[S,S']} \|U_t - U_S\|^p] \leq K(T)\delta^{p/2}. \tag{6}$$

The first estimate is based on Bernstein's inequality for martingales (see e.g. Lemma 4.1 in [6]), and the second one follows from the Burkholder-Davis-Gundy inequalities.

Now, set $\mathcal{E}_i := \mathbf{E}(u(t_{i+1}\wedge\tau, X^N_{t_{i+1}\wedge\tau}) - u(t_i\wedge\tau, X^N_{t_i\wedge\tau})) = \mathbf{E}(\mathbf{1}_{t_i<\tau}\mathbf{E}^{\mathcal{F}_{t_i}}[u(t_{i+1}\wedge\tau, X^N_{t_{i+1}\wedge\tau}) - u(t_i, X^N_{t_i})])$. In view of (2) and (4), the weak error can be decomposed as follows

$$\begin{aligned}\mathbf{E}(f(X^N_T)) - \mathbf{E}(f(X_T)) &= \mathbf{E}(u(T,X^N_T) - u(T\wedge\tau, X^N_{T\wedge\tau}) + u(T\wedge\tau, X^N_{T\wedge\tau}) - u(0,X^N_0))\\ &= \|f\|_\infty O_{\exp}(h) + \sum_{i=0}^{N-1}\mathcal{E}_i\end{aligned}$$

We then need the two following crucial results, that we prove later.

Lemma 2. *Under* **(C)**, **(D)** *and* **(S)**, *for all* $c > 0$, *one has*

$$h\mathbf{E}\left(\sum_{i=0}^{N-1}\mathbf{1}_{t_i<\tau}\exp\big(-c\frac{d^2(X^N_{t_i},\partial D)}{h}\big)\right) \leq K(T)\sqrt{h}.$$

Lemma 3. *Under* **(C)**, **(D)**, **(F)** *and* **(S)**, *for all* $x \in \partial D$ *one has*

$$\mathcal{C}^u(x) := \left(-\nabla u\nabla\gamma a\frac{n}{n.\gamma} + \gamma^* H^u\gamma\frac{n^* a n}{(n.\gamma)^2} - \frac{n^* a H^u \gamma}{n.\gamma}\right)(x) = 0.$$

In view of Tanaka's formula [19], $(X^N_t)_{0\leq t\leq T}$ defined as in the step b) of the algorithm, is a continuous semimartingale for $t \in [t_i, t_{i+1}[$. Easy computations lead to

$$\begin{aligned}dX^N_t =& dY^{N,i}_t + \gamma(Y^{N,i}_t)dL^0_t(F^\gamma(Y^{N,i}_t)) + [F^\gamma(Y^{N,i}_t)]^-\\ &\times\{2\nabla\gamma(Y^{N,i}_t)dY^{N,i}_t + \mathrm{Tr}[H^\gamma(Y^{N,i}_t)a(X^N_{t_i})]dt\}\\ &-\mathbf{1}_{Y^{N,i}_t\notin D}\Big\{2\nabla\gamma(Y^{N,i}_t)a(X^N_{t_i})[\nabla F(Y^{N,i}_t)]^* dt + 2\gamma(Y^{N,i}_t)\nabla F^\gamma(Y^{N,i}_t)dY^{N,i}_t\\ &+\gamma(Y^{N,i}_t)\mathrm{Tr}[H^{F^\gamma}(Y^{N,i}_t)a(X^N_{t_i})]dt\Big\}\end{aligned} \tag{7}$$

since $\{F^\gamma(Y^{N,i}_t) \leq 0\} = \{Y^{N,i}_t \notin D\}$; we have denoted by $\mathrm{Tr}[H^\gamma(Y^{N,i}_t)a(X^N_{t_i})]$ the vector with j-th row equal to $\mathrm{Tr}[H^{\gamma_j}(Y_t^{N,i})a(X_{t_i}^N)]$. Thus, Ito's formula yields

$$
\begin{aligned}
&\mathbf{E}^{\mathcal{F}_{t_i}}(u(t_{i+1}\wedge\tau, X^N_{t_{i+1}\wedge\tau}) - u(t_i, X^N_{t_i})) := A_i^1 + A_i^2,\\
A_i^1 &:= \mathbf{E}^{\mathcal{F}_{t_i}}\big(\int_{t_i}^{t_{i+1}\wedge\tau}[\partial_t u(t,X_t^N)dt + \nabla u(t,X_t^N)dY_t^{N,i} + \frac{1}{2}\mathrm{Tr}(H^u(t,X_t^N)a(X_{t_i}^N))dt]\big),\\
A_i^2 &:= \mathbf{E}^{\mathcal{F}_{t_i}}\big(\int_{t_i}^{t_{i+1}\wedge\tau}[\nabla u(t,X_t^N)(dX_t^N - dY_t^{N,i})\\
&\qquad\qquad + \frac{1}{2}\mathrm{Tr}(H^u(t,X_t^N)(d<X^N_\cdot, X^N_\cdot>_t - a(X_{t_i}^N)dt))]\big).
\end{aligned}
$$

The term A_i^1 is not a surprise for a reader familiar with the approximation of diffusions in the whole space (remind that $d<Y_\cdot^{N,i}, Y_\cdot^{N,i}>_t = a(X_{t_i}^N)dt$); it is actually related to the approximation of $(X_t)_{t\geq 0}$ inside the domain. The term A_i^2 really concerns to the approximation near the boundary.

• <u>TERM A_i^1.</u> Using (4), Itô's formula and simplifications coming from $\partial_t u + Lu = 0$ inside $[0,T]\times\overline{D}$, one easily gets

$$
A_i^1 = \mathbf{E}^{\mathcal{F}_{t_i}}\left(\int_{t_i}^{t_{i+1}} dt \int_{t_i}^{t\wedge\tau}\left[\mathcal{B}_s^{u,1}ds + \mathbf{1}_{Y_s^{N,i}\notin D}\mathcal{B}_s^{u,2}ds + \mathcal{B}_s^{u,3}dL_s^0(F^\gamma(Y_\cdot^{N,i}))\right]\right) + \|f\|^{(5)}O_{\exp}(h)
$$

where the processes $(\mathcal{B}_s^{u,1})_s, (\mathcal{B}_s^{u,2})_s, (\mathcal{B}_s^{u,3})_s$ are continuous, adapted and uniformly bounded by $K(T)\|f\|^{(5)}$ since they can be expressed as a sum of products of spatial derivatives of u (up the order 4) and of coefficients b and σ and their derivatives, each of them being evaluated at point $(s, X_{t_i}^N)$ or (s, X_s^N). Hence, from Tanaka's formula, one gets

$$
\begin{aligned}
|A_i^1| &\leq K(T)\|f\|^{(5)}[h^2 + h\mathbf{E}^{\mathcal{F}_{t_i}}[L^0_{t_{i+1}\wedge\tau}(F^\gamma(Y_\cdot^{N,i})) - L^0_{t_i}(F^\gamma(Y_\cdot^{N,i}))] + O_{\exp}(h)]\\
&\leq K(T)\|f\|^{(5)}(h^2 + h\mathbf{E}^{\mathcal{F}_{t_i}}[|F^\gamma(Y^{N,i}_{t_{i+1}\wedge\tau})| - |F^\gamma(Y^{N,i}_{t_i})|] + O_{\exp}(h))
\end{aligned}
$$

and thus

$$
\left|\mathbf{E}\left(\sum_{i=0}^{N-1}\mathbf{1}_{t_i<\tau}A_i^1\right)\right| \leq K(T)\|f\|^{(5)}h\left(1 + \mathbf{E}\big(\sum_{i=0}^{N-1}|F^\gamma(Y^{N,i}_{t_{i+1}\wedge\tau})| - |F^\gamma(Y^{N,i}_{t_i\wedge\tau})|\big)\right).
$$

On $\{\tau \leq T\}$, the above sum is $O_{\exp}(h)$. On $\{\tau > T\}$, $|F^\gamma(Y^{N,i}_{t_{i+1}})| = |F^\gamma(Y^{N,i+1}_{t_i})|$ because of the symmetry procedure, thus the sum is telescoping: this proves that $\left|\mathbf{E}\left(\sum_{i=0}^{N-1}\mathbf{1}_{t_i<\tau}A_i^1\right)\right| \leq K(T)\|f\|^{(5)}h$.

• <u>TERM A_i^2.</u> When we plug (7) into A_i^2, the integral w.r.t. the local time vanishes because of the Neumann condition (2), while the other contributions can be gathered in a sum A_i^{21} involving the terms in factor of $[F^\gamma(Y_t^{N,i})]^-$ and, a sum A_i^{22} involving the terms in factor of $\mathbf{1}_{Y_t^{N,i}\notin D}$.

Term A_i^{21}. One has $A_i^{21} = \mathbf{E}^{\mathcal{F}_{t_i}}\left(\int_{t_i}^{t_{i+1}\wedge\tau}[F^\gamma(Y_t^{N,i})]^-\mathcal{B}_t^{u,4}dt\right)$ where $(\mathcal{B}_t^{u,4})_t$ has the same properties as $(\mathcal{B}_t^{u,j})_t$ $(j\leq 3)$. The Cauchy-Schwarz inequality combined with estimates (6) and $[F^\gamma(Y^{N,i}_{t_i})]^- = 0$, (5) with $\eta = d(Y^{N,i}_{t_i}, \partial D) = d(X^N_{t_i}, \partial D)$ leads to

$$
\begin{aligned}
|A^{21}_i| &\leq K(T)\|f\|^{(5)} \int_{t_i}^{t_{i+1}} \sqrt{\mathbf{E}^{\mathcal{F}_{t_i}}\big([[F^\gamma(Y^{N,i}_t)]^- - [F^\gamma(Y^{N,i}_{t_i})]^-]^2\big)} \sqrt{\mathbf{P}^{\mathcal{F}_{t_i}}\big(Y^{N,i}_t \notin D\big)} dt \\
&\leq K(T)\|f\|^{(5)} h^{3/2} \exp\big(-\frac{c}{2}\frac{d^2(X^N_{t_i}, \partial D)}{h}\big)
\end{aligned}
$$

for some $c > 0$, and thus, one obtains $\big|\mathbf{E}\big(\sum_{i=0}^{N-1} \mathbf{1}_{t_i<\tau} A^{21}_i\big)\big| \leq K(T)\|f\|^{(5)} h$ using Lemma 2.

Term A^{22}_i. It is equal to $A^{22}_i = \mathbf{E}^{\mathcal{F}_{t_i}}\left(\int_{t_i}^{t_{i+1}\wedge\tau} \mathbf{1}_{Y^{N,i}_t \notin D} \mathcal{B}^{u,5}(t, Y^{N,i}_{t_i}, X^N_t, Y^{N,i}_t) dt\right)$ with

$$
\begin{aligned}
&\mathcal{B}^{u,5}(t, x_i, x, y) \\
=&\nabla u(t,x)\Big\{ - 2\nabla\gamma(y) a(x_i)[\nabla F^\gamma(y)]^* - 2\gamma(y)\nabla F^\gamma(y) b(x_i) - \gamma(y)\mathrm{Tr}[H^{F^\gamma}(y) a(x_i)]\Big\} \\
&+ \frac{1}{2}\mathrm{Tr}\Big\{ H^u(t,x)\big(- 4\gamma(y)\nabla F^\gamma(y) a(x_i) + 4\gamma(y)\nabla F^\gamma(y) a(x_i)[\nabla F^\gamma(y)]^*\gamma^*(y)\big)\Big\}.
\end{aligned}
$$

Now, we notice that this function vanishes when $x_i = x = y \in \partial D$ and $t < T$. Indeed, in view of the Neumann condition in (2), the second and third terms involved in factor of ∇u vanish. In addition, one has $\nabla F^\gamma = \frac{n^*}{n.\gamma}$ on ∂D (see assertion v) in Proposition 1). Thus, for all $z \in \partial D$, we have

$$
\begin{aligned}
\mathcal{B}^{u,5}(t,z,z,z) =& -2\nabla u(t,z)\nabla\gamma(z) a(z)\frac{n(z)}{n(z).\gamma(z)} + 2\mathrm{Tr}\Big\{ H^u(t,z)\big[- \gamma(z)\frac{n^*(z)}{n(z).\gamma(z)} a(z) \\
&+ \gamma(z)\frac{n^*(z)}{n(z).\gamma(z)} a(z)\frac{n(z)}{n(z).\gamma(z)}\gamma^*(z)\big]\Big\}.
\end{aligned}
$$

From easy linear algebra ($\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, etc.), it follows that $\mathcal{B}^{u,5}(t,z,z,z) = 2\mathcal{C}^u(z) = 0$ in view of Lemma 3. Our assertion is proved.

Now, set $\tau_i = \inf\{t \geq t_i : Y^{N,i}_t \notin D\}$: on $\{Y^{N,i}_t \notin D\}$, one has $\tau_i \leq t$, $Y^{N,i}_{\tau_i} \in \partial D$ and $\mathcal{B}^{u,5}(t, Y^{N,i}_{\tau_i}, Y^{N,i}_{\tau_i}, Y^{N,i}_{\tau_i}) = 0$. Since $\mathcal{B}^{u,5}(t,.)$ is continuously differentiable with first derivatives bounded by $K(T)\|f\|^{(5)}$, we easily deduce that

$$
\begin{aligned}
|A^{22}_i| &\leq K(T)\|f\|^{(5)} \mathbf{E}^{\mathcal{F}_{t_i}}\left(\int_{t_i}^{t_{i+1}\wedge\tau} \mathbf{1}_{Y^{N,i}_t \notin D}\big(|Y^{N,i}_{t_i} - Y^{N,i}_{\tau_i}| + |X^N_t - Y^{N,i}_{\tau_i}| + |Y^{N,i}_t - Y^{N,i}_{\tau_i}|\big) dt\right) \\
&\leq K(T)\|f\|^{(5)} \mathbf{E}^{\mathcal{F}_{t_i}}\left(\int_{t_i}^{t_{i+1}\wedge\tau} \mathbf{1}_{Y^{N,i}_t \notin D}\big(|Y^{N,i}_{t_i} - Y^{N,i}_{\tau_i}| + [F^\gamma(Y^{N,i}_t)]^- + |Y^{N,i}_t - Y^{N,i}_{\tau_i}|\big) dt\right),
\end{aligned}
$$

for a constant $K(T)$ changing from line to line. We now apply arguments already used for A^{21}_i. It comes $|A^{22}_i| \leq K(T)\|f\|^{(5)} h^{3/2} \exp\big(-\frac{c}{2}\frac{d^2(X^N_{t_i}, \partial D)}{h}\big)$, and then $\big|\mathbf{E}\big(\sum_{i=0}^{N-1} \mathbf{1}_{t_i<\tau} A^{22}_i\big)\big| \leq K(T)\|f\|^{(5)} h$. The proof of Theorem 1 is complete.

Proof of Lemma 2

If we knew that $X^N_{t_i}$ had a density w.r.t. the Lebesgue measure on D, *uniformly bounded* in N near ∂D, we could easily conclude that $\mathcal{A}_i := \mathbf{E}\big(\mathbf{1}_{t_i<\tau}\exp\big(-c\frac{d^2(X^N_{t_i},\partial D)}{h}\big)\big) \le K(T)\sqrt{h}$. But the desired property on the density of $X^N_{t_i}$ seems difficult to prove, even by using Malliavin calculus tools (because of the $[F^\gamma]^-$ terms).

The idea of our proof is to use the occupation times formula. By iii) and iv) in Proposition 1, for $d(x,\partial D)\le R$ one has $d(x,\partial D)=|F^n(x)|\ge |F^\gamma(x)|/c_1$, and thus

$$\mathcal{A}_{i+1}\le \mathbf{E}\big(\mathbf{1}_{t_{i+1}<\tau}\exp\big(-c\frac{[F^\gamma(X^N_{t_{i+1}})]^2}{c_1^2 h}\big)\big)+O_{\exp}(h).$$

Set $c'=c/(2c_1^2)>0$ and $g(x)=\exp(-2c'x^2/h)$: it is easy to check that $|g(x)|+\sqrt{h}|g'(x)|+h|g''(x)|\le K(T)\exp(-c'x^2/h)$. Hence, for $t\in[t_i,t_{i+1}]$, Itô's formula combined with the decomposition (7) and the estimate (4) yields $\mathbf{E}\big(\mathbf{1}_{t_{i+1}<\tau}\exp\big(-2c'\frac{[F^\gamma(X^N_{t_{i+1}})]^2}{h}\big)\big)\le K(T)\big[\mathbf{E}\big(\mathbf{1}_{t<\tau}\exp\big(-c'\frac{[F^\gamma(X^N_t)]^2}{h}\big)\big)+\frac{1}{h}\int_t^{t_{i+1}}\mathbf{E}\big(\mathbf{1}_{s<\tau}\exp\big(-c'\frac{[F^\gamma(X^N_s)]^2}{h}\big)\big)ds\big]+O_{\exp}(h)$; notice that the local time involved in (7) provides no contribution in the preceding computation because $g'(0)=0$. Integrate this inequality w.r.t. t over $[t_i,t_{i+1}]$ to get

$$h\mathcal{A}_{i+1}\le K(T)\int_{t_i}^{t_{i+1}}\mathbf{E}\big[\mathbf{1}_{s<\tau}\exp\big(-c'\frac{[F^\gamma(X^N_s)]^2}{h}\big)\big]ds+O_{\exp}(h).$$

Observe that for $|F^\gamma(y)|\le R$, $d<F^\gamma(X^N_\cdot),F^\gamma(X^N_\cdot)>_s=\nabla F^\gamma(X^N_s)a(X^N_{t_i})[F^\gamma(X^N_s)]^*ds\ge \sigma_0^2/4\ ds$ using **(E)** and $|\nabla F^\gamma(y)|\ge 1/2$ for $|F^\gamma(y)|\le R$ (we can assume this last property by decreasing R in Proposition 1 if necessary). It readily follows from the occupation times formula that

$$h\mathcal{A}_{i+1}\le K(T)\int_{-R}^{R}dy\exp\big(-c'\frac{y^2}{h}\big)\mathbf{E}\big[L^y_{t_{i+1}\wedge\tau}(F^\gamma(X^N_\cdot))-L^y_{t_i\wedge\tau}(F^\gamma(X^N_\cdot))\big]+O_{\exp}(h).$$

Now,

$$\begin{aligned}
&\frac{1}{2}\mathbf{E}\big[L^y_{t_{i+1}\wedge\tau}(F^\gamma(X^N_\cdot))-L^y_{t_i\wedge\tau}(F^\gamma(X^N_\cdot))\big]\\
=&\mathbf{E}\big[(F^\gamma(X^N_{t_{i+1}\wedge\tau})-y)^+-(F^\gamma(X^N_{t_i\wedge\tau})-y)^+-\int_{t_i\wedge\tau}^{t_{i+1}\wedge\tau}\mathbf{1}_{F^\gamma(X^N_s)\ge y}d(F^\gamma(X^N_s))\big]\\
\le&\mathbf{E}\big[(F^\gamma(X^N_{t_{i+1}\wedge\tau})-y)^+-(F^\gamma(X^N_{t_i\wedge\tau})-y)^+\big]+K(T)h\\
&-\mathbf{E}\big[\int_{t_i\wedge\tau}^{t_{i+1}\wedge\tau}\mathbf{1}_{F^\gamma(X^N_s)\ge y}\nabla F^\gamma(X^N_s)\gamma(Y^{N,i}_s)dL^0_s(F^\gamma(Y^{N,i}_\cdot))\big].
\end{aligned}$$

We have used (7) to get the last inequality. The above integral w.r.t. the local time is non-negative since $\nabla F^\gamma\gamma=1$ on ∂D.
Therefore $\sum_{i=0}^{N-1}\mathbf{E}\big[L^y_{t_{i+1}\wedge\tau}(F^\gamma(Y^{N,i}_\cdot))-L^y_{t_i\wedge\tau}(F^\gamma(Y^{N,i}_\cdot))\big]\le K(T)$ uniformly in $|y|\le R$ since the sum is telescoping. One then concludes that $h\sum_{i=0}^{N-1}\mathcal{A}_{i+1}\le K(T)\sqrt{h}$. The proof of Lemma 2 is complete.

Proof of Lemma 3

In the following, $t < T$ is fixed and we omit it. Since the function $\nabla u\ \gamma$ vanishes on ∂D, one has $n[\nabla(\nabla u\ \gamma)n] = [\nabla(\nabla u\ \gamma)]^*$. Taking into account that $\nabla(\nabla u\ \gamma) = \gamma^* H^u + \nabla u \nabla\gamma$, we derive the following identity on the boundary:

$$H^u\gamma = n(\gamma^* H^u n) + n(\nabla u\nabla\gamma n) - (\nabla u\nabla\gamma)^*.$$

We thus have

$$\begin{aligned}\mathcal{C}^u(x) = &- \nabla u\nabla\gamma a\frac{n}{n.\gamma} + \frac{(n^* a n)}{(n.\gamma)^2}\gamma^*\left[n(\gamma^* H^u n) + n(\nabla u\nabla\gamma n) - (\nabla u\nabla\gamma)^*\right]\\ &- \frac{1}{n.\gamma}n^* a\left[n(\gamma^* H^u n) + n(\nabla u\nabla\gamma n) - (\nabla u\nabla\gamma)^*\right] = -\frac{(n^* a n)}{(n.\gamma)^2}(\nabla u\nabla\gamma\ \gamma)^*.\end{aligned}$$

The right hand side is equal to 0 on ∂D since $\nabla\gamma\ \gamma = 0$ on ∂D: indeed, it follows from $\gamma(x + \lambda\gamma(x)) = \gamma(x)$ for $x \in \partial D$ and $|\lambda| \le R$ (see Proposition 1). We are finished.

4. A numerical example

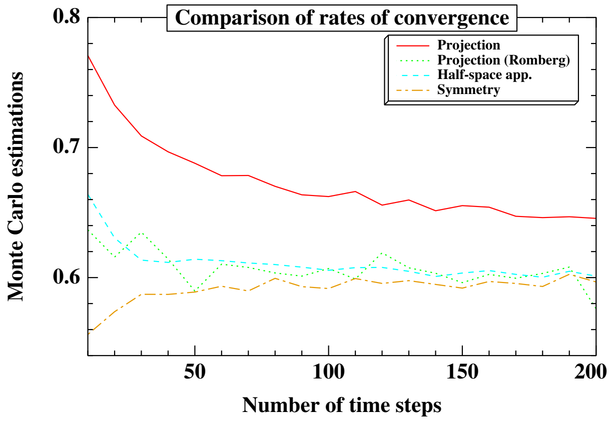

FIGURE 2: Comparison of the weak error for four schemes.

We consider for X a 3-dimensional Brownian motion normally reflected in the unit ball $D = S_3(0,1)$ and say we are interested in the evaluation of $\mathbf{E}(\|X_1\|^2)$: the exact value is unknown as far as we know. To make the experiment more interesting, we compare the scheme of this paper with two other ones: the usual *projected Euler scheme* (see [3]) with order of convergence equal to $\frac{1}{2}$, and the *reflected Euler scheme* on local half-space approximations (see [8]) with order of convergence equal to 1 in this example. We also consider the *Romberg extrapolation* (see [22]) with the projected scheme, assuming that an expansion of the error at order 1/2 is available: it gives $\mathbf{E}\left(\frac{\sqrt{2}f(X_T^N)-f(X_T^{N/2})}{\sqrt{2}-1}\right) = \mathbf{E}(f(X_T)) + o(\sqrt{h})$. The number of simulated paths is $M = 10000$: it provides a width of the 95%-confidence interval essentially equal to 0.03 for each scheme (except for the Romberg extrapolition for which it is larger, that is 0.035).

We plot on Figure 2 the Monte-Carlo estimators w.r.t. the number of time steps N, to get an idea of the efficiency of each procedure. It turns out that procedures with symmetry, half-space approximation and Romberg extrapolation behave both very well. However, the computational time is much smaller for the method presented here because of the simplicity of the symmetry (in fact as simple as the projection method): see table 1.

	Projected scheme	Projected scheme with Romberg extraplolation	Symmetrized scheme	Reflected scheme in local half-space approximation
CPU time	0.92s	1.37s	0.92s	1.52s

TABLE 1: Computational time for each scheme when $N = 50$.

5. Conclusion

We have proved that an Euler scheme with a symmetry procedure yields a accurate approximation of obliquely reflected diffusions when one marginal of the law is evaluated. At last, we give two open issues that we have not been able to handle:

1. how to get an expansion of the error w.r.t. h? it seems that sharper extimates on the law of X^N near the ∂D are needed.
2. While with previous approaches ([3], [8]) it is possible to simulate the local time on ∂D (and hence to evaluate expectations of more complex functionals of type $\mathbf{E}(\int_0^T g(X_t)dk_t)$), we do not have good ideas how to adapt our algorithm to approximate in a satisfactory and accurate way (with first order convergence) these quantities.

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