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Finite difference methods for option pricing in jump-diffusion and exponential Lévy models.

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A finite difference scheme for option pricing in jump diffusion and exponential Lévy models*

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Abstract

We present a finite difference method for solving parabolic partial integro-differential equations with possibly singular kernels which arise in option pricing theory when the random evolution of the underlying asset is driven by a Lévy process or, more generally, a time-inhomogeneous jump-diffusion process. We discuss localization to a finite domain and provide an estimate for the localization error under an integrability condition on the Lévy measure. We propose an explicit-implicit time-stepping scheme to solve the equation and study stability and convergence of the schemes proposed, using the notion of viscosity solution. Numerical tests are performed with smooth and non-smooth initial conditions. Our scheme can be used for European and barrier options, applies in the case of pure-jump models or degenerate diffusion coefficients, and extends to time-dependent coefficients.

Keywords: parabolic integro-differential equations, finite difference methods, Lévy process, jump-diffusion models, option pricing, viscosity solutions.

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The shortcomings of diffusion models in representing the risk related to large market movements have led to the development of various option pricing models with jumps, where large returns are represented as discontinuities of prices as a function of time. Models with jumps allow for more realistic representation of price dynamics and a greater flexibility in modelling and have been the focus of much recent work. A review of financial modelling with jump processes may be found in [12].

Exponential Lévy models, where the market price of an asset is represented as the exponential $S_t = \exp(rt + X_t)$ of a Lévy process X_t, offer analytically tractable examples of positive jump processes which are simple enough to allow a detailed study both in terms of statistical properties and as models for risk-neutral dynamics i.e. option pricing models. Option pricing with exponential Lévy models is discussed in [12, 15, 17, 20, 27]. The flexibility of choice of the Lévy process X allows to calibrate the model to market prices of options and reproduce a wide variety of implied volatility skews/smiles. The Markov property of the price allows us to express option prices as solutions of partial integro-differential equations (PIDEs) which involve, in addition to a (possibly degenerate) second-order differential operator, a non-local integral term which requires specific treatment both at the theoretical and numerical level.

In this paper, we discuss the derivation of such PIDEs for European and barrier options and propose a finite difference scheme for solving them. Our numerical solution is based on splitting the operator into a local and a non-local part: we treat the local term using an implicit step and the non-local term using an explicit step. This idea, previously used for non-linear PDEs [3], allows for an efficient implementation in terms of speed and stability restrictions. Our scheme extends to infinite activity Lévy processes with singular kernels and does not require the diffusion part to be non-degenerate. We study the consistency and stability of this scheme, show its convergence to a viscosity solution of the partial integro-differential equation and study its numerical performance on two examples, the Merton model with Gaussian jumps and the infinite activity Variance Gamma model. Our scheme can be used for European and barrier options and can also be extended to the case of non-constant coefficients.

Such partial integro-differential equations (PIDEs) have been used by other authors to price options in models with jumps [2, 10, 22, 14] but the derivation of the equation is often omitted or incomplete. In section 2, we discuss the derivation of integro-differential equations for European and barrier options, first giving conditions under which option values can be described in terms of classical solutions followed by a more general result using the notion of viscosity solution. The case of American options has been discussed in [23].

Solving such PIDEs by finite difference methods involves several approximations: localization of the equation to a bounded domain, treatment of the singularity due to small jumps, discretization of the equation in space and iteration in time. We discuss localization errors in section 3.1 and provide an estimate for the localization error under an integrability condition on the Lévy measure. In section 3 we propose an explicit-implicit finite difference scheme and study consistency, stability and convergence of the scheme proposed. To study the convergence of the scheme, we use the notion of *viscosity* solution, which allows to show uniform convergence to the solution without requiring a priori smoothness of the solution or non-degeneracy of coefficients.

In section 5, numerical tests are performed for smooth and non-smooth initial conditions to assess the effect of various numerical parameters on the accuracy of the scheme. Our scheme can be used for European and barrier options and extends to the case of time-dependent coefficients.

Various numerical methods for solving such parabolic integro-differential equations have been proposed in the recent literature. In the case where the characteristic function of the log-price is known, Carr & Madan [8] propose a method using the Fast Fourier transform for pricing European options. Our scheme extends this approach to cases where analytical expressions may not available and where barriers / boundary conditions may be present. In the case of jump-diffusion models with finite jump intensity, Andersen & Andreasen [2] propose an operator splitting method where the differential part is treated using a Crank-Nicholson step and the jump integral is computed using an explicit time step. Our method applies more generally to models with infinite activity i.e. singular integral kernels; in addition, we propose an analysis of the convergence of our algorithm, which is absent in [2]. Using a variational formulation of the integro-differential equation, Zhang [30, 31] studied a finite difference scheme in the case of jump-diffusion models with finite intensity and possessing all exponential moments. These conditions rule out all models in the literature except the Merton model: our analysis does not require such restrictive conditions. Variations on this scheme are given in [14].

The variational formulation has been recently extended by Matache et al [22] to the infinite activity case using a wavelet Galerkin method. The relation to the literature is discussed again in section 6.

1 Exponential Lévy models

We consider here the class of models where the risk neutral dynamics of the underlying asset is given by $S_t = \exp(rt+X_t)$ where X_t is a (time-homogeneous) jump-diffusion (Lévy) process.

1.1 Lévy processes: definitions

A Lévy process is a stochastic process X_t with stationary independent increments which is continuous in probability. Without loss of generality we assume that $X_0 = 0$. The characteristic function of X_t has the following form, called the Lévy-Khinchine representation [25]:

$$E[e^{izX_t}] = \exp t\phi(z) \tag{1.1}$$

$$= \exp\{t(-\frac{\sigma^2}{2}z^2 + i\gamma z + \int_{-\infty}^{\infty}(e^{izx} - 1 - izx1_{|x|\leq 1})\nu(x)dx)\} \tag{1.2}$$

where $\sigma > 0$ and b are real constants and ν is a positive measure verifying

$$\int_{-1}^{+1} x^2 \nu(dx) < \infty \qquad \int_{|x|>1} \nu(dx) < \infty \tag{1.3}$$

The random process X can be interpreted as the superposition of a Brownian motion with drift and an infinite superposition of independent (compensated) Poisson processes with various jump sizes x, $\nu(dx)$ being the intensity of jumps of size x.

In general ν is not a finite measure: $\int \nu(dx)$ need not even be finite. In the case where $\lambda = \int \nu(dx) < +\infty$, the measure ν can be normalized to define a *probability measure* μ which can now be interpreted as the distribution of jump sizes:

$$\mu(dx) \quad = \quad \frac{\nu(dx)}{\lambda} \tag{1.4}$$

The jumps of X are then described by a *compound Poisson* process with λ as jump intensity (average number of jumps per unit time) and $\mu(.)$ as jump size distribution. In this case the truncation of small jumps is not needed and the Lévy-Khinchin representation reduces to:

$$E[e^{izX_t}] = \exp\{t(-\frac{\sigma^2 z^2}{2} + ib_0(\nu)z + \int_{-\infty}^{\infty} (e^{izx} - 1 - iux)\nu(x)dx)\} \tag{1.5}$$

and $\gamma_0 t = EX_t$ can be interpreted as the drift of the process.

A Lévy process is a Markov process; its infinitesimal generator $L : f \to Lf$ is an integro-differential operator defined by the expression :

$$\begin{aligned} Lf(x) &= \lim_{t\to 0} \frac{E[f(x+X_t)] - f(x)}{t} \\ &= \gamma\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \int \nu(dy)[f(x+y) - f(x) - y1_{\{|y|<1\}}\frac{\partial f}{\partial x}] \end{aligned} \tag{1.6}$$

which is well defined for $f \in C_0^2(\mathbb{R})$.

1.2 Exponential Lévy models

Let $(S_t)_{t\in[0,T^*]}$ be the price of a financial asset modelled as a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$. Under the hypothesis of absence of arbitrage there exists a measure equivalent to Q under which (S_t) is a martingale. We will assume therefore without loss of generality that Q is already a martingale measure.

In *exponential Lévy* models, the (risk-neutral) dynamics of S_t under Q is represented as the exponential of a Lévy process:

$$S_t = e^{rt+X_t} \tag{1.7}$$

Here X_t is a Lévy process with characteristic triplet (σ,γ,ν) and the interest rate r is included for ease of notation. Different exponential Lévy models proposed in the financial modelling literature simply correspond to different choices for the Lévy measure ν, see [12, Chap. 3] for a review. The absence of arbitrage then imposed that $\hat{S}_t = S_t e^{-rt} = \exp X_t$ is a martingale, which is equivalent to the following conditions on the triplet (b,σ,ν):

$$\int_{|y|>1} \nu(dy) e^y < \infty \tag{1.8}$$

$$\gamma = \gamma(\sigma,\nu) = -\frac{\sigma^2}{2} - \int (e^y - 1 - y 1_{|y|\leq 1}) \nu(dy) \tag{1.9}$$

We will assume this relation holds in the sequel. The infinitesimal generator L then becomes:

$$L.f(x) \quad = \quad \frac{\sigma^2}{2}[\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x}] + \int_{-\infty}^{\infty} \nu(dy)\ [f(x+y) - f(x) - (e^y - 1)\frac{\partial f}{\partial x}(x)]$$

The risk-neutral dynamics of S_t is given by

$$S_t = S_0 + \int_0^t r S_{u-} du + \int_0^t S_{u-} \sigma dW_u + \int_0^t \int_{-\infty}^{\infty} (e^x - 1) S_{u-} \tilde{J}_X(du\ dx) \tag{1.10}$$

where $\tilde{J}_X$ is the compensated random measure describing the jumps of X [24]. We will furthermore assume that

$$\int_{|y|\geq 1} e^{2y} \nu(dy) < \infty \tag{1.11}$$

This condition is equivalent to the existence of a second moment for the price process S_t. Then $\hat{S}_t = \exp X_t$ is a square integrable martingale:

$$\frac{d\hat{S}_t}{\hat{S}_{t-}} = \sigma dW_t + \int_{-\infty}^{\infty} (e^x - 1) \tilde{J}_X(dt\ dx) \qquad \sup_{t\in[0,T]} E[\hat{S}_t^2] < \infty$$

(S_t) is also a Markov process with state space $]0,\infty[$ and infinitesimal generator:

$$L^S f(x) = rx\frac{\partial f}{\partial x}(x) + \frac{\sigma^2 x^2}{2}\frac{\partial^2 f}{\partial x^2}(x) - rf(x) + \int \nu(dy)[f(xe^y) - f(x) - x(e^y - 1)\frac{\partial f}{\partial x}(x)]$$

2 Partial integro-differential equation for option prices

The value of an option with terminal payoff $H(S_T)$ is defined as a discounted conditional expectation of its terminal payoff under risk-neutral probability $\mathbb{Q}$:

$C_t = E[e^{-r(T-t)}H(S_T)|\mathcal{F}_t]$. From the Markov property, $C_t = C(t,S)$ where

$$C(t,S) \quad = \quad E[e^{-r(T-t)}H(S_T)|S_t = S] \tag{2.1}$$

Introducing the change of variable $\tau = T - t$, $x = \ln(S/K) + r\tau$ where K is an arbitrary constant and defining: $h(x) = H(Ke^x)$ and $u(\tau, x) = e^{r(T-t)}C(t,S)$ then

$$u(\tau, x) = E[h(x + X_\tau)] \tag{2.2}$$

If h is in the domain of the infinitesimal generator then we can differentiate with respect to τ to obtain the following integro-differential equation:

$$\frac{\partial u}{\partial \tau} = Lu, \qquad \text{on} \quad (0,T] \times \mathbb{R} \qquad u(0,x) = h(x), \qquad x \in \mathbb{R} \tag{2.3}$$

where L is the infinitesimal generator of X, given by (1.10). Similarly, if u is smooth then using a change of variable we obtain a similar equation for $C(t,S)$:

$$\frac{\partial C}{\partial t}(t,S) + L^S C(t,S) - rC(t,S) = 0 \qquad C(T,S) = H(S) \tag{2.4}$$

These equations are similar to the Black Scholes partial differential equation, except that the second-order differential operator is replaced by the integro-differential operators L or L^S.

However, the above reasoning is heuristic: the payoff function h is usually not in the domain of L and in fact it is usually not even differentiable. For example $h(x) = K(1 - e^x)^+$ for a put option and $h(x) = 1_{x \geq x_0}$ for a binary option. For barrier options, the presence of boundary conditions may lead to additional complications.

We will see nevertheless in section 2.1 that under some conditions, the value of European options $C(t,S), u(\tau,x)$ defined above are classical solutions of the partial integro-differential equations (2.3)-(2.4). However, these conditions are not always verified, especially in pure jump models. This will lead us to consider the notion of viscosity solution; we show in section 2.3 that under more general conditions, values of European or barrier options can be expressed as viscosity solutions of appropriate PIDEs.

2.1 European options

Consider a European option with maturity T and payoff $H(S_T)$. The payoff function H is assumed to verify:

$$|H(x) - H(y)| \leq c|x - y| \tag{2.5}$$

for some $c > 0$. This condition is of course verified by call and put options with $c = 1$. The value C_t of such an option is given by $C_t = C(t, S_t)$ where

$$C(t,S) = e^{-r(T-t)}E[H(S_T)|S_t = S] = e^{-r(T-t)}E[H(Se^{r(T-t)+X_{T-t}})]$$

Proposition 1 (Backward PIDE for European options). *Consider the exponential Lévy model $S_t = S_0 \exp(rt + X_t)$ where X is a Lévy process verifying (1.11). If*

$$\sigma > 0 \quad or \quad \exists \beta \in]0,2[, \qquad \liminf_{\epsilon \downarrow 0} \epsilon^{-\beta} \int_{-\epsilon}^{\epsilon} |x|^2 d\nu(x) > 0 \tag{2.6}$$

then the value of a European option with terminal payoff $H(S_T)$ is given by $C(t,S)$ where:

$$\begin{aligned} C : [0,T]\times]0,\infty] &\mapsto \mathbb{R} \\ (t,S) &\mapsto C(t,S) = E[H(S_T)|S_t = S] \end{aligned}$$

is continuous on $[0,T] \times [0,\infty[$, $C^{1,2}$ on $]0,T[\times]0,\infty[$ and verifies the partial integro-differential equation:

$$\begin{aligned} &\frac{\partial C}{\partial t}(t,S) + rS\frac{\partial C}{\partial S}(t,S) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}(t,S) - rC(t,S) \\ &+ \int \nu(dy)[C(t,Se^y) - C(t,S) - S(e^y - 1)\frac{\partial C}{\partial S}(t,S)] = 0 \end{aligned} \tag{2.7}$$

on $[0,T[\times]0,\infty[$ with the terminal condition:

$$\forall S > 0 \qquad C(T,S) = H(S) \tag{2.8}$$

Proof. The proof involves, as in the Black Scholes case, applying the Itô formula to the martingale $\hat{C}(t,S_t) = e^{-rt}C(t,S_t)$, identifying the drift component and setting it to zero.

Equation (2.6) implies that X_t has a smooth C^2 density with derivatives vanishing at infinity (see [25, proposition 28.3]) so $C(t,S)$ is then a smooth function of S. Smoothness in time can be shown by Fourier methods, see [12, Proposition 12.2].

By construction, $\hat{C}_t = E[H|\mathcal{F}_t]$ is a martingale. Applying the Itô formula to $\hat{C}_t = e^{-rt}C(t,S_t)$ and using equation (1.10) we obtain:

$$\begin{aligned} d\hat{C}_t &= e^{-rt}[-rC_t + \frac{\partial C}{\partial t}(t,S_{t-}) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 C}{\partial S^2}(t,S_{t-})]dt + e^{-rt}\frac{\partial C}{\partial S}(t,S_{t-})dS_t \\ &+ e^{-rt}[C(t,S_{t-}e^{\Delta X_t}) - C(t,S_{t-}) - S_{t-}(e^{\Delta X_t} - 1)\frac{\partial C}{\partial S}(t,S_{t-})] \\ &= a(t)dt + dM_t \qquad\qquad (2.9) \\ \text{where} \quad a(t) &= e^{-rt}[-rC(t,S_{t-}) + \frac{\partial C}{\partial t}(t,S_{t-}) + \frac{\sigma^2 S_{t-}^2}{2}\frac{\partial^2 C}{\partial S^2}(t,S_{t-}) + rS_{t-}\frac{\partial C}{\partial S}(t,S_{t-})] \\ &+ \int_{-\infty}^{\infty} \nu(dx) e^{-rt}[C(t,S_{t-}e^x) - C(t,S_{t-}) - S_{t-}(e^x - 1)\frac{\partial C}{\partial S}(t,S_{t-})] \\ \text{and} \quad dM_t &= e^{-rt}\frac{\partial C}{\partial S}(t,S_{t-})\sigma S_{t-} dW_t + \int_{\mathbb{R}} e^{-rt}[C(t,S_{t-}e^x) - C(t,S_{t-})]\tilde{J}_X(dt\ dx) \end{aligned}$$

Let us now show that M_t is a martingale. Since the payoff function H is Lipschitz, C is also Lipschitz with respect to the second variable:

$$\begin{aligned} C(t,x)-C(t,y) &= e^{-r(T-t)}E[H(xe^{r(T-t)+X_T-X_t})]-e^{-r(T-t)}E[H(ye^{r(T-t)+X_T-X_t})] \\ &\le c|x-y|E[e^{X_{T-t}}]=c|x-y| \end{aligned} \tag{2.10}$$

since e^{X_t} is a martingale. Therefore the predictable random function $\psi(t,x) = C(t,S_{t-}e^x)-C(t,S_{t-})$ verifies

$$\begin{aligned} E[\int_0^T dt\int_{\mathbb{R}}\nu(dx)|\psi(t,x)|^2] &= E[\int_0^T dt\int_{\mathbb{R}}\nu(dx)|C(t,S_{t-}e^x)-C(t,S_{t-})|^2\ ] \\ &\le E[\int_0^T dt\int_{\mathbb{R}}c^2(e^{2x}+1)S_{t-}^2\nu(dx)] \\ \text{using (1.11)} &\le c^2\int_{\mathbb{R}}(e^{2x}+1)\nu(dx)\ E[\int_0^T S_{t-}^2 dt]<\infty \end{aligned}$$

so the compensated Poisson integral

$$\int_0^t\int_{-\infty}^{\infty}e^{-rt}[C(t,S_{t-}e^x)-C(t,S_{t-})]\ \tilde{J}_X(dt\ dx)$$

is a square integrable martingale. Also, since C is Lipschitz, $\partial C/\partial S\ \in L^\infty$ and

$$||\frac{\partial C}{\partial S}(t,.)||_{L^\infty}\le c \quad \text{so} \quad E[\int_0^T S_{t-}^2|\frac{\partial C}{\partial S}(t,S_{t-})|dt]\le c^2E[\int_0^T S_{t-}^2 dt]<\infty$$

Using the isometry relation for compensated Poisson integrals, we obtain that $\int_0^t\sigma S_t\frac{\partial C}{\partial S}(t,S_{t-})dW_t$ is also a square integrable martingale. Therefore M_t is a square integrable martingale. $\hat{C}_t-M_t$ is thus a (square integrable) martingale; but $\hat{C}_t-M_t=\int_0^t a(t)dt$ is also a *continuous* process with finite variation so, by [18, Theorem 4.13-4.50] we must have $a(t)=0$ $\mathbb{Q}$-almost surely which yields the PIDE (2.7). □

The condition (2.6) holds for all jump diffusion models with non-zero diffusion component as well as for Lévy densities behaving near zero as $\nu(x)\sim cx^{1+\beta}$ with $\beta>0$ such as the tempered stable model, but not for the Variance Gamma model [21]. However in the case of the Variance Gamma model, the equation is a first order equation for which one requires only C^1 smoothness.

2.2 Barrier options

Barrier options lead to initial/boundary value problems. Consider for instance an up-and-out call option with maturity T, strike K and (upper) barrier $B>K$. The terminal payoff is given by

$$\begin{aligned} H_T &= (S_T-K)^+ \text{ if } \max_{t\in[0,T]} S_t<B \\ &= 0 \text{ if } \max_{t\in[0,T]} S_t\ge B. \end{aligned}$$

The value of the barrier option at time t can be expressed as discounted expectation of the terminal payoff: $C_t = e^{-r(T-t)}E[H_T|\mathcal{F}_t]$. By construction, $e^{-rt}C_t$ is a martingale. Defining $H(S) = (S-K)^+ 1_{S<B}$ where τ_B is the first exit time from $]0, B]$ we can introduce a deterministic function

$$C(t,S) \quad = \quad e^{-r(T-t)} \; E[H(S_{T\wedge\tau_B})|S_t = S] \tag{2.11}$$

Due to the Markov property of S_t, C_t coincides with $C(t, S_t)$ on the set $\{t \le \tau_B\}$ i.e. before the barrier is crossed. Therefore,

$$C_{t\wedge\tau_B} = C(t\wedge\tau_B, S_{t\wedge\tau_B}) \tag{2.12}$$

for all $t \le T$. Note that outside of the set $\{t \le \tau_B\}$ these two objects are different: if the barrier has already been crossed, C_t will always be zero, but $C(t, S_t)$ may become positive if the stock returns to the region below the barrier.

By analogy with the case of European options one can derive a pricing equation in the following way. Assume that the function $C(t,S) : [0,T]\times]0,\infty[\to]0,\infty[$ is $C^{1,2}$ i.e. smooth enough for the Itô formula to be applied. Applying the Itô formula to $\hat{C}_t = \exp(-rt)C(t,S_t)$ we obtain, as in (2.9):

$$\begin{aligned} d\hat{C}_t &= a(t)dt + dM_t \\ \text{where } dM_t &= e^{-rt}\frac{\partial C}{\partial S}(t,S_{t-})\sigma S_{t-}dW_t + \int_{\mathbb{R}} e^{-rt}[C(t,S_{t-}e^x) - C(t,S_{t-})]\; \tilde{J}_X(dt\; dx) \\ \text{and } a(t) &= e^{-rt}[\frac{\partial C}{\partial t}(t,S_{t-}) - rC(t,S_{t-}) + L^S C(t,S_{t-})] \end{aligned}$$

From (2.12) we know that $\hat{C}_{t\wedge\tau_B}$ is a martingale. Therefore, its drift part must be equal to zero which means that $\int_0^{t\wedge\tau_B} a(t)dt = 0$ for all $t \le T$. This implies that with probability 1, $a(t) = 0$ almost everywhere on $[0, \tau_B]$:

$$\frac{\partial C}{\partial t}(t,S_{t-}) + L^S C(t,S_{t-}) - rC(t,S_{t-}) = 0$$

which should be verified for almost every $t \in]0,\tau_B[$ with probability 1 so:

$$\frac{\partial C}{\partial t}(t,S) + L^S C(t,S) = rC(t,S) \qquad \forall (t,S) \in [0,T]\times]0,B[\tag{2.13}$$

For these arguments to work, we need the smoothness of C which is not obvious, especially at the barrier $S = B$. It can be shown to hold if $\sigma > 0$ [6, Chap. 3]:

Proposition 2 (PIDE for barrier options). *Let $C(t,S)$ be the value of a up-and-out call option defined by (2.11) in an exponential Lévy model with diffusion coefficient $\sigma > 0$ and Lévy measure ν. Then $(t,S) \to C(t,S;T,K)$*

verifies

$$\forall (t,S) \in [0,T[\times]0,\infty[,$$
$$\frac{\partial C}{\partial t}(t,S) + rS\frac{\partial C}{\partial S}(t,S) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}(t,S) - rC(t,S)$$
$$+ \int \nu(dy)[C(t,Se^y) - C(t,S) - S(e^y-1)\frac{\partial C}{\partial S}(t,S)] = 0. \qquad (2.14)$$
$$\forall S \in]0,B[, \qquad C(t=T,S) = (S-K)^+ \qquad (2.15)$$
$$\forall S \geq B, \quad \forall t \in [0,T] \qquad C(t,S) = 0 \qquad (2.16)$$

The equation can be simplified by going to log-variables: defining

$$f(t,y) = e^{r(T-t)}C(t,S_0e^y) \qquad y = \ln\frac{S}{S_0} \qquad b = \ln\frac{B}{S_0} \qquad (2.17)$$

we obtain a PIDE with constant coefficients in terms of f:

$$\forall (t,y) \in [0,T[\times]-\infty,b[\qquad \frac{\partial f}{\partial t}(t,y) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial y^2}(t,y) + (r-\frac{\sigma^2}{2})\frac{\partial f}{\partial y}(t,y)]$$
$$+ \int \nu(dz)[f(t,y+z) - f(t,y) - (e^z-1)\frac{\partial f}{\partial y}(t,y)] = 0 \quad (2.18)$$
$$\forall t \in [0,T[, \forall y \geq b, \quad f(t,y) = 0 \qquad \forall y \in \mathbb{R} \qquad f(T,y) = (S_0e^y - K)^+ 1_{y<b} \quad (2.19)$$

The main difference between this equation and the analogous PDEs for diffusion models is in the "boundary condition": (2.16) not only specifies the behavior of the solution *at* the barrier $S = B$ but also *beyond* the barrier ($S > B$). This is necessary because of the non-local nature of the operator L^S: to compute the integral term we need the function $C(t,.)$ on $]0,\infty[$ and (2.16) extends the function beyond the barrier by zero. In the case of a rebate, the function would be replaced by the value of the rebate in the knock-out region $S > B$.

The case $\sigma > 0$ is not the only case where the barrier price verifies (2.14): more generally, if $C(.,.)$ defined by (2.11) can be shown to be $C^{1,2}$ (or simply $C^{1,1}$ in the case of finite variation models) then C is a solution of the boundary value problem above. Since

$$C(t,S) = e^{-r(T-t)}\int p^B_{t,T}(u)H(u)du$$

where $p^B_{t,T}$ is the transition density of the process $S_{t\wedge T_B}$ stopped at the first exit from the barrier, the smoothness of $(t,S) \mapsto C(t,S)$ is linked to the smoothness of this transition density. In analytical terms $G(t,T,S,u) = e^{-r(T-t)}\rho^B_{t,T}(u)$ is nothing but the *Green function* for the operator L^S with boundary conditions (2.16). In the case of pure jump models where $\sigma = 0$ smoothness is not obvious: the behavior of the option price at the barrier depends on the local behavior of the Lévy process at the barrier. Some smoothness results may be obtained using Fourier methods, see [7].

This point will not be a problem: in the case where $\sigma = 0$ and $\nu(\mathbb{R}) = \infty$, as we shall see in section 3, we will approximate the solution of the boundary value problem by the solution of an auxiliary problem with a "numerical viscosity" term $\sigma > 0$. More generally, if smoothness of $C(.,.)$ is not known it should be seen as a viscosity solution of the PIDE (see below).

2.3 Viscosity solutions

Existence and uniqueness of (classical) solutions for the PIDEs considered above in Sobolev / Hölder spaces have been studied in [6, 16] in the case where the diffusion component is non-degenerate: for a Lévy process this simply means $\sigma > 0$ but more generally these results apply to jump diffusion where the diffusion coefficient is bounded away from zero. However many of the models in the financial modelling literature are pure jump models in which $\sigma = 0$, for which such results are not available. A notion of solution which yields both existence and uniqueness in this case is the notion of viscosity solution, introduced by Crandall & Lions for PDEs (see e.g. [13] for a review) and extended to integro-differential equations of the type considered here in [26, 28, 1, 4, 23].

Consider the following parabolic integro-differential equation on $[0,T] \times \mathbb{R}$ and $O = (a,b) \subset \mathbb{R}$ an open interval:

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= Lu, \qquad \text{on } (0,T] \times O & (2.20)\\ u(\tau,x) &= 0, \qquad (\tau,x) \notin (0,T] \times O & (2.21) \end{aligned}$$

Denote by USC (respectively LSC) the class of upper-semicontinuous (respectively lower semi-continuous) functions $v :]0,T] \times \mathbb{R} \to \mathbb{R}$ and by $C_p([0,T] \times \mathbb{R})$ the set of functions on $[0,T] \times \mathbb{R}$ with polynomial growth of degree p at infinity:

$$\varphi \in C_p([0,T] \times \mathbb{R}) \iff \exists C > 0, |\varphi(t,x)| \leq C(1 + |x|^p) \qquad (2.22)$$

Under a polynomial decay condition on the tails of the Lévy density, $L\varphi$ can be defined for $\varphi \in C^2([0,T] \times \mathbb{R}) \cap C_p([0,T] \times \mathbb{R})$:

$$\begin{aligned} L\varphi(x) &= A\varphi(x) + \int_{|y|\leq 1} \nu(dy)[\varphi(x+y) - \varphi(x) - y\frac{\partial \varphi}{\partial x}] & (2.23)\\ &+ \int_{|y|>1} \nu(dy)[\varphi(x+y) - \varphi(x)] & (2.24) \end{aligned}$$

The terms in (2.23) are well defined for $\varphi \in C^2([0,T] \times \mathbb{R})$ since

$$|\varphi(\tau, x+y) - \varphi(\tau,x) - y\frac{\partial \varphi}{\partial x}(\tau,x)| \leq y^2 \sup_{B(x,1)} |\varphi''(tau,.)|$$

while the term in (2.24) is well defined for $\varphi \in C_p([0,T] \times \mathbb{R})$ if

$$\int_{|y|\leq 1} |x|^p \nu(dx) < +\infty \qquad (2.25)$$

This condition is equivalent to the existence of a moment of order p for the Lévy process X_t. We will assume in the sequel that it holds for $p = 2$.

Definition 1 (Viscosity solution). $v \in USC$ is a viscosity subsolution of (2.20-2.21) iff for any $(\tau, x) \in [0,T] \times \mathbb{R}$ and any (test function) $\varphi \in C^2([0,T] \times \mathbb{R}) \cap C_2([0,T] \times \mathbb{R})$, if if $v - \varphi$ has a global maximum at (τ, x):

$$\forall (t, y) \in [0,T] \times \mathbb{R}, \qquad v(\tau, x) - \varphi(\tau, x) \geq v(t, y) - \varphi(t, y)$$

then

$$\frac{\partial \varphi}{\partial \tau}(\tau, x) - L\varphi(\tau, x) \leq 0, \quad \text{if } (\tau, x) \in (0,T] \times O \quad (2.26)$$

$$\min\{\frac{\partial \varphi}{\partial \tau}(\tau, x) - L\varphi(\tau, x), v(\tau, x) - h(x)\} \leq 0, \quad \text{for } \tau = 0, x \in O \text{ or } x \in \{a, b\}$$

$$\text{and} \qquad v(\tau, x) - h(x) \leq 0 \quad \text{otherwise} \qquad (2.27)$$

$v \in LSC$ is a viscosity supersolution of (2.20-2.21) iff for any $(\tau, x) \in [0,T] \times \mathbb{R}$ and any (test function) $\varphi \in C^2([0,T] \times \mathbb{R}) \cap C_2([0,T] \times \mathbb{R})$, if $v - \varphi$ has a global minimum at (τ, x):

$$\forall (t, y) \in [0,T] \times \mathbb{R}, \qquad v(\tau, x) - \varphi(\tau, x) \leq v(t, y) - \varphi(t, y)$$

then

$$\frac{\partial \varphi}{\partial \tau}(\tau, x) - L\varphi(\tau, x) \geq 0, \quad \text{if } (\tau, x) \in (0,T] \times O,$$

$$\max\{\frac{\partial \varphi}{\partial \tau}(\tau, x) - L\varphi(\tau, x), v(\tau, x) - h(x)\} \geq 0, \quad \text{for } \tau = 0, x \in O \text{ or } x \in \{a, b\}$$

$$\text{and} \qquad v(\tau, x) - h(x) \geq 0 \quad \text{otherwise}$$

A function $u \in C_2([0,T] \times \mathbb{R})$ is called a viscosity solution of (2.20-2.21) if it is both a subsolution and a supersolution. v is then continuous on $]0,T] \times \mathbb{R}$ and verifies (2.22).

Note that a sub-solution / super-solution need not be continuous and the initial condition is verified in a viscosity sense. Several variations on this definition can be found in the articles cited above. First, one can restrict the maximum/mimimum of $u - \varphi$ to be equal to zero:

Lemma 1. *$v \in USC$ is a viscosity subsolution of (2.20-2.21) if and only if for any $(\tau, x) \in [0,T] \times \mathbb{R}$ and any $\varphi \in C^2([0,T] \times \mathbb{R}) \cap C_2([0,T] \times \mathbb{R})$, if $v(\tau, x) = \varphi(\tau, x)$ and $v(\tau, x) \leq \varphi(\tau, x)$ on $[0,T] \times \mathbb{R}$ then φ verifies (2.26)-(2.27).*

Proof. Clearly the definition of a subsolution implies the property above. Inversely, let $\varphi \in C^2([0,T] \times \mathbb{R}) \cap C_2([0,T] \times \mathbb{R})$ be such that $v - \varphi$ has a global maximum at (τ, x). We can modify φ by adding a constant:

$$\varphi(t, y) = \varphi(t, y) + \overline{u}(\tau, x) - \varphi(\tau, x).$$

ψ then satisfies the following properties:

$$\left(\frac{\partial \psi}{\partial \tau} - L\psi\right)(t, y) = \left(\frac{\partial \varphi}{\partial \tau} - L\varphi\right)(t, y), \quad \forall (t, y) \in [0, T] \times \mathbb{R}, \tag{2.28}$$

$$\psi(\tau, x) = \overline{u}(\tau, x), \tag{2.29}$$

and (τ, x) is a strict global maximum point of $\overline{u} - \varphi$. In view of (2.29) it means that

$$\forall (t, y) \in [0, T] \times \mathbb{R}, \quad (t, y) \neq (\tau, x) \quad \Rightarrow \quad \overline{u}(t, y) - \varphi(t, y) < 0. \tag{2.30}$$

□

A similar result holds for supersolutions. Also, as shown in [4], one can replace "maximum" by "strict maximum". Finally, one can require the test functions to be $C^{1,2}$ or C^∞ instead of C^2. The growth condition at infinity $\varphi \in C_p$ on test functions is essential for $L\varphi$ to make sense. It may be replaced by other growth conditions under stronger hypotheses on the decay of the Lévy density.

Since L verifies a maximum principle, one can show that a classical solution $u \in C^{1,2}([0, T] \times \mathbb{R}) \cap C_2([0, T] \times \mathbb{R})$ is also a viscosity solution. However, since the definition above only involves applying derivatives to the test functions φ, a viscosity solution need not be smooth: it is simply required to be continuous on $]0, T] \times \mathbb{R}$.

Remark 1 (Boundary conditions). We noted above that, for classical solutions, "boundary" conditions have to be imposed on $\mathbb{R} - O$ and not only on the boundary $\partial O = \{a, b\}$. This seems not to be the case here since the non-local integral term only involves the test function and not the solution itself so one can be led to think that conditions *on* the boundary are enough (see remark in [23, Sec. 5.1.]). However note that the test functions have verify $\varphi \leq v$ (resp. $\varphi \geq v$) on $[0, T] \times \mathbb{R}$ and not only on $[0, T] \times O$

Existence and uniqueness of viscosity solutions for such parabolic integro-differential equations are discussed in [1] in the case where ν is a finite measure and in [4] and [23] for general Lévy measures. Growth conditions other than $u \in C_2$ can be considered (see e.g. [1, 4]) with additional conditions on the Lévy measure ν. The main tool for showing uniqueness is the comparison principle: if u, v are viscosity solutions and $u(0, x) \geq v(0, x)$ then $\forall \tau \in [0, T], u(\tau, x) \geq v(\tau, x)$. This property can be extended to sub-solutions and super-solutions in the following sense [1, 19]:

Proposition 3 (Comparison principle for semi-continuous solutions[1, 19]). *If $u \in USC$ is a sub-solution and $v \in LSC$ is a super-solution then $u \leq v$ on $]0, T] \times \mathbb{R}$.*

Proofs and extensions can be found in [1] for the case where ν is a bounded measure; the case of a general Lévy measure has been recently treated in [19].

The following result shows that values of European and barrier options can be expressed as viscosity solutions of (2.20-2.21):

Proposition 4 (Option prices as viscosity solutions). *Assume the Lévy density $\nu(.)$ verifies the quadratic decay condition (2.22). Then:*

- *Let $u(\tau, x)$ be the (forward) value of a European option defined by (2.2). If the payoff function H verifies a Lipschitz condition 2.5 and $h(x) = H(S_0 e^x)$ has quadratic growth at infinity, u is the (unique) viscosity solution of the Cauchy problem (2.3).*
- *The forward value of a knockout (double) barrier call or put option $f(t, y)$ defined by (2.17) is the (unique) viscosity solution of (2.18-2.19).*

Proof. u is known to be continuous on $[0, T] \times \mathbb{R}$, see e.g. [23]. Since $|h(x)| \leq c(1 + x^2)$,

$$|u(\tau, x)| = |E[h(x + X_\tau)]| \quad \leq \quad CE[\ |x + X_\tau|^2\ ]$$

Using moment estimates on X_t (see e.g. [25] or [23, Lemma 3.1.]) we obtain $|u(\tau, x)| \leq c(1 + x^2)$ hence $u \in C_2([0, T] \times \mathbb{R})$.

Similarly, f is continuous on $[0, T] \times \overline{O}$ where O is the region defined by the barriers. From the definition of f it is easily seen that $f(T, y) = h(S_0 e^y) 1_O(y)$ and $f(t, y) = 0$ if $(t, y) \notin [0, T] \times O$.

Let us now show that f is a sub-solution of (2.18-2.19). The proof for u is similar so we shall only give the arguments in the case of a bounded domain (barriers). Consider $(t_0, y_0) \in [0, T[\times O$; define $(Y_t)_{t \geq t_0}$ by $Y_t = y_0 + r(t - t_0) + X_{t - t_0}$ and denote by τ the first exit time of Y_t from O:

$$\tau(t_0, y_0) = \inf\{t \geq t_0, y_0 + r(t - t_0) + X_{t - t_0} \notin 0\}$$

and let $\tau = T \wedge \tau(t_0, y_0)$. Consider a test function $\varphi \in C^2([0, T] \times \mathbb{R}) \cap C_2([0, T] \times \mathbb{R})$ such that $\varphi(t_0, y_0) = f(t_0, y_0)$ and

$$\varphi(t, y) \geq f(t, y) \qquad \text{on } [0, T] \times \mathbb{R}$$

Since τ is then a bounded stopping time, applying the sampling theorem to the martingale $M_t = f(t, Y_t)$ we can rewrite, for $0 \leq \delta \leq T - t$:

$$\begin{aligned} f(t_0, y_0) &= E[M_\tau | Y_{t_0} = y_0] = E[f(t_0 + \delta\ \wedge \tau, Y_{(t_0 + \delta) \wedge \tau}) | Y_{t_0} = y_0] \\ &\leq E[\varphi(\ (t_0 + \delta) \wedge \tau, Y_{t_0 + \delta\ \wedge \tau}) | S_{t_0} = S_0] \end{aligned}$$

Note that X can jump over the boundary at the first exit time so we need to impose $\varphi \geq \phi$ not only on $[0, T] \times \overline{O}$ but on $[0, T] \times \mathbb{R}$. Applying the Ito formula to the smooth function $\varphi(t, S_t)$ between t_0 and $(t_0 + \delta) \wedge \tau$ we obtain:

$$f(t_0, y_0) \leq E[\varphi(t_0, y_0) + \int_{t_0}^{(t_0 + \delta) \wedge \tau} (\frac{\partial \varphi}{\partial t} + L^X \varphi + r \frac{\partial \varphi}{\partial y})(t, Y_{t-}) dt\ ] +$$

$$E[\int_{t_0}^{(t_0 + \delta) \wedge \tau} \frac{\partial \varphi}{\partial y}(t, Y_{t-}) \sigma dW_t + \int_{t_0}^{(t_0 + \delta) \wedge \tau} (\varphi(t, Y_{t-} + z) - \varphi(t, Y_{t-})\ )\ J_X(dt\ dz)] \quad (2.31)$$

The stochastic integral in the is a martingale if $E[A_{T\wedge\tau}] < \infty$ where

$$A_s = \int_{t_0}^{s} |\frac{\partial\varphi}{\partial y}(t, Y_{t-})|^2 dt + \int_{t_0}^{s} dt \int_{-\infty}^{\infty} \nu(dz)|\varphi(t, Y_{t-} + z) - \varphi(t, Y_{t-})\ |^2$$

Let us distinguish two cases:

- O is bounded (double barrier option): in this case φ and $\partial\varphi/\partial y$ are bounded on $[0,T] \times O$ so both integrands are bounded a.s. since $Y_{t-} \in O$ for $t_0 \leq t \leq \tau$. Therefore $E[A_{T\wedge\tau}] < \infty$ and the stochastic integral is a martingale with zero expectation:

$$E[\int_{t_0}^{(t_0+\delta)\wedge\tau} \frac{\partial\varphi}{\partial y}(t, Y_{t-})\sigma dW_t + \int_{t_0}^{(t_0+\delta)\wedge\tau} (\varphi(t, Y_{t-}+z) - \varphi(t, Y_{t-})\ ) J_X(dt\, dz)] \leq 0$$

- O is bounded (European or single barrier option): in this case either $a = -\infty$ or $b = \infty$ and φ and $\partial\varphi/\partial y$ are not bounded on $[0,T] \times O$. Consider the sequence of times $\theta_n = \inf\{t \geq t_0, A_t \leq n\ \}$ for $n \geq 1$. Since $(A_t)_{t\geq t_0}$ is left-continuous, $A_{t\wedge\theta_n}$ is bounded by n so the stochastic integral stopped at θ_n is a martingale with zero expectation:

$$E[\int_{t_0}^{(t_0+\delta)\wedge\tau\wedge\theta_n} \frac{\partial\varphi}{\partial y}(t, Y_{t-})\sigma dW_t + \int_{t_0}^{(t_0+\delta)\wedge\tau\wedge\theta_n} (\varphi(t, Y_{t-}+z) - \varphi(t, Y_{t-}) J_X(dt\, dz)] = 0$$

  By Fatou's lemma we conclude that:

$$E[\int_{t_0}^{(t_0+\delta)\wedge\tau} \frac{\partial\varphi}{\partial y}(t, Y_{t-})\sigma dW_t + \int_{t_0}^{(t_0+\delta)\wedge\tau} (\varphi(t, Y_{t-}+z) - \varphi(t, Y_{t-}) J_X(dt\, dz)] \leq 0$$

Therefore, in all cases

$$E[\int_{t_0}^{(t_0+\delta)\wedge\tau} \frac{\partial\varphi}{\partial y}(t, Y_{t-})\sigma dW_t + \int_{t_0}^{(t_0+\delta)\wedge\tau} (\varphi(t, Y_{t-}+z) - \varphi(t, Y_{t-}) J_X(dt\, dz)] \leq 0$$

Since $f(t_0, y_0) = \varphi(t_0, y_0)$ we finally obtain

$$0 \quad \leq \quad E[\int_{t_0}^{t_0+\delta\wedge\tau} (\frac{\partial\varphi}{\partial t} + L^X\varphi + r\frac{\partial\varphi}{\partial y})(t, Y_{t-})dt\ ]$$

Dividing by δ and taking $\delta \to 0^+$ we obtain:

$$\frac{\partial\varphi}{\partial t} + L^X\varphi + r\frac{\partial\varphi}{\partial y} \geq 0 \qquad \text{on} \quad [0,T[\times O$$

u is thus a sub-solution. Similarly, if $\varphi \in C^2([0,T] \times \mathbb{R})$ is such that $\varphi(t_0, y_0) = f(t_0, y_0)$ and $\varphi \leq f(t, y)$ on $[0,T] \times \mathbb{R}$ one can show that

$$\frac{\partial\varphi}{\partial t} + L^X\varphi + r\frac{\partial\varphi}{\partial y} \leq 0 \qquad \text{on} \quad [0,T[\times O$$

which shows that f is a super-solution. $\square$

The hypotheses above on the payoff function applies to put options, single-barrier knockout puts, double barrier knockout options and also to the log-contract. One can then retrieve call options by put-call parity.

3 An explicit-implicit finite difference scheme

In this section we present a numerical procedure for solving the initial/boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= Lu, \qquad (0,T]\times O \\ u(0,x)=h(x), \qquad x\in O \qquad & u(\tau,x)=0, \qquad x\notin O \end{aligned} \tag{3.1}$$

where

$$Lu = \frac{\sigma^2}{2}(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}) + \int \nu(dy)[u(\tau,x+y)-u(\tau,x)-(e^y-1)\frac{\partial u}{\partial x}(\tau,x)].$$

The main idea of our method is based on splitting the operator L into two parts:

$$\frac{\partial u}{\partial \tau} = \mathcal{D}u + \mathcal{J}u,$$

where $\mathcal{D}$ and $\mathcal{J}$ stand for the differential and integral parts of L respectively. We replace $\mathcal{D}u$ with a finite difference approximation, $\mathcal{J}u$ with the trapezoidal quadrature approximation and use the following explicit-implicit time stepping scheme:

$$\frac{u^{n+1}-u^n}{\Delta t} = Du^{n+1} + Ju^n.$$

We treat the integral part in an explicit time stepping in order to avoid the inversion of the dense matrix J. Therefore, the scheme is conditionally stable but we will show that the stability condition is weaker than the usual CFL condition for diffusion equations.

Before applying discretization, the equation must be localized bounded domain; we first discuss estimated for the localization error using a probabilistic approach. We then describe the space discretization and the time-stepping scheme in the case of a jump-diffusion model where the jump intensity is finite. Finally, we discuss how to deal with the singular case $\nu(\mathbb{R}) = +\infty$.

3.1 Localization to a bounded domain

To solve numerically the Cauchy problem (3.1) we first truncate the space domain to a bounded interval $x \in (-A, A)$. Usually, this leads to define some boundary conditions at $x = -A$ and $x = A$. As noted above because of the integral part, the operator is non-local. Thus, we need to extend the function $u(\tau,\cdot)$ outside the domain: more precisely, to $\{x+y \mid x\in(-A,A),\ y\in \operatorname{supp}\nu\}$ which in most examples is equal to the whole real line $\mathbb{R}$.

Extending the solution by zero beyond the boundary will give a localized problem of the form (2.20)-(2.21) and lead to a probabilistic interpretation. Although this may not be the best choice in all cases -another extension is given by the payoff function which is asymptotically close to the solution at infinity- we will see that it already leads to an exponentially decreasing localization error which serves as an upper bound for more refined choices of boundary conditions. Let us define $u_A(\tau,x)$ as the solution of the localized problem with zero boundary conditions:

$$\begin{aligned} &\frac{\partial u_A}{\partial \tau} = Lu_A, && (0,T]\times(-A,A) \qquad (3.2)\\ u_A(0,x) = h(x), \quad & x\in(-A,A) && u_A(\tau,x) = 0 \quad x\notin(-A,A) \end{aligned}$$

The following proposition states that the localization error decays exponentially with the domain size A.

Proposition 5. *Assume that h is bounded $||h||_\infty < \infty$ and*

$$\exists \alpha > 0, \int_{|x|>1} e^{\alpha|x|}\nu(dx) < \infty \qquad (3.3)$$

Let $u(\tau,x)$ and $u_A(\tau,x)$ be the solutions of Cauchy problems (3.1) and (3.2) respectively. Then

$$|u(\tau,x) - u_A(\tau,x)| \quad \leq \quad 2C_{\tau,\alpha}||h||_\infty e^{-\alpha(A-|x|)}, \qquad \forall x\in(-A,A). \qquad (3.4)$$

where the constant $C_{\tau,\alpha}$ does not depend on A.

Proof. The proof is based on the probabilistic representation of the solutions of (3.1) and (3.2). Let us define $M_\tau^x = \sup_{t\in[0,\tau]}|X_t + x|$. Then,

$$\begin{aligned} u(\tau,x) &= \mathbb{E}[h(X_\tau + x)],\\ u_A(\tau,x) &= \mathbb{E}[h(X_\tau + x)1_{\{M_\tau^x < A\}}], \end{aligned}$$

where $\tau(x) = \inf\{t\geq 0, |X_t + x| \geq A\}$ is the first exit time of $X_t + x$ from $[-A,A]$. Subtracting u_A from u gives:

$$\begin{aligned} |u(\tau,x) - u_A(\tau,x)| &= |E[h(X_\tau + x)1_{\{M_\tau^x \geq A\}}]|\\ &\leq 2||h||_\infty \mathbb{Q}(M_\tau^x \geq A) \qquad (3.5) \end{aligned}$$

Theorem 25.18 of [25] together with (3.3) implies

$$C_{\tau,\alpha} = \mathbb{E}e^{\alpha M_\tau^0} < \infty. \qquad (3.6)$$

Therefore, Chebyshev's inequality applies and we obtain

$$\mathbb{Q}(M_\tau^0 \geq A) \leq C_{\tau,\alpha}e^{-\alpha A}. \qquad (3.7)$$

Now, to pass from M_τ^0 to M_τ^x we use the following implications:

$$\begin{aligned} & \sup|X_t + x| && \leq && \sup|X_t| + |x|\\ \Rightarrow\quad & (\sup|X_t + x| \geq A && \Rightarrow && \sup|X_t| + |x| \geq A)\\ \Rightarrow\quad & \mathbb{Q}(M_\tau^x \geq A) && \leq && \mathbb{Q}(M_\tau^0 \geq A - |x|)\\ & && \leq && C_{\tau,\alpha}e^{-\alpha(A-|x|)} \quad \text{by (3.7)} \end{aligned}$$

Combining the last inequality with (3.5) gives the desired result. □

In the case of a put option $\|h\|_\infty = 1$, and the proposition applies. For the call option, the payoff is unbounded but using put-call parity one can transform the problem into pricing a put in order to obtain a smaller localization error.

Remark 2. An exponential bound on localization error in L_2-norm is given in [22] using analytical methods. The advantage of the probabilistic approach is to provide a local (pointwise) estimate. For instance, our estimate (3.4) reflects the intuitive fact that the localization error is more pronounced near the boundary.

The above result implies that, for any point not too close to the boundary i.e. for $|x| \leq (1-\delta)A$ where $0 < \delta < 1$ the localization error decreases with the domain size:

$$|u(\tau, x) - u_A(\tau, x)| \leq k e^{-\alpha\delta A}$$

The hypothesis (3.3) means that the tails of ν have to decrease exponentially, which is true in all examples considered in the option pricing literature (except Carr & Wu's log-stable model [9]). Note that in an exponential Lévy model we already have $\int_1^{+\infty} e^{\alpha x}\nu(\mathrm{d}x) < \infty$ for all $\alpha \leq 1$, because of the martingale condition so (3.3) is a condition on the negative jumps.

3.2 Truncation of the integral

To compute numerically the integral term, we need to reduce the region of integration to a bounded interval; in terms of the jump process, this amounts to the truncation of large jumps. We will now give an estimate for the error resulting from this approximation. Recall that the solution of (3.1) is

$$u(\tau, x) = \mathbb{E}[h(x + X_\tau)], \tag{3.8}$$

where X_τ is a Lévy process with the triplet (γ, σ, ν). Let us define a new process $\tilde{X}_\tau$ characterized by the Lévy triplet $(\tilde{\gamma}, \sigma, \nu\mathbf{1}_{x\in[B_l, B_r]})$, where $\tilde{\gamma}$ is determined by the martingale condition:

$$\tilde{\gamma} = -\frac{\sigma^2}{2} - \int_{B_l}^{B_r} (e^y - 1 - y1_{|y|\leq 1})\nu(dy).$$

We define now:

$$\tilde{u}(\tau, x) = \mathbb{E}[h(x + \tilde{X}_\tau)], \tag{3.9}$$

and we estimate the difference between $\tilde{u}$ and the true solution u.

Proposition 6. *Let $h(x)$ be an almost everywhere differentiable function with $\|h'\|_{L_\infty} < \infty$. Assume that $\exists\ \alpha_r, \alpha_l > 0$, such that $\int_1^\infty e^{(1+\alpha_r)y}\nu(dy) < \infty$, and $\int_{-\infty}^{-1} |y| e^{\alpha_l|y|}\nu(dy) < \infty$.*

If u and $\tilde{u}$ are defined by (3.8) and (3.9) respectively, then

$$|u(\tau, x) - \tilde{u}(\tau, x)| \quad \leq \quad \|h'\|_{L_\infty}\tau(C_1 e^{-\alpha_l|B_l|} + C_2 e^{-\alpha_r|B_r|}). \tag{3.10}$$

Proof. Denoting $R_\tau = X_\tau - \tilde{X}_\tau$ we have

$$\begin{aligned}
|u(\tau,x) - \tilde{u}(\tau,x)| &= |\mathbb{E}[h(x+\tilde{X}_\tau + R_\tau)] - \mathbb{E}[h(x+\tilde{X}_\tau)]| \\
&= |\mathbb{E}R_\tau \int_0^1 h'(x+\tilde{X}_\tau + uR_\tau)du| \\
&\leq \|h'\|_{L_\infty}\mathbb{E}|R_\tau|. \qquad (3.11)
\end{aligned}$$

The Lévy triplet of R_τ is $(\gamma - \tilde{\gamma}, 0, \nu\mathbf{1}_{x\notin[B_l,B_r]})$ with

$$\gamma - \tilde{\gamma} = -\int_{y\notin[B_l,B_r]} (e^y - 1)\nu(dy).$$

One can write $R_\tau = P_\tau + N_\tau$, where P_τ and N_τ are characterized by $(\int_{-\infty}^{B_l}(1 - e^y)\nu(\mathrm{d}y), 0, \nu\mathbf{1}_{x>B_r})$ and $(-\int_{B_r}^{\infty}(e^y-1)\nu(\mathrm{d}y), 0, \nu\mathbf{1}_{x<B_l})$ respectively. We assume without loss of generality that $B_l < -1$, $B_r > 1$[1]. Since P_τ has non-negative drift, no Brownian component, and only positive jumps bounded from below by $B_r > 0$, we have $P_\tau \geq 0$ (recall that $P_0 = 0$). Conversely, N_τ has only negative jumps (bounded from above by $B_l < 0$) and non-positive drift. In consequence, $N_\tau \leq 0$. Therefore,

$$\begin{aligned}
\mathbb{E}|R_\tau| &\leq \mathbb{E}|P_\tau| + \mathbb{E}|N_\tau| \;=\; \mathbb{E}P_\tau - \mathbb{E}N_\tau \\
&= \tau\left[\int_{-\infty}^{B_l}(1-e^y-y)\nu(\mathrm{d}y) + \int_{B_r}^{\infty}(e^y-1+y)\nu(\mathrm{d}y)\right] \\
&\leq \tau\left[2\int_{-\infty}^{B_l}|y|\nu(\mathrm{d}y) + 2\int_{B_r}^{\infty}e^y\nu(\mathrm{d}y)\right]. \qquad (3.12)
\end{aligned}$$

Using the hypotheses on ν we obtain

$$\begin{aligned}
\mathbb{E}|R_\tau| &\leq \tau\left(2e^{-\alpha_l|B_l|}\int_{-\infty}^{B_l}|y|e^{\alpha_l|y|}\nu(\mathrm{d}y) + 2e^{-\alpha_r|B_r|}\int_{B_r}^{\infty}e^{(1+\alpha_r)y}\nu(\mathrm{d}y)\right) \\
&\leq \tau(C_1e^{-\alpha_l|B_l|} + C_2e^{-\alpha_r|B_r|}),
\end{aligned}$$

that we substitute into (3.11). □

Remark 3. The hypotheses on ν in Proposition 3.10 are a little stronger than (3.3). We require them to obtain an exponential decay of the truncation error. However, we can use directly estimate (3.12). In other words, existence of the integrals in (3.12) suffices to obtain a convergence of $\tilde{u}$ to u as $|B_l|$ and $|B_r|$ grow to infinity, but this convergence does not necessarily occur at an exponential rate.

[1] Clearly, if (3.10) is true for such values, it is true for all B_l, B_r, up to change of the constants. On the other hand, this estimate is not useful if ν has a bounded support. For example, if there are only negative jumps, we will take $B_r = 0$, but in this case $\nu\mathbf{1}_{x\leq B_r} = \nu$, and there is no truncation error due to B_r.

Remark 4. The requirements are different for the left tail of ν and the right one. For example, in the Variance Gamma model with $\nu(x) = a\exp(-\eta_\pm|x|)/|x|$ one needs η_+ to be greater than 1, and η_- only positive. Proposition 3.10 then applies with $\alpha_l < \eta_-$ and $\alpha_r < \eta_+ - 1$.

Using Propositions 5 and 3.10 we can fix in advance $[-A, A]$ and $[B_l, B_r]$ to have a given bound on the respective errors. Therefore we will assume this has been done and concentrate ourselves on numerical solving of the localized problem.

3.3 Explicit-implicit scheme: finite activity case

We suppose here that $\nu(\mathbb{R}) = \lambda < +\infty$. Then the integro-differential operator can be written as

$$Lu = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \left(\frac{\sigma^2}{2} + \alpha\right)\frac{\partial u}{\partial x} + \int_{B_l}^{B_r} \nu(dy)u(\tau, x+y) - \lambda u, \tag{3.13}$$

where $\alpha = \int_{B_l}^{B_r}(e^y - 1)\nu(\mathrm{d}y)$.

We introduce a uniform grid on $[0,T] \times \mathbb{R}$: $\tau_n = n\Delta t$, $n = 0\dots M$, $x_i = -A + i\Delta x$, $i \in \{0,..,N\}$, with $\Delta t = T/M$, $\Delta x = 2A/N$. Let $\{u_i^n\}$ be the solution of the numerical scheme, to be defined below.

To approximate the integral terms we use the trapezoidal quadrature rule with the same resolution Δx. Let K_l, K_r be such that $[B_l, B_r] \subset [(K_l - 1/2)\Delta x, (K_r + 1/2)\Delta x]$. Then:

$$\int_{B_l}^{B_r} \nu(dy)u(\tau, x_i + y) \approx \sum_{j=K_l}^{K_r} \nu_j u_{i+j}, \qquad \lambda \approx \hat{\lambda} = \sum_{j=K_l}^{K_r} \nu_j,$$

$$\alpha \approx \hat{\alpha} = \sum_{j=K_l}^{K_r} (e^{y_j} - 1)\nu_j, \qquad \text{where} \qquad \nu_j = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \nu(\mathrm{d}y). \tag{3.14}$$

Notice that for computing the integral term we need to extend the solution to $x \in [-A + B_l, A + B_r]$: in the sequel we assume that it is extended by zero outside $[-A, A]$. For knockout barrier options this is a natural choice; for European options one might choose to extend the solution using the payoff function h on the band $[-A + B_l, -A] \cup [A, A + B_r]$ and by zero beyond.

The space derivatives are discretized using finite differences:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}, \tag{3.15}$$

$$\left(\frac{\partial u}{\partial x}\right)_i \approx \begin{cases} \frac{u_{i+1} - u_i}{\Delta x}, & \text{if } \sigma^2/2 + \hat{\alpha} < 0 \\ \frac{u_i - u_{i-1}}{\Delta x}, & \text{if } \sigma^2/2 + \hat{\alpha} \geq 0. \end{cases} \tag{3.16}$$

The choice of approximation for the first-order derivative is determined by stability requirement and will be discussed later (Section 4.2). Since the two cases are treated similarly, let us suppose without loss of generality that $\sigma^2/2 + \hat{\alpha} < 0$.

Using (3.14)–(3.16) we obtain $Lu \approx Du + Ju$, where

$$(Du)_i = \frac{\sigma^2}{2}\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} - \left(\frac{\sigma^2}{2} + \hat{\alpha}\right)\frac{u_{i+1} - u_i}{\Delta x}, \tag{3.17}$$

$$(Ju)_i = \sum_{j=K_l}^{K_r} \nu_j u_{i+j} - \hat{\lambda} u_i. \tag{3.18}$$

Finally, we replace the problem (3.2) with the following time-stepping scheme:

$$(\mathbf{S}) \quad \begin{array}{ll} \textbf{Initialization}: & \\ u_i^0 = h(x_i), \qquad i \in \mathbb{Z} & (3.19) \\ \textbf{For } \mathbf{n = 0, \ldots, M-1}: & \\ \dfrac{u_i^{n+1} - u_i^n}{\Delta t} = (Du^{n+1})_i + (Ju^n)_i, \quad \text{if } i \in \{0, \ldots, N\} & (3.20) \\ u_i^{n+1} = 0, \qquad i \notin \{0, \ldots, N\}. & (3.21) \end{array}$$

3.4 Explicit-implicit scheme: infinite activity case

If $\nu(\mathbb{R}) = +\infty$ this method cannot be applied directly. The idea is to come down to a non-singular case by approximating the process X_τ by an appropriate finite activity process with a modified diffusion coefficient.

The procedure is similar to the one described in Section 3.2, but this time we deal with small jumps. Given $\varepsilon > 0$ let us define a process X_τ^ε characterized by the Lévy triplet $(\gamma(\varepsilon), \sqrt{\sigma^2 + \sigma^2(\varepsilon)}, \nu \mathbf{1}_{|x| \geq \varepsilon})$, where

$$\sigma^2(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} y^2 \nu(\mathrm{d}y),$$

and $\gamma(\varepsilon)$ is determined by the martingale condition:

$$\gamma(\varepsilon) = -\frac{\sigma^2 + \sigma^2(\varepsilon)}{2} - \int_{|y| \geq \varepsilon} (e^y - 1 - y 1_{|y| \leq 1}) \nu(dy).$$

This means that we replace small jumps of size $\leq \varepsilon$ by a Brownian motion $\sigma(\varepsilon) W_\tau$. Obviously, X_τ^ε has jumps of finite activity.

The function u^ε defined as

$$u^\varepsilon(\tau, x) = \mathbb{E}[h(x + X_\tau^\varepsilon)] \tag{3.22}$$

satisfies the following Cauchy problem:

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial \tau} &= L^\varepsilon u^\varepsilon, \qquad (0, T] \times \mathbb{R} & (3.23) \\ u^\varepsilon(0, x) &= h(x), \qquad x \in \mathbb{R} \end{aligned}$$

where

$$L^\varepsilon f = \frac{\sigma^2 + \sigma^2(\varepsilon)}{2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\sigma^2 + \sigma^2(\varepsilon)}{2} + \alpha(\varepsilon)\right)\frac{\partial f}{\partial x} + \int_{|y|\geq\varepsilon} \nu(dy) f(x+y) - \lambda(\varepsilon) f(x), \quad (3.24)$$

and $\alpha(\varepsilon) = \int_{|y|\geq\varepsilon}(e^y - 1)\nu(\mathrm{d}y), \quad \lambda(\varepsilon) = \int_{|y|\geq\varepsilon}\nu(\mathrm{d}y)$.

The next proposition gives an estimate of the approximation error.

Proposition 7. *Let $h(x)$ be an almost everywhere differentiable function with $\|h'\|_{L_\infty} < \infty$. Let u and u^ε be defined by (3.8) and (3.22) respectively. Then*

$$|u(\tau,x) - u^\varepsilon(\tau,x)| \quad \leq \quad C\|h'\|_{L_\infty}\frac{\int_{-\varepsilon}^{\varepsilon}|y|^3\nu(\mathrm{d}y)}{\sigma^2(\varepsilon)}. \quad (3.25)$$

Proof. We essentially use [12, Proposition 6.2] with the only difference that we also adjust the drift parameter γ of X to preserve the martingale property. Let us define $Y_\tau = X_\tau - (\gamma - \gamma(\varepsilon))\tau$. Then,

$$\begin{aligned}|u(\tau,x) - u^\varepsilon(\tau,x)| &= |\mathbb{E}[h(x+X_\tau)] - \mathbb{E}[h(x+X_\tau^\varepsilon)]| \\ &\leq |\mathbb{E}[h(x+Y_\tau)] - \mathbb{E}[h(x+X_\tau^\varepsilon)]| + \\ &\quad + |\mathbb{E}[h(x+Y_\tau + (\gamma-\gamma(\varepsilon))\tau)] - \mathbb{E}[h(x+Y_\tau)]|. \quad (3.26)\end{aligned}$$

By [12, Proposition 6.2] we have

$$|\mathbb{E}[h(x+Y_\tau)] - \mathbb{E}[h(x+X_\tau^\varepsilon)]| \quad \leq \quad K\|h'\|_{L_\infty}\frac{\int_{-\varepsilon}^{\varepsilon}|y|^3\nu(\mathrm{d}y)}{\sigma^2(\varepsilon)} \quad (3.27)$$

with $K < 16.5$. The second term may be estimated as follows:

$$\begin{aligned}&|\mathbb{E}[h(x+Y_\tau + (\gamma-\gamma(\varepsilon))\tau)] - \mathbb{E}[h(x+Y_\tau)]| = \\ &\qquad |(\gamma-\gamma(\varepsilon))\tau\int_0^1 h'(x+Y_\tau + s(\gamma-\gamma(\varepsilon))\tau)ds| \leq \|h'\|_{L_\infty}|\gamma-\gamma(\varepsilon)|\tau\end{aligned}$$

where

$$\begin{aligned}|\gamma-\gamma(\varepsilon)| &= |\frac{\sigma^2(\varepsilon)}{2} - \int_{|y|<\varepsilon}(e^y - 1 - y)\nu(\mathrm{d}y)| \\ &= |\frac{1}{2}\int_{|y|<\varepsilon}\nu(\mathrm{d}y)\int_0^y e^s(y-s)^2 ds| \leq \frac{e^\varepsilon}{6}\int_{-\varepsilon}^{\varepsilon}|y|^3\nu(\mathrm{d}y). \quad (3.28)\end{aligned}$$

Since $\sigma^2(\varepsilon) \to 0$ as $\varepsilon \to 0$, (3.28) converges faster than (3.27), and therefore may be neglected. $\square$

Remark 5. If $\lim_{x\to 0}\nu(x)|x|^{1+\beta} = a > 0$, with $0 \leq \beta < 2$, then (3.25) gives

$$|u(\tau,x) - u^\varepsilon(\tau,x)| \leq C(\beta)\|h'\|_{L_\infty}\varepsilon$$

so the approximation error is proportional to ε. This case includes all practical examples as variance gamma, stable tempered processes and others.

4 Consistency, stability and convergence

We study in this section some properties of the finite difference scheme presented above. First, we show that the scheme is locally consistent, stable and monotone under a condition on the time step Δt. Under this condition we show that the solution of the scheme converges to the viscosity solution of the PIDE i.e. the option price as $\Delta x,\ \Delta t \to 0$.

4.1 Consistency

Proposition 8 (Consistency). *The finite difference scheme (3.20) is locally consistent with equation (3.2): for all $v \in C_0^\infty([0,T]\times\mathbb{R})$ and for all $(\tau_n, x_i) \in [0,T]\times\mathbb{R}$ we have*

$$\left|\frac{v_i^{n+1}-v_i^n}{\Delta t} - (Dv)_i^{n+1} - (Jv)_i^n - \left(\frac{\partial v}{\partial \tau} - Lv\right)(\tau_n, x_i)\right| = r_i^n(\Delta t, \Delta x) \to 0, \quad (4.1)$$

when $(\Delta t, \Delta x) \to 0$. Moreover:

$$\exists c > 0, \qquad |r_i^n(\Delta t, \Delta x)| \le c(\Delta t + \Delta x)$$

Proof. Using a second order Taylor expansion we obtain:

$$\left|\frac{v_i^{n+1}-v_i^n}{\Delta t} - \frac{\partial v}{\partial \tau}(\tau_n, x_i)\right| = \left|\frac{\Delta t}{2}\frac{\partial^2 v}{\partial \tau^2}(\tilde{\tau}, x_i)\right| \le \frac{\Delta t}{2}\left\|\frac{\partial^2 v}{\partial \tau^2}\right\|_\infty \qquad (4.2)$$

Consider now the terms in Jv.

$$\begin{aligned} |\alpha - \hat{\alpha}| &= \left|\int_{B_l}^{B_r}(e^y - 1)\nu(\mathrm{d}y) - \sum_{j=K_l}^{K_r}(e^{y_j} - 1)\nu_j\right| \\ &= \left|\sum_{j=K_l}^{K_r}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x}(1 - e^{y_j - y})e^y\nu(\mathrm{d}y)\right| \le \Delta x\left|\sum_{j=K_l}^{K_r}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} e^y\nu(\mathrm{d}y)\right| \\ &= \Delta x\int_{B_l}^{B_r} e^y\nu(\mathrm{d}y) \le \Delta x\int e^y\nu(\mathrm{d}y). \qquad (4.3)\end{aligned}$$

We have used that $|1 - e^{y_j - y}| \le \Delta x$ as $\Delta x \le 1$, because $|y_j - y| \le \Delta x/2$.

$$\begin{aligned} &\left|(Dv)_i^{n+1} - (\mathcal{D}v)(\tau_n, x_i)\right| = \left|(Dv)_i^{n+1} - (\mathcal{D}v)(\tau_{n+1}, x_i) + \Delta t(\mathcal{D}v)(\tilde{\tau}, x_i)\right| = \\ &\left|\Delta t(\mathcal{D}v)(\tilde{\tau}, x_i) + \frac{\sigma^2}{2}\frac{\Delta x^2}{24}\left[\frac{\partial^4 v}{\partial x^4}(\tau_{n+1}, \xi_1) + \frac{\partial^4 v}{\partial x^4}(\tau_{n+1}, \eta_1)\right] + \right. \\ &\left.\left(\frac{\sigma^2}{2} + \hat{\alpha}\right)\frac{\Delta x^2}{3}\left[\frac{\partial^3 v}{\partial x^3}(\tau_{n+1}, \xi_2) - \frac{\partial^3 v}{\partial x^3}(\tau_{n+1}, \eta_2)\right] + (\alpha - \hat{\alpha})\frac{\partial v}{\partial x}(\tau_{n+1}, x_i)\right| \\ &\le C_1\Delta t + C_2\Delta x \qquad (4.4)\end{aligned}$$

$$
\begin{aligned}
|(Jv)_i^n - (\mathcal{J}v)(\tau_n, x_i)| &= \left| \sum_{j=K_l}^{K_r} v_{i+j}\nu_j - \int_{B_l}^{B_r} v(\tau_n, x_i + y)\nu(\mathrm{d}y) \right| \\
&= \left| \sum_{j=K_l}^{K_r} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} (v(\tau_n, x_i + y_j) - v(\tau_n, x_i + y))\nu(\mathrm{d}y) \right| \\
&= \left| \sum_{j=K_l}^{K_r} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} (y_j - y)\frac{\partial v}{\partial x}(\tau_n, x_i + \xi)\nu(\mathrm{d}y) \right| \le \frac{\Delta x}{2}\lambda \left\| \frac{\partial v}{\partial x} \right\|_\infty
\end{aligned} \tag{4.5}
$$

therefore

$$
r_i^n(\Delta t, \Delta x) \le C_1\Delta t + C_2\Delta x \to 0. \tag{4.6}
$$

□

4.2 Monotonicity and Stability

Two properties which are important for showing convergence to viscosity solutions are the monotonicity and stability of the scheme.

Definition 2 (Stability). Scheme **(S)** is *stable* iff for a bounded initial condition the solution does exist and is bounded independently on Δt, Δx, uniformly on $[0, T] \times \mathbb{R}$:

$$
\exists C > 0,\ \forall\ \Delta t > 0,\ \Delta x > 0,\ i \in \mathbb{Z},\ n \in \{0, \dots, M\}: \quad |u_i^n| \le C. \tag{4.7}
$$

We will say that vector v (or matrix M) is positive if all its elements are positive. We write $u \ge v$ if $u - v \ge 0$.

Proposition 9 (Stability and discrete comparison principle). *If* $\Delta t \le 1/\hat{\lambda}$, *scheme* **(S)** *is stable and verifies the discrete comparison principle: if* u^0, v^0 *are two bounded initial conditions then*

$$
u^0 \ge v^0 \quad \Rightarrow \quad [\quad \forall n \ge 1, u^n \ge v^n].
$$

Proof. We start with rewriting the equation (3.20) in the following form:

$$
\begin{aligned}
& -c\Delta t u_{i-1}^{n+1} + (1 + a\Delta t)u_i^{n+1} - b\Delta t u_{i+1}^{n+1} = \\
& \qquad = (1 - \Delta t \sum_{j \ne 0} \nu_j)u_i^n + \Delta t \sum_{j \ne 0} \nu_j u_{i+j}^n,
\end{aligned} \tag{4.8}
$$

where[2]

$$\begin{aligned} a &= \frac{\sigma^2}{(\Delta x)^2} - \left(\frac{\sigma^2}{2} + \hat{\alpha}\right)\frac{1}{\Delta x} \geq 0, \\ b &= \frac{\sigma^2}{2(\Delta x)^2} - \left(\frac{\sigma^2}{2} + \hat{\alpha}\right)\frac{1}{\Delta x} \geq 0, \\ c &= \frac{\sigma^2}{2(\Delta x)^2} \geq 0, \end{aligned} \tag{4.9}$$

Notice that $a = b + c$, and $1 - \Delta t \sum_{j\neq 0} \nu_j \;\geq\; 1 - \hat{\lambda}\Delta t \;\geq\; 0$ by the hypothesis.

Stability. We have a tri-diagonal linear system on $\{u_0^{n+1}, \ldots, u_N^{n+1}\}$. It has a unique solution because the main diagonal is dominant: $1 + a\Delta t \geq c\Delta t + b\Delta t$.

We will show that, if h is a bounded initial condition, $\|h\|_\infty < \infty$, then $\forall\, n$,

$$\|u^n\|_\infty \leq \|h\|_\infty. \tag{4.10}$$

Let us proceed by induction. By definition of u^0 we have $\|u^0\|_\infty \leq \|h\|_\infty$. Let (4.10) hold for n. Suppose that $\|u^{n+1}\|_\infty > \|h\|_\infty$. This means that $\exists\, i_0 \in \{0, \ldots, N\}$, such that $|u_{i_0}^{n+1}| = \|u^{n+1}\|_\infty$, and $\forall\, i \in \mathbb{Z}$, $|u_i^{n+1}| \leq |u_{i_0}^{n+1}|$.

Therefore we have

$$\begin{aligned} \|u^{n+1}\|_\infty = |u_{i_0}^{n+1}| &= -c\Delta t|u_{i_0}^{n+1}| + (1 + a\Delta t)|u_{i_0}^{n+1}| - b\Delta t|u_{i_0}^{n+1}| \\ &\leq -c\Delta t|u_{i_0-1}^{n+1}| + (1 + a\Delta t)|u_{i_0}^{n+1}| - b\Delta t|u_{i_0+1}^{n+1}| \\ &\leq |-c\Delta t u_{i_0-1}^{n+1} + (1 + a\Delta t)u_{i_0}^{n+1} - b\Delta t u_{i_0+1}^{n+1}|. \end{aligned}$$

Together with (4.8) this gives

$$\|u^{n+1}\|_\infty \leq (1 - \Delta t \sum_{j\neq 0} \nu_j)\|u^n\|_\infty + \Delta t \sum_{j\neq 0} \nu_j \|u^n\|_\infty \leq \|h\|_\infty, \tag{4.11}$$

which contradicts our assumption. Thus, $\|u^{n+1}\|_\infty \leq \|h\|_\infty$, and the proof is completed.

Monotonicity. Let u^n and v^n be two solutions of **(S)** corresponding to the initial conditions $h(x)$ and $g(x)$ respectively, and $h(x) \geq g(x)$, $\forall\, x \in \mathbb{R}$. Let us define $w^n = u^n - v^n$. We have to show that $w^n \geq 0$, $\forall\, n \geq 0$.

We proceed similarly as in the previous proof. By construction, we have $w_i^0 = h(x_i) - g(x_i) \geq 0$, $\forall\, i \in \mathbb{Z}$. Let $w^n \geq 0$, and suppose that $\inf_{i\in\mathbb{Z}} w_i^{n+1} < 0$. Since $\forall\, i \in \mathbb{Z}\backslash\{0, \ldots, N\}$, $w_i^{n+1} = h(x_i) - g(x_i) \geq 0$, this implies that $\exists\, i_0 \in \{0, \ldots, N\}$, such that $w_{i_0}^{n+1} = \inf_{i\in\mathbb{Z}} w_i^{n+1}$. Using (4.8) we obtain that

$$\begin{aligned} \inf_{i\in\mathbb{Z}} w_i^{n+1} = w_{i_0}^{n+1} &= -c\Delta t w_{i_0}^{n+1} + (1 + a\Delta t)w_{i_0}^{n+1} - b\Delta t w_{i_0}^{n+1} \\ &\geq -c\Delta t w_{i_0-1}^{n+1} + (1 + a\Delta t)w_{i_0}^{n+1} - b\Delta t w_{i_0+1}^{n+1} \\ &= (1 - \Delta t \sum_{j\neq 0} \nu_j) w_{i_0}^n + \Delta t \sum_{j\neq 0} \nu_j w_{i_0+j}^n \geq 0, \end{aligned} \tag{4.12}$$

[2] We recall that the case $\sigma^2/2 + \hat{\alpha} < 0$ is considered. If $\sigma^2/2 + \hat{\alpha} \geq 0$ we change the approximation of the first-order derivative (see (3.16)) to have $a, b, c \geq 0$, which is needed for stability and monotonicity.

which contradicts the assumption. Therefore, $\inf_{i\in\mathbb{Z}} w_i^{n+1} \geq 0$, and, in consequence, $w^{n+1} \geq 0$. □

The discrete comparison principle has an important financial interpretation: it is equivalent to saying that the option values computed using the scheme verify *arbitrage inequalities*: inequalities between payoffs lead to inequalities between values of options. Thus, the discrete comparison principle is a desirable property for our numerical scheme.

The restriction on Δt has an intuitive interpretation: time step cannot be greater than jumps frequency. Otherwise, the process has (on average) more than one jump between τ_n and τ_{n+1}.

Remark 6 (Other notions of stability). Various definitions for the stability of numerical schemes can be found in the literature: for example, the "von-Neumann stability" studied for a similar scheme in [2] represents u^n as a discrete Fourier transform and requires the corresponding Fourier coefficients to be stable. This is essentially equivalent to stability in L^2 norm with periodic boundary conditions and simply controls a global error in the least square sense but does not control the error on the value of a given option. Moreover it does not allow to capture the effect of boundary conditions when they are non-periodic, which is the case here.

4.3 Convergence

In the usual approach to the convergence of finite difference schemes for PDEs, consistency and stability ensure convergence under regularity assumptions on the solution. This approach is not feasible here because, as discussed in sec. 2.3, solutions may be non-smooth and higher order derivatives may not exist. For example, in the Variance Gamma model the value of a call option is C^1 in the price variable but not C^2.

This is where viscosity solutions come to the rescue: in the case of second-order parabolic PDEs verifying a strong uniqueness principle as in proposition (3), Barles & Souganidis [5] show that, for elliptic/parabolic PDEs, any locally consistent, stable and monotone finite difference scheme converges uniformly on each compact subset of $[0,T]\times\mathbb{R}$ to the unique continuous viscosity solution, even when solutions are not smooth. We will now show the convergence of the solution of the scheme considered above to a viscosity solution of (3.1). To make shorter the notations and show the connection with [5] let us rewrite (3.20) in an equivalent form:

$$\frac{1}{\Delta t} S(\Delta t, \Delta x, n+1, i, u_i^{n+1}, \tilde{u}) = 0, \qquad n \geq 0,\ i \in \{0,\dots,N\} \tag{4.13}$$

where $\tilde{u} = \{u_{i-1}^{n+1}, u_{i-1}^{n+1}, u^n\}$, and

$$S(\Delta t, \Delta x, n+1, i, u_i^{n+1}, \tilde{u}) = (1 + a\Delta t) u_i^{n+1} - c\Delta t u_{i-1}^{n+1} - b\Delta t u_{i+1}^{n+1} - \\ - (1 - \Delta t \sum_{j\neq 0} \nu_j) u_i^n - \Delta t \sum_{j\neq 0} \nu_j u_{i+j}^n.$$

Coefficients $a, b, c \geq 0$ depend on Δx and are given by (4.9).

If $1 - \Delta t \sum_{j \neq 0} \nu_j \equiv 1 - \Delta t \hat{\lambda} \geq 0$, we have the **monotonicity** property:

- If $u_i^{n+1} = v_i^{n+1}$, and $\tilde{u} \leq \tilde{v}$, then

$$S(\Delta t, \Delta x, n+1, i, u_i^{n+1}, \tilde{u}) \geq S(\Delta t, \Delta x, n+1, i, v_i^{n+1}, \tilde{v}). \tag{4.14}$$

The **consistency** (4.1) reads as:

- $\forall\, \varphi \in C^\infty([0,T] \times \mathbb{R})$, $\forall\, (\tau, x) \in [0,T] \times [-A, A]$,

$$\frac{1}{\Delta t} S(\Delta t, \Delta x, n+1, i, \varphi_i^{n+1}, \tilde{\varphi}) \to \left(\frac{\partial \varphi}{\partial \tau} - L\varphi \right)(\tau, x), \tag{4.15}$$

  as Δt, $\Delta x \to 0$, and $(\tau_{n+1}, x_i) \to (\tau, x)$.

- Notice also that if ξ is a constant function on the grid, we have

$$S(\Delta t, \Delta x, n+1, i, u_i^{n+1} + \xi, \tilde{u} + \xi) = S(\Delta t, \Delta x, n+1, i, u_i^{n+1}, \tilde{u}). \tag{4.16}$$

  In fact one can see from the definition that the scheme is linear and a constant function is always a solution.

Let us define a piecewise interpolation of the solution to the scheme **(S)**:

$$u^{(\Delta t, \Delta x)}(\tau, x) = u_j^n, \quad \text{if } \tau \in [\tau_n, \tau_{n+1}), \ \ x \in [x_{j-1/2}, x_{j+1/2}). \tag{4.17}$$

Proposition 10 (Convergence of the finite difference scheme). *Let h be a bounded continuous initial condition, $\|h\|_\infty < \infty$. If $\Delta t \leq 1/\hat{\lambda}$, then the solution $u^{(\Delta t, \Delta x)}$ of the numerical scheme converges uniformly on each compact subset of $[0,T] \times \mathbb{R}$ to the unique viscosity solution of the problem (3.1).*

Proof. Define

$$\underline{u}(\tau, x) = \liminf_{(\Delta t, \Delta x) \to 0 \ (t,y) \to (\tau, x)} u^{(\Delta t, \Delta x)}(t, y) \tag{4.18}$$

$$\overline{u}(\tau, x) = \limsup_{(\Delta t, \Delta x) \to 0 \ (t,y) \to (\tau, x)} u^{(\Delta t, \Delta x)}(t, y) \tag{4.19}$$

By construction, $\underline{u} \leq \overline{u}$. We will show that $\overline{u}$ and $\underline{u}$ are respectively viscosity subsolution and supersolution to the problem (3.1). If it is true, then the comparison principle for semi-continuous solutions (Proposition 3) implies that $\underline{u} \geq \overline{u}$. We conclude that $\underline{u} = \overline{u} = u$ is a viscosity solution of (3.1). We will now show that $\overline{u}$ is a viscosity subsolution of (3.1).

First, we remark that $\overline{u}$ is uniformly bounded. In fact, by Proposition 9, $|u^{(\Delta t, \Delta x)}(\tau, x)| \leq \|h\|_\infty$ for all positive Δt, Δx, and all $(\tau, x) \in [0,T] \times \mathbb{R}$. Therefore, by construction, $\overline{u}$ is bounded by the same constant.

We formulate the next property as a lemma. The proofs of lemmas used in this proposition are given in the Appendix.

Lemma 2. *$\overline{u}$ is upper semi-continuous:*

$$\forall(\tau,x), \qquad \limsup_{(t,y)\to(\tau,x)} \overline{u}(t,y) \quad\le\quad \overline{u}(\tau,x).$$

Finally, we will show that $\forall(\tau,x)\in[0,T]\times\mathbb{R}$,

(i) if $x\notin[-A,A]$, we have

$$\overline{u}(\tau,x)=0, \tag{4.20}$$

(ii) if $x\in[-A,A]$, $\varphi\in C^2([0,T]\times\mathbb{R})$ such that $\overline{u}(\tau,x)=\varphi(\tau,x)$ and $\overline{u}<\varphi$ on $[0,T]\times\mathbb{R}-\{(\tau,x)\}$ then

$$\frac{\partial\varphi}{\partial\tau}(\tau,x)-L\varphi(\tau,x)\le 0, \quad \text{if } (\tau,x)\in(0,T]\times(-A,A), \tag{4.21}$$

$$\min\{\frac{\partial\varphi}{\partial\tau}(\tau,x)-L\varphi(\tau,x),\overline{u}(\tau,x)-h(x)\}\le 0, \quad \text{if } \tau=0 \text{ or } x=\pm A, \tag{4.22}$$

which clearly implies (2.26)–(2.27).

Property (i) is an immediate consequence of (3.19) and (3.21). Indeed, if $x\notin[-A,A]$ then by construction $\exists\varepsilon>0$, such that for $\Delta t,\ \Delta x<\varepsilon$ and $|(t,y)-(\tau,x)|<\varepsilon$ we have $u^{(\Delta t,\Delta x)}(t,y)=h(x)$, so (4.20) follows from the definition of the upper limit.

Let us take (τ,x) and φ as in (ii). Let $\Omega\subset\mathbb{R}$ be a bounded interval containing $[-A+B_l,A+B_r]$.[3]

Lemma 3. *There exist Δt_k, $\Delta x_k\to 0$, and n_k, i_k, such that*

$$(\tau_{n_k},x_{i_k})\to(\tau,x), \tag{4.23}$$

$$u^{(\Delta t_k,\Delta x_k)}(\tau_{n_k},x_{i_k})\to\overline{u}(\tau,x), \qquad \textit{as } k\to\infty, \tag{4.24}$$

and for all k, (τ_{n_k},x_{i_k}) is a global maximum point of $u^{(\Delta t_k,\Delta x_k)}(\cdot,\cdot)-\varphi(\cdot,\cdot)$ on the grid inside $[0,T]\times\Omega$:

$$\max_{(\tau_n,x_j)\in[0,T]\times\Omega}\{u^{(\Delta t_k,\Delta x_k)}(\tau_n,x_j)-\varphi(\tau_n,x_j)\}$$
$$=u^{(\Delta t_k,\Delta x_k)}(\tau_{n_k},x_{i_k})-\varphi(\tau_{n_k},x_{i_k}). \tag{4.25}$$

We take a sequence from this lemma. Denoting by ξ_k the right-hand side of (4.25), and $u^{(\Delta t_k,\Delta x_k)}(\tau_n,x_i)\equiv(u^{\Delta_k})_i^n$ we have

$$\xi_k \quad\to\quad 0, \quad \text{as } k\to\infty, \tag{4.26}$$

$$(u^{\Delta_k})_{i_k}^{n_k} \quad=\quad \varphi_{i_k}^{n_k}+\xi_k, \tag{4.27}$$

$$(u^{\Delta_k})_i^n \quad\le\quad \varphi_i^n+\xi_k, \quad \forall(\tau_n,x_j)\in[0,T]\times\Omega. \tag{4.28}$$

[3] By the definition of $[-A,A]$ and $[B_l,B_r]$, the values of any functions outside $[0,T]\times\Omega$ are not involved in our computations.

If $x \in (-A, A)$, from (4.23) it follows that $x_{i_k} \in (-A, A)$ for all k greater than some k_0. Therefore, from the definition of $u^{(\Delta t_k, \Delta x_k)}$ as a solution of the scheme, we have

$$\begin{aligned}
0 &= \frac{1}{\Delta t_k} S(\Delta t_k, \Delta x_k, n_k, i_k, (u^{\Delta_k})^{n_k}_{i_k}, \tilde{u}^{\Delta_k}) \\
&\geq \frac{1}{\Delta t_k} S(\Delta t_k, \Delta x_k, n_k, i_k, \varphi^{n_k}_{i_k} + \xi_k, \tilde{\varphi} + \xi_k) \quad \text{by (4.27)–(4.28) and (4.14)} \\
&= \frac{1}{\Delta t_k} S(\Delta t_k, \Delta x_k, n_k, i_k, \varphi^{n_k}_{i_k}, \tilde{\varphi}) \quad \text{by (4.16)} \\
&\to \left(\frac{\partial \varphi}{\partial \tau} - L\varphi\right)(\tau, x) \quad \text{by the definition of } \Delta t_k, \Delta x_k, n_k, i_k \text{ and (4.15)}
\end{aligned}$$

which is the desired result.

If (τ, x) lies on the boundary, that is $\tau = 0$ or $x = \pm A$, there are two possibilities:

(1) there exists a subsequence of (4.23) such that $\forall k$, $(\tau_{n_k}, x_{i_k}) \in (0, T] \times (-A, A)$;

(2) $\exists k_0$, such that $\forall k > k_0$, $\tau_{n_k} = 0$ or $x_{i_k} = \pm A$.

In the first case we apply the same arguments and obtain $\frac{\partial \varphi}{\partial \tau}(\tau, x) - L\varphi(\tau, x) \leq 0$. In the second case we have $(u^{\Delta_k})^{n_k}_{i_k} = h(x_{i_k})$. Taking limits in (4.27) we obtain

$$h(x) = \varphi(\tau, x) = \overline{u}(\tau, x)$$

Obviously, in the two cases (4.22) is satisfied.

We have proven that $\overline{u}$ is a viscosity subsolution of (3.1). By exactly the same type of arguments it can be shown that $\underline{u}$ is a viscosity supersolution of this problem. By construction $\overline{u} \geq \underline{u}$ so using the comparison principle for semi-continuous solutions [1] we conclude that $\overline{u} = \underline{u} = u$ is a continuous viscosity solution. Using a variation on the proof of Dini's theorem one can show, as in [5], that the convergence is uniform on all compact subsets of $[0, T] \times \mathbb{R}$. $\square$

Readers used to convergence results for finite difference schemes should note that the requirements on the scheme are quite weak: we have only used *local consistency* whereas the usual conditions for convergence using the Lax theorem require *global* consistency i.e. the convergence in (4.1) must not be pointwise but with respect to some global norm, which requires knowing in advance in what function space the solutions live. This shows the flexibility of the notion of viscosity solution. The price to pay is the loss of information about the order of convergence. Of course in cases where smoothness of the solution can be shown by other means, e.g. under condition (2.6), one can discuss the order of convergence using classical methods and higher order schemes can be used. But in absence of information on the smoothness of the solution, the justification of higher-order schemes is not clear.

5 Numerical results

We now illustrate the performance of the scheme proposed above in two examples. The computations were done in variance Gamma models with Lévy density

$$\nu(x) \quad = \quad a\frac{\exp(-\eta_{\pm}|x|)}{|x|}$$

and two sets of parameters $a = 6.25, \eta_- = 14.4$, $\eta_+ = 60.2$ (VG1) and $a = 0.5$, $\eta_- = 2.7$, $\eta_+ = 5.9$ (VG2) and a Merton model with Gaussian jumps in log price with Lévy density

$$\nu(x) = 0.1\frac{e^{-x^2/2}}{\sqrt{2\pi}} \quad \text{and volatility } \sigma = 15\%$$

for a put of maturity of $T = 1$ year. Both of these models satisfy the hypotheses of section 2.3 and 3: the Lévy densities have exponentially decreasing tails. The Black Scholes implied volatilities computed from prices of put options are shown in figure 1: as expected, they display a "smile" (convex) feature as a function of the strike of the option, which flattens out with increasing maturity. These two models have been chosen since in each case there is an alternative method for computing the solution of the PIDE: in the Merton model the solution can be expressed as a series expansion and in the Variance Gamma model a closed form expression for the characteristic function is available so one can use Carr & Madan's Fast Fourier method [8]. Comparing our finite difference solution to these alternative solutions (computed with high precision) allows to study the behavior of various error terms in relation with the parameters of the scheme.

The numerical examples in this section compare the results of the finite difference scheme to these alternative methods in the case of a put option $h(x) = (1 - e^x)^+$. The error metric which is relevant from the point of view of financial applications is the error in terms of Black-Scholes implied volatility:

$$\varepsilon(\tau, x) = |\Sigma^{\text{PIDE}}(\tau, x) - \Sigma^{\text{FFT}}(\tau, x)| \qquad \text{in} \quad \%.$$

where Σ denotes the Black Scholes implied volatility computed by inverting the Black Scholes formula with respect to the volatility parameter and applying it to the computed option price. We have computed both pointwise errors at $x = 0$ (i.e. forward at-the-money options) and uniform errors on the computational range $x \in [2/3,\ 2]$. This range contains all options prices quoted on the market.

The localization error is shown in figure 2 for the Merton model: domain size A is represented in terms of its ratio to the standard deviation of X_T. An acceptable level is obtained as soon as this ratio is of order $\simeq 5$. Figure 3 shows the same analysis for the Variance Gamma model: though the behavior is similar, to achieve the same error level one must choose larger domains since the tails of the Lévy measure decay at a slower rate than in the Merton model.

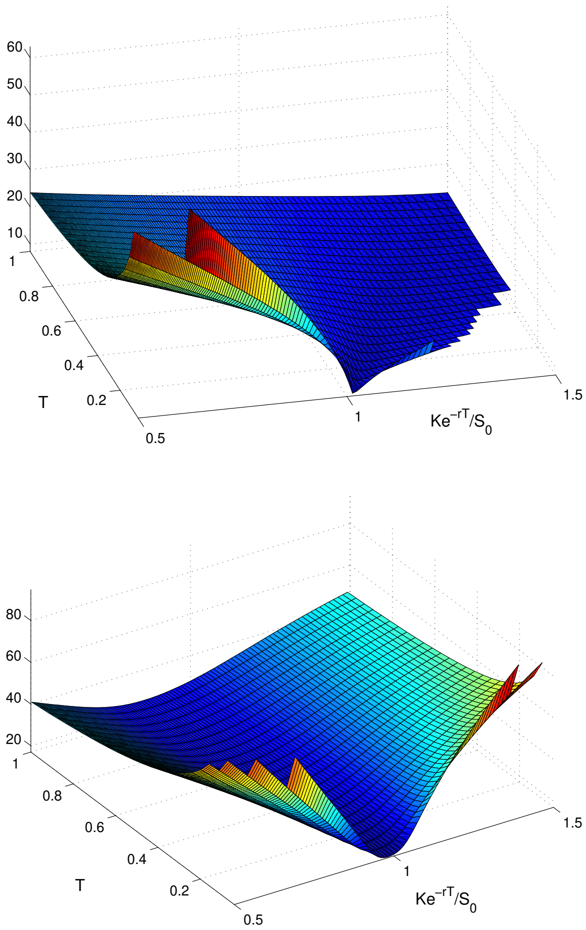

Figure 1: Black-Scholes implied volatilities for call options, as a function of strike and maturity. Above: Variance Gamma model. Below: Merton jump-diffusion model.

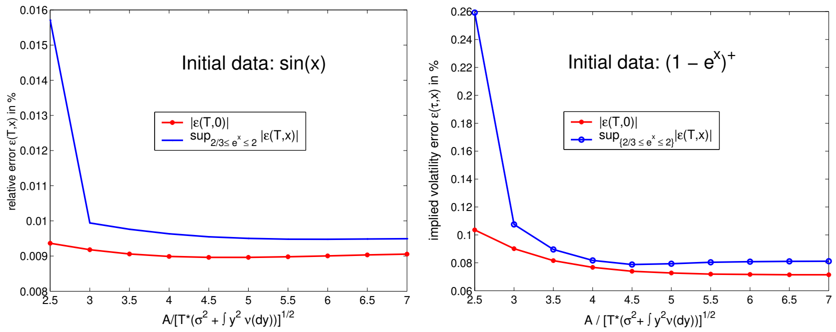

Figure 2: Influence of domain size on localization error for the explicit-implicit finite difference scheme. Left: smooth initial condition $h(x) = \sin(x)$ in Merton jump-diffusion model. Right: put option in Merton jump-diffusion model.

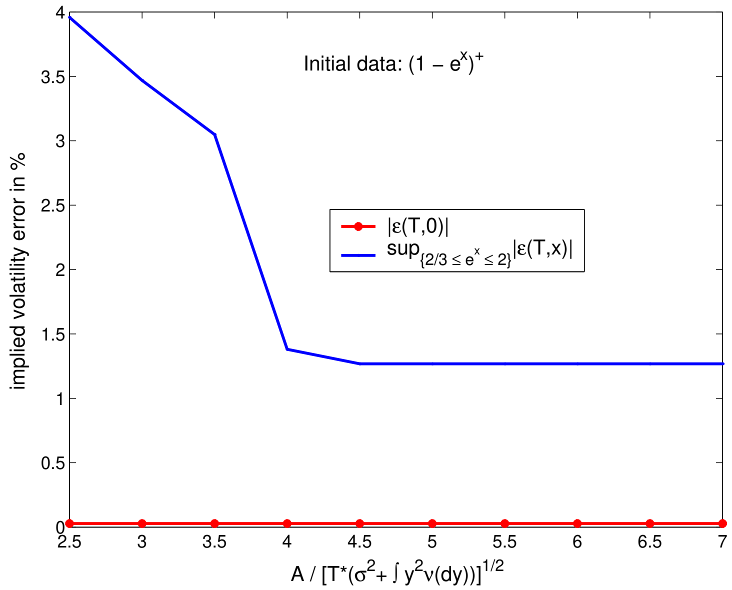

Figure 3: Influence of domain size on localization error for the explicit-implicit finite difference scheme: put option in Variance Gamma model.

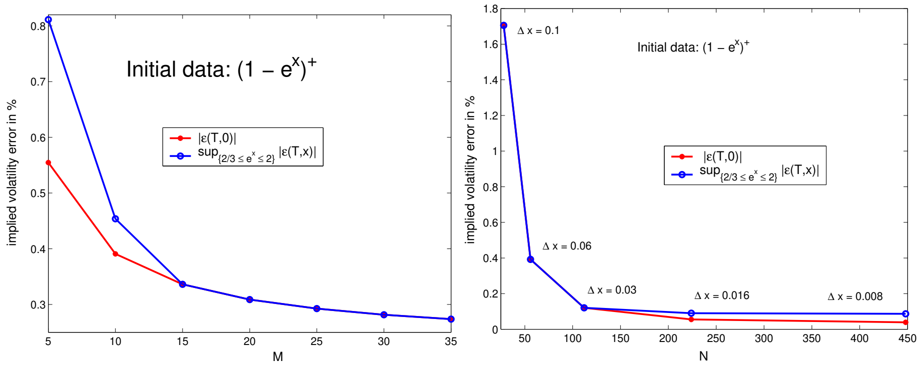

Figure 4: Numerical precision for a put option in the Merton model. Left: Influence of number of time steps M. $\Delta x = 0.05$, $\Delta t = T/M$. Right: influence of number of space steps N, Merton jump-diffusion model.

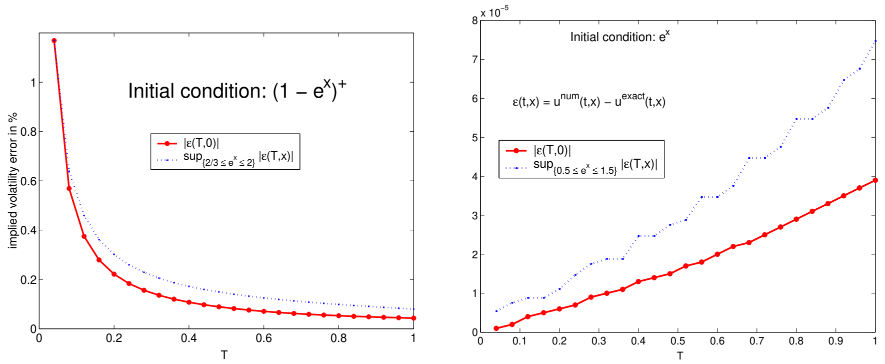

Figure 5: Decrease of error with maturity in the Merton model. Left: non-smooth initial condition (put option) $h(x) = (1 - e^x)^+$. Right: smooth initial condition $h(x) = e^x$.

Figure 4 illustrated the decay of numerical error when $\Delta t, \Delta x \to 0$ i.e. when the number of time steps / space steps is increased. The behavior is quite similar to the case of the Black Scholes model.

Figure 5 illustrated the behavior of the error (for a fixed grid size) as a function of maturity for two initial conditions a smooth one (forward contract) and a non-smooht one (put option). We observe that a non-smooth initial condition leads to a lack of precision for small T. This phenomenon, which is not specific to models with jumps, can be overcome using an irregular time-stepping scheme which exploits the smoothness in time of the solution. Matache et al. [22] have suggested to use irregularly (logarithmically) spaced time stepping, more refined near maturity, in order to improve this convergence. Notice that as soon as this ratio is ≥ 3 the uniform and pointwise errors are quite close to each other, indicating that we are out of the zone of influence of the numerical boundary conditions.

In the case of infinite activity models an additional parameter which influences the solution is the truncation parameter ε for the small jumps. Choosing this parameter as small as possible is not necessarily a good idea: figure 6 shows that, for a given $\Delta x > 0$ the minimal error is obtained for a finite ε which in this case is *larger* than Δx. The optimal choice of ε is not universal and depends on the growth of the Lévy density near zero. In all cases, the numerical diffusion term $\sigma(\varepsilon)$ significantly improves the convergence by introducing a "numerical viscosity" so even when the underlying model is a pure jump Lévy process, this numerical diffusion term plays a regularizing effect.

6 Discussion and extensions

We have discussed in this work the computation of option prices in jump-diffusion and exponential Lévy models via the solution of integro-differential equations involving an integral term with a possibly singular kernel (Lévy density). After establishing a precise relation between the option prices and the solutions of such equations, we have proposed an explicit-implicit finite difference scheme for solving them. Some difficulties arise due to the non-local character of the integral operator, the non-smoothness of initial conditions, the singularity at zero of the integral kernel and the possible degeneracy of the diffusion coefficient. We have proposed approximation which resolve these difficultes and provided error estimates in each case, under hypotheses which are easily verified on the Lévy density.

For European call options, the finite difference method proposed here required more operations than than the Fast Fourier transform method of Carr & Madan [8]; however, our method is applicable in a more general context: it does not require a closed form expression for the characteristic function of the log-price and can handle barrier options.

While other finite difference methods have been proposed in the literature for such equations [2, 14, 29] a comparable analysis of consistency, stability and convergence is absent from these studies. Moreover, by appealing to the

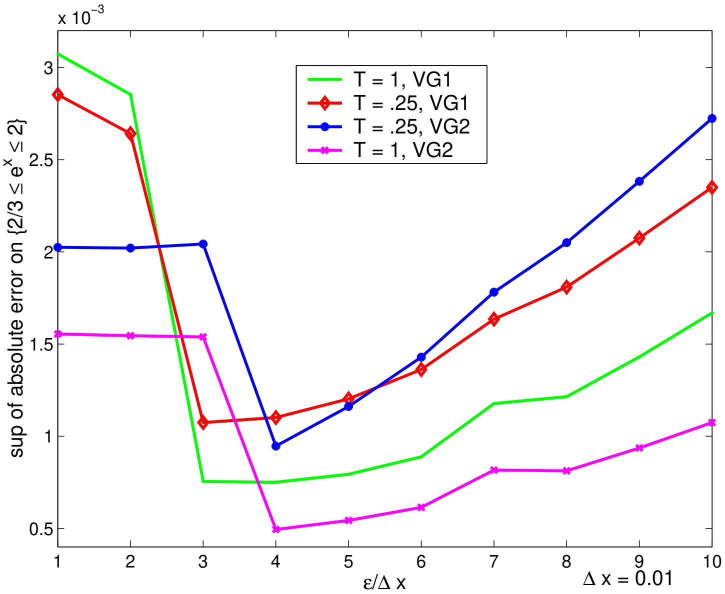

Figure 6: Influence of truncation of small jumps on numerical error in various Variance Gamma models. Put option.

formalism of viscosity solutions, our analysis allows fairly general hypotheses on the model. In particular it does not require the diffusion part to be non-degenerate and thus includes models based on pure-jump Lévy processes such as the Variance Gamma model, hyperbolic models, NIG models etc.

Another approach is based on the variational formulation of the PIDE [6]: studied by Zhang [30] for the Merton model, this approach has been recently extended by Matache et al [22] to the infinite activity case using a wavelet Galerkin method. This method requires to know the analytical form of the Lévy density ν, whereas our finite difference method also accomodates a numerical representation of the Lévy measure; for example, ν can be the result of a calibration to option prices as in [11]. The finite difference scheme is also simpler to implement while retaining the flexibility of separating the local and the non-local part of the operator.

The finite difference method described here is also applicable when the local characteristics $\sigma(t), \nu_t$ of the process are time-dependent, as is the case in time-inhomogeneous jump-diffusion models [12]; it also applied in the case where the diffusion coefficient depends on the price as in [2], without any non-degeneracy requirement on the diffusion coefficient. However empirical studies on performance of risk neutral jump-diffusion models [11] seem to indicate that this feature is redundant with jumps and leads to highly ill-posed calibration problems so the usefulness of combining local volatility models with jumps is far from being clear.

It may be interesting to investigate generalizations of this method to cases where non-linearities are present in the equation such as the valuation of American options or problems with transaction costs.

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A Appendix

Proof of Lemma 2.

Proof. We need to show that $\forall \varepsilon > 0$, $\exists\, U(\tau, x)$ a neighbourhood of (τ, x), such that $\forall (t, y) \in U(\tau, x)$, $\overline{u}(t, y) < \overline{u}(\tau, x) + \varepsilon$. Suppose this is not true, that is

$$\exists \varepsilon > 0,\ \exists (t_k, y_k) \to (\tau, x): \quad \forall k,\ \overline{u}(t_k, y_k) \geq \overline{u}(\tau, x) + \varepsilon. \tag{A.1}$$

For a fixed k we have by construction:

$$\overline{u}(t_k, y_k) \quad = \quad \limsup_{(\Delta t, \Delta x) \to 0\ (s,z) \to (t_k, y_k)} u^{(\Delta t, \Delta x)}(s, z).$$

Therefore, we can find Δt_k, Δx_k, s_k, z_k in the $1/k$-neighborhood of $(0, 0, t_k, y_k)$, such that

$$u^{(\Delta t_k, \Delta x_k)}(s_k, z_k) \quad > \quad \overline{u}(t_k, y_k) - \frac{1}{k}. \tag{A.2}$$

Combining (A.2) with (A.1) we obtain

$$u^{(\Delta t_k, \Delta x_k)}(s_k, z_k) \quad > \quad \overline{u}(\tau, x) + \varepsilon - \frac{1}{k}. \tag{A.3}$$

By construction, we have $(\Delta t_k, \Delta x_k, s_k, z_k) \to (0, 0, \tau, x)$, therefore

$$\limsup_{k \to \infty} u^{(\Delta t_k, \Delta x_k)}(s_k, z_k) \quad \leq \quad \limsup_{(\Delta t, \Delta x) \to 0\ (t,y) \to (\tau, x)} u^{(\Delta t, \Delta x)}(t, y) = \overline{u}(\tau, x).$$

Taking the upper limit in (A.3) as $k \to \infty$ gives

$$\overline{u}(\tau, x) \quad \geq \quad \overline{u}(\tau, x) + \varepsilon,$$

which is impossible. This contradiction implies that (A.1) is not verified, and $\overline{u}(\tau, x)$ is *usc*. $\square$

Proof of Lemma 3.

Proof. The existence of Δt_k, Δx_k, n_k, i_k satisfying (4.23)–(4.24) follows from the definition of the upper limit. Let us take such a sequence and consider for each k the function $u^{(\Delta t_k, \Delta x_k)} - \varphi$. Since there is a finite number of grid points on $[0,T] \times \Omega$, the maximum in (4.25) is reached at a point (τ_{m_k}, x_{j_k}).

Sequence $\{(\tau_{m_k}, x_{j_k})\}_{k\in\mathbb{N}}$ is bounded, therefore one can extract a converging subsequence (for simplicity we will take the same notation): $(\tau_{m_k}, x_{j_k}) \to (\tau^*, x^*)$ as $k \to \infty$. Let us show that $(\tau^*, x^*) = (\tau, x)$. Suppose this is not true. Then, using the definition of $\overline{u}$, the continuity of φ and the fact that $\overline{u} < \varphi$ on on $[0,T] \times \mathbb{R} - \{(\tau, x)\}$ we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \Big(u^{(\Delta t_k, \Delta x_k)}(\tau_{m_k}, x_{j_k}) - \varphi(\tau_{m_k}, x_{j_k}) \Big) \leq \\
&\qquad\leq \limsup_{\substack{(\Delta t, \Delta x)\to 0 \\ (t,y)\to(\tau^*, x^*)}} u^{(\Delta t, \Delta x)}(t,y) - \lim_{k\to\infty} \varphi(\tau_{m_k}, x_{j_k}) = \\
&\qquad\qquad = \overline{u}(\tau^*, x^*) - \varphi(\tau^*, x^*) < 0. \qquad \text{(A.4)}
\end{aligned}$$

On the other hand, from the definition of (τ_{m_k}, x_{j_k}) as a maximum point, and properties (4.24) and (2.29) it follows that

$$\begin{aligned}
&\limsup_{k\to\infty} \Big(u^{(\Delta t_k, \Delta x_k)}(\tau_{m_k}, x_{j_k}) - \varphi(\tau_{m_k}, x_{j_k}) \Big) \geq \\
&\qquad \geq \limsup_{k\to\infty} \Big(u^{(\Delta t_k, \Delta x_k)}(\tau_{n_k}, x_{i_k}) - \varphi(\tau_{n_k}, x_{i_k}) \Big) = \overline{u}(\tau, x) - \varphi(\tau, x) = 0,
\end{aligned}$$

which contradicts (A.4). Therefore, $(\tau^*, x^*) = (\tau, x)$.

From the last inequality we obtain, in particular, that

$$\limsup_{k\to\infty} u^{(\Delta t_k, \Delta x_k)}(\tau_{m_k}, x_{j_k}) \geq \varphi(\tau, x) = \overline{u}(\tau, x).$$

From the definition of $\overline{u}$ we have

$$\limsup_{k\to\infty} u^{(\Delta t_k, \Delta x_k)}(\tau_{m_k}, x_{j_k}) \leq \overline{u}(\tau, x).$$

Therefore, this upper limit is equal to $\overline{u}(\tau, x)$, and there exists a subsequence $u^{(\Delta t_k, \Delta x_k)}(\tau_{m'_k}, x_{j'_k}) \to \overline{u}(\tau, x)$, which satisfies by construction the desired properties (4.23)–(4.25). $\square$