Inf-convolution of risk measures and optimal risk transfer

Pauline Barrieu

L.S.E., Statistics Department, Houghton Street, London, WC2A 2AE, United Kingdom. (e-mail: p.m.barrieu@lse.ac.uk)

Nicole El Karoui

C.M.A.P., Ecole Polytechnique, 91128 Palaiseau Cédex, France. (e-mail: elkaroui@cmapx.polytechnique.fr)

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Nicole El Karoui

C.M.A.P., Ecole Polytechnique, 91128 Palaiseau Cédex, France. (e-mail: elkaroui@cmapx.polytechnique.fr)

Abstract. We develop a methodology to optimally design a financial issue to hedge non-tradable risk on financial markets.The modeling involves a minimization of the risk borne by issuer given the constraint imposed by a buyer who enters the transaction if and only if her risk level remains below a given threshold. Both agents have also the opportunity to invest all their residual wealth on financial markets but they do not have the same access to financial investments. The problem may be reduced to a unique inf-convolution problem involving some transformation of the initial risk measures.

Key words: inf-convolution, risk measure, optimal design, indifference pricing, hedging strategy

JEL Classification: C61, D81, G13, G22

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1 Introduction

Financial markets have witnessed for several years the arrival of a new breed of instruments, depending on non-financial risks and usually considered as falling within the competence of the insurance sector. One may think, for instance, of weather or catastrophic contracts, the flows of which are contingent to the occurrence of certain weather or catastrophic events.
However this global phenomenon of convergence and interplay between insurance and finance raises several questions about the classification of these new products but also about their pricing and management. The characterization of their price is very interesting as it questions the logic of these contracts itself. Indeed, standard techniques for derivatives pricing, using, for instance, replication, are not valid any more because of the specific nature of the underlying risk. Moreover, the determination of the contract structure is a problem in itself: on the one hand, the underlying market related to these risks is extremely illiquid, but on the other hand, the logic of these products itself is closer to that of an insurance policy. Consequently the question of the product design, unusual in finance, is raised.
This paper focuses on these problems in a framework where economic agents may take positions on two types of

risk: a purely financial risk (or market risk) and a (non-financial) non-tradable risk. The optimal structure of a contract depending on the non-tradable risk and its price are determined. Several authors (see, for instance, El Karoui and Rouge (2000), Becherer (2001), Davis (2001) or Musiela and Zariphopoulou (2004)) have been interested in these new products. However, neither their impact on "classical" investments nor their optimal design are mentioned in the literature. As it is usually the case in finance, these papers focus on the pricing rule of these contracts. In that sense, this work presents a very different approach.
The different agents involved in this transaction (i.e. the buyer and the seller of the contract) may invest on the financial market (even when they do not have the same access to it) via a portfolio they optimally choose. They are assumed to determine these financial gain processes simultaneously with the characterization of the non-financial structure. The impact of the non-tradable risk on "classical" financial decisions is obviously not negligible. Since the structure represents a new diversification instrument for any investor, optimal wealth allocation becomes a more complex question and the question of an efficient quantitative risk assessment becomes crucial. Different authors have recently been interested in defining and constructing a coherent, in some sense, risk measure (see, for instance, Artzner *et al.* (1999) or Föllmer and Schied (2002a) and (2002b)), using a systematic axiomatic approach. The framework developer by these authors will be that of this study.
This paper is structured as follows: after having presented some results in an exponential utility framework, where both agents have access to a financial market to reduce their risk, we focus in a third section on a more general framework involving convex risk measures and in particular on the inf-convolution of different convex risk measures. In a fourth section, we focus on the impact of both the financial market and the non-tradable risk on risk measures and give a characterization of the optimal structure, explicitly for a particular family of risk measures and as a necessary and sufficient condition in the general framework. In the fifth section, we focus on the hedging issue and consider the optimality in the inf-convolution problem through two examples. In the last section, we present some concluding remarks.

2 The exponential utility framework

2.1 A simplified approach: the "toy model"

2.1.1 Framework

Two economic agents, respectively denoted A and B, are evolving in an uncertain universe modeled by a probability space $(\Omega, \Im, \mathbb{P})$. At a fixed future date T, agent A is exposed towards a non-tradable risk Θ for an amount $X \triangleq X(\Theta, \omega)$ in the scenario ω. A wants to issue a financial product $F \triangleq F(\Theta, \omega)$ and sell it to agent B for a forward price at time T denoted by π as to reduce her exposure. We assume that X and F belong to the linear space of bounded functions including constant functions, denoted by $\mathcal{X}$.
Both agents are supposed to be risk-averse. We assume that they refer to the same kind of choice criterion, i.e. a (concave, increasing) exponential utility function

$$\forall x \in \mathbb{R} \quad U(x) = -\gamma \exp\left(-\frac{1}{\gamma}x\right)$$

with respective risk tolerance coefficients γ_A and γ_B.
Note that, in this study, we consider risk tolerance parameter instead of the "more usual" risk aversion. It is simply defined as the inverse of the risk aversion. This will be justify later when dealing with general convex risk measure.
Agent A's objective is to choose the optimal structure (F, π) as to maximize the expected utility of her final

wealth

$$\mathbb{E}_{\mathbb{P}}\left[U_A\left(X-(F-\pi)\right)\right] \rightarrow \sup_{F\in\mathcal{X},\pi}$$

Her constraint is then to find a buyer for her issue. Hence, agent B should have an interest in doing this transaction. At least, the F-structure should not worsen her expected utility. Consequently, agent B simply compares two expected utility levels, the first one corresponds to the case where she simply invests her initial wealth in a bank account and the second one to the situation where she enters the F-transaction

$$\mathbb{E}_{\mathbb{P}}\left[U_B\left((F-\pi)+x\right)\right] \geq \mathbb{E}_{\mathbb{P}}\left[U_B\left(x\right)\right]$$

where x is the non-risky forward wealth of agent B before the F-transaction.
Using the definition of the exponential utility function, this program is equivalent to

$$\begin{array}{l} \inf_{F\in\mathcal{X},\pi} \mathbb{E}_{\mathbb{P}}\left[\gamma_A \exp\left(-\frac{1}{\gamma_A}\left(X-(F-\pi)\right)\right)\right] \\ \text{subject to} \quad \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{\gamma_B}\left(F-\pi\right)\right)\right] \leq 1 \end{array} \tag{1}$$

Given the convexity of the program, the constraint is bounded for the optimal structure and the optimal pricing rule $\pi^*(F)$ of the financial product F is entirely determined by the buyer as

$$\pi^*(F) = -\gamma_B \ln \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{1}{\gamma_B}F\right)\right) \triangleq -e_{\gamma_B}(F) \tag{2}$$

She determines the minimal pricing rule, ensuring the existence of the transaction. $\pi^*(F)$ corresponds to the maximal amount agent B is ready to pay to enter the F-transaction and bear the associated risk. In other words, $\pi^*(F)$ corresponds to the *certainty equivalent* of F for the utility function of agent B, or to the *indifference pricing rule.*
Note that, due to the exponential criterion, the forward wealth x does not play any role in the pricing rule and consequently in the optimal structure. Agent B is then indifferent, from her utility point of view, between doing the F-transaction and not doing it.

Remark: *i*) Exponential utility functions have been widely used in the financial literature. Several facts may justify their relative importance compared to other utility functions but, in particular, the absence of constraint on the sign of the future considered cash flows and its relationship with probability measures make them very convenient to use.
ii) The notion of indifference price has been widely studied in the literature, especially when replicating a terminal cash flow using a utility criterion (cf., for instance, the articles of Hodges and Neuberger (1989) or of El Karoui and Rouge (2000)).

2.1.2 Optimal structure

In this simple framework, referred to in the following as the "toy model", the optimal structure is given by the so-called Borch's Theorem, which is presented below in the particular exponential framework. In his paper, Borch (1962) obtained, in a utility framework, optimal exchange of risk, leading in many cases to familiar linear quota-sharing of total pooled losses.

Proposition 2.1 (Borch) *The optimal structure of the Program* (1) *is given as a proportion of the initial exposure X, depending only on the risk tolerance coefficients of both agents:*

$$F^* = \frac{\gamma_B}{\gamma_A+\gamma_B}X \quad \mathbb{P}\ a.s. \qquad \textit{(to within a constant)} \tag{3}$$

Proof:
The convex constrained Program (1) may be solved by introducing a Lagrangian multiplier $\lambda > 0$. The function to be minimized is then

$$\mathbb{E}_{\mathbb{P}}\left[\gamma_A \exp\left(-\frac{1}{\gamma_A}\left(X-(F-\pi)\right)\right)-\lambda\gamma_B\left(1-\exp\left(-\frac{1}{\gamma_B}\left(F-\pi\right)\right)\right)\right]$$

For any scenario ω, the convex function

$$g\left(\triangleq F-\pi\right)\longmapsto\gamma_A \exp\left(-\frac{1}{\gamma_A}\left(X\left(\omega\right)-g\right)\right)-\lambda\gamma_B\left(1-\exp\left(-\frac{1}{\gamma_B}g\right)\right)$$

is minimum at the point g^* satisfying the first order condition

$$\exp\left(-\frac{1}{\gamma_A}\left(X\left(\omega\right)-g^*\right)\right)=\lambda\exp\left(-\frac{1}{\gamma_B}g^*\right)$$

or equivalently

$$g^*\left(\omega\right)=F^*\left(\omega\right)-\pi^*\left(F^*\left(\omega\right)\right)=\frac{\gamma_B}{\gamma_A+\gamma_B}\left(X\left(\omega\right)-c\left(\lambda\right)\right)$$

where $c\left(\lambda\right)$ is given by Equation (2) for $F^*=\frac{\gamma_B}{\gamma_A+\gamma_B}X \quad \mathbb{P}$ a.s. (to within a constant):

$$\begin{aligned} c\left(\lambda\right) &= -\left(\gamma_A+\gamma_B\right)\ln\mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{1}{\gamma_A+\gamma_B}X\right)\right)\triangleq\gamma_C\ e_{\gamma_C}\left(X\right) \\ \text{with } \gamma_C &\triangleq \gamma_A+\gamma_B \end{aligned} \tag{4}$$

□

2.1.3 Formulation in terms of certainty equivalent

Looking at the results of the previous Subsection, the *convex entropic functional*

$$\forall\Psi\in\mathcal{X}\qquad e_\gamma\left(\Psi\right)\triangleq\gamma\ln\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{\gamma}\Psi\right)\right] \tag{5}$$

plays an important role, especially as it characterizes the pricing rule of the structure. It corresponds to the *opposite of the certainty equivalent of* Ψ. The name of this functional will be justified later in Subsection 3.1.1.

One of the key property of the functional e_γ is the translation invariance as

$$\forall m\in\mathbb{R}\quad e_\gamma\left(\Psi+m\right)=e_\gamma\left(\Psi\right)-m \tag{6}$$

Using this functional, the Program (1) may be rewritten as

$$\inf_{F\in\mathcal{X},\pi} e_{\gamma_A}\left(X-\left(F-\pi\right)\right)\qquad\text{subject to}\qquad e_{\gamma_B}\left(\left(F-\pi\right)\right)\leq 0 \tag{7}$$

Using the translation invariance property, we find directly the optimal pricing rule as

$$\pi^*\left(F\right)=-e_{\gamma_B}\left(F\right)$$

Moreover, it is now possible to solve the program without introducing a Lagrangian multiplier since

$$E_{AB}(X) \triangleq \inf_{F \in \mathcal{X}} e_{\gamma_A}(X - (F - \pi(F))) = \inf_{F \in \mathcal{X}} (e_{\gamma_A}(X - F) + e_{\gamma_B}(F)) \tag{8}$$

Given the optimal structure F^* previously obtained in Proposition 2.1, the value functional of this program, $E_{AB}(X)$, may also be expressed in terms of e_γ and the following equality can be easily obtained:

$$E_{AB}(X) = e_{\gamma_C}(X) \qquad \text{with } \gamma_C = \gamma_A + \gamma_B \tag{9}$$

It is simply the opposite of the certainty equivalent of X considering a representative agent with an exponential utility function and a risk tolerance coefficient equal to γ_C.

Remark: *i*) Note also that the composite parameter γ_C is simply equal to the sum of both risk tolerance coefficients γ_A and γ_B. This simply means that the representative agent has a risk tolerance equal to the sum of the risk tolerance of both agents. Note this may justify the use of risk tolerance instead of risk aversion.
ii) The introduction of the functional e_γ enables us to characterize and interpret very easily the value function of the considered program. A direct approach using Subsection 2.1.2 does not lead to such a straightforward result.

2.2 Investment and diversification in a financial market

We now assume that in order to reduce their respective risk exposure, both agents may also invest in a financial market. This market plays a hedging role for the agents. Note that we use the generic terminology "financial markets" but it may cover a more general investment framework, including, for instance, some insurance investments. The introduction of a financial market leads to a much more complicated problem even if, as we will see, the obtained results remain very simple and surprisingly robust.

2.2.1 Hedging portfolios and investment strategies

In the static point of view adopted in Sections 2, 3 and 4, we do not really need to specify the characteristics of the financial investments. We simply consider a set $\mathcal{V}_T$ of bounded terminal gains ξ_T, at time T, resulting from a self-financing investment strategy with a null initial value. More precisely, the net potential gain corresponds to the spread between the terminal wealth resulting from the adopted strategy and the capitalized initial wealth. The key point is that all agents in the market agree on the initial value of these strategies, in other words, the market value at time 0 of any of these strategy is null. In particular, an admissible strategy is associated with a derivative contract with bounded terminal payoff Φ only if its forward market price at time T, $q^m(\Phi)$, is a transaction price for all agents in the market. Then, $\Phi - q^m(\Phi)$ is the bounded terminal gain at time T and is an element of $\mathcal{V}_T$. Typical example of admissible terminal gains ξ_T are then the terminal wealth associated with transactions based on options.
Generally, and especially when adopting a dynamic point of view, it is natural to consider terminal gains associated with dynamic investment strategies. A detailed framework will be introduced, when needed in Section 5.
Moreover, in order to have coherent transaction prices, we assume in the following that the market is arbitrage-free. In our framework, this can be expressed by:

$$\exists \mathbb{Q} \sim \mathbb{P} \qquad \forall \xi_T \in \mathcal{V}_T \quad \mathbb{E}_{\mathbb{Q}}(\xi_T) \geq 0 \tag{10}$$

In particular, considering the financial assets, with a terminal payoff Φ that can be sold and bought, such a condition is written as

$$q^m(\Phi) = \mathbb{E}_{\mathbb{Q}}(\Phi)$$

The probability measure $\mathbb{Q}$ may be viewed as a static version of the classical $\mathcal{V}_T$-martingale measures in a dynamic framework.

2.2.2 Financial properties of $\mathcal{V}_T$ and hedging strategies of both agents

The set $\mathcal{V}_T$, previously defined, has to satisfy some properties to be coherent with some investment principles. The first principle, being the "minimal assumption", is the consistency with the diversification principle. In other words, any convex combination of admissible gains should also be an admissible gain. Hence, the set $\mathcal{V}_T$ is always taken as a *convex set*.
Some additional requirements may be introduced, in particular, if agents are not sensitive to the size of the transactions. In this case, $\mathcal{V}_T$ is assumed to be a *cone*. This assumption is relevant for liquid markets leading to the possibility to make the same order for any quantity. Finally, if agents are not sensitive to the direction of the transactions (buy/sell), then $\mathcal{V}_T$ is a *sub-vector space*. This assumption is consistent with the most liquid part of the market.
Even if there exists a unique large underlying financial market, both agents may not have however the same access to it. In other words, they may differ in the space of financial strategies. Indeed, both agents may be of very different natures *a priori*. The set of hedging products to which they have access may be completely different, because of specific regulations, of usual strategies... We may think for instance of as diverse agents as an insurance company, a reinsurance company, a bank or a private investor having different goals but taking part in the risk transfer process.
The set of admissible strategies for Agent A (resp. Agent B) is also characterized by the associated terminal gains and is denoted by $\mathcal{V}_T^{(A)}$ (resp. $\mathcal{V}_T^{(B)}$). We assume at least that both $\mathcal{V}_T^{(A)}$ and $\mathcal{V}_T^{(B)}$ are convex sets. Some additional assumptions may also be imposed following the previous arguments.
Note that both sets $\mathcal{V}_T^{(A)}$ and $\mathcal{V}_T^{(B)}$ have different interpretations: both agents do not consider indeed the financial investments from the same point of view. For Agent A, the problem is a hedging problem of her remaining risk. In this sense, $\mathcal{V}_T^{(A)}$ corresponds to terminal gains associated with *hedging strategies*. On the other hand, the problem is different for Agent B. She simply wants to make some financial investments and $\mathcal{V}_T^{(B)}$ is then associated with *investment strategies*. In the following, however, we will not make any difference and refer to both types of strategies as hedging strategies.

2.3 Optimization problem

2.3.1 Optimization program

The impact of the financial market concerns above all agent B. Indeed, since she initially invests on financial markets, the F-transaction will have an interest for her if it can increase her expected utility level, taking into account her optimal financial investments. The investor has now a threshold on her hedging strategies. Since the issuer may also invest optimally on the financial market, her problem is simply to maximize the expected utility of her global terminal wealth. In other words, the optimization program is

$$\begin{array}{l} \sup_{\xi_A \in \mathcal{V}_T^{(A)}} \mathbb{E}_{\mathbb{P}} \left[U_A \left(X - (F - \pi) - \xi_A \right) \right] \\ \text{s.t.} \quad \sup_{\xi_B \in \mathcal{V}_T^{(B)}} \mathbb{E}_{\mathbb{P}} \left[U_B \left((F - \pi) + x - \xi_B \right) \right] \geq \sup_{\xi_B \in \mathcal{V}_T^{(B)}} \mathbb{E}_{\mathbb{P}} \left[U_B \left(x - \xi_B \right) \right] \end{array}$$

or equivalently, using the convex entropic functional e_γ previously defined by Equation (5)

$$\begin{array}{l} \inf_{F \in \mathcal{X}, \pi} \inf_{\xi_A \in \mathcal{V}_T^{(A)}} e_{\gamma_A} \left(X - (F - \pi) - \xi_A \right) \\ \text{s.t.} \quad \inf_{\xi_B \in \mathcal{V}_T^{(B)}} e_{\gamma_B} \left((F - \pi) - \xi_B \right) \leq \inf_{\xi_B \in \mathcal{V}_T^{(B)}} e_{\gamma_B} \left(-\xi_B \right) \end{array} \tag{11}$$

Assumption: In the following, we make the following assumptions

$$\inf_{\xi_B \in \mathcal{V}_T^{(B)}} e_{\gamma_B}(-\xi_B) > -\infty \qquad \text{and} \qquad \inf_{\xi_A \in \mathcal{V}_T^{(A)}} e_{\gamma_A}(-\xi_A) > -\infty \tag{$\mathcal{H}$}$$

This condition guarantees that for any $\Psi \in \mathcal{X}$, $\inf_{\xi_i \in \mathcal{V}_T^{(i)}} e_{\gamma_i}(\Psi - \xi_i)$ is finite, for both $i = A, B$. Indeed, as the functional e_γ is decreasing and Ψ is bounded by $-\|\Psi\|_\infty$ and $\|\Psi\|_\infty$, we can write using the cash translation invariance property (Equation (6))

$$-\|\Psi\|_\infty + e_{\gamma_i}(-\xi_i) = e_{\gamma_i}(\|\Psi\|_\infty - \xi_i) \leq e_{\gamma_i}(\Psi - \xi_i) \leq e_{\gamma_i}(-\|\Psi\|_\infty - \xi_i) = e_{\gamma_i}(-\xi_i) + \|\Psi\|_\infty$$

and taking the infimum leads to the result.

2.3.2 Optimal pricing rule

The optimal pricing rule is obtained, as previously, by binding the constraint imposed by the buyer at the optimum and using the cash translation invariance property of the functional e_γ

$$\pi^*(F) = \inf_{\xi_B \in \mathcal{V}_T^{(B)}} e_{\gamma_B}(-\xi_B) - \inf_{\xi_B \in \mathcal{V}_T^{(B)}} e_{\gamma_B}(F - \xi_B) \tag{12}$$

Note that the optimal price is finite for any F in $\mathcal{X}$, using the previous comment.
This optimal pricing rule corresponds to an indifference price since it makes the investor, agent B, indifferent, from her utility point of view, between doing or not doing the F-transaction.
The formulation is less direct than that of the "toy model" (Equation (2)) as it involves optimal investments on financial markets. Note also that, as previously, the exponential utility makes the initial wealth x irrelevant for the pricing rule.

2.3.3 Relationship with the "toy model"

Using the optimal pricing rule and the translation invariance property of the functional e_γ, the optimization Program (11) may be rewritten simply as

$$E_{AB}^m(X) = \inf_{F \in \mathcal{X}} \left(\inf_{\xi_A \in \mathcal{V}_T^{(A)}} e_{\gamma_A}(X - F - \xi_A) + \inf_{\xi_B \in \mathcal{V}_T^{(B)}} e_{\gamma_B}(F - \xi_B) \right) \tag{13}$$

This optimization Program looks very similar to the previous optimization problem (8), referred to as "toy model", when no hedging strategy is available. The only difference comes from the accrued complexity of both functionals $\inf_{\xi_A \in \mathcal{V}_T^{(A)}} e_{\gamma_A}(X - F - \xi_A)$ and $\inf_{\xi_B \in \mathcal{V}_T^{(B)}} e_{\gamma_B}(F - \xi_B)$, replacing $e_{\gamma_A}(X - F)$ and $e_{\gamma_B}(F)$, in the problem we now consider. A first natural choice to solve this problem is therefore to study the functional $\Psi \longmapsto \inf_{\xi \in \mathcal{V}_T} e_\gamma(\Psi - \xi)$. This method is not so easy and not so efficient as the one we choose to present here... But it was our first approach! The nature of these modified functionals $\Psi \longmapsto \inf_{\xi \in \mathcal{V}_T} e_\gamma(\Psi - \xi)$ will be studied in details in the next sections, in reference to the pricing via utility maximization in incomplete markets.
The following Proposition and its proof present some additional simplification that can be made:

Proposition 2.2 *The value functional of the Program* (13), $E_{AB}^m(X)$, *is equal to the value functional of*

$$\inf_{\xi_A \in \mathcal{V}_T^{(A)}, \xi_B \in \mathcal{V}_T^{(B)}} e_{\gamma_C}(X - \xi_A - \xi_B) \qquad \textit{with } \gamma_C = \gamma_A + \gamma_B \tag{14}$$

Proof:
The Program (13) is a succession of three minimizations:

$$E^m_{AB}(X) = \inf_{F\in\mathcal{X},\xi_A\in\mathcal{V}_T^{(A)},\xi_B\in\mathcal{V}_T^{(B)}} \left(e_{\gamma_A}(X-F-\xi_A)+e_{\gamma_B}(F-\xi_B)\right)$$

Hence, working only with infima, we can choose the order of minimization and obtain

$$E^m_{AB}(X) = \inf_{\xi_A\in\mathcal{V}_T^{(A)},\xi_B\in\mathcal{V}_T^{(B)}} \inf_{F\in\mathcal{X}} \left(e_{\gamma_A}(X-F-\xi_A)+e_{\gamma_B}(F-\xi_B)\right)$$

Using a translation of ξ_B and letting $\widetilde{F} \triangleq F-\xi_B \in \mathcal{X}$ enables to rewrite it as

$$E^m_{AB}(X) = \inf_{\xi_A\in\mathcal{V}_T^{(A)},\xi_B\in\mathcal{V}_T^{(B)}} \inf_{\widetilde{F}\in\mathcal{X}} \left(e_{\gamma_A}\left(X-\widetilde{F}-\xi_A-\xi_B\right)+e_{\gamma_B}\left(\widetilde{F}\right)\right) \tag{15}$$

This new Program is closely related to the "toy model". The intermediate optimization program

$$\inf_{\widetilde{F}\in\mathcal{X}} \left(e_{\gamma_A}\left(X-\widetilde{F}-\xi_A-\xi_B\right)+e_{\gamma_B}\left(\widetilde{F}\right)\right) = E_{AB}(X-\xi_A-\xi_B)$$

corresponds indeed to the toy model (Equation (8)) with the initial risk exposure $X-\xi_A-\xi_B$ instead of X and the structure $\widetilde{F}$ to be determined. Hence, using the previous result on the value functional of the toy model problem (see Equation (9)),

$$\inf_{\widetilde{F}\in\mathcal{X}} \left(e_{\gamma_A}\left(X-\widetilde{F}-\xi_A-\xi_B\right)+e_{\gamma_B}\left(\widetilde{F}\right)\right) = e_{\gamma_C}(X-\xi_A-\xi_B)$$

Hence, as a consequence of Equation (15), the value functional of the optimization Program (13) is also given by

$$\inf_{\xi_A\in\mathcal{V}_T^{(A)},\xi_B\in\mathcal{V}_T^{(B)}} e_{\gamma_C}(X-\xi_A-\xi_B)$$

$\square$

2.4 Optimal structure

Considering the right order of minimization, as presented in Proposition 2.2 is crucial since it reduces considerably the difficulties of solving. However, in order to solve completely the different intermediate optimization problems, we have to use the reverse approach, starting from the global hedging problem and then deriving the optimal structure and the individual hedging problems.
More precisely, the first problem to be solved is the "global hedging problem", which is more or less classical. The optimization problem

$$\begin{aligned} E^m_{AB}(X) &= \inf_{\xi_A\in\mathcal{V}_T^{(A)},\xi_B\in\mathcal{V}_T^{(B)}} e_{\gamma_C}(X-\xi_A-\xi_B) \stackrel{def}{=} \inf_{\xi\in\mathcal{V}_T^{(AB)}} e_{\gamma_C}(X-\xi) \qquad (\mathcal{P}_{AB}) \\ \text{with}\quad \xi &= \xi_A+\xi_B \in \mathcal{V}_T^{(AB)} \stackrel{def}{=} \mathcal{V}_T^{(A)}+\mathcal{V}_T^{(B)} \end{aligned}$$

is indeed not so standard. Its originality comes from the relative complexity of the set of admissible financial strategies we consider.
To characterize the optimal structure, we first suppose that the Program $(\mathcal{P}_{AB})$ has an optimal solution $\xi^* \in \mathcal{V}_T^{(A)}+\mathcal{V}_T^{(B)}$, in other words that there exists a decomposition (not necessarily unique) of ξ^* over $\mathcal{V}_T^{(A)}$ and $\mathcal{V}_T^{(B)}$:

$$\xi^* = \eta_A^* + \eta_B^*$$

Theorem 2.3 *Suppose $\xi^* = \eta_A^* + \eta_B^*$ is an optimal solution of the Program $(\mathcal{P}_{AB})$ with $\eta_A^* \in \mathcal{V}_T^{(A)}$ and $\eta_B^* \in \mathcal{V}_T^{(B)}$. Then*

$$F^* = \frac{\gamma_B}{\gamma_A+\gamma_B}X - \frac{\gamma_B}{\gamma_A+\gamma_B}\eta_A^* + \frac{\gamma_A}{\gamma_A+\gamma_B}\eta_B^*$$

is an optimal structure. It characterizes a Pareto-optimal exchange of risk.
Moreover,
i) η_B^* is an optimal investment portfolio for Agent B

$$\frac{1}{\gamma_B}e_{\gamma_B}\left(F^*-\eta_B^*\right) = \frac{1}{\gamma_B}\inf_{\xi_B\in\mathcal{V}_T^{(B)}} e_{\gamma_B}\left(F^*-\xi_B\right) = \frac{1}{\gamma_C}e_{\gamma_C}\left(X-\xi^*\right)$$

ii) η_A^* is an optimal hedging portfolio of $(X-F^*)$ for Agent A

$$\frac{1}{\gamma_A}e_{\gamma_A}\left(X-(F^*+\eta_A^*)\right) = \frac{1}{\gamma_A}\inf_{\xi_A\in\mathcal{V}_T^{(A)}} e_{\gamma_A}\left(X-(F^*+\xi_A)\right) = \frac{1}{\gamma_C}e_{\gamma_C}\left(X-\xi^*\right)$$

We give here a detailed proof of this result which will be used later in Section 4.2 in a more general context (Theorem 4.2).
Proof:
To prove this theorem, we proceed in several steps:
Step 1:
Let us first observe that

$$E_{AB}^m(X) = e_{\gamma_C}\left(X-\xi^*\right) = \inf_{\widetilde{F}\in\mathcal{X}}\left(e_{\gamma_A}\left(X-\widetilde{F}-\xi^*\right)+e_{\gamma_B}\left(\widetilde{F}\right)\right)$$

given Proposition 2.2. Using the "toy model" optimality result (Proposition 2.1), we obtain directly an expression for the optimal "structure" $\widetilde{F}^*$ as:

$$\widetilde{F}^* = \frac{\gamma_B}{\gamma_A+\gamma_B}\left(X-\xi^*\right) = \frac{\gamma_B}{\gamma_C}\left(X-\xi^*\right) \qquad \text{and} \qquad e_{\gamma_B}\left(\widetilde{F}^*\right) = \frac{\gamma_B}{\gamma_C}e_{\gamma_C}\left(X-\xi^*\right)$$

Step 2:
Rewriting in the reverse order the arguments used in the proof of Proposition 2.2, we naturally set $F^* = \widetilde{F}^* + \eta_B^*$. We then want to prove that η_B^* is an optimal investment for agent B.
For the sake of simplicity in our notations, we consider

$$G^X\left(\xi_A,\xi_B,F\right) \triangleq e_{\gamma_A}\left(X-F-\xi_A\right)+e_{\gamma_B}\left(F-\xi_B\right)$$

Given the optimality of $\xi^* = \eta_A^* + \eta_B^*$ and $\widetilde{F}^* = F^* - \eta_B^*$, we have:

$$\begin{aligned} E_{AB}^m(X) &= G^X\left(\eta_A^*,\eta_B^*,F^*\right) \\ &= \inf_{F\in\mathcal{X},\xi_A\in\mathcal{V}_T^{(A)},\xi_B\in\mathcal{V}_T^{(B)}} G^X\left(\xi_A,\xi_B,F\right) \leq \inf_{\xi_B\in\mathcal{V}_T^{(B)}} G^X\left(\eta_A^*,\xi_B,F^*\right) \leq G^X\left(\eta_A^*,\eta_B^*,F^*\right) \end{aligned}$$

Then η_B^* is optimal for the problem $e_{\gamma_B}\left(F-\xi_B\right)\rightarrow \inf_{\xi_B\in\mathcal{V}_T^{(B)}}$.
The optimality of η_A^* can be proved using the same arguments.

Step 3: **Pareto optimality**

Assume that a structure F_A^* improves the situation of agent A

$$e_{\gamma_A}\left(X - F_A^* - \eta_A^*\right) < e_{\gamma_A}\left(X - F^* - \eta_A^*\right)$$

Given the optimality of $(\eta_A^*, \eta_B^*, F^*)$, we have $G^X(\eta_A^*, \eta_B^*, F_A^*) \geq G^X(\eta_A^*, \eta_B^*, F^*)$and then $e_{\gamma_B}\left(F_A^* - \eta_B^*\right) \geq e_{\gamma_B}\left(F^* - \eta_B^*\right)$.

Consequently, if agent A improves her situation, agent B worsens hers, and reciprocally. This is exactly the definition of Pareto-optimality. $\square$

Question of uniqueness:

1. Assume that ξ^* has two distinct decompositions, $\xi^* = \eta_A^* + \eta_B^* = \overline{\eta}_A^* + \overline{\eta}_B^*$, over $\mathcal{V}_T^{(A)} + \mathcal{V}_T^{(B)}$, then it admits an infinity of decompositions, since any convex combination of these decompositions is also an admissible decomposition due to the convexity of both sets $\mathcal{V}_T^{(A)}$ and $\mathcal{V}_T^{(B)}$. Hence there exist an infinity of optimal structures: $\frac{\gamma_B}{\gamma_A+\gamma_B}\left(X - \xi^*\right) + \left(\beta\eta_B^* + (1-\beta)\,\overline{\eta}_B^*\right)$ $(\beta \in [0,1])$.

2. Assume that both $\mathcal{V}_T^{(A)}$ and $\mathcal{V}_T^{(B)}$ are two cones. We write ξ^* as $\xi^* = \eta_A + \eta_B + \kappa_{AB}$ where κ_{AB} is an element of $\mathcal{V}_T^{(A)} \cap \mathcal{V}_T^{(B)}$. Then, another possible decomposition is $\xi^* = \overline{\eta}_A^\alpha + \overline{\eta}_B^\alpha$ considering $\overline{\eta}_A^\alpha = (1-\alpha)\,\eta_A + \alpha\kappa_{AB}$ and $\overline{\eta}_B^\alpha = \alpha\eta_B + (1-\alpha)\,\kappa_{AB}$ for any $\alpha \in [0,1]$.
   In this case, $F^* = \frac{\gamma_B}{\gamma_A+\gamma_B}\left(X - \xi^*\right) + \overline{\eta}_B^\alpha$ is an optimal structure. Choosing $1 - \alpha = \frac{\gamma_B}{\gamma_A+\gamma_B}$ leads to $F^* = \frac{\gamma_B}{\gamma_A+\gamma_B}\left(X - \eta_A - \eta_B\right)$. There is no influence of the common financial market through κ_{AB}.

3. Assume now that both $\mathcal{V}_T^{(A)}$ and $\mathcal{V}_T^{(B)}$ are two vector spaces. Considering two decompositions $\eta_A^* + \eta_B^*$ and $\overline{\eta}_A^* + \overline{\eta}_B^*$ of ξ^*, we obtain $\overline{\eta}_B^* - \eta_B^* = -\left(\overline{\eta}_A^* - \eta_A^*\right) \in \mathcal{V}_T^{(A)} \cap \mathcal{V}_T^{(B)}$ and it is then possible to generate an infinity of optimal structures by simply adding elements of $\mathcal{V}_T^{(A)} \cap \mathcal{V}_T^{(B)}$.

4. Note finally that even if there are an infinity of optimal structures, the terminal wealth of agent B is uniquely determined for any optimal solution ξ^* of the global hedging problem and equal to $\frac{\gamma_B}{\gamma_A+\gamma_B}\left(X - \xi^*\right)$.

The previous Theorem has two corollaries, corresponding to two different particular situations:

Corollary 2.4 (Non-speculative Logic) *Suppose $\mathcal{V}_T^{(A)} = \mathcal{V}_T^{(B)}$ and there is an optimal solution of the Program $(\mathcal{P}_{AB})$,*

$$F^* = \frac{\gamma_B}{\gamma_A + \gamma_B} X$$

is an optimal structure.

Proof:
When $\mathcal{V}_T^{(A)} = \mathcal{V}_T^{(B)} = \mathcal{V}_T$, then $\xi^* = \frac{\gamma_A}{\gamma_A+\gamma_B}\xi^* + \frac{\gamma_B}{\gamma_A+\gamma_B}\xi^*$ is an optimal decomposition where $\frac{\gamma_A}{\gamma_A+\gamma_B}\xi^*$ and $\frac{\gamma_B}{\gamma_A+\gamma_B}\xi^*$ are elements of $\mathcal{V}_T$ since $\mathcal{V}_T$ is a convex set and $0 \in \mathcal{V}_T$. $\square$

When both agents have the same access to the financial market, the underlying logic of the transaction is then *non-speculative* as the issuer has an interest to sell a structure if and only if she is initially exposed (or, more precisely, if her initial exposure differs from that of the buyer). The underlying logic is that of insurance and hedging. This Theorem gives an extension of the classical Borch's Theorem to the situation where an investment alternative is available for the agents.

Corollary 2.5 *Suppose there is an optimal solution of the Program* $(\mathcal{P}_{AB})$ *which may be decomposed over* $\mathcal{V}_T^{(A)}$ *and* $\mathcal{V}_T^{(B)}$*. If Agent A is not initially exposed* ($X \equiv 0$)*, there is still a transaction if both agents have different access to the financial market.*

When both agents do not have the same access to the financial market, a transaction may take place even if the issuer is not initially exposed. This may be indeed an opportunity for both agents to buy some derivative products in the market of the other agent to which they do not have a direct access for trading. The underlying logic may be in this sense *no longer non-speculative*. Both agents can indeed exchange some financial portfolios (of their own market) in a way proportional to their relative risk tolerance. Their own financial market portfolio plays the same role as a non-tradable asset for the other agent.
The question of optimal hedging portfolios will be tackled naturally in the more general framework of convex risk measures where arguments are identical. As a consequence, we leave it to Subsection 5.1.
The results obtained depend neither on the modeling of the financial investment gain processes nor on the distribution of the non-financial risk. In this sense, it is extremely robust. They do however seem to be highly dependent on the entropic choice criterion. Therefore, a natural question is how these results evolve when the choice or risk criterion is generalized. This study is the topic for the next sections.

3 Risk measures: basic properties and new developments

As noticed in the previous Section, the right framework to work with is that of the functional e_γ. This enables to define an entropic risk measure with some key properties as the convexity, the monotonicity but also the cash translation invariance. We will focus in the following on the possible extensions of these results to a more general framework of risk measures holding these properties. We will obtain an extraordinary robustness of the results obtained in the exponential utility framework.
In this section, we define and present the general framework we adopt in the next sections. First, we introduce a general class of risk measures introduced by Föllmer and Schied (2002a) and (2002b) to assess the risk of both agent's exposure. Then, we generate new risk measures as solution of an inf-convolution problem and finally derive the main results which enable us to re-formulate in Section 4 the optimal structure problem into a very simple convex problem.

3.1 Convex risk measures

3.1.1 Definition and properties

We first recall the definition and some key properties of the convex risk measures introduced by Föllmer and Schied (2002a) and (2002b). As previously, $\mathcal{X}$ denotes a linear space of bounded functions including constant functions.

Definition 3.1 *The functional*

$$\begin{aligned} \rho: \quad & \mathcal{X} \rightarrow \mathbb{R} \\ & \Psi \rightarrow \rho(\Psi) \end{aligned}$$

is a convex risk measure in the sense of Föllmer and Schied *if, for any* X *and* Y *in* $\mathcal{X}$*, it satisfies the following properties:*
a) Convexity: $\forall \lambda \in [0,1] \quad \rho(\lambda X + (1-\lambda) Y) \leq \lambda \rho(X) + (1-\lambda)\rho(Y)$*;*
b) Monotonicity: $X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$*;*
c) Translation invariance: $\forall m \in \mathbb{R} \quad \rho(X+m) = \rho(X) - m$.

Intuitively, $\rho(\Psi)$ may be interpreted as the amount the agent has to hold to completely cancel the risk associated with her risky position Ψ

$$\rho(\Psi + \rho(\Psi)) = 0 \tag{16}$$

We may normalize the measure by imposing $\rho(0) = 0$.

The axiomatic approach to risk measures has been first introduced by Artzner et al. (1999). They consider coherent risk measures, satisfying the previous three properties of convexity, monotonicity and translation invariance, together with an positive homogeneity property

$$\forall \Psi \in \mathcal{X}, \forall \lambda \geq 0, \qquad \nu^{\mathcal{H}}(\lambda \Psi) = \lambda \nu^{\mathcal{H}}(\Psi)$$

This simply translates the fact that the size of the transaction or exposure does not have any particular impact. (For more details, please refer to Föllmer and Schied (2002b), Remark 4.13*).*

Example 3.2 *A classical example of convex risk measure is the functional e_γ, defined in the previous section as*

$$\forall \Psi \in \mathcal{X} \qquad e_\gamma(\Psi) = \gamma \ln \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{1}{\gamma}\Psi\right)\right)$$

It is called entropic risk measure*.*

Remark: A risk measure ρ satisfying the three axioms a), b) and c) of the above definition is finite for any $\Psi \in \mathcal{X}$ as soon as $\rho(0)$ is finite.
Indeed, any element of $\mathcal{X}$ is a bounded random variable. So for any $\Psi \in \mathcal{X}$, there exist two real numbers m and M such that

$$m \leq \Psi \leq M$$

Hence, using the monotonicity property of ρ, we have

$$\infty > \rho(m) \geq \rho(\Psi) \geq \rho(M) > -\infty$$

provided that $\rho(0)$ is finite.
This property will be useful in the following, especially when generating new risk measures.

The duality between $\mathcal{X}$ and the set $\mathcal{M}_{1,f}$ of all additive measures on the considered space $(\Omega, \mathcal{F})$ leads to the following dual representation of convex risk measures as presented by Föllmer and Schied (2002b) (Theorem 4.12):

Theorem 3.3 *The dual characterization of the convex risk measure is given in terms of a penalty function, $\alpha(\mathbf{Q})$ taking values in $\mathbb{R} \cup \{+\infty\}$:*

$$\forall \Psi \in \mathcal{X} \quad \rho(\Psi) = \sup_{\mathbf{Q} \in \mathcal{M}_{1,f}} \{\mathbb{E}_{\mathbf{Q}}(-\Psi) - \alpha(\mathbf{Q})\} \tag{17}$$

By duality between $\mathcal{M}_{1,f}$ and $\mathcal{X}$,

$$\forall \mathbf{Q} \in \mathcal{M}_{1,f} \quad \alpha(\mathbf{Q}) = \sup_{\Psi \in \mathcal{X}} \{\mathbb{E}_{\mathbf{Q}}(-\Psi) - \rho(\Psi)\} \qquad (\geq -\rho(0)) \tag{18}$$

Moreover, the supremum is attained in $\mathcal{M}_{1,f}$ and

$$\forall \Psi \in \mathcal{X} \quad \rho(\Psi) = \max_{\mathbf{Q} \in \mathcal{M}_{1,f}} \{\mathbb{E}_{\mathbf{Q}}(-\Psi) - \alpha(\mathbf{Q})\}$$

This last result will be quite important in the following as they ensure the existence of an "optimal" additive measure.

In the following, we are especially interested in risk measures related to probability measures. In general and in the following of the paper, the assumption of decreasing continuity from below

$$\Psi_n \nearrow \Psi \quad \Rightarrow \quad \rho(\Psi_n) \searrow \rho(\Psi) \tag{19}$$

is made and it suffices to imply that the dual formulation of risk measure (Equation (17)) is satisfied for $\mathbb{Q} \in \mathcal{M}_1$, where $\mathcal{M}_1$ is the set of all probability measures on the considered space. In this case, the equation previously obtained concerning the penalty function (Equation 18) still hold replacing $\mathcal{M}_{1,f}$ by $\mathcal{M}_1$. When working with $\mathcal{M}_1$, the supremum is attained under some conditions presented in Theorem 4.22 of Föllmer and Schied (2002b). In this paper, for the sake of simplicity and clarity, we use the notation $\mathbf{Q}$ when dealing with additive measures and $\mathbb{Q}$ when dealing with probability measures.

Example 3.4 *The dual formulation of the functional e_γ is given as follows*

$$\forall \Psi \in \mathcal{X} \qquad e_\gamma(\Psi) = \gamma \ln \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{1}{\gamma}\Psi\right)\right) = \sup_{\mathbb{Q}\in\mathcal{M}_1}\left(\mathbb{E}_{\mathbb{Q}}(-\Psi) - \gamma h(\mathbb{Q}/\mathbb{P})\right)$$

where $h(\mathbb{Q}/\mathbb{P})$ is the relative entropy of $\mathbb{Q}$ with respect to the prior probability $\mathbb{P}$, defined by

$$h(\mathbb{Q}/\mathbb{P}) = \begin{cases} \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right) & \text{if } \mathbb{Q} \ll \mathbb{P} \\ +\infty & \text{otherwise} \end{cases}$$

This justifies why e_γ is referred to as the entropic risk measure.

3.1.2 Risk measure generated by a convex set

Acceptance set and generation of convex risk measures From the definition of the convex risk measure ρ and especially the duality relationship with the penalty function, it is natural to introduce the *acceptance set* related to ρ, $\mathcal{A}_\rho$, defined as the set of all acceptable positions as they carry no positive risk:

$$\mathcal{A}_\rho = \{\Psi \in \mathcal{X}, \quad \rho(\Psi) \leq 0\} \tag{20}$$

It has the following properties:

i) $\mathcal{A}_\rho$ is not empty convex set and $\inf\{m \in \mathbb{R}; m \in \mathcal{A}_\rho\} > -\infty$,

ii) For any $X \in \mathcal{A}_\rho$ and any $Y \in \mathcal{X}$, $Y \geq X \Rightarrow Y \in \mathcal{A}_\rho$,

iii) $\mathcal{A}_\rho$ has a closure property in the sense that for any $X \in \mathcal{A}_\rho$ and any $Y \in \mathcal{X}$,

$$\{\lambda \in [0,1], \text{ such that } \lambda X + (1-\lambda) Y \in \mathcal{A}_\rho\} \text{ is closed in } [0,1] \tag{21}$$

As a direct consequence of Equation (16), another characterization of ρ may be deduced from $\mathcal{A}_\rho$ as

$$\rho(\Psi) = \inf\{m \in \mathbb{R}; m + \Psi \in \mathcal{A}_\rho\} \tag{22}$$

and the associated penalty function is also defined as

$$\forall \mathbf{Q} \in \mathcal{M}_{1,f} \quad \alpha(\mathbf{Q}) = \sup_{\Psi\in\mathcal{A}_\rho} \mathbb{E}_{\mathbf{Q}}(-\Psi) \tag{23}$$

It is possible to consider the relationship (22) between ρ and $\mathcal{A}_\rho$ as a definition of the risk measure ρ and then, to extend it to general convex set in order to generate particular convex risk measures.

Definition 3.5 *Given a non-empty convex subset of $\mathcal{X}$, $\mathcal{H}$, we define*

$$\nu^{\mathcal{H}}(\Psi) = \inf\{m \in \mathbb{R}; \textit{such that } \exists \xi \in \mathcal{H},\ m + \Psi \geq \xi\}$$

If $\nu^{\mathcal{H}}(0) > -\infty$, $\nu^{\mathcal{H}}$ is a convex risk measure and the related acceptance set is defined by

$$\mathcal{A}_{\mathcal{H}} = \{\Psi \in \mathcal{X}, \exists \xi \in \mathcal{H}, \quad \Psi \geq \xi\}$$

The associated penalty function $l^{\mathcal{H}}$ is given by:

$$\forall \mathbf{Q} \in \mathcal{M}_{1,f} \quad l^{\mathcal{H}}(\mathbf{Q}) = \sup_{H \in \mathcal{H}} \mathbb{E}_{\mathbf{Q}}(-H)$$

When $\mathcal{H}$ is a cone, the penalty function associated with $\nu^{\mathcal{H}}$ is the indicator function of the cone

$$\mathcal{M}_{\mathcal{H}} = \{\mathbf{Q} \in \mathcal{M}_{1,f}; \forall \xi \in \mathcal{H},\ \mathbb{E}_{\mathbf{Q}}(\xi) \geq 0\}$$

in the sense of the convex analysis (see Rockafellar (1970)):

$$l^{\mathcal{H}}(\mathbf{Q}) = \delta(\mathbf{Q} \mid \mathcal{M}_{\mathcal{H}}) = \begin{cases} 0 & \textit{if } \mathbf{Q} \in \mathcal{M}_{\mathcal{H}} \\ +\infty & \textit{otherwise} \end{cases}$$

The risk measure $\nu^{\mathcal{H}}$ is then coherent and its dual formulation is simply given by

$$\forall \Psi \in \mathcal{X} \qquad \nu^{\mathcal{H}}(\Psi) = \sup_{\mathbf{Q} \in \mathcal{M}_{\mathcal{H}}} \mathbb{E}_{\mathbf{Q}}(-\Psi)$$

Note that the set $\mathcal{M}_{\mathcal{H}}$ is close to the familiar notion of "equivalent martingale measures".

Interpretation in terms of buyer's price The risk measure generated by $\mathcal{H}$ may have another interpretation. Considering any $\xi \in \mathcal{H}$ as a hedging strategy, $\nu^{\mathcal{H}}(\Psi)$ corresponds indeed to the opposite of the buyer price of Ψ. The buyer of Ψ is satisfied by a strategy (x, ξ) such that $\Psi \geq x + \xi$. For a given ξ, the buyer always considers the worst case, corresponding to the maximal amount x such that $\Psi \geq x + \xi$:

$$\pi_b(\Psi) = \sup\{x \in \mathbb{R}, \exists \xi \in \mathcal{H}, \quad \Psi \geq x + \xi\}$$

Consequently, the arg sup is the maximal price the buyer is ready to pay for Ψ. In this sense, it may be seen as the equivalent for the buyer of the super-replicating price for the seller. Given that $\nu^{\mathcal{H}}$ is defined by:

$$\nu^{\mathcal{H}}(\Psi) = \inf\{m \in \mathbb{R}, \exists \xi \in \mathcal{H}, \quad \Psi + m \geq \xi\}$$

we finally obtain that the risk measure of Ψ corresponds to the opposite of the "super buyer's price" of Ψ:

$$\nu^{\mathcal{H}}(\Psi) = -\pi_b(\Psi)$$

We will come back later to this point when considering a financial market.

In a very general framework of a convex risk measure ρ, $p(\Psi) \triangleq -\rho(\Psi)$ may also be interpreted as a price. It corresponds indeed to the (capitalized) "indifference" buyer's price which lead the agent to be indifferent between buying Ψ for a price p and doing nothing since

$$\rho(\Psi - p(\Psi)) = \rho(\Psi) + p(\Psi) = \rho(\Psi) - \rho(\Psi) = 0$$

3.2 Inf-convolution of risk measures

3.2.1 Main results

Föllmer and Schied (2002a) and (2002b) have been especially interested in the supremum of a sequence of convex risk measures. Considering a sequence a convex risk measures $(\rho_i)_{i\in I}$ with respective penalty function α_i, such that $\sup_{i\in I}\rho_i(0)<\infty$, they have obtained (see Proposition 4.15 in Föllmer and Schied (2002b)) that

$$\forall\Psi\in\mathcal{X}\qquad \rho(\Psi)\triangleq\sup_{i\in I}\rho_i(\Psi)$$

is a convex risk measure and the associated penalty function is given by

$$\forall\mathbf{Q}\in\mathcal{M}_{1,f}\qquad \alpha(\mathbf{Q})=\inf_{i\in I}\alpha_i(\mathbf{Q})$$

The following Theorem gives another stability property of convex risk measures and their respective penalty function:

Theorem 3.6 *Let ρ_1 and ρ_2 be two convex risk measures with respective penalty functions α_1 and α_2. Let $\rho_{1,2}$ be the inf-convolution of ρ_1 and ρ_2 defined as*

$$\Psi\to\rho_{1,2}(\Psi)\triangleq\rho_1\square\rho_2(\Psi)=\inf_{H\in\mathcal{X}}\{\rho_1(\Psi-H)+\rho_2(H)\}$$

and assume that $\rho_{1,2}(0)>-\infty$.
Then $\rho_{1,2}$ is a convex risk measure, which is finite for all $\Psi\in\mathcal{X}$. Moreover, if ρ_1 is continuous from below, then $\rho_{1,2}$ is also continuous from below.
The associated penalty function is given by

$$\forall\mathbf{Q}\in\mathcal{M}_{1,f}\quad \alpha_{1,2}(\mathbf{Q})=\alpha_1(\mathbf{Q})+\alpha_2(\mathbf{Q})$$

The related acceptance set $\mathcal{A}_{\rho_{1,2}}$ is the "pseudo-closure" of $\mathcal{A}_{\rho_1}+\mathcal{A}_{\rho_2}$ (in the sense of Föllmer and Schied (2002b) Proposition 4.5)

Note that the convex risk measure $\rho_{1,2}$ may also be defined as the value functional of the program

$$\rho_{1,2}(\Psi)=\inf\{\rho_1(\Psi-H),H\in\mathcal{A}_{\rho_2}\}$$

Either formulations will be used indifferently.

An immediate corollary may be obtained as:

Corollary 3.7 *Let $\mathcal{H}$ be a convex subset of $\mathcal{X}$ and ρ be a convex risk measure with penalty function α such that* $\inf\{\rho(-H),H\in\mathcal{H}\}>-\infty$.
The inf-convolution of ρ and $\nu^{\mathcal{H}}$

$$\rho^{\mathcal{H}}(\Psi)\triangleq\rho\square\nu^{\mathcal{H}}(\Psi)=\inf\{\rho(\Psi-H),H\in\mathcal{H}\}\tag{24}$$

is a convex risk measure with penalty function

$$\forall\mathbf{Q}\in\mathcal{M}_{1,f}\quad \alpha^{\mathcal{H}}(\mathbf{Q})=\alpha(\mathbf{Q})+l^{\mathcal{H}}(\mathbf{Q})$$

If $\mathcal{H}$ is a cone, $\rho^{\mathcal{H}}$ has the penalty function

$$\alpha^{\mathcal{H}}(\mathbf{Q})=\begin{cases}\alpha(\mathbf{Q}) & \text{if } \mathbf{Q}\in\mathcal{M}^{\mathcal{H}}\\ +\infty & \text{otherwise}\end{cases}$$

Proof:
The only point to be proved is the following equality

$$\rho^{\mathcal{H}}(\Psi) = \inf\left\{\rho(\Psi - H), H \in \mathcal{H}\right\}$$

By definition, $\rho^{\mathcal{H}}$ is indeed defined as

$$\rho^{\mathcal{H}}(\Psi) \triangleq \rho \Box \nu^{\mathcal{H}}(\Psi) = \inf_{\Phi}\left\{\rho(\Psi - \Phi) + \nu^{\mathcal{H}}(\Phi)\right\} = \inf\left\{\rho(\Psi - \Phi); \Phi \in \mathcal{A}_{\mathcal{H}}\right\}$$

But for any $\Phi \in \mathcal{A}_{\mathcal{H}}$, there exists $H \in \mathcal{H}$ such that $\Phi \geq H$ and so $\rho(\Psi - \Phi) \geq \rho(\Psi - H)$ since ρ is decreasing. Hence

$$\inf\left\{\rho(\Psi - \Phi); \Phi \in \mathcal{A}_{\mathcal{H}}\right\} \geq \inf\left\{\rho(\Psi - H); H \in \mathcal{H}\right\}$$

The reverse inequality is immediate since $\mathcal{H} \subset \mathcal{A}_{\mathcal{H}}$. $\Box$

Proof of Theorem 3.6:
i) The monotony and translation invariance properties of $\rho_{1,2}$ are immediate.
ii) The convexity simply comes from the fact that, for any Ψ_1, Ψ_2, H_1 and H_2 in $\mathcal{X}$ and any $\lambda \in [0,1]$, the following inequalities hold as ρ_1 and ρ_2 are convex risk measures

$$\begin{aligned} \rho_1\left[\lambda\Psi_1 + (1-\lambda)\Psi_2 - (\lambda H_1 + (1-\lambda) H_2)\right] &\leq \lambda\rho_1(\Psi_1 - H_1) + (1-\lambda)\rho_1(\Psi_2 - H_2) \\ \rho_2\left[\lambda H_1 + (1-\lambda) H_2\right] &\leq \lambda\rho_2(H_1) + (1-\lambda)\rho_2(H_2) \end{aligned}$$

By adding both inequalities and taking the infimum in H_1 and H_2 on the left-hand side and in H_1 on the right-hand side, we obtain:

$$\rho_1 \Box \rho_2(\lambda\Psi_1 + (1-\lambda)\Psi_2) \leq \lambda\rho_1 \Box \rho_2(\Psi_1) + (1-\lambda)(\rho_1(\Psi_2 - H_2) + \rho_2(H_2))$$

Taking then the infimum in H_2 on the right-hand side yields the convexity inequality for $\rho_{1,2}$.
iii) The continuity from below is directly obtained considering an increasing sequence of $(\Psi_n) \in \mathcal{X}$ converging to Ψ. Using the monotonicity property, we have

$$\begin{aligned} \inf_n \rho_1 \Box \rho_2(\Psi_n) &= \inf_n \inf_H \left\{\rho_1(\Psi_n - H) + \rho_2(H)\right\} \\ &= \inf_H \inf_n \left\{\rho_1(\Psi_n - H) + \rho_2(H)\right\} = \inf_H \left\{\rho_1(\Psi - H) + \rho_2(H)\right\} \\ &= \rho_1 \Box \rho_2(\Psi) \end{aligned}$$

iv) The assumption $\rho_{1,2}(0) > -\infty$ guarantees that $\rho_{1,2}(\Psi)$ is finite for any $\Psi \in \mathcal{X}$, as previously mentioned.
v) Using Equation (18), the associated penalty function is given, for any $\mathbf{Q} \in \mathcal{M}_{1,f}$, by

$$\begin{aligned} \alpha_{1,2}(\mathbf{Q}) &= \sup_{\Psi \in \mathcal{X}}\left\{\mathbb{E}_{\mathbf{Q}}(-\Psi) - \rho_{1,2}(\Psi)\right\} = \sup_{\Psi \in \mathcal{X}}\left\{\mathbb{E}_{\mathbf{Q}}(-\Psi) - \inf_{H \in \mathcal{X}}\left\{\rho_1(\Psi - H) + \rho_2(H)\right\}\right\} \\ &= \sup_{\Psi \in \mathcal{X}} \sup_{H \in \mathcal{X}}\left\{\mathbb{E}_{\mathbf{Q}}(-(\Psi - H)) + \mathbb{E}_{\mathbf{Q}}(-H) - \rho_1(\Psi - H) - \rho_2(H)\right\} \end{aligned}$$

Then, letting $\widetilde{\Psi} \triangleq \Psi - H \in \mathcal{X}$ and knowing that $\mathcal{X}$ is the set of all bounded random variables, it becomes

$$\begin{aligned} \alpha_{1,2}(\mathbf{Q}) &= \sup_{\widetilde{\Psi} \in \mathcal{X}} \sup_{H \in \mathcal{X}}\left(\mathbb{E}_{\mathbf{Q}}\left(-\widetilde{\Psi}\right) - \rho_1\left(\widetilde{\Psi}\right) + \mathbb{E}_{\mathbf{Q}}(-H) - \rho_2(H)\right) \\ &= \sup_{H \in \mathcal{X}}\left[\mathbb{E}_{\mathbf{Q}}(-H) - \rho_2(H) + \sup_{\widetilde{\Psi} \in \mathcal{X}}\left(\mathbb{E}_{\mathbf{Q}}\left(-\widetilde{\Psi}\right) - \rho_1\left(\widetilde{\Psi}\right)\right)\right] \\ &= \sup_{H \in \mathcal{X}}\left[\mathbb{E}_{\mathbf{Q}}(-H) - \rho_2(H) + \alpha_1(\mathbf{Q})\right] \end{aligned}$$

If $\alpha_1(\mathbf{Q}) = +\infty$, then $\alpha_{1,2}(\mathbf{Q}) = +\infty$. In the case where $\alpha_1(\mathbf{Q}) < +\infty$, then

$$\alpha_{1,2}(\mathbf{Q}) = \alpha_1(\mathbf{Q}) + \sup_{H \in \mathcal{X}} \left[\mathbb{E}_{\mathbf{Q}}(-H) - \rho_2(H)\right] = \alpha_1(\mathbf{Q}) + \alpha_2(\mathbf{Q})$$

using Equation (23).
vi) As a consequence of Equation (23), the acceptance set of the new risk measure $\rho_{1,2}$ may be characterized by

$$\Psi \in \mathcal{A}_{\rho_{1,2}} \Leftrightarrow \forall \mathbf{Q} \in \mathcal{M}_{1,f} \quad \alpha_{1,2}(\mathbf{Q}) = \alpha_1(\mathbf{Q}) + \alpha_2(\mathbf{Q}) \geq \mathbb{E}_{\mathbf{Q}}(-\Psi)$$

But, we also know that $\forall \mathbf{Q} \in \mathcal{M}_{1,f}$

$$\alpha_{1,2}(\mathbf{Q}) = \sup_{\Psi_1 \in \mathcal{A}_{\rho_1}} \mathbb{E}_{\mathbf{Q}}(-\Psi_1) + \sup_{\Psi_2 \in \mathcal{A}_{\rho_2}} \mathbb{E}_{\mathbf{Q}}(-\Psi_2) \geq \mathbb{E}_{\mathbf{Q}}(-(\Psi_1 + \Psi_2)) \quad \forall (\Psi_1, \Psi_2) \in \mathcal{A}_{\rho_1} \times \mathcal{A}_{\rho_2}$$

Hence

$$\alpha_{1,2}(\mathbf{Q}) \geq \sup_{\Psi \in \mathcal{A}_{\rho_1} + \mathcal{A}_{\rho_2}} \mathbb{E}_{\mathbf{Q}}(-\Psi)$$

So $\mathcal{A}_{\rho_1} + \mathcal{A}_{\rho_2} \subseteq \mathcal{A}_{\rho_{1,2}}$.
More precisely, let us consider two sequences (Ψ_1^n) and (Ψ_2^n) such that $\mathbb{E}_{\mathbf{Q}}(-\Psi_i^n)$ converges to $\sup_{\Psi_i \in \mathcal{A}_{\rho_i}} \mathbb{E}_{\mathbf{Q}}(-\Psi_i)$ for $i = 1, 2$. Then

$$\begin{aligned} \alpha_1(\mathbf{Q}) + \alpha_2(\mathbf{Q}) &= \lim_n \mathbb{E}_{\mathbf{Q}}(-\Psi_1^n) + \lim_n \mathbb{E}_{\mathbf{Q}}(-\Psi_2^n) \\ &= \lim_n \mathbb{E}_{\mathbf{Q}}(-(\Psi_1^n + \Psi_2^n)) \leq \sup_{\Psi \in \mathcal{A}_{\rho_1} + \mathcal{A}_{\rho_2}} \mathbb{E}_{\mathbf{Q}}(-\Psi) \end{aligned}$$

Hence, $\alpha_1(\mathbf{Q}) + \alpha_2(\mathbf{Q}) = \sup_{\Psi \in \mathcal{A}_{\rho_1} + \mathcal{A}_{\rho_2}} \mathbb{E}_{\mathbf{Q}}(-\Psi)$.
We are now interested in the relationships between both sets $\mathcal{A}_{\rho_1} + \mathcal{A}_{\rho_2}$ and $\mathcal{A}_{\rho_{12}}$. Both are convex sets. However, $\mathcal{A}_{\rho_1} + \mathcal{A}_{\rho_2}$ does not necessarily the property of closure mentioned in Equation 21. $\square$

3.2.2 Dilatation of convex risk measures and semi-group properties

In this subsection, we present an example of risk measure transformation which is stable by inf-convolution and satisfies a dilatation property with respect to the size of the position.

Definition 3.8 *Let ρ be a convex risk measure with penalty function α and $\gamma > 0$ a real parameter called the risk tolerance coefficient.*
The dilated risk measure ρ_γ, associated with ρ and γ, is by definition

$$\forall \Psi \in \mathcal{X} \qquad \rho_\gamma(\Psi) = \gamma \rho\left(\frac{1}{\gamma}\Psi\right)$$

The associated penalty function is

$$\forall \mathbf{Q} \in \mathcal{M}_{1,f} \qquad \alpha_{\rho_\gamma}(\mathbf{Q}) = \gamma \alpha(\mathbf{Q})$$

Note that the entropic functional $e_\gamma(\Psi) = \gamma \ln \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{1}{\gamma}\Psi\right)\right)$ is the dilated risk measure associated with the convex risk measure e_1. In this entropic case, this dilattion property has been referred to as volume scaling by Becherer (2003).

Moreover, as seen in the first section, when studying the "toy model", the inf-convolution of two entropic risk measures is again an entropic risk measure. The risk tolerance coefficient of the latter is simply equal to the sum of both risk tolerance coefficient (see Equation (9)). Hence, the following stability property holds for entropic risk measures for any (γ,γ'), strictly positive:

$$e_\gamma \Box e_\gamma = e_{\gamma+\gamma'}$$

This last property still holds for general dilated risk measure:

Theorem 3.9 *Let $(\rho_\gamma, \gamma > 0)$ be the family of ρ-dilated risk measures. Then, the following properties hold:*
i) For any $\gamma, \gamma^{'} > 0$, $\rho_\gamma \Box \rho_{\gamma'} = \rho_{\gamma+\gamma'}$,
ii) Moreover, $F^* = \frac{\gamma^{'}}{\gamma+\gamma^{'}} X$ is an optimal structure for the minimization program:

$$\rho_{\gamma+\gamma'}(X) = \rho_\gamma \Box \rho_{\gamma'}(X) = \inf_F \{\rho_\gamma(X-F) + \rho_{\gamma'}(F)\} = \rho_\gamma(X-F^*) + \rho_{\gamma'}(F^*)$$

iii) Let ρ and ρ' be two convex risk measures.
Then, for any $\gamma > 0$, $\rho_\gamma \Box \rho'_\gamma = (\rho \Box \rho')_\gamma$.

Proof:
Both i) and iii) are immediate consequences of the definition and characterization of dilated risk measures.
ii) Let us search for the optimal structure in the family $\{\alpha X; \alpha \in \mathbb{R}\}$. Then,

$$\rho_\gamma((1-\alpha)X) + \rho_{\gamma'}(\alpha X) = \gamma\rho\left(\frac{1-\alpha}{\gamma}X\right) + \gamma^{'}\rho\left(\frac{\alpha}{\gamma^{'}}X\right) = (\gamma+\gamma').\rho\left(\frac{1}{\gamma+\gamma'}X\right)$$

A natural candidate is then obtained for $\frac{1-\alpha}{\gamma}X = \frac{\alpha}{\gamma}X = \frac{1}{\gamma+\gamma'}X$. Hence the result.$\Box$
Moreover, the following asymptotic properties hold for dilated risk measures, extending the entropic framework (see for instance, El Karoui-Rouge (2000) (Theorem 5.2) and Becherer (2003) (Proposition 3.2)).

Proposition 3.10 *i) ρ is a coherent risk measure if and only if $\rho_\gamma \equiv \rho$*
ii) Suppose that $\rho(0) = 0$. Then, $\rho_\infty \triangleq \lim_{\gamma\to\infty} \rho_\gamma$ is a coherent risk measure and

$$\rho_\infty(\Psi) = \sup_{\substack{\mathbf{Q}\in\mathcal{M}_{1,f} \\ \alpha(\mathbf{Q})=0}} \mathbb{E}_\mathbf{Q}(-\Psi)$$

iii) On the other hand, $\rho_0 \triangleq \lim_{\gamma\to 0} \rho_\gamma$ is simply the "super-pricing rule" of $-\Psi$:

$$\rho_0(\Psi) = \sup_{\substack{\mathbf{Q}\in\mathcal{M}_{1,f} \\ \alpha(\mathbf{Q})<\infty}} \mathbb{E}_\mathbf{Q}(-\Psi)$$

Proof:
i) comes immediately from the definition and characterization of both coherent risk measures and dilated risk measures.
ii) Let us first observe that ρ_γ is a decreasing function of γ. This monotonicity property comes from the dual representation of convex risk measures together with the expression of the penalty function of dilated risk measure.
The risk measure corresponding to an infinite risk tolerance, $\rho_\infty \triangleq \lim_{\gamma\to\infty} \downarrow \rho_\gamma$, is a coherent risk measure since:

$$\gamma\rho_\infty\left(\frac{1}{\gamma}\Psi\right) = \gamma \lim_{c\to\infty}\left(\rho_c\left(\frac{1}{\gamma}\Psi\right)\right) = \gamma \lim_{c\to\infty}\left(c\rho\left(\frac{1}{\gamma c}\Psi\right)\right) = \rho_\infty(\Psi)$$

Moreover, since

$$\forall \mathbf{Q} \in \mathcal{M}_{1,f} \qquad \alpha_\infty(\mathbf{Q}) = \sup_\Psi \{\mathbb{E}_\mathbf{Q}(-\Psi) - \rho_\infty(\Psi)\} \geq \sup_\Psi \{\mathbb{E}_\mathbf{Q}(-\Psi) - \rho_\gamma(\Psi)\} = \alpha_\gamma(\mathbf{Q}) = \gamma.\alpha(\mathbf{Q})$$

then $\alpha_\infty(\mathbf{Q}) = \infty$ if $\alpha(\mathbf{Q}) > 0$. Hence

$$\rho_\infty(\Psi) = \sup_{\substack{\mathbf{Q}\in\mathcal{M}_{1,f} \\ \alpha(\mathbf{Q})=0}} \mathbb{E}_\mathbf{Q}(-\Psi)$$

iii) By monotonicity,

$$\begin{aligned} \rho_0(\Psi) = \lim\nolimits_{\gamma\to 0} \uparrow \rho_\gamma(\Psi) \quad &= \sup\nolimits_\gamma \sup\nolimits_{\{\mathbf{Q};\alpha(\mathbf{Q})<\infty\}} \{\mathbb{E}_\mathbf{Q}(-\Psi) - \gamma\alpha(\mathbf{Q})\} \\ &= \sup\nolimits_{\mathcal{Q}_\alpha} \sup\nolimits_\gamma \{\mathbb{E}_\mathbf{Q}(-\Psi) - \gamma\alpha(\mathbf{Q})\} \\ &= \sup\nolimits_{\mathcal{Q}_\alpha} \mathbb{E}_\mathbf{Q}(-\Psi) \end{aligned}$$

where $\mathcal{Q}_\alpha = \{\mathbf{Q} \in M_{1,f}; \alpha(\mathbf{Q}) < \infty\}$. $\square$

4 Optimal design problem

This Section is dedicated to our initial problem to characterize the optimal issue written on the non-tradable risk in the general framework presented above.

4.1 Framework

In the following, we come back to our initial problem of optimal transaction between agent A and agent B described in Subsection 2.1.1:
At a fixed future date T, agent A is exposed towards a non-tradable risk Θ for an amount X. To reduce her exposure, she wants to issue a financial product F and sell it to agent B for a forward price at time T denoted by π. Both agents now assess the risk associated with their respective positions by a *convex risk measure*, denoted by ρ_A and ρ_B (with associated penalty functions α_A and α_B).

Just previously, both agents may reduce their risk by also investing in the financial market, choosing optimally their financial investments via, in general, two convex sets $\mathcal{V}_T^{(A)}$ and $\mathcal{V}_T^{(B)}$ as previously described in Subsection 2.2.

1. This opportunity to invest optimally in a financial market reduces the risk of both agents. To assess their respective risk exposure, they now refer to a market modified risk measure defined by $\inf_{\xi_A\in\mathcal{V}_T^{(A)}} \rho_A(\Psi - \xi_A) \triangleq \rho_A^m(\Psi)$ and $\inf_{\xi_B\in\mathcal{V}_T^{(B)}} \rho_B(\Psi - \xi_B) \triangleq \rho_B^m(\Psi)$. As usually, we assume that

$$\rho_A^m(0) > -\infty \quad \text{and } \rho_B^m(0) > -\infty \tag{25}$$

   Given Corollary 3.7, we introduce the risk measures generated by both convex sets $\mathcal{V}_T^{(A)}$ and $\mathcal{V}_T^{(B)}$, denoted respectively by ν^A and ν^B. Then, ρ_A^m and ρ_B^m correspond to the inf-convolution between the initial risk measures and the risk measures generated by the financial markets:

$$\rho_A^m(\Psi) = \rho_A \square \nu^A(\Psi) \qquad \text{and} \qquad \rho_B^m(\Psi) = \rho_B \square \nu^B(\Psi)$$

2. Consequently, the optimization program related to the F-transaction is simply

$$\inf_{F\in\mathcal{X},\pi} \rho_A^m\left(X-F+\pi\right) \qquad \text{subject to} \qquad \rho_B^m\left(F-\pi\right)\leq \rho_B^m\left(0\right)$$

As previously, using the cash translation invariance property and binding the constraint at the optimum, the pricing rule of the F-structure is fully determined by the buyer as

$$\pi^*\left(F\right)=\rho_B^m\left(0\right)-\rho_B^m\left(F\right) \tag{26}$$

It corresponds to an "indifference" pricing rule from the point of view of agent B's market modified risk measure.

3. Using again the cash translation invariance property, the optimization program simply becomes

$$\inf_{F\in\mathcal{X}}\left(\rho_A^m\left(X-F\right)+\rho_B^m\left(F\right)-\rho_B^m\left(0\right)\right)$$

We are almost in the framework of Theorem 3.6, apart from the constant $\rho_B^m\left(0\right)$. To deal with it, noticing that the value functional obtained in this case should be translated by the constant $-\rho_B^m\left(0\right)$ in order to obtain the value function of the previous program, we consider the reduced program

$$\begin{aligned} R_{AB}^m\left(X\right) &= \inf_{F\in\mathcal{X}}\left(\rho_A^m\left(X-F\right)+\rho_B^m\left(F\right)\right) \\ &= \rho_A^m\Box\rho_B^m\left(X\right)=\rho_A\Box\nu^A\Box\rho_B\Box\nu^B\left(X\right) \end{aligned} \tag{27}$$

The value functional R_{AB}^m of this program, resulting from the inf-convolution of four different risk measures, may be interpreted as the *residual risk measure* after all transactions.

4. Using the previous Theorem 3.6 on the stability of convex risk measure, $R_{AB}^m\left(X\right)$ is a convex risk measure with the penalty function

$$\forall\mathbf{Q}\in\mathcal{M}_{1,f} \quad \alpha_{AB}^m\left(\mathbf{Q}\right)=\alpha_A^m\left(\mathbf{Q}\right)+\alpha_B^m\left(\mathbf{Q}\right)=\alpha_A\left(\mathbf{Q}\right)+\alpha_B\left(\mathbf{Q}\right)+l^A\left(\mathbf{Q}\right)+l^B\left(\mathbf{Q}\right)$$

Note that the financial market plays exactly the same role as an intermediate agent imposing some constraint on the considered agent. As a consequence, we end up with four different risk measures, two per agents.

4.2 Dilated risk measures and Borch's Theorem

Our problem is to construct optimal structures. We have already almost solved it completely in the entropic framework (assuming the solution of the hedging problem). In that case, the existence of a solution is ensured. In the general case, it may be more of a problem. However, in the particular case when both agents have the same type of risk measures but differ in their respective risk tolerance, everything becomes very simple as we will see in the following.

Hence, we consider the situation where both agents have dilated initial risk measures, ρ_A and ρ_B. In this sense, we may say that the framework is symmetrical for both agents.

The residual risk measure $R_{AB}^m\left(X\right)$ may be simplified using the commutativity property of the inf-convolution and the semi-group property of dilated risk measures:

$$R_{AB}^m\left(X\right)=\rho_A\Box\nu^A\Box\rho_B\Box\nu^B\left(X\right)=\rho_A\Box\rho_B\Box\nu^A\Box\nu^B\left(X\right)\triangleq\rho_C\Box\nu^A\Box\nu^B\left(X\right) \tag{28}$$

where ρ_C is the dilated risk measure associated with the risk tolerance coefficient $\gamma_C = \gamma_A + \gamma_B$.

We present two different results depending on the access both agents have to the financial markets. Both proofs will be presented in the next section, when some general results on optimality in inf-convolution problems.

4.2.1 Borch's Theorem

We first assume that both agents have the same access to the financial market via a *cone* $\mathcal{H}$. Given the fact that the risk measure generated by $\mathcal{H}$ is coherent and thus invariant by dilatation, the market modified risk measures of both agents are dilated from $\rho \square \nu^{\mathcal{H}}$ as

$$\rho_A \square \nu^{\mathcal{H}} = \rho_A \square \nu^{\mathcal{H}}_{\gamma_A} = \left(\rho \square \nu^{\mathcal{H}}\right)_{\gamma_A} \quad \text{and} \quad \rho_B \square \nu^{\mathcal{H}} = \rho_B \square \nu^{\mathcal{H}}_{\gamma_B} = \left(\rho \square \nu^{\mathcal{H}}\right)_{\gamma_B}$$

Hence, using Equation (28), we have

$$R^m_{AB}\left(X\right) = \left(\rho \square \nu^{\mathcal{H}}\right)_{\gamma_C}$$

Using Theorem 3.9, we find again the so-called Borch's theorem:

Proposition 4.1 *If both agents have dilated risk measures and have the same access to the financial market via a cone, then an optimal structure, solution of the minimization Program* (27) *is given by:*

$$F^* = \frac{\gamma_B}{\gamma_A + \gamma_B} X$$

4.2.2 Different access to hedging strategies

In a more general framework, when both agents have different access to the financial market, we may use the same arguments as in the entropic framework, after some transformation in the presentation of the residual risk measure. According to Equation (28), we have

$$R^m_{AB}\left(X\right) = \rho_C \square \nu^A \square \nu^B \left(X\right)$$

In particular, using the properties of the inf-convolution,

$$\begin{aligned} R^m_{AB}\left(X\right) &= \inf_{\xi_B \in \mathcal{V}^{(B)}_T} \inf_{\xi_A \in \mathcal{V}^{(A)}_T} \rho_C\left(X - \xi_A - \xi_B\right) = \inf_{\xi_A + \xi_B \in \mathcal{V}^{(A)}_T + \mathcal{V}^{(B)}_T} \rho_C\left(X - \xi_A - \xi_B\right) \\ &\triangleq \inf_{\xi \in \mathcal{V}^{(A)}_T + \mathcal{V}^{(B)}_T} \rho_C\left(X - \xi\right) \end{aligned}$$

Then, $R^m_{AB}\left(X\right)$ is very similar to the residual risk measure in the entropic framework, $E^m_{AB}\left(X\right)$, given as the value functional of the Program $(\mathcal{P}_{AB})$. As a consequence, the following result is very similar to Theorem 2.3. The proof, consisting of three main steps, has been detailed in Subsection 2.1.2. It does not use the explicit formulation of the entropic risk measure and can be directly extended to this general framework.

Theorem 4.2 *Suppose $\xi^* = \eta^*_A + \eta^*_B$ is an optimal solution of the Program*

$$\inf_{\xi \in \mathcal{V}^{(A)}_T + \mathcal{V}^{(B)}_T} \rho_C\left(X - \xi\right)$$

with $\eta^*_A \in \mathcal{V}^{(A)}_T$ and $\eta^*_B \in \mathcal{V}^{(B)}_T$. Then

$$F^* = \frac{\gamma_B}{\gamma_A + \gamma_B} X - \frac{\gamma_B}{\gamma_A + \gamma_B} \eta^*_A + \frac{\gamma_A}{\gamma_A + \gamma_B} \eta^*_B$$

is an optimal structure.
Moreover,
i) η_B^* *is an optimal investment portfolio for Agent B*

$$\frac{1}{\gamma_B}\rho_B\left(F^* - \eta_B^*\right) = \frac{1}{\gamma_B}\inf_{\xi_B \in \mathcal{V}_T^{(B)}}\rho_B\left(F^* - \xi_B\right) = \frac{1}{\gamma_C}\rho_C\left(X - \xi^*\right)$$

ii) η_A^* *is an optimal hedging portfolio of* $(X - F^*)$ *for Agent A*

$$\frac{1}{\gamma_A}\rho_A\left(X - (F^* + \eta_A^*)\right) = \frac{1}{\gamma_A}\inf_{\xi_A \in \mathcal{V}_T^{(A)}}\rho_A\left(X - (F^* + \xi_A)\right) = \frac{1}{\gamma_C}\rho_C\left(X - \xi^*\right)$$

Standard diversification will also occur in exchange economies as soon as agents have proportional penalty functions. This extends the results obtained in the entropic framework in Section 2. The regulator has to impose very different rules on agents as to generate risk measures with non-proportional penalty functions if she wants to increase the diversification in the market. In other words, diversification occurs when agents are very different one from the other. This result supports for instance the intervention of reinsurance companies on financial markets in order to increase the diversification on the reinsurance market.

4.3 Characterization of the optimal structure in the general framework

We now consider a very general framework where the problem is to find an optimal structure F^* realizing the minimum of the Program (27)

$$R_{AB}^m(X) = \inf_F\left\{\rho_A^m(X - F) + \rho_B^m(F)\right\}$$

Let us first introduce two definitions of optimality and precise the dual relationship between exposure and additive measure:

Definition 4.3 *Given a convex risk measure* ρ *and its associated penalty function* α*, we say*
i) that the additive measure $\mathbf{Q}_\rho^\Psi$ *is optimal for* (Ψ, ρ) *if*

$$\rho(\Psi) = \sup_{\mathbf{Q} \in \mathcal{M}_{1,f}}\left\{\mathbb{E}_{\mathbf{Q}}(-\Psi) - \alpha(\mathbf{Q})\right\} = \mathbb{E}_{\mathbf{Q}_\rho^\Psi}(-\Psi) - \alpha\left(\mathbf{Q}_\rho^\Psi\right)$$

ii) that the exposure Ψ *is optimal for* $(\mathbf{Q}, \alpha)$ *if*

$$\alpha(\mathbf{Q}) = \sup_{\Phi \in \mathcal{X}}\left\{\mathbb{E}_{\mathbf{Q}}(-\Phi) - \rho(\Phi)\right\} = \mathbb{E}_{\mathbf{Q}}(-\Psi) - \rho(\Psi)$$

iii) a sequence (Ψ_n) *is maximizing for* $(\mathbf{Q}, \alpha)$ *if*

$$\sup_n\left\{\mathbb{E}_{\mathbf{Q}}(-\Psi_n) - \rho(\Psi_n)\right\} = \sup_{\Phi \in \mathcal{X}}\left\{\mathbb{E}_{\mathbf{Q}}(-\Phi) - \rho(\Phi)\right\}$$

Remark: Quite obviously, if Ψ is optimal for $(\mathbf{Q}, \alpha)$ then for any $I \in \mathcal{I}_\rho$, $\Psi + I$ is optimal for $(\mathbf{Q}, \alpha)$. Moreover, if $\mathbf{Q}$ is optimal for (Ψ, ρ), then Ψ is optimal for $(\mathbf{Q}, \alpha)$.
Then, the following result is obtained:

Theorem 4.4 *The necessary and sufficient condition to have an optimal solution F^* to the inf-convolution program*

$$R_{AB}^m(X) = \inf_F \{\rho_A^m(X-F) + \rho_B^m(F)\}$$

is that there exists an optimal additive measure $\mathbf{Q}_{AB}^X$ for (X, R_{AB}^m) such that F^* is optimal for $\left(\mathbf{Q}_{AB}^X, \alpha_B^m\right)$ and $X - F^*$ is optimal for $\left(\mathbf{Q}_{AB}^X, \alpha_A^m\right)$.
More generally, (F_n) is a minimizing sequence for the inf-convolution problem if and only if (F_n) is a maximizing sequence for $\left(\mathbf{Q}_{AB}^X, \alpha_B^m\right)$ and $(X - F_n)$ is a maximizing sequence for $\left(\mathbf{Q}_{AB}^X, \alpha_A^m\right)$.

Note that everything relies upon the existence of an optimal additive measure $\mathbf{Q}_{AB}^X$ for $R_{AB}^m(X)$. As mentioned in Subsection 3.1.1, the existence of such an additive measure is guaranteed as soon as the penalty function is defined by (23). When working with probability measures $\mathcal{M}_1$, the supremum

$$\rho(\Psi) = \sup_{\mathbb{Q}\in\mathcal{M}_1} \{\mathbb{E}_{\mathbb{Q}}(-\Psi) - \alpha(\mathbb{Q})\}$$

is attained under some topological conditions obtained by Föllmer and Schied (2002b) (Theorem 4.22).

It may be worth noticing however that if one of the risk measures involved in the inf-convolution is continuous from below, then the optimal additive measure of the inf-convolution is in fact σ-additive.

Proof:
In the proof, we denote by Ψ^c, the centered random variable Ψ with respect to the given additive measure $\mathbf{Q}_{AB}^X$ optimal for (X, R_{AB}^m):

$$\Psi^c = \Psi - \mathbb{E}_{\mathbf{Q}_{AB}^X}(\Psi)$$

So, by definition,

$$\begin{aligned} -R_{AB}^m(X^c) &= \alpha_A^m\left(\mathbf{Q}_{AB}^X\right) + \alpha_B^m\left(\mathbf{Q}_{AB}^X\right) \\ &= \sup_{F\in\mathcal{X}} \{-\rho_A^m(X^c - F^c)\} + \sup_{F\in\mathcal{X}} \{-\rho_B^m(F^c)\} \\ &\geq -\inf_{F\in\mathcal{X}} \{\rho_A^m(X^c - F^c) + \rho_B^m(F^c)\} = -R_{AB}^m(X^c) \end{aligned}$$

In particular, all inequalities are equalities and

$$\sup_{F\in\mathcal{X}} \{-\rho_A^m(X^c - F^c)\} + \sup_{F\in\mathcal{X}} \{-\rho_B^m(F^c)\} = \sup_{F\in\mathcal{X}} \{-\rho_A^m(X^c - F^c) - \rho_B^m(F^c)\}$$

Hence, F^* is optimal for the inf-convolution problem, or equivalently for the program on the right-hand side of this equality, if and only if F^* is optimal for both problems $\sup_{F\in\mathcal{X}} \{-\rho_B^m(F^c)\}$ and $\sup_{F\in\mathcal{X}} \{-\rho_A^m(X^c - F^c)\}$. More generally, the same argument holds for any minimizing sequence for the inf-convolution problem, (F_n), such that there exists $\varepsilon > 0$ and

$$-\rho_A^m(X^c - F_n^c) + (-\rho_B^m(F_n^c)) + \varepsilon \geq -R_{AB}^m(X^c) = \sup_{F\in\mathcal{X}} \{-\rho_A^m(X^c - F^c)\} + \sup_{F\in\mathcal{X}} \{-\rho_B^m(F^c)\}$$

and similarly

$$-\rho_B^m(F_n^c) + \varepsilon \geq \sup_{F\in\mathcal{X}} \{-\rho_B^m(F^c)\}$$

Then, (F_n) is a maximizing sequence for both problems $\sup_{F\in\mathcal{X}} \{-\rho_B^m(F^c)\}$ and $\sup_{F\in\mathcal{X}} \{-\rho_A^m(X^c - F^c)\}$. The converse is obvious. $\square$

Optimal hedge Let us now illustrate this theorem via the particular question of hedging. Let $\mathcal{H}$ be a cone of bounded variables.
We introduce the generated coherent risk measure $\nu^{\mathcal{H}}$ as in Definition 3.5 and its acceptance set

$$\mathcal{A}_{\mathcal{H}} = \{\Psi; \exists H \in \mathcal{H} \quad \Psi \geq H\}$$

The penalty function of the risk measure $\nu^{\mathcal{H}}$ is the indicator function (in the sense of the convex analysis) of the set

$$\mathcal{M}^{\mathcal{H}} = \{\mathbf{Q} \in \mathcal{M}_{1,f}; \forall H \in \mathcal{H} \quad \mathbb{E}_{\mathbf{Q}}(H) \geq 0\}$$

i) Let ρ be a convex risk measure and $\rho^{\mathcal{H}}$ its inf-convolution with $\nu^{\mathcal{H}}$. By Corollary 3.7:

$$\rho^{\mathcal{H}}(X) = \rho \Box \nu^{\mathcal{H}}(X) = \inf_{H \in \mathcal{H}} \rho(X - H)$$

Let $\mathbb{Q}^*$ be an optimal additive measure for $\left(X, \rho^{\mathcal{H}}\right)$. Given the fact that $\alpha^{\mathcal{H}}(\mathbb{Q})$ is finite if and only if $\mathbb{Q} \in \mathcal{M}^{\mathcal{H}}$, we immediately obtain that $\mathbb{Q}^* \in \mathcal{M}^{\mathcal{H}}$.
Using Theorem 4.4, we obtain the following characterization of the optimal structure G^* for the inf-convolution problem:
G^* is optimal if and only if:
1) $G^* \in \mathcal{H}$,
2) $\mathbb{E}_{\mathbf{Q}^*}(G^*) = 0$,
3) $\alpha(\mathbb{Q}^*) = \sup_G(-\rho(X^c - G^c)) = -\rho(X^c - G^*)$.

Proof:
By Theorem 4.4, G^* is optimal if and only if $G^* \in \mathcal{H}$ and

$$\alpha(\mathbb{Q}^*) = -\rho^{\mathcal{H}}(X^c) = -\rho(X^c - G^{*c})$$

Both the facts that $\forall H \in \mathcal{H}$, $\mathbb{E}_{\mathbf{Q}^*}(H) \geq 0$ and

$$-\rho^{\mathcal{H}}(X^c) = \sup_{H \in \mathcal{H}}(-\rho(X^c - H)) = \sup_{H \in \mathcal{H}}(-\rho(X^c - H^c) - \mathbb{E}_{\mathbf{Q}^*}(H))$$

imply that $\mathbb{E}_{\mathbf{Q}^*}(G^*) = 0$. $\Box$

ii) When ρ is the entropic risk measure with the penalty function $\gamma h(\mathbb{Q}/\mathbb{P})$, an optimal hedge Ψ^* satisfies the three properties obtained above. More precisely,
1) $\Psi^* \in \mathcal{H}$,
2) $\mathbb{E}_{\mathbf{Q}^*}(\Psi^*) = 0$,
3) $\gamma h(\mathbb{Q}^*/\mathbb{P}) = -e_\gamma(X^c - \Psi^*)$.
Equivalently,

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \frac{1}{k} \exp - \left(\frac{\Psi^* - X}{\gamma}\right).$$

Necessarily, an optimal probability measure is equivalent to $\mathbb{P}$ and has a finite relative entropy with respect to $\mathbb{P}$. We will come back to this particular question in Subsection 5.2. We will then prove that under some additional assumption, this condition is also sufficient for the optimality.

5 Optimality in the inf-convolution problem: some examples

We now study the solving of the inf-convolution problem, in particular to the existence of a solution to this problem. We consider first a general framework of convex risk measure and then come back to the hedging problem in the entropic framework. This question has been widely studied in the literature under the name of "hedging in incomplete markets and pricing via utility maximization" in some particular framework. Most of the studies have considered exponential utility functions. Among the numerous papers, we may quote the papers by Frittelli (2000), El Karoui-Rouge (2000), Delbaen *et al.* (2002), Kabanov-Stricker (2002) or the PhD dissertation of Becherer (2001).

5.1 Some existence results for the hedging problem

We are interested here in the solving of the following inf-convolution problem:

$$\inf_{\xi \in \mathcal{V}_T} \rho(X - \xi) \qquad (\mathcal{P}) \tag{29}$$

where $\mathcal{V}_T$ is a convex set of bounded variables and ρ is a convex risk measure, continuous from below.

Preliminary results The existence of a solution to this problem is closely related to the following properties of the functional ρ:

$$\begin{array}{ll} \star & \xi \to \rho(\xi) \text{ is convex and decreasing} \\ \star & \text{If } \xi_n \uparrow \xi\ , \quad \rho(\xi_n) \downarrow \rho(\xi) \qquad \text{and if } \xi_n \downarrow \xi\ , \quad \rho(\xi) \uparrow \rho(\xi) \end{array} \tag{30}$$

In the following, we will assume that for any elements $(X, Y) \in \mathcal{X}^2$ such that $X = Y \ \ \mathbb{P}\ a.s.$

$$\rho(X) = \rho(Y)$$

Hence, the properties given by Equation (30) are also true when considering almost surely convergence.

Moreover, a key argument in the proof of the existence of a solution relies upon the following version of the Komlos Theorem (Komlos (1967)):

Theorem 5.1 (Komlos) *Let (ϕ_n) be a sequence in $L^1(\mathbb{P})$ such that*

$$\sup_n \mathbb{E}_{\mathbb{P}}(|\phi_n|) < +\infty$$

Then there exists a subsequence $(\phi_{n'})$ of (ϕ_n) and a function $\phi^* \in L^1(\mathbb{P})$ such that for every further subsequence $(\phi_{n''})$ of (ϕ_n)

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n''=1}^{N} \phi_{n''}(\omega) = \phi^*(\omega) \qquad \textit{for almost every } \omega \in \Omega$$

Existence Theorem Now we are able to present the following Theorem of existence:

Theorem 5.2 *Assume* $\inf_{\xi \in \mathcal{V}_T} \rho(\xi) > -\infty$.
i) Let $\mathcal{V}_T$ be a convex set, bounded in $L^1(\mathbb{P})$, of bounded random variables ξ.
The infimum of the hedging program

$$\rho^m(X) \triangleq \inf_{\xi \in \mathcal{V}_T} \rho(X - \xi)$$

is "attained" for a random variable ξ^* in $L^1(\mathbb{P})$, belonging to the closure of $\mathcal{V}_T$ with respect to the a.s. convergence.
ii) When $\mathcal{V}_T = \mathcal{V}_T^{(A)} + \mathcal{V}_T^{(B)}$ with $\mathcal{V}_T^{(A)}$ and $\mathcal{V}_T^{(B)}$ convex and bounded in $L^1(\mathbb{P})$, the infimum of the hedging program

$$\rho^m(X) \triangleq \inf_{\eta_A \in \mathcal{V}_T^{(A)}, \eta_B \in \mathcal{V}_T^{(B)}} \rho(X - \eta_A - \eta_B)$$

is attained for $\xi^* = \eta_A^* + \eta_B^*$ where η_A^* and η_B^* belong to the a.s. closure of $\mathcal{V}_T^{(A)}$ and $\mathcal{V}_T^{(B)}$.

Note in particular that Theorem 4.2 still holds in this case.

Proof:
Note first that the proof of this Theorem relies on arguments similar to those used by Kabanov and Stricker (2002).
i) Let $(\xi_n \in \mathcal{V}_T)$ be a minimizing sequence for the hedging program $\rho^m(X) \triangleq \inf_{\xi \in \mathcal{V}_T} \rho(X - \xi)$ such that $\rho(X - \xi_n)$ tends to $\rho^m(X)$.
Given the assumption that (ξ_n) is a L^1-bounded sequence, we can apply Komlos Theorem (Theorem 5.1). Therefore, there exists a subsequence $(\xi_{j_k} \in \mathcal{V}_T)$ such that the Cesaro-means, $\widetilde{\xi}_n \triangleq \frac{1}{n}\sum_{k=1}^{n} \xi_{j_k}$ converges almost surely to $\xi^* \in L^1(\mathbb{P})$.
Note that $\widetilde{\xi}_n$ is an element of $\mathcal{V}_T$ as a convex combination of elements of $\mathcal{V}_T$. So ξ^* belongs to the *a.s.* closure of $\mathcal{V}_T$.

Since ρ is decreasing and stable by monotone convergence,

$$\limsup_n \rho\left(X - \widetilde{\xi}_n\right) \leq \rho(X - \xi^*) = \rho\left(\lim_n \left(X - \widetilde{\xi}_n\right)\right) \leq \liminf_n \rho\left(X - \widetilde{\xi}_n\right)$$

Then,

$$\rho^m(X) \leq \rho(X - \xi^*) \leq \liminf_n \rho\left(\frac{1}{n}\sum_{k=1}^{n}(X - \xi_{j_k})\right) \leq \liminf_n \frac{1}{n}\sum_{k=1}^{n}\rho(X - \xi_{j_k})$$

by Jensen inequality. Finally, given the convergence of $\rho(X - \xi_{j_k})$ to $\rho^m(X)$,

$$\rho(X - \xi^*) = \inf_{\xi \in \mathcal{V}_T} \rho(X - \xi)$$

ii) Suppose now that the convex space $\mathcal{V}_T = \mathcal{V}_T^{(A)} + \mathcal{V}_T^{(B)}$ where $\mathcal{V}_T^{(A)}$ and $\mathcal{V}_T^{(B)}$ are bounded in $L^1(\mathbb{P})$. Using the same arguments, we can select step by step a sequence $(\xi_n = \eta_A^n + \eta_B^n)$ converging almost surely, a Cesaro subsequence $\left(\widetilde{\xi}_n\right)$ converging almost surely to ξ^*, then two new Cesaro subsequences $(\widetilde{\eta}_A^n)$ and $(\widetilde{\eta}_B^n)$ such that $(\widetilde{\eta}_A^n)$ converges almost surely η_A^*. This implies that $\left(\widetilde{\eta}_B^n = \widetilde{\xi}_n - \widetilde{\eta}_A^n\right)$ also converges almost surely $\eta_B^* = \xi^* - \eta_A^*$. The rest of the proof lies on the same arguments as *i*). Finally

$$\rho(X - \xi^*) \triangleq \inf_{\eta_A \in \mathcal{V}_T^{(A)}, \eta_B \in \mathcal{V}_T^{(B)}} \rho(X - \eta_A - \eta_B) = \rho(X - \eta_A^* - \eta_B^*)$$

□

5.2 Dynamic hedging in the hedging framework

We now consider the global hedging problem in the entropic framework when the set of admissible gains $\mathcal{V}_T$ is related to dynamic strategies. In this case, solving directly the primal problem $(\mathcal{P})$ may be very tricky. It is easier to work with its dual formulation. We will recall some classical results of the literature, which are useful for our problem.

$\mathcal{V}_T$ as a set of dynamic financial strategies The framework we now consider is general but standard (see, for instance Delbaen-Schachermayer (1994)). The basic financial assets are evaluated by their *forward* price at time T denoted by S. The process $(S_t; t \in [0,T])$ is assumed to be a vector $(\mathbb{P} - \Im_t)$-semi-martingale, locally bounded, where $(\Im_t; t \in [0,T])$ is a filtration on $(\Omega, \Im, \mathbb{P})$ satisfying the usual conditions of right-continuity and completeness. In particular, S may be a discontinuous vector process, with bounded jumps.

Several sets of probability measures are therefore important

$$\begin{aligned}
\mathcal{P}_a &\triangleq \{\mathbb{Q}, \mathbb{Q} \ll \mathbb{P},\ S \text{ is a } (\mathbb{Q}, \Im_t)\text{-local martingale}\} \\
\mathcal{P}_e &\triangleq \{\mathbb{Q}, \mathbb{Q} \sim \mathbb{P},\ S \text{ is a } (\mathbb{Q}, \Im_t)\text{-local martingale}\} \\
\mathcal{P}_f &\triangleq \{\mathbb{Q} \in \mathcal{P}_a,\ h(\mathbb{Q}/\mathbb{P}) < \infty\}
\end{aligned}$$

In the literature, the following assumption, implying no-arbitrage, is made $\mathcal{P}_e \cap \mathcal{P}_f \neq \emptyset$.

The self-financing strategies, in which agents invest, are predictable processes, ϕ, such that their stochastic integrals with respect to S are well-defined and bounded from below at any time t, $t \in [0,T]$.
We consider the following set of admissible strategies:

$$\Phi_M = \{(\phi) \text{ admissible } ; \phi.S \text{ is a } (\mathbb{Q}, \Im_t)\text{-martingale for all } \mathbb{Q} \in \mathcal{P}_f\}$$

The associated set of terminal gains is denoted by $\mathcal{G}_{\Phi_M}$

$$\mathcal{G}_{\Phi_M} = \left\{ \xi_T \ ; \ \xi_T = \int_0^T \langle \phi_t, dS_t \rangle \ ; \ (\phi) \in \Phi_M \right\}$$

The set $\mathcal{V}_T$ we consider is defined as

$$\mathcal{V}_T = \mathcal{G}_{\Phi_M} \cap \mathcal{X}$$

Hence, $\mathcal{V}_T$ satisfies some key properties essential to solve completely the hedging problem. It is not however the minimal set of terminal gains. Several authors have studied in details the question of the minimal space of admissible strategies and the dual formulation of the hedging problem. For more details, please refer in particular to the detailed study of Delbaen *et al.* (2002).

Optimal entropic probability measure and global hedging portfolio When considering a dynamic presentation of the financial market, we may obtain more accurate results on the optimal hedging strategy. More precisely, Becherer (2003) proposes a clear formulation of some results of the literature in Proposition 2.2, as presented in the following Proposition.

Proposition 5.3 *Assuming that*
$H1)$ S is a locally bounded semi-martingale.
$H2)$ $\mathcal{P}_e \cap \mathcal{P}_f \neq \emptyset$. In other words, there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that S is a $\mathbb{Q}$-local martingale and the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ is finite.

H3) The random variable X is bounded, i.e. $X \in \mathcal{X}$.
H4) The following duality property

$$\inf_{\xi \in \mathcal{V}_T} \gamma \ln \mathbb{E}_{\mathbb{P}} \left(\exp \left(-\frac{1}{\gamma} (\xi + X) \right) \right) = \sup_{\mathbb{Q} \in \mathcal{P}_a} \left(\mathbb{E}_{\mathbb{Q}} (-X) - \gamma . h (\mathbb{Q}/\mathbb{P}) \right)$$

holds for any $X \in \mathcal{X}$ and any $\gamma > 0$.
Then,
i) there exists a unique probability measure $\mathbb{Q}^X \in \mathcal{P}_e \cap \mathcal{P}_f$, such that

$$\sup_{\mathbb{Q} \in \mathcal{P}_f} \left(\mathbb{E}_{\mathbb{Q}} (-X) - \gamma . h (\mathbb{Q}/\mathbb{P}) \right) = \mathbb{E}_{\mathbb{Q}^X} (-X) - \gamma . h \left(\mathbb{Q}^X / \mathbb{P} \right)$$

ii) The density of $\mathbb{Q}^X$ is given by

$$\frac{d\mathbb{Q}^X}{d\mathbb{P}} = c. \exp \left(-\frac{1}{\gamma} \left(\int_0^T \left\langle \phi_s^X, dS_s \right\rangle + X \right) \right)$$

where c is a normalizing constant, $\int_0^T \left\langle \phi_s^X, dS_s \right\rangle \triangleq \xi_X \in \mathcal{V}_T$ and $\phi^X \in \Phi_M$.
iii) Moreover, the following duality result holds:

$$\gamma \ln \mathbb{E}_{\mathbb{P}} \left(\exp \left(-\frac{1}{\gamma} (\xi_X + X) \right) \right) = \mathbb{E}_{\mathbb{Q}^X} (-X) - \gamma . h \left(\mathbb{Q}^X / \mathbb{P} \right)$$

Note that assumption ($H4$) is not very strong when $\mathcal{V}_T$ is a cone.

Reinterpretation in terms of previous results This Proposition 5.3 can be reinterpreted in terms of the previous result obtained in Subsection 4.3, when considering an application of Theorem 4.4 to the question of optimal hedge. Thus, $\xi_X = \int_0^T \left\langle \phi_s^X, dS_s \right\rangle$ is the optimal hedge while $\mathbb{Q}^X$ is the optimal probability measure for (X, e_γ). Note that the properties of the optimal hedge, obtained in Subsection 4.3, are also found here: in particular, $\mathbb{E}_{\mathbb{Q}^X} \left(\int_0^T \left\langle \phi_s^X, dS_s \right\rangle \right) = 0$ since $\phi^X.S$ is a $\mathbb{Q}^X$-martingale.

Decomposition As already mentioned in Section 2, the global hedging problem to be solved is

$$\inf_{\xi_A + \xi_B \in \mathcal{V}_T^{(AB)}} e_{\gamma_C} (X + \xi_A + \xi_B) \triangleq \inf_{\xi \in \mathcal{V}_T^{(AB)}} e_{\gamma_C} (X + \xi) \qquad \mathcal{P}_{AB} \tag{31}$$

where $\mathcal{V}_T^{(AB)} \triangleq \mathcal{V}_T^{(A)} + \mathcal{V}_T^{(B)}$.
In this particular dynamic framework, it is not a restrictive assumption to consider that agent A has access to a particular set of financial assets $\mathcal{S}^A$ whereas agent B has access to a set $\mathcal{S}^B$. We only consider financial assets, at least one of the agents has access to. In others words, the set of basic financial assets is $\mathcal{S} = \mathcal{S}^A \cup \mathcal{S}^B$. These assets have a forward price process $S = \left(S^A, S^B \right)'$ with obvious notations. Note that if there are some common components, they are not repeated.

The program $(\mathcal{P}_{AB})$ is first solved under the assumptions of Proposition 5.3. In particular, assumption $(H4)$ holds for $\mathcal{V}_T = \mathcal{V}_T^{(AB)} = \mathcal{V}_T^{(A)} + \mathcal{V}_T^{(B)}$.

It simply remains to decompose the vector process $\phi^X \in \Phi_M$ into two components over the respective sets of assets $\mathcal{S}^A$ and $\mathcal{S}^B$ after having noticed that the set of admissible strategies associated with S^A (resp. S^B) is included in Φ_M.

Comment on the literature The Proposition 5.3, obtained by Becherer (2003), is very close to the results of Delbaen *et al.* (2002), Kabanov-Stricker (2002) and to the pioneer papers of Frittelli (2000a) and (2000b). Bellini-Frittelli (2002), Grandits-Rheinländer (2002) or Schachermayer (2000) are also papers of interest for this question in particular.
Another family of papers use quadratic BSDEs to solve this problem when asset prices are continuous semi-martingales. The first paper is due to El Karoui-Rouge (2000) when the strategies belong to a cone. More recently, Sekine (2004) uses first order condition to state the quadratic BSDE related to the problem when considering a convex subset of $\mathbb{R}^n$ for the space of strategies. Mania *et al. (2003)*, and very recently, Hu *et al.* (2003), solve the problem under BMO-assumptions.

6 Comments

The framework of convex risk measures enables to set additional constraints or opportunities to economic agents without changing the general framework's characteristics. In particular, a constraint imposed by another agent or the opportunity to invest on a financial market are technically equivalent as they simply lead to a transformation of the initial risk measure of the considered agent into another convex risk measure: both corresponds indeed to the solution of an inf-convolution problem. The penalty function of the generated risk measure is simply made of the sum of the penalty of the initial risk measure and the penalty associated with the constraint.

This ability to generate familiar risk measures is very interesting for the sake of economic interpretation. Modifications in the investment framework of an agent change her perception of risk and consequently generate a new risk measure. The fact that this risk measure still holds the key properties of monotonicity, convexity and translation invariance is consistent with the notion of risk measure itself.

In the optimal risk transfer problem we consider, the pricing rule of the structure is fully determined by the buyer as it binds her constraint at the optimum. This price may be related to an indifference price, usually obtained in the problems of replicating a terminal cash flow using a utility criterion (cf., for instance, the articles of Hodges and Neuberger (1989) or of El Karoui and Rouge (2000)). All the parameters of the framework of this study, especially the risk measures, are probably revealed during the trade talks preceding the transaction, where both agents will reveal some information concerning their anticipation (prior, exposure...) just as their attitudes towards risk. Note that the negotiation takes place at a double level: not only the price is at stake but also the structure (or equivalently, in some ways, the amount). This will lead to a higher probability to reach an agreement between both agents.

The optimal structure is explicitly derived when agents have dilated risk emasures, generalizing the results obtained in the entropic framework. This optimal structure is always equal to a certain proportion of the issuer's initial exposure, this proportionality factor being constant and corresponding to the relative risk tolerance coefficient of the buyer. When both agents differ in their access to other investment opportunities aither for hedging or diversification purposes, there is an additional term, taking into account these differences and leading

both agents to more comparable profiles after the transaction.
Therefore, when both agents have the same access to the financial market, the underlying logic of the transaction is non-speculative in the sense that the issuer has an interest to sell a structure if an only if she is initially exposed towards the non-tradable risk. When they have different access, however, there may be a transaction even if the issuer is not initially exposed but it will not involve any exchange of the non-tradable risk.

These results are especially interesting from a regulation point of view: standard diversification (i.e. simple quota sharing of the risk) will occur in exchange economies as soon as agents have dilated risk measures, or equivalently when they assess their respective risk exposure using the same family of risk measures and simply differ in their risk tolerance. The regulator has to impose very different rules on agents as to generate different risk measures and to increase diversification in the market.

In a general framework, when agents have risk measure of different type, an explicit derivation of the optimal structure is not possible any more. Some necessary and sufficient conditions to its existence are then obtained. The use of dynamic programing techniques, in particular Backward Stochastic Differential Equations (BSDEs) and non-linear Partial Differential Equations (PDEs), may help to study risk measures defined by their local specifications as in Barrieu-El Karoui (2004). This particular question is the area of further research.

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