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Abstract

We discuss the behaviour of a model of ferroelectric material represented by a thin cylinder with small thickness $\nu > 0$. This model is described by a couple of Maxwell equations satisfied by the electromagnetic field (H, E) and electric polarization field P. We give a complete description of the limit model as $\nu \to 0$ in the linear case when (E, H) and P satisfy a Silver-Müller type boundary condition. When the potential is nonlinear and P satisfies the boundary condition $P \times n = 0$ we prove the strong convergence of the polarization field which allows to give the description of the behaviour of the nonlinear problem when the thickness ν tends to 0. We observe that the behaviour is very sensitive to the choice of the boundary conditions satisfied by the different fields.

Key words. Maxwell equations, Ferroelectric material, Thin cylinders.
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1 Introduction

We shall discuss the model equations of ferroelectric materials introduced by Greenberg et al. see [8] and also [7]. The characteristic feature of ferroelectric cristal is the appearance of a spontaneous electric dipole. It can be reversed, with no net change in magnitude, by an applied electric field. The current density j of the ferroelectric domain Ω is driven by the difference between the electric equilibrium field $\widehat{E}(P)$ and the electric field E where P is the spontaneous electric polarization). If one denotes by m the internal magnetic field then the model equations introduced in [8] takes the form in $\mathbb{R}^+ \times \Omega$

$$\begin{cases} \epsilon(\partial_t P + \theta\, j) = \operatorname{curl} m \\ \mu(\partial_t m + \theta\alpha\, m) = -\operatorname{curl} P \\ \partial_t j + \theta\alpha\, j = \gamma\theta(\widehat{E}(P) - E). \end{cases} \tag{1}$$

This set of equations is completed by initial conditions $P(0) = P^0$, $m(0) = m^0$, $j(0) = j^0$ and boundary conditions which will be discussed later. Eliminating the variables j and m we get the following Maxwell equation satisfied by P

$$\partial_t^2 P + (\epsilon\mu)^{-1}\operatorname{curl}^2 P + a\partial_t P = -\gamma\theta(\widehat{E}(P) - E) \tag{2}$$

where $a = \theta\alpha$. We set $\text{curl}^2 P = \text{curl}(\text{curl} P)$. The parameters $\epsilon > 0$ and $\mu > 0$ are the permititivity and the magnetic permeabilty of the vaccum and the others ones are some physical constants. The equilibrium electric field is given by $\widehat{E}(P) = \phi'(|P|^2)P$ where ϕ is a two wells potential satisfying some hypotheses given later. The electric displacement D is linked to the electric and polarization fields E and P by the law $D = \epsilon(E + P)$. Hence the electromagnetic field (H, E) satisfies in $\mathbb{R}^+ \times \Omega$ the Maxwell equations

$$\mu\partial_t H - \text{curl}\, E = 0,\ \ \epsilon\partial_t(E + P) + \text{curl}\, H + \sigma E = 0 \tag{3}$$

where $\sigma > 0$ is the constant conductivity of Ω. The initial contitions are $E(0) = E^0$, $H(0) = H^0$. The boundary conditions satisfied by E and P take an important place in the characterization of the thin limit behaviour of the model. If m satisfies the boundary condition $m \times n = 0$ on $\partial\Omega$ where n is the unit outward normal to Ω, one deduces by using (1) that P satisfies the boundary condition

$$\text{curl}\, P \times n = 0. \tag{4}$$

This condition was proposed in [8]. If more generally m satisfies a Silver-Müller type boundary condition like $n \times m + \rho\, n \times (P \times n) = 0$ with $\rho \geq 0$ is a function defined on $\partial\Omega$ then,we obtain directly from (1) the boundary condition for P

$$\text{curl}\, P \times n + \rho\mu\ n \times ((\partial_t + a)P \times n) = 0. \tag{5}$$

Formally, we have the Green formula

$$\begin{aligned} &\textstyle\int_\Omega \text{curl}^2 P \cdot \partial_t P dx = \frac{1}{2}\frac{d}{dt}(\int_\Omega |\text{curl}\, P|^2 dx + \\ &+ \mu a \textstyle\int_{\partial\Omega} \rho |P \times n|^2 d\alpha) + \mu \int_{\partial\Omega} \rho |\partial_t P \times n|^2 d\alpha \end{aligned} \tag{6}$$

This boundary condition will be used in our work only for the linear case when ϕ is given by $\phi(|P|^2) = k|P|^2/2$. For the nonlinear case, we will use the boundary condition

$$P \times n = 0 \tag{7}$$

instead of (5). It can be deduced from the boundary condition $\text{curl}\, m \times n = \theta j \times n$. The reason we use (7) is that we can prove the $\mathbb{H}^1$-regularity of the polarization field P which allow to control the nonlinear term $\widehat{E}(P)$. We refer to the work of [1] where the boundary condition (5) is considered. For the linear as well as the nonlinear case we use for the electric field E, the Silver-Müller boundary condition

$$H \times n + \beta\ n \times (E \times n) = 0,\ \ P \times n = 0 \tag{8}$$

where $\beta \geq 0$ is some function defined on $\partial\Omega$. We have formally the Green formula

$$\int_\Omega (\text{curl}\, H \cdot E - \text{curl}\, E \cdot H) dx = \int_{\partial\Omega} H \times n \cdot E d\alpha = \beta \int_{\partial\Omega} |E \times n|^2 d\alpha. \tag{9}$$

The equilibrium electric field $\widehat{E}(P)$ is given by $\widehat{E}(P) = \phi'(|P|^2)P$ where $\phi : \mathbb{R} \to \mathbb{R}$ is a C^2 potential function, defined in [8], such that $\phi(0) = 0$, $r_0^2 > 0$ with $\phi(r_0^2) < 0$ is the location of the unique minimum of $\phi(r^2)$ and $\phi(r^2) > 0$ for $r^2 \geq r_1^2$ and ϕ satisfies the following hypotheses

$$\phi(s) \sim C_0 s \text{ for } s \to +\infty,\ |\phi'(s)| \leq C_1,\ s\phi^{(2)}(s) \leq C_2 \text{ for } s \geq 0. \tag{10}$$

where $C_0, C_1 > 0$ and $C_2 > 0$ are constants depending only of ϕ. It follows that there exists $C_* > 0$ depending only of C_1 and C_2 such that

$$|(s\phi^{(2)}(s^2))'| \le C_* \text{ for } s \ge 0. \tag{11}$$

Using (11) we get

$$|A\phi'(|A|^2) - B\phi'(|B|^2)| \le C_*|A - B|. \tag{12}$$

for all $A, B \in \mathbb{R}^3$.

Let us precise the models we shall discuss. To simplify the presentation we equate to 1 all the parameters appearing in the model except $a > 0$ and σ to measure the size of the dissipation process. Let $\nu > 0$ be fixed representing the thickness of the cylinder $\Omega^\nu = \widehat{\Omega} \times (0, \nu)$ with cross section $\widehat{\Omega} \subset \mathbb{R}^2$ assumed to be an open bounded and regular domain. We denote by n the outward unit normal to Ω^ν. The generic point $x \in \Omega^\nu$ is denoted by $x = (\widehat{x}, x_3)$ where $\widehat{x} = (x_1, x_2) \in \widehat{\Omega}$ and $0 < x_3 < \nu$. The electromagnetic field (H^ν, E^ν) satisfies in $\mathbb{R}^+ \times \Omega^\nu$ the problem

$$\begin{cases} \partial_t H^\nu - \operatorname{curl} E^\nu = 0, \ \partial_t(E^\nu + P^\nu) + \operatorname{curl} H^\nu + \sigma E^\nu = 0 \\ H^\nu(0) = H^0, \ E^\nu(0) = E^0 \text{ in } \Omega^\nu \\ H^\nu \times n + \beta^\nu(x_3) n \times (E^\nu \times n) = 0 \text{ in } \mathbb{R}^+ \times \partial\Omega^\nu \end{cases} \tag{13}$$

coupled to the polarization equation

$$\begin{cases} \partial_t^2 P^\nu + \operatorname{curl}^2 P^\nu + a\, \partial_t P^\nu + \phi'(|P^\nu|^2)P^\nu = E^\nu \text{ in } \mathbb{R}^+ \times \Omega^\nu \\ P^\nu(0) = P^0, \ \partial_t P^\nu(0) = P^1 \text{ in } \Omega^\nu \\ P^\nu \times n = 0 \text{ in } \mathbb{R}^+ \times \partial\Omega^\nu. \end{cases} \tag{14}$$

For the linear case we have $\phi'(|P|^2) = k$ where $k > 0$ is a constant. The polarization P satisfies the equation

$$\begin{cases} \partial_t^2 P^\nu + \operatorname{curl}^2 P^\nu + a\, \partial_t P^\nu + kP^\nu = E^\nu \text{ in } \mathbb{R}^+ \times \Omega^\nu \\ P^\nu(0) = P^0, \ \partial_t P^\nu(0) = P^1 \text{ in } \Omega^\nu \\ \operatorname{curl} P^\nu \times n + \rho^\nu(x_3) n \times ((\partial_t + a)P^\nu \times n) = 0 \text{ in } \mathbb{R}^+ \times \partial\Omega^\nu \end{cases} \tag{15}$$

where β^ν and ρ^ν are two functions depending of the variable x_3 and the thickness parameter ν.

Let $\mathcal{O}$ be an open, bounded and regular domain of $\mathbb{R}^3$ or $\mathbb{R}^2$. We denote by $\mathbb{L}^2(\mathcal{O})$ the Lebesgues space $(L^2(\mathcal{O}))^3$ equipped with the usual norm denoted by $|\cdot|$ and the scalar product $(\cdot\,;\,\cdot)$. The norm of the Sobolev space $\mathbb{H}^1(\mathcal{O})$ is denoted by $|\cdot|_{\mathbb{H}^1}$. Let $\mathcal{H}(\operatorname{curl}, \mathcal{O})$ be the usual Hilbert space used in the theory of Maxwell equations equipped with the norm $|\cdot|_{\mathcal{H}}$. We also use the Banach space $L^p(\mathbb{R}^+; \mathbb{L}^2(\mathcal{O}))$ for $p \ge 1$, $p \ne 2$ and the Hilbert space $L^2(\mathbb{R}^+; \mathbb{L}^2(\mathcal{O}))$ with norms denoted respectively by $||\cdot||_p$ and $||\cdot||$.

The existence and regularity of solutions (H^ν, E^ν, P^ν) to problem (13)-(14) as well as to (13)-(15) has been proved in [2] and [9] in the nonlinear case with the boundary conditions $E^\nu \times n = 0$ and either $\operatorname{curl} P^\nu \times n = 0$ or $P^\nu \times n = 0$. The Silver-Muller boundary condition satisfied by E^ν and H^ν is usual in the existence theory of solutions to Maxwell equations. Following the lines of the proof given in [2] we way prove the same results in the case of Silver-Müller type boundary conditions used in our problem (13)-(15). We have for the linear problem (13)-(15) the result,

Theorem 1.1 *(***The linear case.***) Let $\rho^\nu, \beta^\nu \in L^\infty(0, \nu)$ be fixed such that $\rho^\nu(x_3) \ge 0$ and $\beta^\nu(x_3) \ge 0$ a.e. We assume that the initial data are such that*

$$\left\{\begin{array}{l} H^0, E^0, P^1 \in \mathbb{L}^2(\Omega^\nu),\ \ P^0 \in \mathcal{H}(\mathrm{curl}\,, \Omega^\nu) \\ P^0 \times n \in \mathbb{L}^2(\partial\Omega^\nu). \end{array}\right. \tag{16}$$

Then, there exists a unique weak solution (H^ν, E^ν, P^ν) *to problem* (13)-(15) *such that* H^ν,E^ν $\in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega^\nu))$ *and* $P^\nu \in L^\infty(\mathbb{R}^+; \mathcal{H}(\mathrm{curl}\,, \Omega^\nu))$. *The tangential traces* $H^\nu \times n$, $E^\nu \times n$, $\partial_t P^\nu \times n$ *belong to* $L^2(\mathbb{R}^+; \mathbb{L}^2(\partial\Omega^\nu))$ *and* $P^\nu \times n \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\partial\Omega^\nu))$. *Moreover for all* $t \geq 0$, *we have the energy inequality*

$$\begin{array}{l} \mathcal{E}^\nu(t) + 2\int_0^t (a|\partial_t P^\nu(s)|^2 + \sigma|E^\nu(s)|^2 + \\ |\sqrt{\beta^\nu} E^\nu \times n|^2 + |\sqrt{\rho^\nu}\partial_t P^\nu(s) \times n|^2) ds \leq \mathcal{E}_0^\nu \end{array} \tag{17}$$

where the energy at the time t *is defined by*

$$\begin{array}{l} \mathcal{E}^\nu(t) = |\partial_t P^\nu(t)|^2 + k|P^\nu(t)|^2 + |\mathrm{curl}\, P^\nu(t)|^2 \\ +\ a|\sqrt{\rho^\nu} P^\nu(t) \times n|^2 + |H^\nu(t)|^2 + |E^\nu(t)|^2 \end{array} \tag{18}$$

and the initial energy $\mathcal{E}_0^\nu$ *is given by*

$$\mathcal{E}_0^\nu = |P^1|^2 + k|P^0|^2 + |\mathrm{curl}\, P^0|^2 +\ a|\sqrt{\rho^\nu} P^0 \times n|^2 + |H^0|^2 + |E^0|^2. \tag{19}$$

For the nonlinear problem (13)-(14) we have

Theorem 1.2 *(***The nonlinear case***) We assume that* ϕ *satisfies hypotheses* (10) *and the initial data are such that*

$$H^0, E^0, P^1 \in \mathbb{L}^2(\Omega^\nu),\ \ P^0 \in \mathcal{H}(\mathrm{curl}\,, \Omega^\nu) \tag{20}$$

Then, there exists a unique weak solution (H^ν, E^ν, P^ν) *to problem* (13)-(14) *such that* H^ν,E^ν $\in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega^\nu))$ *and* $P^\nu \in L^\infty(\mathbb{R}^+; \mathcal{H}(\mathrm{curl}\,, \Omega^\nu))$. *The tangential traces* $H^\nu \times n$, $E^\nu \times n$ *belong to* $L^2(\mathbb{R}^+; \mathbb{L}^2(\partial\Omega^\nu))$. *Moreover for all* $t \geq 0$, *we have the energy inequality*

$$\mathcal{E}^\nu(t) + 2\int_0^t (a|\partial_t P^\nu(s)|^2 + \sigma|E^\nu(s)|^2 + |\sqrt{\beta^\nu} E^\nu \times n|^2) ds \leq \mathcal{E}_0^\nu \tag{21}$$

where the energy at the time t *is defined by*

$$\mathcal{E}^\nu(t) = |\partial_t P^\nu(t)|^2 + \int_{\Omega^\nu} \phi(|P^\nu(t)|^2) dx + |\mathrm{curl}\, P^\nu(t)|^2 + |H^\nu(t)|^2 + |E^\nu(t)|^2 \tag{22}$$

and the initial energy $\mathcal{E}_0^\nu$ *is given by*

$$\mathcal{E}_0^\nu = |P^1|^2 + \int_{\Omega^\nu} \phi(|P^0|^2) dx + |\mathrm{curl}\, P^0|^2 + |H^0|^2 + |E^0|^2. \tag{23}$$

Using hypotheses (10)- (11) satisfied by ϕ, we get for all $t \geq 0$

$$|P^\nu(t)|^2 \leq C(\int_{\Omega^\nu} \phi(|P^\nu(t)|^2) dx + |\Omega^\nu|) \tag{24}$$

where $C > 0$ depends only of ϕ and, $|\Omega^\nu| = \nu|\widehat{\Omega}|$ is the Lebesgue measure of Ω^ν. Finally we have the regularity result

Lemma 1.1 *(***Regularity***) Let (H^ν, E^ν, P^ν) be a weak solution of either problem* (13)-(14) *or problem* (13)-(15)*. If in both cases (H^0, E^0, P^0, P^1) satisfies*

$$H^0, E^0, P^1, \operatorname{curl} P^0 \in \mathcal{H}(\operatorname{curl}, \Omega^\nu), \;\; P^0 \times n \in \mathbb{L}^2(\partial\Omega^\nu), \tag{25}$$

and moreover for the linear case we have $P^1 \times n \in \mathbb{L}^2(\partial\Omega^\nu)$ then $\partial_t H^\nu$, $\partial_t E^\nu$ and $\partial_t^2 P^\nu \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega^\nu))$. Further, H^ν, E^ν belong to $L^\infty(\mathbb{R}^+; \mathcal{H}(\operatorname{curl}, \Omega^\nu))$ and $\operatorname{curl} P^\nu, \partial_t P^\nu \in L^\infty(\mathbb{R}^+; \mathcal{H}(\operatorname{curl}, \Omega^\nu))$.

The following regularity result is devoted to the nonlinear problem (13)-(14).

Proposition 1.1 *(***$\mathbb{H}^1$-regularity***) We assume that the open and bounded domain $\widehat{\Omega}$ is convex, the data (H^0, E^0, P^0) are independent of the variable x_3 satisfying* (20) *with $\widehat{\operatorname{div}} H^0$, $\widehat{\operatorname{div}} E^0$, $\widehat{\operatorname{div}} P^0$, $\widehat{\operatorname{div}} P^1$ are in $L^2(\widehat{\Omega})$. Then, for all fixed $T > 0$ there exists $C_T > 0$ such that for all $\nu > 0$ the solution (H^ν, E^ν, P^ν) of problem* (13)-(14) *satisties for all $T > 0$, the uniform bound*

$$||P^\nu||_{W^{1,\infty}(0,T;\mathbb{H}^1(\Omega^\nu))} + ||\operatorname{div} H^\nu||^2 + ||\operatorname{div} E^\nu||^2 \leq \nu C_T. \tag{26}$$

Proof. Let (H^ν, E^ν, P^ν) associated with the initial data satisfying the hypotheses stated in the proposition. The initial energy $\mathcal{E}_0^\nu$ of the system satisfies

$$\mathcal{E}_0^\nu = O(\nu). \tag{27}$$

Using the estimate $|P^\nu(t)|^2 \leq C(\int_{\Omega^\nu} \phi(|P^\nu(t)|^2)dx + |\Omega^\nu|)$ and the energy inequality we deduce that $|P^\nu(t)|^2 + |\operatorname{curl} P^\nu(t)|^2 = O(\nu)$, $|H^\nu(t)|^2 + |E^\nu(t)|^2 = O(\nu)$. To prove the proposition, we follow the lines of the proof given in [2] and [9] and we use a classical result of the regularity of Maxwell fields [3]. For ν fixed, let $(H^\epsilon, E^\epsilon, P^\epsilon)$ be the solution of the problem (13)-(14) when we replace $\phi'(|P|^2)P$ by $\phi'(|\rho_\epsilon \star P|^2)(\rho_\epsilon \star P)$ where ρ_ϵ is a nonnegative regularizing sequence with unit mass in $\mathbb{R}^3$. We shall prove the estimate (26) for P^ϵ (indeed $P^{\nu,\epsilon}$) then we deduce the whished result by letting $\epsilon \to 0$. Of course $(H^\epsilon, E^\epsilon, P^\epsilon)$ satisfies the same energy estimate as (H^ν, E^ν, P^ν). Moreover the initial energy estimate satisfies also $\mathcal{E}_0^\nu = O(\nu)$ and we have $|\rho^\epsilon \star P^\epsilon(t)|^2 = O(\nu)$ uniformly with respect ϵ. Let us consider the compatibility equations associated with the regularized problem of (13)-(14). Setting $u^\epsilon = \operatorname{div} H^\epsilon$, $v^\epsilon = \operatorname{div} E^\epsilon$ and $w^\epsilon = \operatorname{div} P^\epsilon$ we get

$$\begin{cases} \partial_t u^\epsilon = 0, \\ \partial_t(v^\epsilon + w^\epsilon) + \sigma v^\epsilon = 0 \\ \partial_t^2 w^\epsilon + a\partial_t w^\epsilon - v^\epsilon = F^\epsilon \end{cases} \tag{28}$$

with $F^\epsilon = \operatorname{div}(\phi'(|\rho_\epsilon \star P^\epsilon|^2)(\rho_\epsilon \star P^\epsilon)) = \phi'(|\rho_\epsilon \star P^\epsilon|^2)\rho^\epsilon \star \operatorname{div} P^\epsilon + 2\phi^{(2)}(|\rho_\epsilon \star P^\epsilon|^2)\sum_{k,j}(\rho_\epsilon \star P_k^\epsilon)(\rho_\epsilon \star P_j^\epsilon)(\rho_\epsilon \star \partial_k P^\epsilon)$. Setting $W^\epsilon = (v^\epsilon, w^\epsilon, \partial_t w^\epsilon)$ we get the equation

$$\partial_t W^\epsilon + \mathcal{C}W^\epsilon = \mathcal{S}^\epsilon(P^\epsilon), \; W^\epsilon(0) = W_0 \tag{29}$$

where $W_0 = (\widehat{\operatorname{div}} E^0, \widehat{\operatorname{div}} P^0, \widehat{\operatorname{div}} P^1)$ and

$$\mathcal{C} = \begin{pmatrix} \sigma & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & a \end{pmatrix}, \; \mathcal{S}^\epsilon(P^\epsilon) = \begin{pmatrix} 0 \\ 0 \\ F^\epsilon \end{pmatrix} \tag{30}$$

Notice that we have $|W^0| = O(\nu)$. Using the hypotheses (10)-(11), there exists $C > 0$ which is independent of ν and ϵ such that

$$|\mathcal{S}^\epsilon(P^\epsilon(t))| \leq C|\nabla P^\epsilon(t)| \tag{31}$$

for all $t \geq 0$. Consequently W^ϵ satisfies the estimate

$$|W^\epsilon(t)|^2 \leq e^{\delta t}(|W_0|^2 + C\int_0^t |\nabla P^\epsilon(s)|^2 ds) \tag{32}$$

where the constant $\delta > 0$ and $C > 0$ depend only of ϕ (but not of ν and ϵ). Recall that we have $|W_0|^2 = \nu |W_0|^2_{\mathbb{L}^2(\widehat{\Omega})}$. Since Ω^ν is a cylinder with convex cross section, then using the estimate given in lemma 2.17 of [3] we get

$$|\nabla P^\epsilon(t)|^2 \leq |\mathrm{curl}\, P^\epsilon(t)|^2 + |\mathrm{div}\, P^\epsilon(t)|^2 \tag{33}$$

for all $t \geq 0$. Let $T > 0$ be fixed. For $t \in [0, T]$, we deduce (by using $|w^\epsilon(t)|^2 \leq |W^\epsilon(t)|^2$) the estimate

$$|W^\epsilon(t)|^2 \leq \nu C_T + Ce^{\delta t}\int_0^t (|\mathrm{curl}\, P^\epsilon(s)|^2 + |W^\epsilon(s)|^2) ds \tag{34}$$

where $C_T > 0$ is independent of ν and ϵ. Finally the estimate of $||\mathrm{curl}\, P^\epsilon||^2 \leq \nu C$ implies the inequality $|W^\epsilon(t)|^2 \leq \nu C_T + C_T \int_0^t |W^\epsilon(s)|^2 ds$ and finally we obtain

$$|\nabla P^\epsilon(t)|^2 + |\mathrm{div}\, P^\epsilon(t)|^2 + |\mathrm{div}\, \partial_t P^\epsilon(t)|^2 \leq \nu C_T. \tag{35}$$

Since $P^\epsilon \to P^\nu$ strongly in $L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega^\nu))$ as $\epsilon \to 0$ see [2], [9] then, using the convexity of the L^2-norm we deduce that (E^ν, P^ν) satisfies the estimate $|\nabla P^\nu(t)|^2 + |\mathrm{div}\, E^\nu(t)|^2 + |\mathrm{div}\, \partial_t P^\nu(t)|^2 \leq \nu C_T$ and $|P^\nu(t)|^2 = O(\nu)$ on $[0, T]$. This concludes the proof of the proposition.

The content of this work is the following. In the next section we introduce the change of variable and the new problems (linear and nonlinear) setted in the fixed cylinder Ω^1 with thickness 1 which is denoted Ω in the sequel. We prove uniform bounds with respect to the parameter ν for the scaled solution (h^ν, e^ν, p^ν) of the new problems. In section 3, we discuss the behaviour of the solution of the linear problem and section 5, we give the behaviour of the solutions of the nonlinear problem.

2 Uniform bounds for the scaled solutions

In the sequel Ω denotes the cylinder with thickness $\nu = 1$. Let (u_1, u_2, u_3) be the canonical basis of $\mathbb{R}^3$. The generic point x of Ω^ν is denoted by $x = (\widehat{x}, x_3) \in \widehat{\Omega} \times (0, \nu)$ and $\widehat{x} = (x_1, x_2)$. For a vector valued function we set $f = (\widehat{f}, f_3)$ with $\widehat{f} = (f_1, f_2)$ and $f_3 = f \cdot u_3$. The partial derivative of a function g with respect to the variable x_j is denoted $\partial_j g$. The curl operator of a vector valued function f in $\mathbb{R}^3$ or in $\mathbb{R}^2$ is defined by $\mathrm{curl}\, f = (\partial_2 f_3 - \partial_3 f_2, \partial_3 f_1 - \partial_1 f_3, \partial_1 f_2 - \partial_2 f_1)$ or $\widehat{\mathrm{curl}}\, \widehat{f} = \partial_1 f_2 - \partial_2 f_1 = \mathrm{div}\,(f \times u_3)$ and the curl of a scalar function is defined by $\mathrm{Curl}\,(f \cdot u_3) = (\partial_2 f_3, -\partial_1 f_3, 0)$. The div operator of a vector valued function in $\mathbb{R}^3$ or in $\mathbb{R}^2$ is defined by $\mathrm{div}\, f = \partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3$ or by $\widehat{\mathrm{div}}\, \widehat{f} = \partial_1 f_1 + \partial_2 f_2$. Finally the gradient operator is written as $\nabla = \widehat{\nabla} + \partial_{x_3} u_3$. Notice that $\mathrm{curl}\, \widehat{f}$ is a vector of $\mathbb{R}^3$ and $\mathrm{curl}\, \widehat{f} \cdot u_3 = \widehat{\mathrm{curl}}\, \widehat{f}$. Moreover we have $f \times \widehat{n} \in \mathbb{R}^3$ and $(f \times \widehat{n}) \cdot u_3 = \widehat{f} \times \widehat{n} = f_1 n_2 - f_2 n_1 \in \mathbb{R}$.

Since this work concerns the reduction of the dimension of problems (13)-(15) and (13)-(14) then we assume that the initial data H^0, E^0, P^0, P^1 are independent of the variable x_3. Hence, for the linear problem we assume that

$$\begin{cases} H^0 = H^0(\widehat{x}), E^0 = E^0(\widehat{x}), P^0 = P^0(\widehat{x}), P^1 = P^1(\widehat{x}) \\ P^0, \mathrm{curl}\, P^0, P^1, H^0, E^0 \in \mathcal{H}(\mathrm{curl}\,, \Omega) \\ P^0 \times \widehat{n},\ P^1 \times \widehat{n} \in \mathbb{L}^2(\partial\widehat{\Omega}) \end{cases} \tag{36}$$

Notice for example that we have $\operatorname{curl} P^0 = \operatorname{Curl} P_3^0 + (\widehat{\operatorname{curl}}\, \widehat{P}^0)u_3$. The generic point x of Ω is denoted by $x = (\widehat{x}, z)$ with $0 < z < 1$. We set $Q = \mathbb{R}^+ \times \Omega$, $\widehat{Q} = \mathbb{R}^+ \times \widehat{\Omega}$ and for $T > 0$ fixed we set $\Omega_T = \Omega \times (0, T)$ and $\widehat{\Omega}_T = \widehat{\Omega} \times (0, T)$. We introduce the change of variable $z = x_3/\nu$. For a vector valued function $F^\nu(\widehat{x}, x_3)$ defined in Ω^ν we introduce the scaled function $f^\nu(\widehat{x}, z)$ defined in Ω associated with F^ν by setting $F^\nu(\widehat{x}, x_3) = f^\nu(\widehat{x}, x_3/\nu)$. Notice that we have $\widehat{\nabla} F^\nu = \widehat{\nabla} f^\nu$ and $\partial_3 F^\nu = \frac{1}{\nu}\partial_z f^\nu$. More generally we get

$$\begin{cases} \operatorname{curl} F^\nu = -\frac{1}{\nu}\partial_z(f^\nu \times u_3) + \operatorname{Curl}(f^\nu \cdot u_3) + (\widehat{\operatorname{curl}}\, \widehat{f}^\nu)u_3 \\ \operatorname{div} F^\nu = \widehat{\operatorname{div}}\, \widehat{f}^\nu + \frac{1}{\nu}\partial_z f_3^\nu. \end{cases} \tag{37}$$

In the sequel we use the notations

$$\begin{cases} \operatorname{curl}_\nu f^\nu = (\theta^\nu, \partial_1 f_2^\nu - \partial_2 f_1^\nu) \\ \theta^\nu = (\partial_2 f_3^\nu - \frac{1}{\nu}\partial_z f_2^\nu, \frac{1}{\nu}\partial_z f_1^\nu - \partial_1 f_3^\nu) \\ \operatorname{div}_\nu f^\nu = \widehat{\operatorname{div}}\, \widehat{f}^\nu + \frac{1}{\nu}\partial_z f_3^\nu. \end{cases} \tag{38}$$

Notice that $\theta^\nu \cdot (\partial_1 f_2^\nu - \partial_2 f_1^\nu)u_3 = 0$.

Let (H^ν, E^ν, P^ν) be the global solution of either problem (13)-(15) or (13)-(14) associated with the initial data (H^0, E^0, P^0, P^1) satisfying the hypotheses (20)-(36). Let $h^\nu(t, \widehat{x}, z) = H^\nu(t, \widehat{x}, \nu z)$, $e^\nu(t, \widehat{x}, z) = E^\nu(t, \widehat{x}, \nu z)$ and $p^\nu(t, \widehat{x}, z) = P^\nu(t, \widehat{x}, \nu z)$ be the scaled solution defined in $\mathbb{R}^+ \times \Omega$ and associated with (H^ν, E^ν, P^ν).

2.1 The linear problem

The scaled solution (h^ν, e^ν, p^ν) satisfies in $\mathbb{R}^+ \times \Omega$ the problem

$$\begin{cases} \partial_t h^\nu - \operatorname{curl}_\nu e^\nu = 0 \ in\ \mathbb{R}^+ \times \Omega \\ \partial_t(e^\nu + p^\nu) + \operatorname{curl}_\nu h^\nu + \sigma e^\nu = 0 \ in\ \mathbb{R}^+ \times \Omega \\ h^\nu(0) = H^0(\widehat{x}),\ e^\nu(0) = E^0(\widehat{x}) \ in\ \Omega \\ h^\nu \times n + \beta^\nu n \times (e^\nu \times n) = 0 \ in\ \mathbb{R}^+ \times \partial\Omega \end{cases} \tag{39}$$

coupled to the polarization equation

$$\begin{cases} \partial_t^2 p^\nu + \operatorname{curl}_\nu^2 p^\nu + a\ \partial_t p^\nu + k\ p^\nu = e^\nu \ in\ \mathbb{R}^+ \times \Omega \\ p^\nu(0) = P^0(\widehat{x}),\ \partial_t p^\nu(0) = P^1(\widehat{x}) \ in\ \Omega \\ \operatorname{curl}_\nu p^\nu \times n + \rho^\nu n \times ((\partial_t + a)p^\nu \times n) = 0 \ in\ \mathbb{R}^+ \times \partial\Omega. \end{cases} \tag{40}$$

The energy inequality becomes

$$\mathcal{E}^\nu(t) + \int_0^t (a|\partial_t p^\nu(s)|^2 + \sigma|e^\nu(s)|^2 + |\sqrt{\beta^\nu} e^\nu \times n|^2 + |\sqrt{\rho^\nu}\partial_t p^\nu(s) \times n|^2) ds \le \mathcal{E}_0^\nu \tag{41}$$

where the energy at the time t is given by

$$\begin{aligned} \mathcal{E}^\nu(t) &= |\partial_t p^\nu(t)|^2 + k|p^\nu(t)|^2 + |\operatorname{curl}_\nu p^\nu(t)|^2 \\ &+ a|\sqrt{\rho^\nu} p^\nu(t) \times n|^2 + |h^\nu(t)|^2 + |e^\nu(t)|^2 \end{aligned} \tag{42}$$

and the initial energy $\mathcal{E}_0^\nu$, by using hypotheses (20)-(36), becomes

$$
\begin{aligned}
\mathcal{E}_0^\nu &= |P^1|^2 + k|P^0|^2 + |\widehat{\mathrm{curl}}\,\widehat{P}^0|^2 + |\mathrm{Curl}\,P_3^0|^2 \\
&+ a|\sqrt{\rho^\nu}P^0 \times n|^2 + |H^0|^2 + |E^0|^2.
\end{aligned}
\tag{43}
$$

Notice that the initial energy $\mathcal{E}_0^\nu$ is uniformly bounded with respect to the parameter ν if ρ^ν is bounded uniformly in $L^\infty(0,1)$.

We assume that the functions β^ν and ρ^ν are given by

$$
\begin{cases}
\beta^\nu(z) = \beta \geq 0 \text{ if } 0 < z < 1,\ \beta^\nu(0) = \nu\beta_0 > 0,\ \beta^\nu(1) = \nu\beta_1 > 0 \\
\rho^\nu(z) = \rho \geq 0 \text{ if } 0 < z < 1,\ \rho^\nu(0) = \nu\rho_0 > 0,\ \rho^\nu(1) = \nu\rho_1 > 0
\end{cases}
\tag{44}
$$

Then, the energy inequality implies the following uniform bounds

Lemma 2.1 *Under hypotheses* (20)-(36) *and* (44) *there exists* $C > 0$ *which is independent of* ν *such that the solution* (h^ν, e^ν, p^ν) *of problem* (39)-(40) *satisfies the nestimates*

$$
\begin{cases}
||h^\nu||_\infty^2 + ||e^\nu||_\infty^2 + ||p^\nu||_\infty^2 + ||e^\nu||^2 + ||\partial_t p^\nu||^2 \leq C \\
||\partial_t p^\nu||_\infty^2 + ||\partial_2 p_1^\nu - \partial_1 p_2^\nu||_\infty^2 + ||\theta^\nu||_\infty^2 \leq C
\end{cases}
\tag{45}
$$

moreover we have

$$
\begin{cases}
||(e^\nu \times u_3)_{z=0,1}||^2_{L^2(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} + ||\widehat{e}^\nu \times \widehat{n}||^2_{L^2(\mathbb{R}^+;L^2(\partial\widehat{\Omega}\times(0,1)))} \leq C \\
||(h^\nu \times u_3)_{|z=0,1}||^2_{L^2(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} \leq C\nu,\ ||\widehat{h}^\nu \times \widehat{n}||^2_{L^2(\mathbb{R}^+;L^2(\partial\widehat{\Omega}\times(0,1)))} \leq C.
\end{cases}
\tag{46}
$$

and

$$
\begin{cases}
||(p^\nu \times u_3)_{|z=0,1}||^2_{L^\infty(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} + ||\widehat{p}^\nu \times \widehat{n}||^2_{L^\infty(\mathbb{R}^+;L^2(\partial\widehat{\Omega}\times(0,1)))} \leq C \\
||(\partial_t p^\nu \times u_3)_{|z=0,1}||^2_{L^2(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} + ||\partial_t \widehat{p}^\nu \times \widehat{n}||^2_{L^2(\mathbb{R}^+;L^2(\partial\widehat{\Omega}\times(0,1)))} \leq C
\end{cases}
\tag{47}
$$

Notice that from the boundary condition satisfied by p^ν and the previous estimates we deduce that

$$
||(\mathrm{curl}_\nu p^\nu \times u_3)_{|z=0,1}||^2_{L^2_{loc}(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} \leq C\nu.
\tag{48}
$$

We assume that the initial data are as in lemma 1.1. We get

Lemma 2.2 *Under hypotheses* (25) *the solution has a time regularity property (see theorem 1) which implies the following estimates*

$$
\begin{cases}
||\partial_t h^\nu||_\infty^2 + ||\partial_t e^\nu||_\infty^2 + ||\partial_t^2 p^\nu||_\infty^2 \leq C \\
||\mathrm{curl}_\nu h^\nu||_\infty^2 + ||\mathrm{curl}_\nu e^\nu||_\infty^2 + ||\mathrm{curl}_\nu^2 p^\nu||_\infty^2 + ||\mathrm{curl}_\nu \partial_t p^\nu||_\infty^2 \leq C
\end{cases}
\tag{49}
$$

where C *is independent of* ν.

2.2 The nonlinear problem

The scaled solution (h^ν, e^ν, p^ν) satisfies in $\mathbb{R}^+ \times \Omega$ the problem (39) coupled to the polarization equation

$$
\begin{cases}
\partial_t^2 p^\nu + \mathrm{curl}_\nu^2 p^\nu + a\ \partial_t p^\nu + \phi'(|p^\nu|^2)p^\nu = e^\nu \text{ in } \mathbb{R}^+ \times \Omega \\
p^\nu(0) = P^0(\widehat{x}),\ \partial_t p^\nu(0) = \widehat{P}^1(\widehat{x}) \text{ in } \Omega \\
p^\nu \times n = 0 \text{ in } \mathbb{R}^+ \times \partial\Omega.
\end{cases}
\tag{50}
$$

The energy inequality becomes

$$\mathcal{E}^\nu(t) + 2\int_0^t (a|\partial_t p^\nu(s)|^2 + \sigma|e^\nu(s)|^2 + |\sqrt{\beta^\nu} e^\nu \times n|^2 \leq \mathcal{E}_0^\nu \tag{51}$$

where the energy at the time t is given by

$$\mathcal{E}^\nu(t) = |\partial_t p^\nu(t)|^2 + \int_\Omega \phi(|p^\nu(t)|^2)dx + |\mathrm{curl}\,_\nu p^\nu(t)|^2 + |h^\nu(t)|^2 + |e^\nu(t)|^2 \tag{52}$$

and the initial energy $\mathcal{E}_0^\nu$, by using hypotheses (20)-(36), becomes

$$\mathcal{E}_0^\nu = |P^1|^2 + \int_\Omega \phi(|P^0|^2)dx + |\widehat{\mathrm{curl}}\,\widehat{P}^0|^2 + |\mathrm{Curl}\,P_3^0|^2 + |H^0|^2 + |E^0|^2. \tag{53}$$

Notice that the initial energy $\mathcal{E}_0^\nu$ is uniformly bounded if β^ν is bounded uniformly in $L^\infty(0,1)$. From the energy inequality we deduce the following uniform bounds

Lemma 2.3 *If β^ν is given as in* (44) *then, under hypotheses* (20)-(36)*, there exists $C > 0$ which is independent of ν such that the solution (h^ν, e^ν, p^ν) of problem* (39)-(50) *satisfies the estimates*

$$\begin{cases} ||h^\nu||_\infty^2 + ||e^\nu||_\infty^2 + ||p^\nu||_\infty^2 + ||e^\nu||^2 + ||\partial_t p^\nu||^2 \leq C \\ ||\partial_t p^\nu||_\infty^2 + ||\partial_2 p_1^\nu - \partial_1 p_2^\nu||_\infty^2 + ||\theta^\nu||_\infty^2 \leq C \end{cases} \tag{54}$$

moreover we have the bounds

$$\begin{cases} ||(e^\nu \times u_3)_{|z=0,1}||^2_{L^2(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} + ||\widehat{e}^\nu \times \widehat{n}||^2_{L^2(\mathbb{R}^+;L^2(\partial\widehat{\Omega}\times(0,1)))} \leq C \\ ||(h^\nu \times u_3)_{|z=0,1}||^2_{L^2(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} \leq C\nu,\ ||\widehat{h}^\nu \times \widehat{n}||^2_{L^2(\mathbb{R}^+;L^2(\partial\widehat{\Omega}\times(0,1)))} \leq C. \end{cases} \tag{55}$$

Assuming the initial data to be more regular as in lemma 1.1 then we have

Lemma 2.4 *Under hypotheses* (25) *the solution of* (39)-(50) *satisfies estimates*

$$\begin{cases} ||\partial_t h^\nu||_\infty^2 + ||\partial_t e^\nu||_\infty^2 + ||\partial_t^2 p^\nu||_\infty^2 \leq C \\ ||\mathrm{curl}\,_\nu h^\nu||_\infty^2 + ||\mathrm{curl}\,_\nu e^\nu||_\infty^2 + ||\mathrm{curl}\,_\nu^2 p^\nu||_\infty^2 + ||\mathrm{curl}\,_\nu \partial_t p^\nu||_\infty^2 \leq C \end{cases} \tag{56}$$

where C is independent of ν.

2.3 Weak convergences

For a subsequence still denoted (h^ν, e^ν, p^ν), where (h^ν, e^ν, p^ν) is either the solution of the linear problem (39)-(40) or the nonlinear problem (39)-(50), the following convergences hold.

$$\begin{cases} (h^\nu, e^\nu, p^\nu) \rightharpoonup (h, e, p) \text{ in } L^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega)) \text{ weakly} - \star, \\ (e^\nu, \partial_t p^\nu) \rightharpoonup (e, \partial_t p) \text{ in } L^2(\mathbb{R}^+;\mathbb{L}^2(\Omega)) \text{ weakly}, \\ \partial_t p^\nu \rightharpoonup \partial_t p \text{ in } L^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega)) \text{ weakly} - \star \end{cases} \tag{57}$$

The estimate in $L^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ of the $\mathrm{curl}\,_\nu$ of the solution gives the weak-$\star$ convergences

$$\begin{cases} \widehat{\mathrm{curl}}\,\widehat{p}^\nu \rightharpoonup \widehat{\mathrm{curl}}\,\widehat{p},\ \theta^\nu \rightharpoonup \theta \\ \widehat{\mathrm{curl}}\,\widehat{h}^\nu \rightharpoonup \widehat{\mathrm{curl}}\,\widehat{h},\ \widehat{\mathrm{curl}}\,\widehat{e}^\nu \rightharpoonup \widehat{\mathrm{curl}}\,\widehat{e} \end{cases} \tag{58}$$

where $\theta \in L^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ is some vector valued function. Since we have $\widehat{\mathrm{curl}}\,\widehat{h}$, $\widehat{\mathrm{curl}}\,\widehat{e}$, $\widehat{\mathrm{curl}}\,\widehat{p} \in L^\infty(\mathbb{R}^+;L^2(\Omega))$ we deduce that the traces $\widehat{h}\times\widehat{n}$, $\widehat{e}\times\widehat{n}$ and $\widehat{p}\times\widehat{n}$ are well defined in $L^\infty(\mathbb{R}^+;H^{-1/2}(\partial\widehat{\Omega}\times(0,1))$. Hence the weak-$\star$ convergence of that traces holds at least in $L^\infty(\mathbb{R}^+;H^{-1/2}(\partial\widehat{\Omega}\times(0,1))$.

Now let us consider the bounds of the traces of e^ν, h^ν and p^ν. First of all, using estimate (49) we deduce that h^ν, e^ν, p^ν and $\partial_t p^\nu$ are uniformly bounded in $W^{1,\infty}(\mathbb{R}^+;\mathbb{L}^2(\Omega))$. Hence, h, e, p and $\partial_t p$ belong to $W^{1,\infty}(\mathbb{R}^+;\mathbb{L}^2(\Omega))$. Finally we get

Lemma 2.5 *(**Initial data**) The traces of h, e, p and $\partial_t p$ at $t = 0$ make sense in $\mathbb{L}^2(\Omega)$ and we have*

$$h(0) = H^0,\ e(0) = E^0,\ p(0) = P^0,\ \partial_t p(0) = P^1 \tag{59}$$

Next, from (46) we get the convergences

Lemma 2.6 *(**Convergence of the traces**) The traces of the solutions (h^ν, e^ν) of either (39)-(40) or (39)-(50) satisfy*

$$\begin{cases} (e^\nu \times u_3)_{|z=0,1} \rightharpoonup A_{0,1} \ \textit{in}\ L^2(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))\ \textit{weak} \\ (h^\nu \times u_3)_{|z=0,1} \rightarrow 0 \ \textit{in}\ L^2(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))\ \textit{strong.} \end{cases} \tag{60}$$

The tangential traces of the polarization p^ν for the linear problem (39)-(40) satisfy

$$\begin{cases} (p^\nu \times u_3)_{|z=0,1} \rightharpoonup B_{0,1} \ \textit{in}\ L^\infty(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))\ \textit{weak} - \star, \\ (\partial_t p^\nu \times u_3)_{|z=0,1} \rightharpoonup \partial_t B_{0,1} \ \textit{in}\ L^2(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))\ \textit{weak} \\ (\mathrm{curl}_\nu p^\nu \times u_3)_{|z=0,1} \rightarrow 0 \ \textit{in}\ L^2_{loc}(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))\ \textit{strong} \end{cases} \tag{61}$$

where A_0, A_1, B_0 and B_1 are some vector valued functions belonging to the spaces defined by the weak convergences.

Before we pass to the limit in problem (39)-(40) and (39)-(50) let us give the remark

Remark 2.1 *Assume that the function β^ν and ρ^ν are independent of ν say $\beta^\nu = \beta > 0$ and $\rho^\nu = \rho > 0$ then the energy inequality (41) and (51) imply, in particular, the following bounds*

$$\begin{cases} ||(p^\nu \times u_3)_{|z=0,1}||^2_{L^\infty(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} + ||(\partial_t p^\nu \times u_3)_{|z=0,1}||^2_{L^2(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} \le C\nu \\ ||(e^\nu \times u_3)_{|z=0,1}||^2_{L^\infty(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} \le C\nu. \end{cases} \tag{62}$$

and then $A_{0,1} = B_{0,1} = 0$.

We shall use for the solution (h^ν, e^ν, p^ν) of (39)-(40) as well as for (39)-(50) the following convergence results. We dedote by v^ν one of the fields h^ν, e^ν, p^ν or $. : rot_\nu p^\nu$.

Proposition 2.1 *Let v^ν be a uniformly bounded sequence in $L^\infty(\mathbb{R}^+;\mathcal{H}(\mathrm{curl}_\nu,\Omega)$ such that the tangential trace $v^\nu \times n$ is uniformly bounded in $L^p(\mathbb{R}^+;\mathbb{L}^2(\partial\Omega))$ with $p = \infty$ or $p = 2$. Then there exists a subsequence such that $v^\nu \rightharpoonup u$, $\widehat{\mathrm{curl}}\, v^\nu \rightharpoonup \widehat{\mathrm{curl}}\, v$ weakly-$\star$ in $L^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$, $\widehat{v}^\nu \times \widehat{n} \rightharpoonup \widehat{v} \times \widehat{n} = v_1 n_2 - v_2 n_1$ weakly in $L^p(\mathbb{R}^+;\mathbb{H}^{-1/2}(\partial\widehat{\Omega} \times (0,1))$ satisfying the properties*

$$\begin{cases} \widehat{v}\ \textit{is independent of}\ z,\ \widehat{v} \in L^\infty(\mathbb{R}^+;\mathcal{H}(\widehat{rot},\widehat{\Omega})) \\ (v^\nu \times u_3)_{|z=0,1} \rightharpoonup (v \times u_3)_{|z=0,1}\ \textit{weakly in}\ L^p(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega})) \\ \int_0^1 (v^\nu \times \widehat{n}) \cdot u_3 dz \rightharpoonup (v \times \widehat{n}) \cdot u_3\ \textit{weakly in}\ L^p(\mathbb{R}^+;\mathbb{L}^2(\partial\widehat{\Omega})) \end{cases} \tag{63}$$

where $(v \times \widehat{n}) \cdot u_3 = v_1 n_2 - v_2 n_1$ (which is independent of z).

We use the following Green formula

$$\begin{array}{l}\int_Q \operatorname{curl}_\nu v^\nu \cdot \varphi dxdt = \int_Q v^\nu \cdot \operatorname{curl}_\nu \varphi dxdt - \int_{\mathbb{R}^+\times\partial\widehat{\Omega}\times(0,1)} (v^\nu \times \widehat{n})\cdot \varphi d\alpha dzdt \\ -\frac{1}{\nu}\int_{\widehat{Q}} (v^\nu \times u_3)_{|z=1} \cdot \varphi_{|z=1} d\widehat{x}dt + \frac{1}{\nu}\int_{\widehat{Q}} (v^\nu \times u_3)_{|z=0} \cdot \varphi_{|z=0} d\widehat{x}dt\end{array} \tag{64}$$

for all test function $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \overline{\widehat{\Omega}} \times [0,1])$. Choosing $\varphi = (\nu\varphi_1, \nu\varphi_2, 0)$ with $\varphi_j \in \mathcal{D}(\mathbb{R}^+ \times \widehat{\Omega} \times (0,1))$ for $j = 1,2$. We have $\operatorname{curl}_\nu \varphi = -\partial_z(\varphi \times u_3) + \nu \operatorname{Curl} \varphi_3 + \nu(\widehat{\operatorname{curl} \widehat{\varphi}})u_3$. Passing to the limit in the Green formula we obtain $\int_Q (-v_1\partial_z\varphi_2 + v_2\partial_z\varphi_1)dxdt = 0$ and then $\partial_z \widehat{v} = 0$ in the sense of distributions. Next let $V_j \in L^p(\mathbb{R}^+; \mathbb{L}^2(\widehat{Q}))$ be the weak limit of $(v^\nu \times u_3)_{|z=j}$ for $j = 1,2$. Next, we choose in the Green formula $\varphi = (\nu z\varphi_1(t,\widehat{x}), \nu z\varphi_2(t,\widehat{x}), 0)$ and pass to the limit we get $\int_{\widehat{Q}} (-v_1\varphi_2 + v_2\varphi_1) d\widehat{x}dt - \int_{\widehat{Q}} V_1 \cdot \varphi d\widehat{x}dt = 0$ which implies that $V_1 = u \times u_3$. Finally with the test function $\varphi = (\nu(1-z)\varphi_1(t,\widehat{x}), \nu(1-z)\varphi_2(t,\widehat{x}), 0)$ we get $\int_{\widehat{Q}} (v_1\varphi_2 - v_2\varphi_1) d\widehat{x}dt - \int_{\widehat{Q}} V_0 \cdot \varphi d\widehat{x}dt = 0$ and then $V_0 = -v \times u_3$. Let $g \in L^p(\mathbb{R}^+; L^2(\mathbb{R}^+ \times \partial\widehat{\Omega} \times (0,1))$ be the weak limit of the sequence $(v^\nu \times \widehat{n}) \cdot u_3$. We choose in the Green formula $\varphi = (0,0,\varphi_3(t,\widehat{x}))$ with $\varphi_3 \in \mathcal{D}(\mathbb{R}^+ \times \overline{\widehat{\Omega}})$. We have $\operatorname{curl}_\nu \varphi = \operatorname{Curl} \varphi_3$. Then passing to the limit in the Green formula we get $\int_{\widehat{Q}} (\partial_1 v_2 - \partial_2 v_1)\varphi_3 d\widehat{x}dt = \int_{\widehat{Q}} (v_1\partial_2\varphi_3 - v_2\partial_1\varphi_3) d\widehat{x}dt - \int_{\mathbb{R}^+\times\partial\widehat{\Omega}} \int_0^1 gdz\varphi_3 d\alpha dt$. Since we have $\widehat{v} \in L^\infty(\mathbb{R}^+; \mathcal{H}(\widehat{\operatorname{curl}}, \widehat{\Omega})$ then we deduce that $\int_0^1 gdz = v_1n_2 - v_2n_1$.

Remark 2.2 *Notice $(\widehat{h}, \widehat{e}, \widehat{p})$ is independent of z. For $v^\nu = e^\nu$ we have $e \times u_3 = A_0 = A_1$, for $v^\nu = h^\nu$ we have $h \times u_3 = 0$ and then $\widehat{h} = 0$. For $v^\nu = p^\nu$ we have $p \times u_3 = B_0 = B_1$. Moreover we have $\operatorname{curl}_\nu p^\nu, \operatorname{curl}_\nu(\operatorname{curl}_\nu p^\nu) \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ and $(\operatorname{curl}_\nu p^\nu \times n)_{|z=0,1} = O(\nu)$. Let θ be the weak-$\star$ limit of θ^ν defined by $\operatorname{curl}_\nu p^\nu = \theta^\nu + \widehat{\operatorname{curl} \widehat{p}^\nu} u_3$ then θ is independent of z and $\theta = 0$.*

Proposition 2.2 *Let v^ν be a uniformly bounded sequence in $L^\infty(\mathbb{R}^+; \mathcal{H}(\operatorname{curl}_\nu, \Omega))$ satisfying $v^\nu \times n = 0$ on $\mathbb{R}^+ \times \partial\Omega$. Then θ satisfies $\int_0^1 \theta dz = \int_0^1 \operatorname{Curl} v_3 dz$ in $\mathbb{R}^+ \times \widehat{\Omega}$ and $\int_0^1 v_3 dz = 0$ on $\mathbb{R}^+ \times \partial\widehat{\Omega}$. Moreover we have $\widehat{v} = 0$*

Proof. Since we have $\operatorname{curl}_\nu v^\nu = \theta^\nu + \widehat{\operatorname{curl} \widehat{v}^\nu} u_3$, the Green formula gives

$$\begin{array}{l}\int_Q \theta^\nu \cdot \widehat{\varphi} dxdt + \int_Q (\partial_1 v_2^\nu - \partial_2 v_1^\nu)\varphi_3 dxdt = \int_Q v_1^\nu(\partial_2\varphi_3 - \frac{1}{\nu}\partial_z\varphi_2)dxdt \\ +\int_Q v_2^\nu(\frac{1}{\nu}\partial_z\varphi_1 - \partial_1\varphi_3)dxdt + \int_Q v_3^\nu(\partial_1\varphi_2 - \partial_2\varphi_1)dxdt\end{array} \tag{65}$$

for all test function $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \overline{\widehat{\Omega}} \times (0,1])$. Passing to the limit we get with $\varphi = (\widehat{\varphi}(t,\widehat{x}), 0)$ we get

$$\int_Q \theta \cdot \widehat{\varphi} dxdt = \int_Q v_3(\partial_1\varphi_2 - \partial_2\varphi_1)dxdt \tag{66}$$

integrating by parts we get $\int_Q \theta \cdot \widehat{\varphi} dxdt = \int_Q \operatorname{Curl} v_3 \cdot \widehat{\varphi} dxdt - \int_{\mathbb{R}^+\times\partial\widehat{\Omega}\times(0,1)} v_3(n_1\varphi_2 - n_2\varphi_1) d\alpha dzdt$. It follows that $\int_0^1 \operatorname{Curl} v_3 dz = \int_0^1 \theta dz$ which belongs to $L^\infty(\mathbb{R}^+; \mathbb{L}^2(\widehat{\Omega}))$. The trace of v_3 on $\partial\widehat{\Omega}$ makes sense in $L^\infty(\mathbb{R}^+; \mathbb{H}^{-1/2}(\partial\widehat{\Omega}))$ and we have $\int_0^1 v_3 dz = 0$ on $\mathbb{R}^+ \times \partial\widehat{\Omega}$. Next writting $rot_\nu v^\nu = -\frac{1}{\nu}\partial_z(v^\nu \times u_3) + \operatorname{Curl} v_3^\nu + \widehat{\operatorname{curl} \widehat{v}^\nu} u_3$ and using the Green formulation with $\varphi = \nu\psi$ we get $O(\nu) = \int_Q (-v_1^\nu\partial_z\psi_2 + v_2^\nu\partial_z\psi_1)dxdt + O(\nu)$. Passing to the limit we get $\int_Q (-v_1\partial_z\psi_2 + v_2\partial_z\psi_1)dxdt = 0$. It follows that $\widehat{v}$ is independent of the variable z and using the previous proposition and the boundary condition $(v^\nu \times n)_{|z=0,1} = 0$ we deduce that $\widehat{v} = 0$.

3 Convergence of the linear problem

We shall prove the following result.

Theorem 3.1 *Let (H^0, E^0, P^0, P^1) be initial data which are independent of the variable z and satisfying the hypotheses (36). We assume that the functions β^ν, ρ^ν are as in lemma 2.1. Let (h^ν, e^ν, p^ν) be the solution of problem (39)-(40) associated with the initial data (H^0, E^0, P^0, P^1). Then there exists a subsequence still denoted (h^ν, e^ν, p^ν) converging weakly-$\star$ in $L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ to (h, e, p) which is independent of the variable z and $\widehat{h} = 0$. The weak star limit (h, e, p) is the solution in $\mathbb{R}^+ \times \widehat{\Omega}$ of the problem*

$$\begin{cases} \partial_t h_3 - \widehat{\operatorname{curl}}\,\widehat{e} = 0, \ \partial_t(\widehat{e} + \widehat{p}) + \operatorname{Curl} h_3 + (\sigma + \beta_1 + \beta_0)\widehat{e} = 0 \\ (\partial_t^2 + a\partial_t + k)\widehat{p} + \operatorname{Curl} \widehat{\operatorname{curl}}\,\widehat{p} + (\rho_1 + \rho_0)(\partial_t + a)p = \widehat{e} \\ \widehat{e}(0) = \widehat{E}^0, \ \widehat{p}(0) = P^0, \ \partial_t \widehat{p}(0) = \widehat{p}^1, \ h_3(0) = H_3^0 \ \textit{in } \widehat{\Omega}, \\ h_3 = \beta(e_1 n_2 - e_2 n_1), \ \widehat{\operatorname{curl}}\,\widehat{p} = \rho(\partial_t + a)(p_1 n_2 - p_2 n_1) \ \textit{on } \mathbb{R}^+ \times \partial\widehat{\Omega} \end{cases} \tag{67}$$

and the system of o.d.e

$$\begin{cases} \partial_t(e_3 + p_3) + \sigma e_3 = 0, \ (\partial_t^2 + a\partial_t + k)p_3 = e_3 \\ e_3(0) = E_3^0, \ p_3(0) = P_3^0, \ \partial_t p_3(0) = P_3^1 \ \textit{in } \widehat{\Omega}. \end{cases} \tag{68}$$

Moreover, we have $h_3 \in L^\infty(\mathbb{R}^+; H^1(\Omega))$, $\widehat{e} \in L^\infty(\mathbb{R}^+; \mathcal{H}(\widehat{\operatorname{curl}}, \widehat{\Omega}))$, $p \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\widehat{\Omega}))$ and $\widehat{\operatorname{curl}}\,\widehat{p} \in L^\infty; H^+(\widehat{\Omega}))$.

Let (h^ν, e^ν, p^ν) be the solution associated with (H^0, E^0, P^0, P^1) satisfying the hypotheses of the theorem and let (h, e, p) be the weak-$\star$ limit of a subsequence. Applying proposition 2.1 to h^ν, e^ν and p^ν we deduce that (h, e, p) satisfies the conclusions of the proposition. The weak formulations associated with problem (39)-(40) take the forms

$$\begin{cases} -\int_Q h^\nu \cdot \partial_t \varphi dx dt - \int_Q e^\nu \cdot \operatorname{curl}_\nu \varphi dx dt + \int_{\mathbb{R}^+ \times \partial\widehat{\Omega} \times (0,1)} e^\nu \times \widehat{n} \cdot \varphi d\alpha dz dt \\ + \frac{1}{\nu} \int_{\mathbb{R}^+ \times \widehat{\Omega}} (e^\nu \times u_3)_{|z=1} \cdot \varphi_{|z=1} d\widehat{x} dt - \frac{1}{\nu} \int_{\mathbb{R}^+ \times \widehat{\Omega}} (e^\nu \times u_3)_{|z=0} \cdot \varphi_{|z=0} d\widehat{x} dt \\ = -\int_\Omega H^0 \varphi(0) dx \end{cases} \tag{69}$$

and,

$$\begin{cases} -\int_Q (e^\nu + p^\nu) \cdot \partial_t \phi dx dt + \int_Q h^\nu \cdot \operatorname{curl}_\nu \phi dx dt \\ + \beta \int_{\mathbb{R}^+ \times \partial\widehat{\Omega} \times (0,1)} e^\nu \times \widehat{n} \cdot \phi \times \widehat{n} d\alpha dz dt \\ + \beta_1 \int_{\mathbb{R}^+ \times \widehat{\Omega}} (e^\nu \times u_3)_{|z=1} \cdot (\phi \times u_3)_{|z=1} d\widehat{x} dt \\ - \beta_0 \int_{\mathbb{R}^+ \times \widehat{\Omega}} (e^\nu \times u_3)_{|z=0} \cdot (\phi \times u_3)_{|z=0} d\widehat{x} dt + \ \sigma \int_Q e^\nu \cdot \phi dx dt = \\ -\int_\Omega (E^0 + P^0)\phi(0) dx. \end{cases} \tag{70}$$

for all regular test functions φ, ϕ defined on $\overline{Q}$. Here we used the boundary condition $h^\nu \times n + \beta^\nu n \times (e^\nu \times n) = 0$. The polarization field p^ν satisfies

$$\begin{cases} \int_Q p^\nu \cdot (\partial_t^2 - a\partial_t + k)\psi dx dt + \int_Q \operatorname{curl}_\nu p^\nu \cdot \operatorname{curl}_\nu \psi dx dt - \int_Q e^\nu \cdot \psi dx dt \\ + \rho \int_{\mathbb{R}^+ \times \partial\widehat{\Omega} \times (0,1)} (\partial_t + a) p^\nu \times \widehat{n} \cdot \psi \times \widehat{n} d\alpha dz dt \\ + \rho_1 \int_{\mathbb{R}^+ \times \widehat{\Omega}} ((\partial_t + a) p^\nu \times u_3)_{|z=1} \cdot (\psi \times u_3)_{|z=1} d\widehat{x} dt \\ - \rho_0 \int_{\mathbb{R}^+ \times \widehat{\Omega}} ((\partial_t + a) p^\nu \times u_3)_{|z=0} \cdot (\psi \times u_3)_{|z=0} d\widehat{x} dt \\ = -\int_\Omega (P^1 \cdot \psi(0) - P^0 \cdot (\partial_t \psi(0) - a\psi(0)) dx. \end{cases} \tag{71}$$

for all test function ψ defined in $\overline{Q}$. Here we used the boundary condition $\operatorname{curl}_\nu p^\nu \times b + \rho^\nu n \times ((\partial_t + a)p^\nu \times n) = 0$. We have the convergences

$$\left\{ \begin{array}{l} \int_{\mathbb{R}^+ \times \partial\Omega} h^\nu \times n^\nu \cdot \phi d\alpha dt \to \beta \int_{\mathbb{R}^+ \times \partial\widehat{\Omega} \times (0,1)} e \times \widehat{n} \cdot \phi \times \widehat{n} d\alpha dz dt \\ +\beta_1 \int_{\mathbb{R}^+ \times \widehat{\Omega}} A_1 \cdot \phi_{|z=1} \times u_3 d\widehat{x} dt - \beta_0 \int_{\mathbb{R}^+ \times \widehat{\Omega}} A_0 \cdot \phi_{|z=0} \times u_3 d\widehat{x} dt \end{array} \right. \tag{72}$$

appearing in the weak formulation (70) and

$$\left\{ \begin{array}{l} \int_{\mathbb{R}^+ \times \partial\Omega} \operatorname{curl}_\nu p^\nu \times n^\nu \cdot \psi d\alpha dt \to \rho \int_{\mathbb{R}^+ \times \partial\widehat{\Omega} \times (0,1)} (\partial_t + a) p^\nu \times \widehat{n} \cdot \psi \times \widehat{n} d\alpha dz dt \\ + \int_{\mathbb{R}^+ \times \widehat{\Omega}} (\rho_1 (\partial_t + a) B_1 \cdot \psi_{z=1} \times u_3 - \rho_0 (\partial_t + a) B_0 \cdot \psi_{z=0} \times u_3) d\widehat{x} dt. \end{array} \right. \tag{73}$$

appearing in the third weak formulation (71).

The compatibilty conditions for problem (39)- (40) (obtained by using $\operatorname{div}_\nu(\operatorname{curl}_\nu) = 0$) can be written as

$$\left\{ \begin{array}{l} \partial_t(\widehat{\operatorname{div}}\, \widehat{h}^\nu + \frac{1}{\nu}\partial_z h_3^\nu) = 0 \\ \partial_t(\widehat{\operatorname{div}}\, (\widehat{e}^\nu + \widehat{p}^\nu) + \frac{1}{\nu}\partial_z(e_3^\nu + p_3^\nu)) + \sigma(\widehat{\operatorname{div}}\, \widehat{e}^\nu + \frac{1}{\nu}\partial_z h_3^\nu) = 0 \\ (\partial_t^2 + a\partial_t + k)(\widehat{\operatorname{div}}\, \widehat{p}^\nu + \frac{1}{\nu}\partial_z p_3^\nu) - (\widehat{\operatorname{div}}\, \widehat{e}^\nu + \frac{1}{\nu}\partial_z e_3^\nu) = 0. \end{array} \right. \tag{74}$$

passing to the limit in the sense of distributions we get the result

Lemma 3.1 *The functions h_3, e_3 and p_3 are independent of the variable z*

Let us consider in (69) and (70) test functions of the form $\varphi = (\widehat{\varphi}, \varphi_3)$ and $\phi = (\widehat{\phi}, \phi_3)$ respectively which are independent of the variable z. One observes that $\operatorname{curl}_\nu \varphi = (\partial_2\varphi_3, -\partial_1\varphi_3, \partial_1\varphi_2 - \partial_2\varphi_1)$. Hence, we obtain

$$\left\{ \begin{array}{l} -\int_Q h^\nu \cdot \partial_t \varphi dx dt - \int_Q e^\nu \cdot (\operatorname{Curl} \varphi_3 + \widehat{\operatorname{curl}}\, \widehat{\varphi} u_3) dx dt \\ + \int_{\mathbb{R}^+ \times \partial\widehat{\Omega} \times (0,1)} e^\nu \times \widehat{n} \cdot \varphi d\alpha dz dt + \\ \frac{1}{\nu} \int_{\mathbb{R}^+ \widehat{\Omega}} (e^\nu \times u_3)_{|z=1} \cdot \varphi_{|z=1} d\widehat{x} dt - \frac{1}{\nu} \int_{\mathbb{R}^+ \widehat{\Omega}} (e^\nu \times u_3)_{|z=0} \cdot \varphi_{|z=0} d\widehat{x} dt = \\ -\int_{\widehat{\Omega}} H^0 \cdot \varphi(0) d\widehat{x} \end{array} \right. \tag{75}$$

and

$$\left\{ \begin{array}{l} -\int_Q (e^\nu + p^\nu) \cdot \partial_t \phi dx dt + \int_Q h^\nu \cdot (\operatorname{Curl} \phi_3 + \widehat{\operatorname{curl}}\, \widehat{\phi} u_3) dx dt \\ +\beta \int_{\mathbb{R}^+ \times \partial\widehat{\Omega} \times (0,1)} e^\nu \times \widehat{n} \cdot \phi \times \widehat{n} d\alpha dz dt \\ -\int_{\mathbb{R}^+ \times \widehat{\Omega}} (\beta_1 (e^\nu \times u_3)_{|z=1} \cdot \phi_{|z=1} \times u_3 + \beta_0 (e^\nu \times u_3)_{|z=0} \cdot \phi_{|z=0} \times u_3) d\widehat{x} dt \\ +\sigma \int_Q e^\nu \cdot \phi dx dt = -\int_\Omega (E^0 + P^0) \cdot \phi(0) d\widehat{x}. \end{array} \right. \tag{76}$$

Recalling that $\widehat{h} = 0$ then we choose $\varphi = (0, 0, \varphi_3)$ in (75) we get $-\int_Q h_3^\nu \partial_t \varphi_3 dx dt - \int_Q \widehat{e^\nu} \cdot \operatorname{Curl} \varphi_3 dx dt + \int_{\mathbb{R}^+ \times \partial\widehat{\Omega} \times (0,1)} (n_2 e_1^\nu - n_1 e_2^\nu) \varphi_3 d\alpha dz dt = -\int_{\widehat{\Omega}} H_3^0 \varphi_3(0) d\widehat{x}$ and passing to the limit we get $-\int_{\widehat{Q}} h_3 \partial_t \varphi_3 d\widehat{x} dt - \int_{\widehat{Q}} \widehat{e} \cdot \operatorname{Curl} \varphi_3 d\widehat{x} dt + \int_{\mathbb{R}^+ \times \partial\widehat{\Omega}} (n_2 e_1 - n_1 e_2) \varphi_3 d\alpha dt = -\int_{\widehat{\Omega}} H_3^0 \varphi_3(0) d\widehat{x}$. Hence, the magnetic field $h = (\widehat{0}, h_3)$ satisfies in the sense of distributions the problem

$$\left\{ \begin{array}{l} \partial_t h_3 - \widehat{\operatorname{curl}}\, \widehat{e} = 0 \ in\ \mathbb{R}^+ \times \widehat{\Omega} \\ h_3(0) = H_3^0 \ in\ \widehat{\Omega}, \end{array} \right. \tag{77}$$

Next, we pass to the limit in (76) with test function ϕ which are independent of z and use the result $A_0 = A_1 = e \times u_3$. We get

$$
\begin{cases}
-\int_{\widehat{Q}}(e+p)\cdot\partial_t\phi d\widehat{x}dt+\int_{\widehat{Q}}h_3\cdot\widehat{\operatorname{curl}}\,\widehat{\phi}d\widehat{x}dt\\
+\beta\int_{\mathbb{R}^+\times\partial\widehat{\Omega}\times}e\times\widehat{n}\cdot\phi\times\widehat{n}d\alpha dt\\
+(\beta_1+\beta_0)\int_{\mathbb{R}^+\times\widehat{\Omega}}e\times u_3\cdot\phi\times u_3d\widehat{x}dt+\sigma\int_{\widehat{Q}}e\cdot\phi d\widehat{x}dt\\
=-\int_{\widehat{\Omega}}(E^0+P^0)\phi(0)d\widehat{x}.
\end{cases}
\tag{78}
$$

Integrating by parts we get (notice that $h_3\in L^\infty(\mathbb{R}^+;H^1(\widehat{\Omega}))$)

$$
\begin{cases}
\partial_t(\widehat{e}+\widehat{p})+\operatorname{Curl}h_3+(\sigma+\beta_1-\beta_0)\widehat{e}=0\\
\partial_t(e_3+p_3)+\sigma e_3=0\\
h_3+\beta(e_1n_2-e_2n_1n)=0\\
e(0)+p(0)=E^0+P^0\text{ in }\widehat{\Omega}
\end{cases}
\tag{79}
$$

Let us consider the weak formulation (71) associated with the polarization equation (40). Let us recall the notation used $\operatorname{curl}_\nu p^\nu=\theta^\nu+\widehat{\operatorname{curl}}\,\widehat{p}^\nu u_3$ where $\theta^\nu=-\frac{1}{\nu}\partial_z(p^\nu\times u_3)+\operatorname{Curl}p_3^\nu$. Using proposition 2.1 and lemma 2.1 we deduce that $p\times u_3=B_0=B_1$ and we have that p_3 is independent of z). Passing to the limit we get

$$
\begin{cases}
\int_{\widehat{Q}}p\cdot(\partial_t^2-a\partial_t+k)\psi dxdt+\int_Q\theta\cdot\operatorname{Curl}\psi_3d\widehat{x}dzdt+\\
\int_{\widehat{Q}}(\widehat{\operatorname{curl}}\,\widehat{p})(\widehat{\operatorname{curl}}\,\widehat{\psi})dxdt-\int_Q e\cdot\psi d\widehat{x}dt\\
+\rho\int_{\mathbb{R}^+\times\partial\widehat{\Omega}}(\partial_t+a)p\times\widehat{n}\cdot\psi\times\widehat{n}d\alpha dt\\
-\rho_1\int_{\mathbb{R}^+\times\widehat{\Omega}}(\partial_t+a)B_1\cdot\psi\times u_3d\widehat{x}dt\\
+\rho_0\int_{\mathbb{R}^+\times\widehat{\Omega}}(\partial_t+a)B_0\cdot\psi\times u_3d\widehat{x}dt\\
=-\int_{\widehat{\Omega}}(P^1\cdot\psi(0)-P^0\cdot(\partial_t\psi(0)-a\psi(0))dx.
\end{cases}
\tag{80}
$$

In proposition 2.1 and remark 2.1 we have shown that $\theta=0$. Integrating by parts in (80), we get in $\mathbb{R}^+\times\widehat{\Omega}$

$$
\begin{cases}
(\partial_t^2+a\partial_t+k)\widehat{p}+\operatorname{Curl}\widehat{\operatorname{curl}}\,\widehat{p}+(\rho_1+\rho_0)(\partial_t+a)\widehat{p}=\widehat{e}\\
\widehat{\operatorname{curl}}\,\widehat{p}=\rho((\partial_t+a)(p_1n_2-p_2n_1))\ on\ \mathbb{R}^+\times\partial\widehat{\Omega}\\
\widehat{p}(0)=\widehat{P}^0,\ \partial_t\widehat{p}(0)=\widehat{P}^1\text{ in }\widehat{\Omega}
\end{cases}
\tag{81}
$$

and

$$
\begin{cases}
(\partial_t^2+a\partial_t+k)p_3=e_3\text{ in }\mathbb{R}^+\times\widehat{\Omega}\\
p_3(0)=P_3^0,\ \partial_tp_3(0)=P_3^1\ in\ \widehat{\Omega}.
\end{cases}
\tag{82}
$$

Hence the main theorem is proved.

4 Convergence of the nonlinear problem

In the sequel we assume that $\widehat{\Omega}$ is a bounded, regular and convex domain of $\mathbb{R}^2$ and the initial data H^0, E^0, P^0 and P^1 which are independent of the variable x_3 satisfy the hypothesis

$$
\begin{cases}
H^0=(0,0,H_3^0),\ E^0=(\widehat{E}^0,E_3^0),\ P^0=(0,0,P_3^0),\ P^1=(0,0,P_3^1)\\
H_3^0,\ P_3^0,\ P_3^1\in H^1(\widehat{\Omega}),\ \widehat{\operatorname{div}}\,E^0\in L^2(\widehat{\Omega})
\end{cases}
\tag{83}
$$

We shall prove the following result

Theorem 4.1 *Under hypotheses* (83)-(44)*, there exists a subsequence* (h^ν, e^ν, p^ν) *of solutions to the nonlinear problem* (39)-(50) *such that* $h^\nu \rightharpoonup (0, h_3)$*,* $e^\nu \rightharpoonup e$*,* $p^\nu \rightharpoonup (0, p_3)$ *weakly-*$\star$ *in* $L^\infty(\mathbb{R}^+; \mathbb{L}^2(\widehat{\Omega}))$ *and* (h_3, e, p_3) *is independent of the variable* z*. Moreover* $p^\nu \to p$ *strongly in* $L^2(0,T; \mathbb{L}^2(\widehat{\Omega}))$*,* $\operatorname{div} h^\nu \rightharpoonup \partial_z h_3$ *and* $\operatorname{div} e^\nu \rightharpoonup \operatorname{div} e$ *weakly-*$\star$ *in* $L^\infty(\mathbb{R}^+; L^2(\widehat{\Omega}))$*. Further,* (h_3, e, p_3) *satisfies in* $\mathbb{R}^+ \times \widehat{\Omega}$ *the limit problem*

$$\begin{cases} \partial_t h_3 - \widehat{\operatorname{curl} \widehat{e}} = 0, \\ \partial_t \widehat{e} + \operatorname{Curl} h_3 + (\sigma - \beta_1 + \beta_0)\widehat{e} = 0, \\ \partial_t(e_3 + p_3) + \sigma e_3 = 0 \\ \partial_t^2 p_3 + a\partial_t p_3 - \widehat{\Delta} p_3 + \phi'(|p_3|^2)p_3 = e_3 \\ h_3(0) = H_3^0,\ \widehat{e}(0) = \widehat{E}^0,\ e_3(0) = E_3^0,\ p_3(0) = P_3^0,\ \partial_t p_3(0) = P_3^1 \\ p_3 = 0,\ h_3 = \beta(n_1 e_2 - n_2 e_1) \text{ on } \mathbb{R}^+ \times \partial\Omega. \end{cases} \tag{84}$$

The proof of the theorem is essentially proved in section 3, at the end of the proof of theorem 3.1 and by the strong convergence result given in proposition 1.1. Let us precise tha convergences we obtain in this case.
Let (h^ν, e^ν, p^ν) be the associated scalled solution of (39)-(50). Proposition 1.1 implies the following uniform bound in $L^\infty(0,T,L^2(\Omega))$,

$$||\nabla_\nu p^\nu||_\infty^2 + ||\nabla_\nu \partial_t p^\nu||_\infty^2 + ||\operatorname{div}_\nu h^\nu||_\infty^2 + ||\operatorname{div}_\nu e^\nu||_\infty^2 \leq C_T. \tag{85}$$

combining this bound, the results of propositions 2.1 and 2.2 and the energy inequality associated with problem (39)-(50) we get

Lemma 4.1 *For a subsequence (still denoted* (h^ν, e^ν, p^ν)*) we have* $(h^\nu, e^\nu, p^\nu) \to (h, e, p)$ *weakly-*$\star$ *in* $L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ *where* (h, e, p) *is independent of the variable* z*. Moreover*

$$\begin{cases} p^\nu \to p,\ \phi'(|p^\nu|^2)p^\nu \to \phi'(|p|^2)p \text{ in } L^\infty(0,T; \mathbb{L}^2(\Omega)) \text{ strong} \\ \widehat{\nabla}\widehat{p}^\nu \rightharpoonup \widehat{\nabla}\widehat{p},\ \widehat{\operatorname{div} \widehat{h}^\nu} \rightharpoonup 0,\ \widehat{\operatorname{div} \widehat{e}^\nu} \rightharpoonup \widehat{\operatorname{div} \widehat{e}} \text{ in } L^\infty(0,T; \mathbb{L}^2(\Omega)) \text{ weakly}-\star \end{cases} \tag{86}$$

for all $T > 0$*. Furthermore, we have*

$$\widehat{h} = 0,\ \widehat{p} = 0,\ \theta = \operatorname{Curl} p_3 \text{ and } p_3 = 0 \text{ on } \mathbb{R}^+ \times \partial\widehat{\Omega}. \tag{87}$$

,

Let us consider problem (39)-(50) with the boundary condition $p^\nu \times n = 0$ on $\mathbb{R}^+ \times \partial\Omega$. The convergence for the Maxwell equation follows the lines of the proof given in theorem 2.1. The weak formulation of the polarization equation becomes

$$\begin{cases} \int_Q p^\nu \cdot (\partial_t^2 - a\partial_t)\psi dxdt + \int_Q \operatorname{curl}_\nu p^\nu \cdot \operatorname{curl}_\nu \psi dxdt - \int_Q e^\nu \cdot \psi dxdt = \\ = -\int_Q \phi'(|p^\nu(t)|^2)p^\nu \cdot \psi dxdt - \int_\Omega (P^1 \cdot \psi(0) - P^0 \cdot (\partial_t \psi(0) - a\psi(0))dx. \end{cases} \tag{88}$$

for all test function ψ satisfying the boundary condition $\psi \times n = 0$ on $\mathbb{R}^+ \times \partial\Omega$. Since $\widehat{p} = 0$ we use test functions of the form $\psi = (0, 0, \psi_3)$ which are independent of the variable z. Then passing to the limit by using lemma 4.1 we get the result stated in theorem 4.1.

Remark 4.1 *If we consider problem* (39)-(50) *with the Silver-Müller boundary condition* $\operatorname{curl}_\nu p^\nu \times n + \rho^\nu n \times (p^\nu \times n) = 0$ *on* $\mathbb{R}^+ \times \partial\Omega$ *then all the results obtained for problem* (39)-(40) *remain true. The main point we have to prove is that* $\phi'(|p^\nu|^2)p^\nu \rightharpoonup \phi'(|p|^2)p$ *weakly star in* $L^\infty(\mathbb{R}^+; \mathbb{L}^2(\widehat{\Omega}))$*. Of course the weak-*$\star$ *convergence follows from the bounds given by the energy inequality but, to* *identify the weak-$\star$ limit the energy inequality is not sufficient to do that. In [1], it is proved an uniform bound (with respect to ν) in $L^\infty(0,T;\mathbb{H}^{1/2}(\Omega))$ for p^ν which allows to pass to the limit in the nonlinear term and to identify the zero thickness limit of the problem.*

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