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A functional central limit theorem in equilibrium for a large network in which customers join the shortest of several queues

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Abstract. We consider N single server infinite buffer queues with service rate β. Customers arrive at rate $N\alpha$, choose L queues uniformly, and join the shortest one. The stability condition is $\alpha < \beta$. We study in equilibrium the fraction of queues of length at least $k \geq 0$. We prove a functional central limit theorem on an infinite-dimensional Hilbert space with its weak topology, with limit a stationary Ornstein-Uhlenbeck process. We use ergodicity and justify the inversion of limits $\lim_{N\to\infty} \lim_{t\to\infty} = \lim_{t\to\infty} \lim_{N\to\infty}$ by a compactness-uniqueness method. The main tool for proving tightness of the ill-known invariant laws and ergodicity of the limit is a global exponential stability result for the nonlinear dynamical system obtained in the functional law of large numbers limit.

Key-words: Mean-field interaction, ergodicity, equilibrium fluctuations, birth and death processes, spectral gap, global exponential stability, nonlinear dynamical systems

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1 Introduction

1.1 The queuing network, and some notation

Customers arrive at rate $N\alpha$ on a network constituted of $N \geq L \geq 1$ infinite buffer single server queues. Each customer is allocated L distinct queues uniformly at random and joins the shortest, ties being resolved uniformly. Servers work at rate β. Inter-arrival times, allocations, and services are independent and memoryless. For $L = 1$ we have N i.i.d. $M_\alpha/M_\beta/1/\infty$ queues, and for $L \geq 2$ the interaction structure depends only on sampling from the empirical measure of L-tuples of queue states. In statistical mechanics terminology, this system is in L-body mean-field interaction.

The process $(X_i^N)_{1\leq i\leq N}$, where $X_i^N(t)$ denotes the length of queue i at time $t \geq 0$, is Markov. Its empirical measure μ^N with samples in $\mathcal{P}(\mathbb{D}(\mathbb{R}_+, \mathbb{N}))$ and its marginal process $\bar{X}^N = (\bar{X}_t^N)_{t\geq 0}$ with sample paths in $\mathbb{D}(\mathbb{R}_+, \mathcal{P}(\mathbb{N}))$ are given by

$$\mu^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i^N}\,, \qquad \bar{X}_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i^N(t)}\,.$$

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We are interested in the tails of the distributions $\bar{X}_t^N$. We consider

$$\mathcal{V} = \left\{ (v(k))_{k\in\mathbb{N}} : v(0) = 1,\ v(k) \geq v(k+1),\ \lim_{k\to\infty} v(k) = 0 \right\} \subset c_0\,, \quad \mathcal{V}^N = \mathcal{V} \cap \frac{1}{N}\mathbb{N}^{\mathbb{N}}\,,$$

with the uniform topology. Note that the uniform and the product topology coincide on $\mathcal{V}$. We consider the process $R^N = (R_t^N)_{t\geq 0}$ with sample paths in $\mathbb{D}(\mathbb{R}_+, \mathcal{V}^N)$ given by

$$R_t^N(k) = \frac{1}{N}\sum_{i=1}^N \mathbb{I}_{X_i^N(t)\geq k}\,, \qquad k \in \mathbb{N}\,,$$

the fraction of queues at time t of length at least k.

We have $R_t^N(k) = \bar{X}_t^N([k,\infty[)$ and $X_t^N\{k\} = R_t^N(k) - R_t^N(k+1)$ using the classical homeomorphism between $\mathcal{P}(\mathbb{N})$ and $\mathcal{V}$, which maps the subspace of probability measures with finite first moment onto $\mathcal{V} \cap \ell_1$ corresponding to having a finite number of customers. The symmetry structure implies that $\bar{X}^N$ and R^N are Markov processes.

The network is ergodic if and only if $\alpha < \beta$ (Theorem 5 (a) in [12], Theorem 4.2 in [6]). The proofs use non-constructive ergodicity criteria, and we lack information and controls on the invariant laws (stationary distributions). We study the large N asymptotics in the stationary regime using an indirect approach involving ergodicity in appropriate transient regimes and an inversion of limits for large N and large times. Law of large numbers (LLN) results are already known, and we shall obtain a functional central limit theorem (CLT).

General notation. We denote by c_0^0 and ℓ_p^0 for $p \geq 1$ the subspaces of sequences vanishing at 0 of the classical sequence spaces c_0 (with limit 0) and ℓ_p (with summable p-th power). The diagonal matrix with successive diagonal terms given by the sequence a is denoted by $\mathrm{diag}(a)$. When using matrix notations, sequences vanishing at 0 are often identified with infinite column vectors indexed by $\{1, 2, \cdots\}$. Sequence inequalities, etc., should be interpreted termwise. Empty sums are equal to 0 and empty products to 1. Constants such as K may vary from line to line. We denote by $g_\theta = (\theta^k)_{k\geq 1}$ the geometric sequence of reason θ.

1.2 Previous results: laws of large numbers

We relate results found in essence in Vvedenskaya et al. [12]. Graham [6] extended some of these results, and also considered the empirical measures on path space μ^N, yielding chaoticity results (asymptotic independence of queues). (The rates ν and λ in [6] are replaced here by α and β.)

Consider the mappings with values in c_0^0 given for v in c_0 by

$$F_+(v)(k) = \alpha\big(v(k-1)^L - v(k)^L\big)\,, \quad F_-(v)(k) = \beta(v(k) - v(k+1))\,, \qquad k \geq 1\,, \qquad (1.1)$$

and $F = F_+ - F_-$ and the nonlinear differential equation $\dot{u} = F(u)$ on $\mathcal{V}$ given for $t \geq 0$ by

$$\dot{u}_t(k) = F(u_t)(k) = \alpha\big(u_t(k-1)^L - u_t(k)^L\big) - \beta(u_t(k) - u_t(k+1))\,, \qquad k \geq 1\,. \qquad (1.2)$$

This is the infinite system of scalar differential equations (1.6) in [12] (where the arrival rate is λ and service rate 1) and (3.9) in [6]. Note that F_- is linear.

Theorem 1.1 *There exists a unique solution $u = (u_t)_{t\geq 0}$ taking values in $\mathcal{V}$ for (1.2), and u is in $C(\mathbb{R}_+, \mathcal{V})$. If u_0 is in $\mathcal{V} \cap \ell_1$ then u takes values in $\mathcal{V} \cap \ell_1$.*

Proof. We use Theorem 3.3 and Proposition 2.3 in [6]. These exploit the homeomorphism between $\mathcal{P}(\mathbb{N})$ with the weak topology and $\mathcal{V}$ with the product topology. Then (1.2) corresponds to a non-linear forward Kolmogorov equation for a pure jump process with uniformly bounded (time-dependent) jump rates. Uniqueness within the class of bounded measures and existence of a probability-measure valued solution are obtained using the total variation norm. Theorem 1 (a) in [12] yields existence (and uniqueness) in $\mathcal{V} \cap \ell_1$. $\square$

Firstly, a functional LLN for initial conditions satisfying a LLN is part of Theorem 3.4 in [6] and can be deduced from Theorem 2 in [12].

Theorem 1.2 *Assume that $(R_0^N)_{N\geq L}$ converges in law to u_0 in $\mathcal{V}$. Then $(R^N)_{N\geq L}$ converges in law in $\mathbb{D}(\mathbb{R}_+, \mathcal{V})$ to the unique solution $u = (u_t)_{t\geq 0}$ starting at u_0 for (1.2).*

Secondly, the limit equation (1.2) has a globally attractive stable point $\tilde{u}$ in $\mathcal{V} \cap \ell_1$.

Theorem 1.3 *For $\rho = \alpha/\beta < 1$ the equation (1.2) has a unique stable point $\tilde{u}$ in $\mathcal{V}$ given by*

$$\tilde{u} = (\tilde{u}(k)_{k\in\mathbb{N}}\,, \qquad \tilde{u}(k) = \rho^{(L^k-1)/(L-1)} = \rho^{L^{k-1}+L^{k-2}+\cdots+1}\,,$$

and the solution u of (1.2) starting at any u_0 in $\mathcal{V} \cap \ell_1$ is such that $\lim_{t\to\infty} u_t = \tilde{u}$.

Proof. Theorem 1 (b) in [12] yields that $\tilde{u}$ is globally asymptotically stable in $\mathcal{V} \cap \ell_1$. A stable point u in $\mathcal{V}$ satisfies $\beta u(k+1) - \alpha u(k)^L = \beta u(k) - \alpha u(k-1)^L = \cdots = \beta u(1) - \alpha$ and converges to 0, hence $u(1) = \alpha/\beta$ and $u(2), u(3), \ldots$ are successively determined uniquely. $\square$

Lastly, a compactness-uniqueness method justifying the inversion of limits $\lim_{N\to\infty} \lim_{t\to\infty} = \lim_{t\to\infty} \lim_{N\to\infty}$ yields a result in equilibrium. This method was used by Whitt [13] for the star-shaped loss network, and is described in detail in Graham [5] Sections 9.5 and 9.7.3. The following functional LLN in equilibrium (Theorem 4.4 in [6]) can be deduced from [12], but is not stated there as such; it implies using uniform integrability bounds that under the invariant laws $\lim_{N\to\infty} \mathbf{E}(R_0^N(k)) = \tilde{u}(k)$ for $k \in \mathbb{N}$, a result stated in Theorem 5 (c) in [12].

Theorem 1.4 *Let $\rho = \alpha/\beta < 1$ and the networks of size $N \geq L$ be in equilibrium. Then $(R^N)_{N\geq L}$ converges in probability in $\mathbb{D}(\mathbb{R}_+, \mathcal{V})$ to $\tilde{u}$.*

Note that $\tilde{u}(k)$ decays hyper-exponentially in k for $L \geq 2$ instead of the exponential decay ρ^k corresponding to i.i.d. queues in equilibrium ($L = 1$). The asymptotic large queue sizes are dramatically decreased by this simple choice.

We seek rates of convergence and confidence intervals. Theorem 3.5 in [6] gives convergence bounds when $(X_i^N(0))_{1\leq i\leq N}$ are i.i.d. for the variation norm on $\mathcal{P}(\mathbb{D}([0,T],\mathbb{N}^k))$ using results in Graham and Méléard [7]. This can be extended if the initial laws satisfy a priori controls, but it is not so in equilibrium, where on the contrary controls are obtained using the network evolution.

1.3 The outline of this paper

We consider the process R^N with values in $\mathcal{V}^N$, a solution $u = (u_t)_{t\geq 0}$ for (1.2) in $\mathcal{V}$, and the empirical fluctuation processes $Z^N = (Z_t^N)_{t\geq 0}$ with sample paths in c_0^0 given by

$$Z^N = N^{1/2}(R^N - u)\,, \qquad Z_t^N = N^{1/2}(R_t^N - u_t)\,. \tag{1.3}$$

We are interested in particular in the stationary regime, which defines *implicitly* the initial data: the law of R_0^N is the invariant law for R^N and $u_0 = \tilde{u}$.

Our main result is a functional CLT: in equilibrium $(Z^N)_{N\geq L}$ converges in law to a stationary Ornstein-Uhlenbeck process, which we characterize. This *implies* a CLT for the marginal laws: under the invariant laws $(Z_0^N)_{N\geq L}$ converges to the invariant law for this Gaussian process. This important result seems very difficult to obtain directly. We use ergodicity of Z^N for fixed N and intricate fine studies of the long-time behavior of the nonlinear dynamics appearing at the large N limit, simply in order to prove tightness bounds for $(Z_0^N)_{N\geq L}$ under the invariant laws and ergodicity for the Ornstein-Uhlenbeck process.

Section 2 introduces the main theorems, which are proved in subsequent sections. Section 3 considers arbitrary u_0 and R_0^N and derives martingales of interest and the limit Ornstein-Uhlenbeck process. We consider the stationary regime whenever possible for simplicity, but the infinite-horizon bounds used for the control of the invariant laws are obtained considering *transient* regimes.

We study the Ornstein-Uhlenbeck process in Section 4. We give a spectral representation for the linear operator in the drift term, and prove the existence of a spectral gap. A main difficulty is that the Hilbert space in which this operator is self-adjoint is *not* large enough (its norm is *too* strong) for the limit non-linear dynamical system and for the invariant laws for finite N. We obtain results of global exponential stability in appropriate Hilbert spaces in which it is *not* self-adjoint.

In Section 5 we prove that $\tilde{u}$ is globally exponentially stable for the non-linear dynamical system in appropriate Hilbert spaces. In Section 6, uniformly for large N, we obtain bounds for the processes Z^N on $[0,T]$ using martingale properties, and then for Z_t^N uniformly for $t \geq 0$ using the above result on the dynamical system in order to iterate the bounds on intervals of length T. Bounds on the invariant laws of Z^N follow using ergodicity. We then prove the functional CLT by a compactness-uniqueness method and martingale characterizations. We consider the non-metrizable weak topology on the Hilbert spaces, and use adapted tightness criteria and the above bounds.

2 The functional central limit theorem in equilibrium

We assume that $\rho = \alpha/\beta < 1$ and $u_0 = \tilde{u} = u$, but could easily likewise study appropriate transient regimes. We quickly introduce notation and state the main results, leaving most proofs for later.

2.1 Preliminaries

For any sequence $w = (w(k))_{k\geq 1}$ such that $w > 0$ we define the Hilbert spaces

$$L_2(w) = \left\{ x \in \mathbb{R}^{\mathbb{N}} : x(0) = 0\,,\ \|x\|^2_{L_2(w)} = \sum_{k\geq 1} \left(\frac{x(k)}{w(k)}\right)^2 w(k) = \sum_{k\geq 1} x(k)^2 w(k)^{-1} < \infty \right\}$$

and in matrix notation $(x, y)_{L_2(w)} = x^* \mathrm{diag}(w^{-1}) y$. We consider the elements of $L_2(w)$ as measures identified with their densities with respect to the reference measure w. Then $L_1(w) = \ell_1^0$ and if w is summable then $\|x\|_1 \leq \|w\|_1^{1/2} \|x\|_{L_2(w)}$ and $L_2(w) \subset \ell_1^0$. Using $L_2(1) = \ell_2^0$ as a pivot space, for bounded w we have the Gelfand triplet of Hilbert spaces $L_2(w) \subset \ell_2^0 \subset L_2(w)^* = L_2(w^{-1})$.

Lemma 2.1 *If $w = O(v)$ and $v = O(w)$ then the $L_2(v)$ and $L_2(w)$ norms are equivalent.*

Proof. This follows from obvious computations. $\square$

We give a refined existence result for (1.2). We recall that $g_\theta = (\theta^k)_{k\geq 1}$.

Theorem 2.2 *Let $w > 0$ be such that there exists $c > 0$ and $d > 0$ with*

$$cw(k+1) \leq w(k) \leq dw(k+1)\,, \qquad k \geq 1\,.$$

Then in $\mathcal{V} \cap L_2(w)$ the mappings F, F_+ and F_- are Lipschitz for the $L_2(w)$ norm and there is existence and uniqueness for (1.2). The assumptions and conclusions hold for $w = g_\theta$ for $\theta > 0$.

Proof. The identity $x^L - y^L = (x-y)(x^{L-1} + x^{L-2}y + \cdots + y^{L-1})$ yields

$$\begin{aligned}
\left(u(k-1)^L - v(k-1)^L\right)^2 w(k)^{-1} &\leq \left(u(k-1) - v(k-1)\right)^2 L^2 d w(k-1)^{-1}\,,\\
\left(u(k)^L - v(k)^L\right)^2 w(k)^{-1} &\leq \left(u(k) - v(k)\right)^2 L^2 w(k)^{-1}\,,\\
\left(u(k+1) - v(k+1)\right)^2 w(k)^{-1} &\leq \left(u(k+1) - v(k+1)\right)^2 c^{-1} w(k+1)^{-1}\,,
\end{aligned}$$

hence we have the Lipschitz bounds $\|F_+(u) - F_+(v)\|^2_{L_2(w)} \leq 2\alpha^2 L^2 (d+1) \|u - v\|^2_{L_2(w)}$ and $\|F_-(u) - F_-(v)\|^2_{L_2(w)} \leq 2\beta^2 (c^{-1} + 1) \|u - v\|^2_{L_2(w)}$ and existence and uniqueness follows by a classical Cauchy-Lipschitz method. We have $\theta^{-1}\theta^{k+1} \leq \theta^k \leq \theta^{-1}\theta^{k+1}$ for $k \geq 1$. $\square$

2.2 The Ornstein-Uhlenbeck process

We consider the linear operator $\mathcal{K} : x \in c_0^0 \mapsto \mathcal{K}x \in c_0^0$ given by

$$\begin{aligned}\mathcal{K}x(k) &= \alpha L\tilde{u}(k-1)^{L-1}x(k-1) - \left(\alpha L\tilde{u}(k)^{L-1} + \beta\right)x(k) + \beta x(k+1) \\ &= \beta L\rho^{L^{k-1}}x(k-1) - \left(\beta L\rho^{L^k} + \beta\right)x(k) + \beta x(k+1)\,, \qquad k \ge 1\,, \end{aligned} \tag{2.1}$$

which we identify with its infinite matrix in the canonical basis $(0,1,0,0\ldots), (0,0,1,0\ldots), \ldots$

$$\mathcal{K} = \begin{pmatrix} -\left(\beta L\rho^{L} + \beta\right) & \beta & 0 & 0 & \cdots \\ \beta L\rho^{L} & -\left(\beta L\rho^{L^2} + \beta\right) & \beta & 0 & \cdots \\ 0 & \beta L\rho^{L^2} & -\left(\beta L\rho^{L^3} + \beta\right) & \beta & \cdots \\ 0 & 0 & \beta L\rho^{L^3} & -\left(\beta L\rho^{L^4} + \beta\right) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{2.2}$$

used identifying the sequence $x = (0, x(1), x(2), \ldots)$ with its coordinates in the canonical basis $(x(1), x(2), \ldots)$ taken as a column vector.

Note that $\mathcal{K} = \mathcal{A}^*$ where $\mathcal{A}$ is the infinitesimal generator of a sub-Markovian birth and death process. We shall develop this point of view and obtain a spectral decomposition for $\mathcal{K}$ in Section 4.2, to which we give a few anticipated references below. The potential coefficients of $\mathcal{A}$ given by

$$\pi = (\pi(k))_{k\ge 1}\,, \qquad \pi(k) = L^{k-1}\rho^{(L^k-L)/(L-1)} = \rho^{-1}L^{k-1}\tilde{u}(k)\,,$$

solve the detailed balance equations $\pi(k+1) = L\rho^{L^k}\pi(k)$ with $\pi(1) = 1$.

The linearization of (1.2) around its stable point $\tilde{u}$ is the linearization of the equation satisfied by $z = u - \tilde{u}$ and is given for $t \ge 0$ by the forward Kolmogorov equation

$$\dot{z}_t = \mathcal{K}z_t\,. \tag{2.3}$$

Let $B = (B(k))_{k\in\mathbb{N}}$ be independent Brownian motions such that $B(0) = 0$ and $\mathrm{var}(B_1(k)) = \mathbf{E}(B_1(k)^2) = \tilde{v}(k)$ where $\tilde{v}$ in c_0^0 is given by

$$\tilde{v}(k) = 2\beta\left(\tilde{u}(k) - \tilde{u}(k+1)\right) = 2\beta\rho^{(L^k-1)/(L-1)}\left(1-\rho^{L^k}\right), \qquad k \ge 1\,.$$

The infinitesimal covariance matrix of B is given by $\mathrm{diag}(\tilde{v})$.

Theorem 2.3 *The process B is an Hilbertian Brownian motion in $L_2(w)$ if and only if*

$$\sum_{k\ge 1}\tilde{u}(k)w(k)^{-1} = \sum_{k\ge 1}\rho^{(L^k-1)/(L-1)}w(k)^{-1} < \infty\,. \tag{2.4}$$

This is true for $w = \pi$ and $w = g_\theta$ for $\theta > 0$ when $L \ge 2$ or for $w = g_\theta$ for $\theta > \rho$ when $L = 1$.

Proof. This follows from obvious computations. □

The Ornstein-Uhlenbeck process $Z = (Z(k))_{k\in\mathbb{N}}$ solves the affine SDE given for $t \ge 0$ by

$$Z_t = Z_0 + \int_0^t \mathcal{K} Z_s\, ds + B_t \tag{2.5}$$

which is a Brownian perturbation of (2.3).

Theorem 2.4 *Let $w > 0$ be such that there exists $c > 0$ and $d > 0$ with*

$$cw(k+1) \le w(k) \le d\rho^{-2L^k} w(k+1)\,, \qquad k \ge 1\,.$$

(a) In $L_2(w)$, the operator $\mathcal{K}$ is bounded, equation (2.3) has a unique solution $z_t = \mathrm{e}^{\mathcal{K}t} z_0$ where $\mathrm{e}^{\mathcal{K}t}$ has a spectral representation given by (4.1), and there is uniqueness of solutions for the SDE (2.5). The assumptions and conclusions hold for $w = \pi$ and $w = g_\theta$ for $\theta > 0$.
(b) In addition let w satisfy (2.4). The SDE (2.5) has a unique solution $Z_t = \mathrm{e}^{\mathcal{K}t} Z_0 + \int_0^t \mathrm{e}^{\mathcal{K}(t-s)}\, dB_s$ in $L_2(w)$, further explicited in (4.2). The assumptions and conclusions hold for $w = \pi$ and $w = g_\theta$ for $\theta > 0$ when $L \ge 2$ or for $w = g_\theta$ for $\theta > \rho$ when $L = 1$.

Theorem 2.5 *(Spectral gap.) The operator $\mathcal{K}$ is bounded self-adjoint in $L_2(\pi)$. The least point γ of the spectrum of $\mathcal{K}$ is such that $0 < \gamma \le \beta$. The solution $z_t = \mathrm{e}^{\mathcal{K}t} z_0$ for (2.3) in $L_2(\pi)$ satisfies $\|z_t\|_{L_2(\pi)} \le \mathrm{e}^{-\gamma t} \|z_0\|_{L_2(\pi)}$.*

The $L_2(\pi)$ norm is too strong for studying the CLT. Indeed, $\mathbf{P}(X_1^N + \cdots + X_N^N \ge Nk) \le \mathbf{P}(X_1^N \ge k) + \cdots + \mathbf{P}(X_N^N \ge k)$ and since the total service rate in the system cannot exceed $N\beta$, by comparison with an $M_{N\alpha}/M_{N\beta}/1$ queue, in equilibrium

$$\mathbf{E}(R_t^N(k)) = \mathbf{P}(X_i^N(t) \ge k) \ge \frac{1}{N}\rho^{Nk}$$

decreases at most exponentially in $k \ge 0$. Further, the mapping F_+ is not Lipschitz in $\mathcal{V} \cap L_2(\pi)$ for the $L_2(\pi)$ norm, see Theorem 2.2 and the contrasting assumptions and proof of Theorem 2.4. We prove global exponential stability in appropriate spaces.

Theorem 2.6 *Let $0 < \theta < 1$ when $L \ge 2$ or $\rho \le \theta < 1$ when $L = 1$. There exists $\gamma_\theta > 0$ and $C_\theta < \infty$ such that the solution $z_t = \mathrm{e}^{\mathcal{K}t} z_0$ for (2.3) in $L_2(g_\theta)$ satisfies $\|z_t\|_{L_2(g_\theta)} \le \mathrm{e}^{-\gamma_\theta t} C_\theta \|z_0\|_{L_2(g_\theta)}$.*

We deduce exponential ergodicity for the Ornstein-Uhlenbeck process, valid for any w satisfying the conclusions of Theorems 2.4 and 2.6.

Theorem 2.7 *Let $w = \pi$ or $w = g_\theta$ with $0 < \theta < 1$ when $L \ge 2$ or let $w = g_\theta$ with $\rho < \theta < 1$ when $L = 1$. Any solution for the SDE (2.5) in $L_2(w)$ converges in law for large times to its unique invariant law (exponentially fast). This law is the law of $\int_0^\infty \mathrm{e}^{\mathcal{K}t} dB_t$ which is Gaussian centered with covariance matrix $\int_0^\infty \mathrm{e}^{\mathcal{K}t} \mathrm{diag}(\tilde{v}) \mathrm{e}^{\mathcal{K}^* t} dt$, further explicited in (4.3) and (4.4). There is a unique stationary Ornstein-Uhlenbeck process solving the SDE (2.5) in $L_2(w)$.*

2.3 Global exponential stability for the dynamical system and tightness estimates

Global exponential stability of the dynamical system allows control of the invariant laws using the long time behavior. We need uniformity over the state space, and Theorems 2.5 or 2.6 are useless for this purpose (except in the linear case $L = 1$). Such a result does *not* hold in $L_2(\pi)$ for $L \geq 2$.

Theorem 2.8 *Let $\rho \leq \theta < 1$ and u be the solution of (1.2) starting at u_0 in $\mathcal{V} \cap L_2(g_\theta)$. There exists $\gamma_\theta > 0$ and $C_\theta < \infty$ such that $\|u_t - \tilde{u}\|_{L_2(g_\theta)} \leq \mathrm{e}^{-\gamma_\theta t} C_\theta \|u_0 - \tilde{u}\|_{L_2(g_\theta)}$.*

The following finite-horizon bounds yield tightness estimates for the processes $(Z^N)_{N\geq L}$ provided the initial laws are known to satisfy similar bounds.

Lemma 2.9 *For $\theta > 0$ and $T \geq 0$ we have*

$$\limsup_{N\geq L} \mathbf{E}\left(\|Z_0^N\|^2_{L_2(g_\theta)}\right) < \infty \Rightarrow \limsup_{N\geq L} \mathbf{E}\left(\sup_{0\leq t\leq T} \|Z_t^N\|^2_{L_2(g_\theta)}\right) < \infty\,.$$

Theorem 2.8 is an essential ingredient in the proof of the following infinite-horizon bound for the marginal laws of the processes.

Lemma 2.10 *Let $\rho \leq \theta < 1$ when $L \geq 2$ or $\rho < \theta < 1$ when $L = 1$. Then*

$$\limsup_{N\geq L} \mathbf{E}\left(\|Z_0^N\|^2_{L_2(g_\theta)}\right) < \infty \Rightarrow \limsup_{N\geq L} \sup_{t\geq 0} \mathbf{E}\left(\|Z_t^N\|^2_{L_2(g_\theta)}\right) < \infty\,.$$

This yields control of the long time limit of the marginals, the invariant law, which in turn will enable us to use Lemma 2.9 to prove tightness of the processes in equilibrium.

Lemma 2.11 *Let $\rho \leq \theta < 1$ when $L \geq 2$ or $\rho < \theta < 1$ when $L = 1$. Then under the invariant laws*

$$\limsup_{N\geq L} \mathbf{E}\left(\|Z_0^N\|^2_{L_2(g_\theta)}\right) < \infty\,.$$

2.4 The main result: the functional CLT in equilibrium

This result is obtained by a compactness-uniqueness method. We refer to Jakubowski [8] for the Skorokhod topology for the non-metrizable weak topology on infinite-dimensional Hilbert spaces.

Theorem 2.12 *Let the networks of size $N \geq L$ be in equilibrium. For $L \geq 2$ consider $L_2(g_\rho)$ with its weak topology and $\mathbb{D}(\mathbb{R}_+, L_2(g_\rho))$ with the corresponding Skorokhod topology. Then $(Z^N)_{N\geq L}$ converges in law to the unique stationary Ornstein-Uhlenbeck process solving the SDE (2.5), which is continuous and Gaussian, in particular $(Z_0^N)_{N\geq L}$ converges in law to the invariant law for this process (see Theorem 2.7). For $L = 1$ the same result holds in $L_2(g_\theta)$ for $\rho < \theta < 1$.*

3 The derivation of the limit Ornstein-Uhlenbeck process

Let $(x)_k = x(x-1)\cdots(x-k+1)$ for $x \in \mathbb{R}$ denote the Jordan or falling factorial of degree $k \in \mathbb{N}$. Considering (1.1), let the mappings F^N and F_+^N with values in c_0^0 be given for v in c_0 by

$$F^N(v) = F_+^N(v) - F_-(v)\,, \qquad F_+^N(v)(k) = \alpha\,\frac{(Nv(k-1))_L - (Nv(k))_L}{(N)_L}\,, \quad k \geq 1\,.$$

The process R^N is Markov on $\mathcal{V}^N$, and when in state r has jumps in its k-th coordinate, $k \geq 1$, of size $1/N$ at rate $NF_+^N(r)(k)$ and size $-1/N$ at rate $NF_-(r)(k)$.

Lemma 3.1 *Let R_0^N be in $\mathcal{V}^N$, u solve (1.2) starting at u_0 in $\mathcal{V}$, and Z^N be given by (1.3). Then*

$$Z_t^N = Z_0^N + \int_0^t N^{1/2}\left(F^N(R_s^N) - F(u_s)\right) ds + M_t^N \tag{3.1}$$

defines an independent family of square integrable martingales $M^N = (M^N(k))_{k\in\mathbb{N}}$ independent of R_0^N with Doob-Meyer brackets given by

$$\left\langle M^N(k)\right\rangle_t = \int_0^t \left\{F_+^N(R_s^N)(k) + F_-(R_s^N)(k)\right\} ds\,. \tag{3.2}$$

Proof. This follows from a classical application of the Dynkin formula. □

The first following combinatorial identity shows that it is indifferent to choose the L queues with or without replacement at this level of precision. The second one is a linearization formula.

Lemma 3.2 *For $N \geq L$ and a in $\mathbb{R}$ we have*

$$A^N(a) := \frac{(Na)_L}{(N)_L} - a^L = \sum_{j=1}^{L-1}(a-1)^j a^{L-j} \sum_{1\leq i_1<\cdots<i_j\leq L-1} \frac{i_1\cdots i_j}{(N-i_1)\cdots(N-i_j)}$$

and $A^N(a) = N^{-1}O(a)$ uniformly for a in $[0,1]$. We have $A^N(a) \leq 0$ for a in $\{0, N^{-1}, 2N^{-1}, \ldots, 1\}$.

Proof. We have

$$\frac{(Na)_L}{(N)_L} = \prod_{i=0}^{L-1}\frac{Na-i}{N-i} = \prod_{i=0}^{L-1}\left(a + (a-1)\frac{i}{N-i}\right)$$

and by developing the product we obtain the first identity. Direct inspection of the right-hand side of the identity shows that $A^N(a) = N^{-1}O(a)$ uniformly for a in $[0,1]$. For a in $\{0, N^{-1}, 2N^{-1}, \ldots, 1\}$ the product either is composed of terms which are positive and do not exceed a or contains a term equal to 0, and hence does not exceed a^L. □

Lemma 3.3 *For $N \geq L$ and a and h in $\mathbb{R}$ we have*

$$B(a,h) := (a+h)^L - a^L - La^{L-1}h = \sum_{i=2}^{L}\binom{L}{i}a^{L-i}h^i$$

with $B(a,h) = 0$ for $L = 1$ and $B(a,h) = h^2$ for $L = 2$. For $L \geq 2$ we have $0 \leq B(a,h) \leq h^L + \left(2^L - L - 2\right) ah^2$ for a and $a+h$ in $[0,1]$.

Proof. Newton's binomial formula yields the identity. For a and $a+h$ in $[0,1]$ and $L \ge 2$

$$B(a,h) \le h^L + \sum_{i=2}^{L-1} \binom{L}{i} a h^2 = h^L + \left(2^L - L - 2\right) a h^2 \,.$$

A convexity argument yields $B(a,h) \ge 0$. □

We define the functions G^N mapping v in c_0 to $G^N(v)$ in c_0^0 given by

$$G^N = F^N - F = F_+^N - F_+ \,, \quad G^N(v)(k) = \alpha A^N(v(k-1)) - \alpha A^N(v(k)) \,, \quad k \ge 1 \,, \tag{3.3}$$

and $\mathbf{K}$ and H mapping (v,x) in $c_0 \times c_0^0$ to $\mathbf{K}(v)x$ and $H(v,x)$ in c_0^0 given by

$$\begin{aligned} \mathbf{K}(v)x(k) &= \alpha L v(k-1)^{L-1} x(k-1) - (\alpha L v(k)^{L-1} + \beta) x(k) + \beta x(k+1) \,, \quad k \ge 1 \,, \\ H(v,x)(k) &= \alpha B(v(k-1), x(k-1)) - \alpha B(v(k), x(k)) \,, \quad k \ge 1 \,. \end{aligned} \tag{3.4}$$

For v and $v+x$ in $\mathcal{V}$ we may use the bounds in Lemmas 3.2 and 3.3. We have

$$F(v+x) - F(v) = F_+(v+x) - F_+(v) + F_-(x) = \mathbf{K}(v)x + H(v,x) \,. \tag{3.5}$$

We derive a limit equation for the fluctuations from (3.1) and (3.2) using (3.3), (3.5), and Lemmas 3.2 and 3.3. Let u solve (1.2) in $\mathcal{V}$ and $(M(k))_{k\in\mathbb{N}}$ be independent real continuous centered Gaussian martingales, determined in law by their deterministic Doob-Meyer brackets given by

$$\langle M(k) \rangle_t = \int_0^t \left\{ F_+(u_s)(k) + F_-(u_s)(k) \right\} ds \,.$$

The processes $M = (M(k))_{k\ge 0}$ and $\langle M \rangle = (\langle M(k) \rangle)_{k\in\mathbb{N}}$ have sample paths with values in c_0^0, and $\mathbf{K}(u_t) : z \mapsto \mathbf{K}(u_t)z$ are linear operators on c_0^0. The natural limit equation for the fluctuations is the inhomogeneous affine SDE given for $t \ge 0$ by

$$Z_t = Z_0 + \int_0^t \mathbf{K}(u_s) Z_s \, ds + M_t \,.$$

We set $\mathcal{K} = \mathbf{K}(\tilde{u})$. For $u_0 = \tilde{u}$, (1.1) and $F_+(\tilde{u}) = F_-(\tilde{u})$ yield the formulation in Section 2.2.

4 Main properties of the Ornstein-Uhlenbeck process

4.1 Proof of Theorem 2.4

Considering (2.1) and convexity bounds we have

$$\begin{aligned} \|\mathcal{K}z\|_{L_2(w)}^2 &= \beta^2 \sum_{k\ge 1} \left(L\rho^{L^{k-1}} z(k-1) - (L\rho^{L^k} + 1) z(k) + z(k+1) \right)^2 w(k)^{-1} \\ &\le \beta^2 (2L+2) \Bigg(L \sum_{k\ge 1} \rho^{2L^{k-1}} z(k-1)^2 w(k)^{-1} + L \sum_{k\ge 1} \rho^{2L^k} z(k)^2 w(k)^{-1} \end{aligned}$$

$$
\begin{aligned}
&\qquad + \sum_{k\geq 1} z(k)^2 w(k)^{-1} + \sum_{k\geq 1} z(k+1)^2 w(k)^{-1}\Big) \\
&\leq \beta^2(2L+2)\Big(Ld\sum_{k\geq 2} z(k-1)^2 w(k-1)^{-1} + (L\rho^{2L}+1)\sum_{k\geq 1} z(k)^2 w(k)^{-1} \\
&\qquad + c^{-1}\sum_{k\geq 1} z(k+1)^2 w(k+1)^{-1}\Big) \\
&\leq \beta^2(2L+2)\left(L\rho^{2L}+Ld+c^{-1}+1\right)\|z\|^2_{L_2(w)}\,.
\end{aligned}
$$

The Gronwall Lemma yields uniqueness. For $k \geq 1$ we have

$$
(L\rho^L)^{-1}\pi(k+1) \leq \pi(k) = (L\rho^{L^k})^{-1}\pi(k+1) \leq L^{-1}\rho^L\rho^{-2L^k}\pi(k+1)\,,
$$

$$
\theta^{-1}\theta^{k+1} \leq \theta^k \leq \theta^{-1}\rho^L\rho^{-2L^k}\theta^{k+1}\,.
$$

When B is an Hilbertian Brownian motion, the formula for Z is well-defined and solves the equation.

4.2 A related birth and death process, and the spectral decomposition

Considering (2.2), $\mathcal{A} = \mathcal{K}^*$ is the infinitesimal generator of the sub-Markovian birth and death process on the irreducible class $(1, 2, \ldots)$ with birth rates $\lambda_k = \beta L\rho^{L^k}$ and death rates $\mu_k = \beta$ for $k \geq 1$ (killed at rate $\mu_1 = \beta$ at state 1). The process is well-defined since the rates are bounded.

Karlin and McGregor [10, 11] give a spectral decomposition for such processes, used by Callaert and Keilson [1, 2] and van Doorn [3] to study exponential ergodicity properties. The state space in these works is $(0, 1, 2, \ldots)$, possibly extended by an absorbing barrier or graveyard state at -1. We consider $(1, 2, \ldots)$ and adapt their notations to this simple shift.

The potential coefficients ([10] eq. (2.2), [3] eq. (2.10)) are given by

$$
\pi(k) = \frac{\lambda_1\cdots\lambda_{k-1}}{\mu_2\cdots\mu_k} = L\rho^{L^1}\cdots L\rho^{L^{k-1}} = L^{k-1}\rho^{(L^k-L)/(L-1)}, \qquad k \geq 1\,,
$$

and solve the detailed balance equations $\mu_{k+1}\pi(k+1) = \lambda_k\pi(k)$ with $\pi(1) = 1$.

The equation $\mathcal{A}Q(x) = -xQ(x)$ for an eigenvector $Q(x) = (Q_n(x))_{n\geq 1}$ of eigenvalue $-x$ yields $\lambda_1 Q_2(x) = (\lambda_1 + \mu_1 - x)Q_1(x)$ and $\lambda_n Q_{n+1}(x) = (\lambda_n + \mu_n - x)Q_n(x) - \mu_n Q_{n-1}(x)$ for $n \geq 2$. With the natural convention $Q_0 = 0$ and choice $Q_1 = 1$, we obtain inductively Q_n as the polynomial of degree $n-1$ satisfying

$$
-xQ_n(x) = \beta Q_{n-1}(x) - \left(\beta L\rho^{L^n} + \beta\right)Q_n(x) + \beta L\rho^{L^n}Q_{n+1}(x)\,, \qquad n \geq 1\,.
$$

These recursions correspond to [10] eq. (2.1) and [3] eq. (2.15). As stated there, such a sequence of polynomials is orthogonal with respect to a probability measure ψ on $\mathbb{R}_+$ and

$$
\int_0^\infty Q_i(x)^2\,\psi(dx) = \pi(i)^{-1}\,, \quad \int_0^\infty Q_i(x)Q_j(x)\,\psi(dx) = 0\,, \qquad i,j \geq 1\,,\ i \neq j\,,
$$

or in matrix notation $\int_0^\infty Q(x)Q(x)^*\,\psi(dx) = \mathrm{diag}(\pi^{-1})$.

Let $P_t = (p_t(i,j))_{i,j\geq 1}$ denote the sub-stochastic transition matrix for $\mathcal{A}$. The adjoint matrix P_t^* is the fundamental solution for the forward Kolmogorov equation $\dot{z}_t = \mathcal{A}^* z_t = \mathcal{K} z_t$. The representation formula of Karlin and McGregor [10, 11] (see (1.2) and (2.18) in [3]) yields

$$\mathrm{e}^{\mathcal{K}t} = P_t^* = (p_t^*(i,j))_{i,j\geq 1}\,, \qquad p_t^*(i,j) = p_t(j,i) = \pi(i)\int_0^\infty \mathrm{e}^{-xt} Q_i(x)Q_j(x)\,\psi(dx)\,, \quad (4.1)$$

or in matrix notation $\mathrm{e}^{\mathcal{K}t} = \mathrm{diag}(\pi)\int_0^\infty \mathrm{e}^{-xt}Q(x)Q(x)^*\,\psi(dx)$.

The probability measure ψ is called the spectral measure, its support S is called the spectrum, and we set $\gamma = \min S$. The Ornstein-Uhlenbeck process in Theorem 2.4 (b) and its invariant law and its covariance matrix in Theorems 2.7 and 2.12 can be written

$$Z_t = \mathrm{diag}(\pi)\int_S \mathrm{e}^{-xt}Q(x)^*\left(Z_0 + \int_0^t \mathrm{e}^{xs}\,dB_s\right)Q(x)\,\psi(dx)\,, \quad (4.2)$$

$$\int_0^\infty \mathrm{e}^{\mathcal{K}t}\,dB_t = \mathrm{diag}(\pi)\int_S \left(Q(x)^*\int_0^\infty \mathrm{e}^{-xt}\,dB_t\right)Q(x)\,\psi(dx)\,, \quad (4.3)$$

$$\int_0^\infty \mathrm{e}^{\mathcal{K}t}\mathrm{diag}(\tilde{v})\mathrm{e}^{\mathcal{K}^*t}\,dt = \mathrm{diag}(\pi)\int_{S^2}\frac{Q(x)^*\mathrm{diag}(\tilde{v})Q(y)}{x+y}\,Q(x)Q(y)^*\,\psi(dx)\psi(dy)\,\mathrm{diag}(\pi). \quad (4.4)$$

4.3 The spectral gap, exponential stability, and ergodicity

Proof of Theorem 2.5. The potential coefficients $(\pi(k))_{k\geq 1}$ solve the detailed balance equations for $\mathcal{A}$ and hence $\mathcal{K} = \mathcal{A}^*$ is self-adjoint in $L_2(\pi)$.

For the spectral gap, we follow Van Doorn [3], Section 2.3. The orthogonality properties imply that for $n \geq 1$, Q_n has $n-1$ distinct zeros $0 < x_{n,1} < \ldots < x_{n,n-1}$ such that $x_{n+1,i} < x_{n,i} < x_{n+1,i+1}$ for $1 \leq i \leq n-1$. Hence $\xi_i = \lim_{n\to\infty} x_{n,i} \geq 0$ exists, $\xi_i \leq \xi_{i+1}$, and $\sigma = \lim_{i\to\infty}\xi_i$ exists in $[0,\infty]$. Theorem 5.1 in [3] establishes that $\gamma > 0$ if and only if $\sigma > 0$, Theorem 5.3 (i) in [3] that $\sigma = \beta > 0$, and Theorem 3.3 in [3] that $\gamma = \xi_1 \leq \sigma$. (Estimating ξ_1 is impractical.)

For the exponential stability, we have $\|z_t\|^2_{L_2(\pi)} = \left(\mathrm{e}^{\mathcal{K}t}z_0, \mathrm{e}^{\mathcal{K}t}z_0\right)_{L_2(\pi)}$. The fact that $\mathrm{e}^{\mathcal{K}t}$ is self-adjoint in $L_2(\pi)$ and the spectral representation (4.1) yield

$$\begin{aligned}\left(\mathrm{e}^{\mathcal{K}t}z_0, \mathrm{e}^{\mathcal{K}t}z_0\right)_{L_2(\pi)} &= \left(z_0, \mathrm{e}^{2\mathcal{K}t}z_0\right)_{L_2(\pi)} = \int_S \mathrm{e}^{-2xt}z_0^*Q(x)Q(x)^*z_0\,\psi(dx)\\ &\leq \mathrm{e}^{-2\gamma t}\int_S z_0^*Q(x)Q(x)^*z_0\,\psi(dx) = \mathrm{e}^{-2\gamma t}\,(z_0,z_0)_{L_2(\pi)}\,.\end{aligned}$$

We refer to Callaert and Keilson [2] Section 10 for related results.

Proof of Theorem 2.6 (non self-adjoint case). It is similar to and simpler than the proof for Theorem 2.8 in the interactive case $L \geq 2$, and we wait till that point to give it.

Proof of Theorem 2.7. We use the uniqueness result and explicit formula for Z in Theorem 2.4, and Theorem 2.5 or 2.6.

5 Exponential stability for the nonlinear system

5.1 Some comparison results

Considering (3.5), $\mathcal{K} = \mathbf{K}(\tilde{u})$ and $F(\tilde{u}) = 0$, if u is a solution of (1.2) in $\mathcal{V}$ starting at u_0 then $y = u - \tilde{u}$ is a solution to the recentered equation starting at $y_0 = u_0 - \tilde{u}$ given by

$$\begin{aligned}\dot{y}_t(k) &= \mathcal{K}y_t(k) + H(\tilde{u}, y_t)(k)\\ &= \beta L\rho^{L^{k-1}} y_t(k-1) + \alpha B(\tilde{u}(k-1), y_t(k-1))\\ &\qquad - \Big(\beta L\rho^{L^k} y_t(k) + \alpha B(\tilde{u}(k), y_t(k)) + \beta y_t(k)\Big) + \beta y_t(k+1)\,, \qquad k \geq 1\,, \end{aligned} \tag{5.1}$$

and if u_0 is in $\mathcal{V} \cap \ell_1$ then u is in $\mathcal{V} \cap \ell_1$ and hence y is in ℓ_1^0 and for $k \geq 1$

$$\dot{y}_t(k) + \dot{y}_t(k+1) + \cdots = \beta L\rho^{L^{k-1}} y_t(k-1) + \alpha B(\tilde{u}(k-1), y_t(k-1)) - \beta y_t(k)\,. \tag{5.2}$$

Reciprocally, if y is a solution to the recentered equation (5.1) starting at y_0 such that $y_0 + \tilde{u}$ is in $\mathcal{V}$, then $u = y + \tilde{u}$ is a solution of (1.2) in $\mathcal{V}$ starting at $u_0 = y_0 + \tilde{u}$. Then $-\tilde{u} \leq y \leq 1 - \tilde{u}$ and $-1 < y < 1$. For $y_0 + \tilde{u}$ in $\mathcal{V} \cap \ell_1$ we have y in ℓ_1^0.

Lemma 5.1 *Let u and v be two solutions for (1.2) in $\mathcal{V}$ such that $u_0 \leq v_0$. Then $u_t \leq v_t$ for $t \geq 0$. Let $y_0 + \tilde{u}$ be in $\mathcal{V}$ and y solve (5.1). If $y_0 \geq 0$ then $y_t \geq 0$ and if $y_0 \leq 0$ then $y_t \leq 0$ for $t \geq 0$.*

Proof. Lemma 6 in [12] yields the result for (1.2) (the proof written for $L = 2$ is valid for $L \geq 1$). The result for (5.1) follows by consideration of the solutions $u = y + \tilde{u}$ and $\tilde{u}$ for (1.2). $\square$

We shall compare solutions of the nonlinear equation (5.1) and of certain linear equations.

Lemma 5.2 *Let $\hat{\mathcal{A}}$ be the generator of the sub-Markovian birth and death process with birth rate $\hat{\lambda}_k \geq 0$ and death rate β at $k \geq 1$. Let $\sup_k \hat{\lambda}_k < \infty$. In ℓ_1^0 the linear operator*

$$\hat{\mathcal{A}}^* x(k) = \hat{\lambda}_{k-1} x(k-1) - (\hat{\lambda}_k + \beta)x(k) + \beta x(k+1)\,, \qquad k \geq 1\,,$$

is bounded and there exists a unique $z = (z_t)_{t\geq 0}$ given by $z_t = \mathrm{e}^{\hat{\mathcal{A}}^* t} z_0$ solving the forward Kolmogorov equation $\dot{z} = \hat{\mathcal{A}}^* z$. If $z_0 \geq 0$ then $z_t \geq 0$ and if $z_0 \leq 0$ then $z_t \leq 0$. For $k \geq 1$, $\dot{z}_t(k) + \dot{z}_t(k+1) + \cdots = \hat{\lambda}_{k-1} z_t(k-1) - \beta z_t(k)$.

Proof. The operator norm in ℓ_1^0 of $\hat{\mathcal{A}}^*$ is bounded by $2(\sup_k \hat{\lambda}_k + \beta)$, hence existence and uniqueness. Uniqueness and linearity imply that if $z_0 = 0$ then $z_t = 0$ and else if $z_0 \geq 0$ then $z_t \|z_0\|_1^{-1}$ is the instantaneous law of the process starting at $z_0 \|z_0\|_1^{-1}$ and hence $z_t \geq 0$. If $z_0 \leq 0$ then $-z$ solves the equation starting at $-z_0 \geq 0$ and hence $-z_t \geq 0$. $\square$

Lemma 5.3 *Let $L \ge 2$ and $y = (y_t)_{t\ge 0}$ solve (5.1) with $y_0 + \tilde{u}$ in $\mathcal{V} \cap \ell_1$. Under the assumptions of Lemma 5.2, let $z = (z_t)_{t\ge 0}$ solve $\dot{z} = \hat{\mathcal{A}}^* z$ with z_0 in ℓ_1^0 and $h = (h_t)_{t\ge 0}$ be given by*

$$h = (h(k))_{k\ge 1}\,, \qquad h(k) = z(k) + z(k+1) + \cdots - (y(k) + y(k+1) + \cdots).$$

(a) Let $\hat{\lambda}_k \ge \beta L \rho^{L^k} + \alpha\big(1 + \big(2^L - L - 2\big)\,\tilde{u}(k)\big)$ for $k \ge 1$, $y_0 \ge 0$, and $h_0 \ge 0$. Then $h_t \ge 0$ for $t \ge 0$.

(b) Let $\hat{\lambda}_k \ge \beta L \rho^{L^k}$ for $k \ge 1$, $y_0 \le 0$, and $h_0 \le 0$. Then $h_t \le 0$ for $t \ge 0$.

Proof. We prove (a). For $\varepsilon > 0$ let $\hat{\mathcal{A}}_\varepsilon^*$ correspond to $\hat{\lambda}_k^\varepsilon = \hat{\lambda}_k + \varepsilon$. The operator norm in ℓ_1^0 of $\hat{\mathcal{A}}_\varepsilon^* - \hat{\mathcal{A}}^*$ is bounded by 2ε, hence $\lim_{\varepsilon\to 0} \mathrm{e}^{\hat{\mathcal{A}}_\varepsilon^* t} z_0 = z_t$ in ℓ_1^0 and we may assume that $\hat{\lambda}_k > \beta L \rho^{L^k} + \alpha\big(1 + \big(2^L - L - 2\big)\,\tilde{u}(k)\big)$ for $k \ge 1$. Since $z_t = \mathrm{e}^{\hat{\mathcal{A}}^* t} z_0$ depends continuously on z_0 in ℓ_1^0 we may assume $h_0 > 0$.

Let $\tau = \inf\{t \ge 0 : \{k \ge 1 : h_t(k) = 0\} \ne \emptyset\}$ be the first time when $h(k) = 0$ for some $k \ge 1$. Then $\tau > 0$ and if $\tau = \infty$ the proof is ended. Else, Lemma 5.2 and (5.2) yield

$$\begin{aligned}\dot{h}_\tau(k) = {}& \hat{\lambda}_{k-1} y_\tau(k-1) - \beta L \rho^{L^{k-1}} y_\tau(k-1) - \alpha B(\tilde{u}(k-1), y_\tau(k-1)) \\ & + \hat{\lambda}_{k-1}(z_\tau(k-1) - y_\tau(k-1)) - \beta(z_\tau(k) - y_\tau(k))\,.\end{aligned}$$

Lemma 5.1 yields $y \ge 0$ and Lemma 3.3 and $y \le 1$ yield

$$\begin{aligned} B(\tilde{u}(k-1), y(k-1)) &\le y(k-1)^L + \big(2^L - L - 2\big)\,\tilde{u}(k-1) y(k-1)^2 \\ &\le \big(1 + \big(2^L - L - 2\big)\,\tilde{u}(k-1)\big)\, y(k-1)\,,\end{aligned}$$

hence $\hat{\lambda}_{k-1} y(k-1) - \beta L \rho^{L^{k-1}} y(k-1) - \alpha B(\tilde{u}(k-1), y(k-1)) \ge 0$ with equality only when $y(k-1) = 0$. For k in $\mathcal{K} = \{k \ge 1 : h_\tau(k) = 0\} \ne \emptyset$ we have

$$z_\tau(k-1) - y_\tau(k-1) = h_\tau(k-1) \ge 0\,, \quad z_\tau(k) - y_\tau(k) = -h_\tau(k+1) \le 0\,,$$

with equality if only if $k-1$ is in $\mathcal{K} \cup \{0\}$ and $k+1$ is in $\mathcal{K}$. Hence $\dot{h}_\tau(k) \ge 0$. Moreover $h_t(k) > 0$ for $t < \tau$ and $h_\tau(k) = 0$ imply $\dot{h}_\tau(k) \le 0$, hence $\dot{h}_\tau(k) = 0$, and the above signs and equality cases yield that $z_\tau(k-1) = y_\tau(k-1) = 0$ and $k-1$ is in $\mathcal{K} \cup \{0\}$ and $k+1$ is in $\mathcal{K}$. By induction $z_\tau(i) = y_\tau(i) = 0$ for $i \ge 1$ which implies $z_t = y_t = 0$ for $t \ge \tau$.

The proof for (b) is similar and involves obvious changes of sign. We may assume $\hat{\lambda}_k > \beta L \rho^{L^k}$ which suffices to conclude since Lemma 3.3 yields $B(\tilde{u}(k-1), y(k-1)) \ge 0$. $\square$

Lemma 5.4 *For any $0 < \theta < 1$ there exists $K_\theta < \infty$ such that for x in $L_2(g_\theta) \subset \ell_1^0$*

$$\|(x(k) + x(k+1) + \cdots)_{k\ge 1}\|_{L_2(g_\theta)} \le K_\theta \|x\|_{L_2(g_\theta)}\,.$$

Proof. Using a classical convexity inequality

$$\begin{aligned}
&\sum_{k\ge1}(x(k)+x(k+1)+\cdots)^2\theta^{-k}\\
&\quad\le \sum_{k\ge1} n\big(x(k)^2+x(k+1)^2+\cdots+x(k+n-2)^2+(x(k+n-1)+x(k+n)+\cdots)^2\big)\,\theta^{-k}\\
&\quad\le n\big(1+\theta+\cdots+\theta^{n-2}\big)\sum_{k\ge1}x(k)^2\theta^{-k}+n\,\theta^{n-1}\sum_{k\ge1}(x(k)+x(k+1)+\cdots)^2\theta^{-k}.
\end{aligned}$$

We take n large enough that $n\theta^{n-1}<1$ and $K_\theta^2=(1-n\theta^{n-1})^{-1}n(1-\theta^{n-1})(1-\theta)^{-1}$. □

5.2 Proofs of Theorems 2.8 and 2.6

Proof of Theorem 2.8 for $L\ge 2$. Let u_0 be in $\mathcal{V}\cap L_2(g_\theta)$. Then $u_0^-=\min\{u_0,\tilde u\}$ and $u_0^+=\max\{u_0,\tilde u\}$ are in $\mathcal{V}\cap L_2(g_\theta)$. Theorem 2.2 yields that the corresponding solutions u^- and u^+ for (1.2) are in $\mathcal{V}\cap L_2(g_\theta)$. Lemma 5.1 yields that $u_t^-\le u_t\le u_t^+$ and $u_t^-\le\tilde u\le u_t^+$ for $t\ge0$. Then

$$y=u-\tilde u\,,\qquad y^+=u^+-\tilde u\ge0\,,\qquad y^-=u^--\tilde u\le0\,,$$

solve (5.1), and termwise

$$|y_0|=\max\{y_0^+,-y_0^-\}\,,\qquad |y_t|\le\max\{y_t^+,-y_t^-\}\,,\quad t\ge0\,. \tag{5.3}$$

We consider the birth and death process with generator $\hat{\mathcal{A}}$ defined in Lemma 5.2 with

$$\hat\lambda_k=\max\left\{\beta L\rho^{L^k}+\alpha\big(1+\big(2^L-L-2\big)\,\tilde u(k)\big)\,,\beta\theta\right\}\,,\qquad k\ge1\,,$$

which satisfies the assumptions of Lemma 5.3 (a) and (b). We perform the same spectral study as in Sections 4.2 and 4.3, all notions being similar and denoted using a hat.

For $\rho\le\theta<1$ we have $\alpha\le\beta\theta$ and hence $\hat\lambda_k$ is equivalent to $\beta\theta$ for large k, hence Theorem 5.3 (i) in [3] yields that $0<\hat\gamma\le\hat\sigma=\big(\sqrt{\beta}-\sqrt{\beta\theta}\big)^2=\beta\left(1-\sqrt{\theta}\right)^2$, and moreover

$$\theta^{k-1}\le\hat\pi(k)=\theta^{k-1}\prod_{i=1}^{k-1}\max\left\{\theta^{-1}L\rho^{L^k}+\theta^{-1}\rho\big(1+\big(2^L-L-2\big)\,\tilde u(k)\big)\,,1\right\}$$

and the product converges using simple criteria. Hence $\hat\pi(k)=O(\theta^k)$ and $\theta^k=O(\hat\pi(k))$ and Lemma 2.1 yields that there exists $c>0$ and $d>0$ such that $c^{-1}\|\cdot\|_{L_2(\hat\pi)}\le\|\cdot\|_{L_2(g_\theta)}\le d\|\cdot\|_{L_2(\hat\pi)}$. The version of Theorem 2.5 for the the above process yields that if z solves $z=\hat{\mathcal{A}}^*z$ in $L_2(g_\theta)$ then

$$\|z_t\|_{L_2(g_\theta)}\le d\|z_t\|_{L_2(\hat\pi)}\le \mathrm{e}^{-\hat\gamma t}d\|z_0\|_{L_2(\hat\pi)}\le \mathrm{e}^{-\hat\gamma t}cd\|z_0\|_{L_2(g_\theta)}\,.$$

Hence if z^+ solves $z^+=\hat{\mathcal{A}}^*z^+$ starting at $z_0^+=y_0^+\ge0$ then Lemma 5.3 (a) and Lemma 5.4 yield

$$\begin{aligned}
\|y_t^+\|_{L_2(g_\theta)}&\le\|(y_t^+(k)+y_t^+(k+1)+\cdots)_{k\ge1}\|_{L_2(g_\theta)}\\
&\le\|(z_t^+(k)+z_t^+(k+1)+\cdots)_{k\ge1}\|_{L_2(g_\theta)}\\
&\le K_\theta\|z_t^+\|_{L_2(g_\theta)}\le \mathrm{e}^{-\hat\gamma t}cdK_\theta\|y_0^+\|_{L_2(g_\theta)}\,,
\end{aligned}$$

and similarly if z^- solves $z^- = \hat{\mathcal{A}}^* z^-$ starting at $z_0^- = y_0^- \le 0$ then Lemma 5.3 (b) and Lemma 5.4 yield $\|y_t^-\|_{L_2(g_\theta)} \le \mathrm{e}^{-\hat{\gamma} t} cdK_\theta \|y_0^-\|_{L_2(g_\theta)}$. We set $\gamma_\theta = \hat{\gamma}$ and $C_\theta = cdK_\theta$. Considering (5.3),

$$\|y_t\|^2_{L_2(g_\theta)} \le \|y_t^+\|^2_{L_2(g_\theta)} + \|y_t^-\|^2_{L_2(g_\theta)} \le \mathrm{e}^{-2\gamma_\theta t} C_\theta^2 \left(\|y_0^+\|^2_{L_2(g_\theta)} + \|y_0^-\|^2_{L_2(g_\theta)} \right)$$

and we complete the proof by remarking that for $k \ge 1$, either $y_0^+(k) = y_0(k)$ and $y_0^-(k) = 0$ or $y_0^-(k) = y_0(k)$ and $y_0^+(k) = 0$, and hence $\|y_0^+\|^2_{L_2(g_\theta)} + \|y_0^-\|^2_{L_2(g_\theta)} = \|y_0\|^2_{L_2(g_\theta)}$.

Proof of Theorem 2.6 and of Theorem 2.8 for $L = 1$. The linearization (2.3) of Equation (1.2) is obtained from Equation (5.1) by replacing the nonlinear functions B and H by 0, and coincides with (5.1) for $L = 1$. Likewise, the equation for (2.3) corresponding to (5.2) is obtained by omitting the term $\alpha B(\tilde{u}(k-1), y_t(k-1))$. We obtain a result for the linear equation (2.3) corresponding to Lemma 5.3 (a) and (b) under the sole assumption $\hat{\lambda}_k \ge \beta L \rho^{L^k}$ for $k \ge 1$. The proof proceeds as for Theorem 2.8 for $L \ge 2$ with the difference that $\hat{\lambda}_k = \max\left\{\beta L \rho^{L^k}, \beta\theta\right\}$. We have $\hat{\lambda}_k$ equal to $\beta\theta$ for large k for $0 < \theta < 1$ when $L \ge 2$ and for $\rho \le \theta < 1$ when $L = 1$.

6 Tightness estimates and the functional central limit theorem

6.1 Finite horizon bounds for the process: proof of Lemma 2.9

We use Lemma 3.1. Considering (3.1) and (3.3),

$$Z_t^N = Z_0^N + M_t^N + N^{1/2} \int_0^t G^N(R_s^N)\, ds + \int_0^t N^{1/2} \left(F(R_s^N) - F(\tilde{u}) \right) ds \tag{6.1}$$

where Lemma 3.2 yields that

$$G^N(R_s^N)(k) = \alpha\big(A^N\left(R_s^N(k-1)\right) - A^N\left(R_s^N(k)\right)\big) = N^{-1} O\left(R_s^N(k-1) + R_s^N(k)\right)$$

and hence for some $K < \infty$

$$\left\|G^N(R_s^N)\right\|_{L_2(g_\theta)} \le N^{-1} K \left\|R_s^N\right\|_{L_2(g_\theta)} \tag{6.2}$$

where

$$\left\|R_s^N\right\|_{L_2(g_\theta)} \le \|\tilde{u}\|_{L_2(g_\theta)} + N^{-1/2} \left\|Z_s^N\right\|_{L_2(g_\theta)}. \tag{6.3}$$

The mapping F being Lipschitz (Theorem 2.2), the Gronwall Lemma yields that for some $K_T < \infty$

$$\sup_{0 \le t \le T} \left\|Z_t^N\right\|_{L_2(g_\theta)} \le K_T \left(\left\|Z_0^N\right\|_{L_2(g_\theta)} + \sup_{0 \le t \le T} \left\|M_t^N\right\|_{L_2(g_\theta)} + N^{-1/2} \|\tilde{u}\|_{L_2(g_\theta)} \right).$$

We conclude using the Doob inequality, (3.2), (3.3), the bounds (6.2) and (6.3), and (see Theorem 2.2)

$$\left\|F_+(R_s^N) + F_-(R_s^N)\right\|_{L_2(g_\theta)} \le K \left\|R_s^N\right\|_{L_2(g_\theta)}. \tag{6.4}$$

6.2 Infinite horizon bounds for the marginals: proof of Lemma 2.10

Let $U_h(v)$ be the solution of (1.2) at time $h \geq 0$ with initial value v in $\mathcal{V}$, in particular $\tilde{u} = U_h(\tilde{u})$, and $Z^N_{t_0,h} = N^{1/2}\left(R^N_{t_0+h} - U_h(R^N_{t_0})\right)$ for $t_0 \geq 0$. We have $Z^N_{t_0+h} = Z^N_{t_0,h} + N^{1/2}\left(U_h(R^N_{t_0}) - \tilde{u}\right)$ and Theorem 2.8 yields that

$$\left\|Z^N_{t_0+h}\right\|_{L_2(g_\theta)} \leq \left\|Z^N_{t_0,h}\right\|_{L_2(g_\theta)} + \mathrm{e}^{-\gamma_\theta h} C_\theta \left\|Z^N_{t_0}\right\|_{L_2(g_\theta)} . \tag{6.5}$$

The conditional law of $(Z^N_{t_0,h})_{h\geq 0}$ given $R^N_{t_0} = r$ is the law of Z^N started with $R^N_0 = u_0 = r$, the empirical fluctuation process centered on $U(r)$ and starting at 0. We reason as in Section 6.1, using additionally (6.5) on the bound (6.3) with $s = t_0 + h$. We obtain that for some $K_T < \infty$

$$\sup_{0\leq h\leq T} \left\|Z^N_{t_0,h}\right\|_{L_2(g_\theta)} \leq K_T\left(N^{-1}C_\theta \left\|Z^N_{t_0}\right\|_{L_2(g_\theta)} + \sup_{0\leq h\leq T}\left\|M^N_{t_0+h} - M^N_{t_0}\right\|_{L_2(g_\theta)} + N^{-1/2}\left\|\tilde{u}\right\|_{L_2(g_\theta)}\right)$$

and then that for some $L_T < \infty$ we have for $0 \leq h \leq T$

$$\mathbf{E}\left(\left\|Z^N_{t_0+h}\right\|^2_{L_2(g_\theta)}\right) \leq L_T + 2(K_T N^{-1} + \mathrm{e}^{-\gamma_\theta h})^2 C_\theta^2\, \mathbf{E}\left(\left\|Z^N_{t_0}\right\|^2_{L_2(g_\theta)}\right) . \tag{6.6}$$

We fix T large enough for $8\mathrm{e}^{-2\gamma_\theta T} C_\theta^2 \leq \varepsilon < 1$. Uniformly for $N \geq K_T \mathrm{e}^{\gamma_\theta T}$, for $m \in \mathbb{N}$

$$\mathbf{E}\left(\left\|Z^N_{(m+1)T}\right\|^2_{L_2(g_\theta)}\right) \leq L_T + \varepsilon\, \mathbf{E}\left(\left\|Z^N_{mT}\right\|^2_{L_2(g_\theta)}\right)$$

and by induction

$$\mathbf{E}\left(\left\|Z^N_{mT}\right\|^2_{L_2(g_\theta)}\right) \leq L_T \sum_{j=1}^{m} \varepsilon^{j-1} + \varepsilon^m\, \mathbf{E}\left(\left\|Z^N_0\right\|^2_{L_2(g_\theta)}\right) \leq \frac{L_T}{1-\varepsilon} + \mathbf{E}\left(\left\|Z^N_0\right\|^2_{L_2(g_\theta)}\right) ,$$

and (6.6) yields

$$\sup_{0\leq h\leq T} \mathbf{E}\left(\left\|Z^N_{mT+h}\right\|^2_{L_2(g_\theta)}\right) \leq L_T + 8C_\theta^2\, \mathbf{E}\left(\left\|Z^N_{mT}\right\|^2_{L_2(g_\theta)}\right) ,$$

hence

$$\sup_{t\geq 0} \mathbf{E}\left(\left\|Z^N_t\right\|^2_{L_2(g_\theta)}\right) \leq L_T + 8C_\theta^2\left(\frac{L_T}{1-\varepsilon} + \mathbf{E}\left(\left\|Z^N_0\right\|^2_{L_2(g_\theta)}\right)\right) .$$

6.3 Bounds on the invariant laws: proof of Lemma 2.11

Ergodicity and the Fatou Lemma yield that for Z^N_∞ distributed according to the invariant law

$$\mathbf{E}\left(\left\|Z^N_\infty\right\|^2_{L_2(g_\theta)}\right) \leq \liminf_{t\geq 0} \mathbf{E}\left(\left\|Z^N_t\right\|^2_{L_2(g_\theta)}\right) \leq \sup_{t\geq 0} \mathbf{E}\left(\left\|Z^N_t\right\|^2_{L_2(g_\theta)}\right)$$

and considering Lemma 2.10 the proof will be complete as soon as we show that we can choose R^N_0 in $\mathcal{V}^N$ such that

$$\limsup_{N\geq L} \mathbf{E}\left(\left\|Z^N_0\right\|^2_{L_2(g_\theta)}\right) < \infty . \tag{6.7}$$

We consider $L \geq 2$, the case $L = 1$ being similar. Let $R_0^N = (R_0^N(k))_{k\in\mathbb{N}}$ with

$$R_0^N(k) = iN^{-1} \text{ for } -(2N)^{-1} < \tilde{u}(k) - iN^{-1} \leq (2N)^{-1}, \qquad i \in \{0, 1, \ldots, N\},$$

and

$$k(N) = \inf\{k \geq 1 : R_0^N(k) = 0\} = \inf\{k \geq 1 : \tilde{u}(k) \leq (2N)^{-1}\}.$$

Since for $x \geq 0$ and $0 < y \leq 1$

$$\begin{aligned} y = \rho^{(L^x-1)/(L-1)} &\Leftrightarrow x = \log\left(1 + (L-1)\log y / \log\rho\right) / \log L \\ &\Leftrightarrow \theta^{-x} = \left(1 + (L-1)\log y / \log\rho\right)^{-\log\theta/\log L} \end{aligned}$$

we have $k(N) = \inf\left\{k \in \mathbb{N} : k \geq \log\left(1 + (L-1)\log\left((2N)^{-1}\right)/\log\rho\right)/\log L\right\}$. Then

$$\left\|Z_0^N\right\|^2_{L_2(g_\theta)} = N\sum_{k=1}^{k(N)-1}\left(R_0^N(k) - \tilde{u}(k)\right)^2\theta^{-k} + N\sum_{k\geq k(N)}\tilde{u}(k)^2\theta^{-k},$$

$$N\sum_{k=1}^{k(N)-1}\left(R_0^N(k) - \tilde{u}(k)\right)^2\theta^{-k} \leq (4N)^{-1}\frac{\theta^{-k(N)} - \theta^{-1}}{\theta^{-1} - 1} = O\left(N^{-1}(\log N)^{-\log\theta/\log L}\right),$$

and for large enough N (and hence $k(N)$)

$$\begin{aligned} N\sum_{k\geq k(N)}\tilde{u}(k)^2\theta^{-k} &= N\sum_{k\geq k(N)}\rho^{2(L^k-1)/(L-1)}\theta^{-k} \\ &= N\rho^{2(L^{k(N)}-1)/(L-1)}\sum_{k\geq k(N)}\rho^{2(L^k-L^{k(N)})/(L-1)}\theta^{-k} \\ &\leq (4N)^{-1}\sum_{j\geq 0}\rho^{2L^{k(N)}(L^j-1)/(L-1)}\theta^{-(j+k(N))} \\ &\leq (4N)^{-1}\sum_{j\geq 0}\rho^{L^{k(N)}(L^j-1)/(L-1)} = o(N^{-1}). \end{aligned}$$

Hence (6.7) holds and the proof is complete.

6.4 The functional CLT: Proof of Theorem 2.12

Lemma 2.11 and the Markov inequality imply that in equilibrium $(Z_0^N)_{N\geq L}$ is asymptotically tight for the weak topology of $L_2(g_\rho)$, for which all bounded sets are relatively compact. We consider a subsequence of $N \geq L$. Let $(N_j)_{j\geq 1}$ denote a further subsequence such that $(Z_0^{N_j})_{j\geq 1}$ converges in law to some square-integrable Z_0^∞ in $L_2(g_\rho)$. We decompose the rest of the proof in three steps.

Step 1. We prove that $(Z^{N_j})_{j\geq 1}$ is tight in $\mathbb{D}(\mathbb{R}_+, L_2(g_\rho))$ with the Skorokhod topology, where $L_2(g_\rho)$ is considered with its non-metrizable weak topology. The compact subsets of $L_2(g_\rho)$ are metrizable and hence Polish, a fact yielding tightness criteria. We easily deduce from Theorem 4.6 and 3.1 in Jakubowski [8], which considers completely regular Hausdorff spaces (Tychonoff spaces) of which $L_2(g_\rho)$ with its weak topology is an example, that a sufficient condition is that

1. For each $T \geq 0$ and $\varepsilon > 0$ there is a (weakly) compact subset $K_{T,\varepsilon}$ of $L_2(g_\rho)$ such that

$$\mathbf{P}\left(Z^{N_j} \in \mathbb{D}([0,T], K_{T,\varepsilon})\right) > 1 - \varepsilon\,, \qquad j \geq 1\,. \tag{6.8}$$

2. For each $d \geq 1$, the d-dimensional processes $(Z^{N_j}(1), \dots, Z^{N_j}(d))_{j\geq 1}$ are tight.

Lemma 2.11 implies that the assumptions of Lemma 2.9 hold, and (6.8) follows considering the Markov inequality. We use (6.1) (derived from (3.1)) and (3.2), and the bounds (6.2), (6.3) and (6.4). The uniform bounds in Lemma 2.9 and the fact that $Z^N(k)$ has jumps of size $N^{-1/2}$ imply classically that $(Z^{N_j}(1), \dots, Z^{N_j}(d))_{j\geq 1}$ is tight, see for instance Ethier-Kurtz [4] Theorem 4.1 p. 354 or Joffe-Métivier [9] Proposition 3.2.3 and their proofs.

Step 2. The tightness result for $(Z^{N_j})_{j\geq 1}$ implies it converges in law along some further subsequence to some Z^∞ with initial law given by the law of Z_0^∞. Considering (3.5), we have in (6.1)

$$N^{1/2}\left(F(R_s^N)(k) - F(\tilde{u})(k)\right) = \mathcal{K}Z_s^N + N^{1/2}H\left(\tilde{u}, N^{-1/2}Z_s^N\right). \tag{6.9}$$

We likewise consider (3.2). We use again the bounds (6.2), (6.3) and (6.4), the uniform bounds in Lemma 2.9, and additionally (3.4) and Lemma 3.3. We deduce by a martingale characterization that Z^∞ has the law of the Ornstein-Uhlenbeck process unique solution for (2.5) in $L_2(g_\rho)$ starting at Z_0^∞, see Theorem 2.4. The drift vector is given by the limit for (3.1) and (6.1) considering (6.9), and the diffusion matrix by the limit for (3.2). See for instance Ethier-Kurtz [4] Theorem 4.1 p. 354 or Joffe-Métivier [9] Theorem 3.3.1 and their proofs for details.

Step 3. The limit in law of a sequence of stationary processes is stationary (see Ethier-Kurtz [4] p. 131, Lemma 7.7 and Theorem 7.8). Hence the law of Z^∞ is the unique law of the stationary Ornstein-Uhlenbeck process given by (2.5), see Theorem 2.7. We deduce that from every subsequence we can extract a further subsequence converging in law to this process. Hence $(Z^N)_{N\geq L}$ converges in law to this process.

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