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Vicent Caselles and Antonin Chambolle

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Abstract

We study in this paper the evolution by mean curvature, and in particular by anisotropic mean curvature, of convex sets in $\mathbb{R}^N$ (without driving forces). If the anisotropy is smooth, we show that the evolution remains convex. If the anisotropy is crystalline, we build a convex evolution which satisfies an equation which is a weak form of the crystalline curvature motion equation.

1 Introduction

In this paper, we study the evolution of a convex set whose boundary moves with a velocity equal to its anisotropic mean curvature. This so-called *curvature-driven* flow models the evolution of a crystal under the influence of its surface tension and possibly some external potential, that we will not consider in the present study. We refer to [35, 46] for the physical motivations of the problem. A particularly interesting case is the *crystalline case*, when the *interfacial energy* (or surface tension) $\varphi^\circ(\nu)$ (a convex, even, one–homogeneous function that depends on the normal ν to the surface of the set) is nonsmooth, that is, when the *Frank diagram* $\{\varphi^\circ \leq 1\}$ is a convex polytope.

From a mathematical point of view, the motion by anisotropic mean curvature of the boundary ∂E corresponds to the evolution of a bounded set $E \subset \mathbb{R}^N$ along the gradient flow of the surface energy functional $P_\varphi(E) = \int_{\partial E} \varphi^\circ(\nu^E)\, d\mathcal{H}^{N-1}$. The surface tension φ° corresponds to an anisotropic density with respect to the usual perimeter. The "$^\circ$" refers to the fact that it is the polar of an anisotropic norm φ in $\mathbb{R}^N$ (that defines an anisotropic metric), that is, $\varphi^\circ(\xi) = \sup_{\varphi(\eta)\leq 1} \eta \cdot \xi$ (and *vice-versa*). We will follow this point of view, which is described, in all generality, in [18] and is studied in many subsequent papers by Bellettini, Novaga and Paolini.

In [2], Almgren, Taylor and Wang have proposed a variational approach for constructing evolutions at all time. The idea can be described in the framework of Minimizing Movements

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of De Giorgi (see [4] for a nice description). It is based on a time discretization, and on a minimization problem for computing the surface at time $(k+1)h$ from the surface at time kh, $k \in \mathbb{N}$, $h > 0$. More precisely, if $T_h(E)$ is a solution of

$$\min_F P_\varphi(F) + \frac{1}{h} \int_{F \triangle E} d(x, \partial E)\, dx \tag{1}$$

where $d(x, \partial E)$ denotes the distance from x to ∂E, then the discrete evolution of the set E_0 is $T_h(t)^{[t/h]}(E_0)$. The distance d can be the Euclidean distance, or, as in [18], the distance induced by the norm φ. The idea of Almgren, Taylor and Wang (also proposed by Luckhaus and Sturzenhecker in the isotropic (Euclidean) case $\varphi = \varphi^\circ = |\cdot|$ [38]), is that the Euler equation of this minimization problem corresponds to an implicit time-discretization of the curvature-driven flow, with time-step h. They actually show that if $P_\varphi(E) < +\infty$, the iterates $T_h^{[t/h]}E$ ($[\cdot]$ denotes the integer part) will converge to some flow $E(t)$ (Hölder-continuous in time in the L^1 topology), that they call the *flat φ°-curvature flow*. When E and φ, φ° are smooth they can show that $E(t)$ coincides, for small t, with the classical definition of the flow.

In a recent paper [19], the second author has found that a way to build a solution T_hE to (1) is by letting $T_hE = [u < 0] := \{x \,:\, u(x) < 0\}$, where u is a minimizer of

$$\int_\Omega \varphi^\circ(Du) \, + \, \frac{1}{2h} \int_\Omega (u(x) - d_E(x))^2 \, dx \,, \tag{2}$$

Ω being an arbitrary open subset of $\mathbb{R}^N$ "large" enough with respect to E, and d_E, this time, being the *signed* distance function to ∂E, that is, $-d_{\partial E}$ inside E and $+d_{\partial E}$ outside. In this case, T_h is *monotone*, that is, $E \subset E' \Rightarrow T_hE \subset T_hE'$. This is very interesting, since, together with the above-mentionned consistance result of [2], it allows to show that $E(t)$ coincides with the generalized motion by mean curvature starting from E, in the sense of *barriers* or *viscosity solutions*, as long as this motion is unique [24, 20, 31].

In this paper, we essentially follow the same approach, except that we consider a u satisfying the Euler equation of (2) in the whole space $\mathbb{R}^N$, namely,

$$-h \operatorname{div} \partial\varphi^\circ(\nabla u) \; + \; u \; - \; d_E \; \ni \; 0 \; \text{ in } \mathbb{R}^N \,. \tag{3}$$

This is particularly interesting when the initial set E is convex: indeed, in this case, we are able to show that also u is convex, hence the evolved set T_hE. In the smooth case, it yields the convexity, up to extinction, of the set $E(t)$, in any dimension of space. In the isotropic case, this fact was proven first by Gage and Hamilton in 2D [26, 25], then by Huisken [36] in arbitrary dimension. The same result was then given by Evans and Spruck in the framework of vicosity solutions [24]. In the anisotropic case, Angenent and Gurtin [8, 7] have studied very general evolutions (including with nonconvex potentials) in dimension 2. They show, for evolutions defined by local equations (which excludes the completely singular crystalline case, but allows some particular smooth but nonconvex φ°s), local existence of a flow which preserves convexity. Our result, although valid only for convex interfacial energies, shows that convexity is actually preserved in any dimension.

In the crystalline case, a notion of (non local) evolution can be defined in dimension 2 (see [8, 45, 44, 32, 29, 27, 28]). The convexity of evolutions was studied by McCann, still in dimension 2: he shows that convexity is preserved if the initial set is balanced and convex, following an approach to which ours is related: indeed, he uses the static approximation (1) to deduce the convexity of the flat flow. This yields the result also for the classical flow, since it is shown in [1] that both notions are consistent, in the 2-dimensional crystalline case. Our approach shows the convexity of the flat flow in any dimension, without further hypothesis on the initial convex shape.

Moreover, our construction shows that the approximate flat flow $\partial T_h^{[t/h]}E$ converges in the Hausdorff distance to a convex evolution $\partial E(t)$ (up to some time $T > 0$ after which $E(t)$ vanishes or becomes $(N-1)$–dimensional) which satisfies, in all cases including crystalline, a weak form of the curvature flow equation: indeed, the distance function d_E is a weak subsolution inside E, and supersolution outside E, of the equation $\partial_t d - \operatorname{div} \partial\varphi^\circ(\nabla d) \ni 0$. In the framework of viscosity solutions (hence whenever φ° is smooth), it was observed by Soner that these conditions characterize the motion [43].

In the crystalline case in three or more dimensions, for a general initial set, the situation is very complicated, in particular because of the "facet-breaking" phenomena: facets of an evolving polyhedron may break or may bend during its evolution [17]. While the first phenomenon has been rigorously proved, the second is not completely proved but has been confirmed in numerical experiments [41]. These phenomena are related to the difficulties to prove the existence of φ-regular motions in the sense of [16]. In dimension 3, under the facet-stays-as-facet assumption, local existence in time results for crystalline evolution have been proved for crystalline surfaces having a triple junction at each corner [33, 30]. As it has been suggested in [33], when more than three faces meet at a corner a new facet probably emerges and, as far as we know, no existence result has been proved in this case. We conjecture that the flat-flow that we build is in fact a φ-regular crystalline curvature flow: this is the subject of future studies [12]. Let us mention that in this case, uniqueness holds according to the result in [16].

Eventually, we would like to make a comment on the choice of the distance function in (1) (and (2)). In our paper we have adopted the convention in [18] and defined the distances in $\mathbb{R}^N$ by means of the function φ, polar of φ° (that is, $d(x, \partial E) = \min_{y\in\partial E} \varphi(x-y)$, etc.) This is particularly interesting since in this case, the solution of (2) for E the *Wulff shape* $\{x : \varphi(x) \leq 1\}$ is explicit (see Appendix B). However, this is not very general. As mentionned in [39], the choice of this distance governs the choice of the *mobility* in the curvature equation: our choice corresponds to a normal velocity $\varphi^\circ(\nu)\times$(anisotropic φ-curvature) (hence mobility $\varphi^\circ(\nu)$), while Almgren, Taylor and Wang's choice of the Euclidean distance corresponds to a normal velocity equal to the anisotropic mean curvature (mobility 1). In fact, any other convex, one homogeneous function ψ° (polar of a norm ψ in $\mathbb{R}^N$) could be considered for the mobility. All of our results still hold in this case, except those related to the explicit evolution of the Wulff shape. Concerning the latter, estimates can still be shown, of the minimal and maximal speed at which it decreases (see Appendix D).

Let us explain the plan of the paper. In Section 2 we collect some preliminary results on functions with bounded variation, anisotropic total variation and its subdifferential with Dirichlet boundary conditions. General facts about the characterization of the solutions of (3) are stated in Section 3. Then, in Section 4 we prove that (3) has a convex solution as soon as d_E is replaced with any convex and Lipschitz function. This includes in particular the case of the distance function to a convex set. In Section 5 we relate problem (3) with the variational formulation (1). We can deduce the uniqueness of the solution of (1) when E is convex (for h small enough). Section 6, together with Appendix B, is devoted to the explicit computation of the evolution of the Wulff shape. Section 7 contains our main result, namely, the proof of the convergence of Almgren, Taylor and Wang's algorithm to a convex evolution, when the initial set is convex. For the reader's convenience we include in Appendix A some basic results about the Hausdorff distance and the convergence of sets. In Appendix C, we prove a general uniqueness and comparison principle for the solutions of the PDE (13), which is a general form of (3) with a L^1_{loc} data. This gives an alternative approach to the uniqueness result of Section 5, that may prove useful in future studies.

2 Preliminaries

2.1 BV functions and sets of finite perimeter

Let Ω be an open subset of $\mathbb{R}^N$. A function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a (vector valued) Radon measure with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. The total variation of Du on Ω turns out to be

$$\sup\left\{\int_\Omega u \,\mathrm{div}\, z \; dx : z \in C_0^\infty(\Omega;\mathbb{R}^N), \|z\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |z(x)| \le 1\right\}, \tag{4}$$

(where for a vector $v = (v_1,\dots,v_N) \in \mathbb{R}^N$ we set $|v|^2 := \sum_{i=1}^N v_i^2$) and will be denoted by $|Du|(\Omega)$ or by $\int_\Omega |Du|$. It turns out that the map $u \to |Du|(\Omega)$ is $L^1_{\mathrm{loc}}(\Omega)$-lower semicontinuous. $BV(\Omega)$ is a Banach space when endowed with the norm $\int_\Omega |u|\; dx + |Du|(\Omega)$. We recall that $BV(\mathbb{R}^N) \subseteq L^{N/(N-1)}(\mathbb{R}^N)$. The total variation of u on a Borel set $B \subseteq \Omega$ is defined as $\inf\{|Du|(A) : A \text{ open }, B \subseteq A \subseteq Q\}$. We denote by $BV_{\rm loc}(\Omega)$ the space of functions $w \in L^1_{\rm loc}(\Omega)$ such that $w\varphi \in BV(\Omega)$ for all $\varphi \in C_0^\infty(\Omega)$. For results and informations on functions of bounded variation we refer to [5, 23].

A measurable set $E \subseteq \mathbb{R}^N$ is said to be of finite perimeter in Ω if (4) is finite when u is substituted with the characteristic function χ_E of E. The perimeter of E in Ω is defined as $P(E,\Omega) := |D\chi_E|(\Omega)$, and $P(E,\Omega) = P(\mathbb{R}^N \setminus E, \Omega)$. We shall use the notation $P(E) := P(E,\mathbb{R}^N)$. For sets of finite perimeter E one can define the essential boundary $\partial^* E$, which is countably $(N-1)$ rectifiable with finite $\mathcal{H}^{N-1}$ measure, and compute the outer unit normal $\nu^E(x)$ at $\mathcal{H}^{N-1}$ almost all points x of $\partial^* E$, where $\mathcal{H}^{N-1}$ is the $(N-1)$ dimensional Hausdorff measure. Moreover, $|D\chi_E|$ coincides with the restriction of $\mathcal{H}^{N-1}$ to $\partial^* E$.

Given $u \in BV(\Omega)$, we denote by $\nabla u(x)\,dx$ is the absolutely continuous part of the derivative Du, $D^s u$ will denote its singular part (with respect to the Lebesgue measure dx).

If μ is a (possibly vector valued) Radon measure and f is a Borel function, the integration of f with respect to μ will be denoted by $\int f d\mu$. When μ is the Lebesgue measure, the symbol dx will be often omitted.

2.2 Anisotropic Total Variation

Let us consider a convex function $F : \mathbb{R}^N \to \mathbb{R}$ satisfying the conditions

$$F(t\xi) = |t| F(\xi) \quad \forall \xi \in \mathbb{R}^N, \forall t \in \mathbb{R}, \tag{5}$$

$$m|\xi| \le F(\xi) \le M|\xi| \quad \forall \xi \in \mathbb{R}^N, \tag{6}$$

for some positive constants m, M. The polar function F° is defined by

$$F^\circ(\xi) := \sup\{\langle \xi, x\rangle : F(x) \le 1\}.$$

The polar function satisfies also assumptions (5), (6).

Let Ω be an open subset of $\mathbb{R}^N$ with Lipschitz boundary. Let $u \in BV(\Omega)$. We define the anisotropic total variation of u with respect to F in Ω ([3]) as

$$\int_\Omega F(Du) = \sup\Big\{\int_\Omega u \,\mathrm{div}\, \sigma : \sigma \in \mathcal{C}_F\Big\} \tag{7}$$

where

$$\mathcal{C}_F := \{\sigma \in C_0^1(\Omega;\mathbb{R}^N) : F^\circ(\sigma(x)) \le 1, \forall x \in \Omega\}.$$

If $E \subseteq \mathbb{R}^N$ has finite perimeter in Ω, we set

$$P_{F^\circ}(E,\Omega) := \int_\Omega F(D\chi_E)$$

and we have ([3])

$$P_{F^\circ}(E,\Omega) = \int_{\partial^* E} F(\nu^E(x))\, d\mathcal{H}^{N-1}.$$

Recall that, since F is homogeneous, $F(Du)$ coincides with the nonnegative Radon measure in $\mathbb{R}^N$ given by $F(Du) = F(\nabla u(x))\, dx + F\left(\frac{D^s u}{|D^s u|}\right) |D^s u|$.

2.3 A generalized Green's formula

Let Ω be an open subset of $\mathbb{R}^N$. Following [9], let

$$X_2(\Omega) := \{z \in L^\infty(\Omega;\mathbb{R}^N) : \text{div } z \in L^2(\Omega)\}.$$

If $z \in X_2(\Omega)$ and $w \in L^2(\Omega) \cap BV(\Omega)$ we define the functional $(z, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$< (z, Dw), \varphi >:= - \int_\Omega w\, \varphi \,\text{div }\ z\, dx - \int_\Omega w\, z \cdot \nabla\varphi\, dx.$$

Then (z, Dw) is a Radon measure in Ω,

$$\int_\Omega (z, Dw) = \int_\Omega z \cdot \nabla w\, dx \qquad \forall w \in L^2(\Omega) \cap W^{1,1}(\Omega),$$

and

$$\left| \int_B (z, Dw) \right| \le \int_B |(z, Dw)| \le \|z\|_\infty \int_B |Dw| \qquad \forall B \text{ Borel set } \subseteq \Omega.$$

If no confusion arises we shall write $z \cdot Dw$ instead of (z, Dw). We denote by $\theta(z, Dw) \in L^\infty_{|Dw|}(\Omega)$ the density of (z, Dw) with respect to $|Dw|$, that is

$$(z, Dw)(B) = \int_B \theta(z, Dw)\ d|Dw| \qquad \forall \text{ Borel set } B \subseteq \Omega. \tag{8}$$

We recall that [9] (see also [6], Corollary C.16) if $z \in X_2(\Omega)$, $w \in L^2(\Omega) \cap BV(\Omega)$, and $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous increasing function then

$$\theta(z, D(p \circ w), x) = \theta(z, w, x) \quad |Du|\text{-a.e..} \tag{9}$$

We recall the following result proved in [9].

Theorem 1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Let $u \in BV(\Omega) \cap L^2(\Omega)$ and $z \in X_2(\Omega)$. Then there exists a function $[z \cdot \nu^\Omega] \in L^\infty(\partial\Omega)$ such that $\|[z \cdot \nu^\Omega]\|_{L^\infty(\partial\Omega)} \le \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}$, and*

$$\int_\Omega u \ \text{div}\, z\ dx + \int_\Omega (z, Du) = \int_{\partial\Omega} [z \cdot \nu^\Omega] u\ d\mathcal{H}^{N-1}.$$

When $\Omega = \mathbb{R}^N$ we have the following integration by parts formula [9], for $z \in X_2(\mathbb{R}^N)$ and $w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} w \text{ div } z \, dx + \int_{\mathbb{R}^N} (z, Dw) = 0. \tag{10}$$

In particular, if $z \in X_2(\mathbb{R}^N)$ and Q is bounded and has finite perimeter in $\mathbb{R}^N$, from (10) and (8) it follows

$$\int_Q \text{div } z \, dx = \int_{\mathbb{R}^N} (z, -D\chi_Q) = \int_{\partial^* Q} \theta(z, -D\chi_Q) \, d\mathcal{H}^{N-1}. \tag{11}$$

If additionally, Q is a bounded open set with Lipschitz boundary, then $\theta(z, -D\chi_Q)$ coincides with $[z \cdot \nu^Q]$.

2.4 The subdifferential of the anisotropic TV

Let us consider a convex function $F : \mathbb{R}^N \to \mathbb{R}$ satisfying (5) and (6). As usual, we shall denote by $\partial F(\xi)$ the subdifferential of F at $\xi \in \mathbb{R}^N$ (defined by $\eta \in \partial F(\xi)$ iff $F(\xi') \geq F(\xi) + \eta \cdot (\xi' - \xi)$ for any ξ'). Since F is homogeneous of degree 1, for any $\eta \in \partial F(\xi)$ we have

$$F(\xi) = \eta \cdot \xi.$$

We also observe that for any $\xi \in \mathbb{R}^N$, $\partial F(\xi) \subseteq \partial F(0) \subseteq B(0, M)$. We have $F(\xi) \geq \eta \cdot \xi$ for any $\eta \in \partial F(0)$, in fact, $F(\xi) = \max_{\eta \in \partial F(0)} \eta \cdot \xi$ for any $\xi \in \mathbb{R}^N$.

Let Ω be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, and $h \in L^1(\partial\Omega)$. Let $\Psi_{F,h} : L^2(\Omega) \to (-\infty, +\infty]$ be the functional defined by

$$\Psi_{F,h}(u) := \begin{cases} \displaystyle\int_\Omega F(Du) + \int_{\partial\Omega} F(\nu^\Omega)|u - h| & \text{if} \quad u \in L^2(\Omega) \cap BV(\Omega) \\ \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \tag{12}$$

The functional $\Psi_{F,h}$ is convex and lower semicontinuous in $L^2(\Omega)$, hence $\partial\Psi_{F,h}$ is a maximal monotone operator in $L^2(\Omega)$.

Let us recall the following result proved in [40].

Proposition 2.1 *Let $u, v \in L^2(\Omega)$. The following conditions are equivalent*

(i) $v \in \partial\Psi_{F,h}(u)$

(ii) $u \in BV(\Omega)$, *and there exists a vector field* $z \in X_2(\Omega)$, $z(x) \in \partial F(\nabla u(x))$ *a.e., satisfying*

$$z \cdot Du = F(Du) \quad \textit{as measures in } \Omega$$

$$v = -\text{div } z \quad \textit{in } \mathcal{D}'(\Omega),$$

and the Dirichlet boundary condition holds in a relaxed way, i.e.,

$$[z(x), \nu^\Omega] \in \text{sign}(h - u|_{\partial\Omega}) F(\nu^\Omega(x)) \quad \textit{a.e. in } \partial\Omega.$$

Here "sign" is the subdifferential of $|\cdot|$ (that is, $x \mapsto -1$ if $x < 0$, $[-1, 1]$ if $x = 0$, 1 if $x > 0$).

3 Some preliminary results

Let us consider the following partial differential equation

$$u - \text{div } \partial F(\nabla u) \ni g \quad \text{in } \mathbb{R}^N. \tag{13}$$

Definition 1 *We say that $u \in BV_{\text{loc}}(\mathbb{R}^N) \cap L^2_{\text{loc}}(\mathbb{R}^N)$ is a minimizing solution to (13) if for any $R > 0$ and any $v \in C^1_c(B_R)$ (with $B_R = B(0, R)$),*

$$\int_{B_R} F(Du) \; + \; \int_{B_R} \frac{(u(x) - g(x))^2}{2} \, dx$$
$$\leq \int_{B_R} F(D(u + v)) \; + \; \int_{B_R} \frac{(u(x) + v(x) - g(x))^2}{2} \, dx. \tag{14}$$

Definition 2 *Let $g \in L^2_{\text{loc}}(\mathbb{R}^N)$. We say that $u \in BV_{\text{loc}}(\mathbb{R}^N) \cap L^2_{\text{loc}}(\mathbb{R}^N)$ is a solution of (13) if there exists a vector field $z \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ with $z(x) \in \partial F(\nabla u(x))$ almost everywhere,* $\text{div}\, z \in L^2_{\text{loc}}(\mathbb{R}^N)$, *such that*

(a) $u - \text{div } z = g$ *in* $\mathcal{D}'(\mathbb{R}^N)$.

(b) $z \cdot Du = F(Du)$ *locally as measures in* $\mathbb{R}^N$.

Proposition 3.1 *Assume that $u, g \in L^2_{\text{loc}}(\mathbb{R}^N)$. The following assertions are equivalent*

(i) u is a solution of (13).

(ii) u is a minimizing solution of (13).

(iii) for all $R > 0$, u is the solution of

$$\text{Min} \int_{B_R} F(Dw) + \frac{1}{2} \int_{B_R} |w - g|^2 \tag{15}$$

on the class of functions $w \in L^2(B_R) \cap BV(B_R)$ such that $w|_{\partial B_R} = u|_{\partial B_R}$.

(iv) for all $R > 0$, u is the solution of

$$\text{Min} \int_{B_R} F(Dw) + \frac{1}{2} \int_{B_R} |w - g|^2 + \int_{\partial B_R} F(\nu^{B_R}) \, |w - u|_{\partial B_R}| \tag{16}$$

on the class of functions $w \in L^2(B_R) \cap BV(B_R)$.

By Proposition 2.1, the solution w of (16) is characterized by the equation

$$\begin{cases} w - \operatorname{div} \partial F(Dw) \ni g & \text{in } B_R \\ w = u|_{\partial B_R} & \text{on } \partial B_R \end{cases} \tag{17}$$

i.e., there exists a vector field $z \in X_2(B_R)$, $z(x) \in \partial F(\nabla w(x))$ a.e., satisfying $z \cdot Dw = F(Dw)$ as measures in B_R, $w - \operatorname{div} z = g$ in $\mathcal{D}'(\Omega)$, and the Dirichlet boundary condition in a relaxed way

$$[z(x), \nu^{B_R}] \in \operatorname{sign}(u - w|_{\partial B_R}) F(\nu^{B_R}(x)) \quad \text{a.e. in } \partial B_R.$$

Proof. Since any function $v \in L^2(B_R) \cap BV(B_R)$ with $v|_{\partial B_R} = 0$ can be approximated in $L^2(B_R)$ by a sequence of functions $v_n \in C_c^1(B_R)$ satisfying $\int_{B_R} F(Du+Dv_n) \to \int_{B_R} F(Du+Dv)$ it follows that *(ii)* $\to$ *(iii)*. The implications *(iii)* $\to$ *(iv)* and *(iv)* $\to$ *(ii)* are immediate.

Assume that u solves (13), and let us prove that $u|_{B_R}$ is a minimizing solution, hence *(ii)*. Let $v \in C_c^1(B_R)$. Multiplying (13) by v and integrating by parts, and using (b) of definition 2, we obtain

$$\begin{aligned} \int_{B_R} (u-g) v \, dx &= -\int_{B_R} z \cdot Dv = -\int_{B_R} z \cdot D(u+v) - \int_{B_R} F(Du) \\ &\leq \int_{B_R} F(D(u+v)) - \int_{B_R} F(Du) \end{aligned}$$

and this implies (14).

Now, assume that u satisfies *(iv)*. Then for any $R > 0$ there exists $z_R \in L^\infty(B_R; \mathbb{R}^N)$ such that $z_R \in \partial F(\nabla u(x))$ on B_R,

$$u - \operatorname{div} z_R = g \quad \text{in } \mathcal{D}'(B_R) \tag{18}$$

$$z_R \cdot Du = F(Du) \quad \text{as measures in } B_R$$

$$[z_R, \nu^{B_R}] \in \operatorname{sign}(u - u|_{\partial B_R}) F(\nu^{B_R}(x)) \quad \text{a.e. in } \partial B_R.$$

Take $\psi \in C_0^\infty(\mathbb{R}^N)$, multiply (18) by $u\psi$ and, after integrating by parts, we obtain

$$\int u^2 \psi + \int F(Du)\psi + \int z_R \cdot \nabla \psi u = \int g u \psi.$$

Letting $R \to \infty$, and assuming, by extracting a subsequence, if necessary, that $z_R \to z$ weakly* in $L^\infty(\mathbb{R}^N, \mathbb{R}^N)$, we obtain

$$\int u^2 \psi + \int F(Du)\psi + \int z \cdot \nabla \psi u = \int g u \psi. \tag{19}$$

Similarly, multiplying (18) by ψ, integrating by parts, and letting $R \to \infty$, we obtain

$$\int u \psi + \int z \cdot \nabla \psi = \int g \psi, \tag{20}$$

hence

$$u - \mathrm{div}\ z = g \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Now, multiplying this equation by $u\psi$ and integrating by parts, after comparing with (19) we obtain that $z \cdot Du = F(Du)$ locally as measures in $\mathbb{R}^N$. In particular, we have $z(x) \cdot \nabla u(x) = F(\nabla u(x))$ a.e. in $\mathbb{R}^N$, hence $z(x) \in \partial F(\nabla u(x))$ a.e.. Thus u is a solution of (13). $\square$

Remark 3.2 If u is a solution of 13, then we have that $z \cdot Dp(u) = F(Dp(u))$ locally as measures in $\mathbb{R}^N$ for any function $p \in \mathcal{P}$ where

$$\mathcal{P} := \{p \in W^{1,\infty}(\mathbb{R}) : \, p' \geq 0, \ \mathrm{supp}(p') \text{ compact}\}.$$

First, we observe that since $z \cdot Du = F(Du)$ locally as measures in $\mathbb{R}^N$ we have that

$$\theta(z, Du)|Du| = F(Du)$$

locally as measures in $\mathbb{R}^N$. Second, we have

$$F(Dp(u)) = F(\nabla p(u)) + F\Big(\frac{D^s p(u)}{|D^s p(u)|}\Big)|D^s p(u)| = F(\nabla p(u)) + F\Big(\frac{D^s u}{|D^s u|}\Big)|D^s p(u)|$$

Finally, recall the chain rule for functions in BV,

$$Dp(u) = \overline{p}_u Du$$

where $\overline{p}_u$ denotes Vol'pert's average superposition, $\overline{p}_u = p'(u)$ a.e. [5]. Now, for any $\psi \in C_0^\infty(\mathbb{R}^N)$, using (9), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} z \cdot Dp(u)\psi &= \int_{\mathbb{R}^N} \theta(z, Dp(u))\psi \, |Dp(u)| = \int_{\mathbb{R}^N} \theta(z, Du)\psi \, |Dp(u)| \\
&= \int_{\mathbb{R}^N} \theta(z, Du)\overline{p}_u |Du|\psi = \int_{\mathbb{R}^N} \theta(z, Du)[p'(u)\nabla u + \overline{p}_u |D^s u|]\psi \\
&= \int_{\mathbb{R}^N} [F(\nabla u)p'(u) + F\Big(\frac{D^s u}{|D^s u|}\Big)|D^s u|\overline{p}_u]\psi \\
&= \int_{\mathbb{R}^N} F(Dp(u))\psi.
\end{aligned}$$

Hence $z \cdot Dp(u) = F(Dp(u))$ locally as measures in $\mathbb{R}^N$.

Even if it is not necessary to prove our main result, let us state the following comparison principle whose proof will be included in Appendix C.

Theorem 2 *Let $u, \overline{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ be two solutions of (13) corresponding to the right hand sides $g, \overline{g} \in L^\alpha_{\mathrm{loc}}(\mathbb{R}^N)$, respectively. Assume that $\alpha > \max(N, 2)$. If $g \leq \overline{g}$, then $u \leq \overline{u}$.*

4 Convexity properties

Let $F : \mathbb{R}^N \to [0, +\infty)$ be a convex real valued function, satisfying $F(p) \geq c|p| - c'$ for some positive c, c' and $F(0) = 0$. We consider the problem

$$u - \operatorname{div} \partial F(\nabla u) \ \ni \ g \tag{21}$$

where $g : \mathbb{R}^N \to \mathbb{R}$ is a convex function with $\limsup_{|x|\to\infty} g(x)/|x| = L < +\infty$ (in particular g is L–Lipschitz). We prove the following result.

Theorem 3 *There exists u a convex, L–Lipschitz minimizing solution (*cf.* Def. 1) to (21), with $u \geq g$.*

Proof. The proof follows a similar proof by N. Korevaar, on bounded domains [37]. For all $\varepsilon > 0$ we let $F^\varepsilon(p) = \rho_\varepsilon * F(p) + \varepsilon|p|^2/2$, where ρ is a smoothing kernel, and $a^\varepsilon_{i,j}(p) = \partial^2_{i,j} F^\varepsilon(p)$.

We let, then, u^ε be the unique viscosity solution of

$$-a^\varepsilon_{i,j}(\nabla u)\partial^2_{i,j} u^\varepsilon \ + \ u^\varepsilon \ = \ g \ \text{ in } \mathbb{R}^N. \tag{22}$$

For the notion of viscosity solution and the results needed in this section we refer to [21], and in particular to Section 5.D. and Thm. 5.1. In particular, this theorem yields existence and uniqueness of u^ε in the class of the uniformly continuous functions that grow at most linearly at infinity. We observe that the function u^ε is of class $C^{2,\alpha}$ for some $\alpha < 1$ [34, Thm 15.18–15.19].

In the first two steps of this proof, we will show that u^ε is convex and L–Lipschitz for any $\varepsilon > 0$ (the fact it is L–Lipschitz is also easily deduced from the comparison principle for vicosity solutions, see [21, Sec. 5.D.]). In a third step, we will deduce that the limit of the u^ε as $\varepsilon \to 0$ is a convex solution to the original problem. For simplicity, in Steps 1 and 2, we drop the superscripts ε.

Step 1 Let us first assume that there exists $R > 0$ and $\lambda > 0$ such that if $|x| > R$, $g(x) = \lambda|x|$. Consider the $C^{1,1}$ function

$$u_c(x) \ = \begin{cases} \lambda\left(|x| + \dfrac{c}{|x|}\right) & \text{if } |x| > \sqrt{c}, \\ 2\lambda\sqrt{c} & \text{if } |x| \leq \sqrt{c}. \end{cases}$$

We show that, for c large enough, u_c is a supersolution of the equation, that is

$$-a_{i,j}(\nabla u_c)\partial^2_{i,j} u_c \ + \ u_c \ - \ g \ \geq \ 0 \ \text{ in } \mathbb{R}^N.$$

We must find c such that it is a classical supersolution (hence a viscosity supersolution) for $\{|x| \neq \sqrt{c}\}$, and a viscosity supersolution on $\{|x| = \sqrt{c}\}$.

First we assume $c > R^2$. If $|x| = \sqrt{c}$, $\nabla u_c = 0$. If X is a symmetric $N \times N$ matrix such that $u_c(y) \geq u_c(x) + (1/2)(X(y-x)) \cdot (y-x) + o(|y-x|)^2$ for all y near x, one must have $X \leq 0$. Hence u_c is a viscosity supersolution as soon as $-a_{i,j}(0)X_{i,j} + 2\lambda\sqrt{c} - g(x) \geq 0$ for all nonpositive matrix X. This is true since $2\lambda\sqrt{c} - g(x) = 2\lambda\sqrt{c} - \lambda|x| = \lambda\sqrt{c} \geq 0$. If $|x| < \sqrt{c}$, by convexity of g we still have $g \leq 2\lambda\sqrt{c}$, hence u_c is a classical supersolution.

If $|x| > \sqrt{c}$, then

$$\nabla u_c \;=\; \lambda\left(1 - \frac{c}{|x|^2}\right)\frac{x}{|x|}$$

while

$$D^2 u_c \;=\; \frac{\lambda}{|x|}\left[\left(1 - \frac{c}{|x|^2}\right)\left(I - \frac{x \otimes x}{|x|^2}\right) + \frac{2c}{|x|^2}\frac{x \otimes x}{|x|^2}\right].$$

Hence, $|\nabla u_c| \leq \lambda$, and $|D^2 u_c| \leq \lambda\sqrt{N+3}/|x|$ in the Euclidean (or Frobenius) norm. Letting $M = \max_{|p|\leq\lambda} |(a_{i,j}(p))_{i,j=1}^N|$ (Euclidean norm), we find that on $\{|x| > \sqrt{c}\}$,

$$-a_{i,j}(\nabla u_c)\partial^2_{i,j} u_c + u_c - g \;\geq\; -\frac{\lambda M\sqrt{N+3}}{|x|} + \frac{\lambda c}{|x|} \;\geq\; 0$$

as soon as $c > M\sqrt{N+3}$. Hence u_c is a supersolution, provided c is chosen large enough.

We deduce (from the comparison principle in [21, Sec. 5.D.]) that $u \leq u_c$.

On the other hand, g, being convex, is trivially a subsolution (that is, $-a_{i,j}(p)X_{i,j} + g(x) - g(x) \leq 0$ for all x, p and X such that $g(y) \leq g(x) + p\cdot(y-x) + (1/2)(X(y-x))\cdot(y-x) + o(|y-x|^2)$, which yields in particular $X \geq 0$). Hence $g \leq u$.

We find that for $|x|$ large enough,

$$0 \;\leq\; u(x) - g(x) \;=\; u(x) - \lambda|x| \;\leq\; \frac{\lambda c}{|x|} \tag{23}$$

Hence $u(x) - g(x) = u(x) - \lambda|x| \to 0$ uniformly as $|x| \to \infty$.

Now let us introduce, as in [37], the quantity

$$C(x,y,t) \;=\; u(tx + (1-t)y) - tu(x) - (1-t)u(y),$$

with $x, y \in \mathbb{R}^N$ and $t \in [0,1]$. In order to show that u is convex, we need to show that $C \leq 0$ everywhere. Let $(x_n, y_n, t_n)_{n\geq 1}$ be a maximizing sequence for C and assume $m = \sup C > 0$. We will let also $z_n = t_n x_n + (1-t_n)y_n$. We consider the two following cases:

1) Let us first assume that x_n and y_n are bounded: we may assume there exists a maximum (x,y,t) where $C(x,y,t) = m$, and we let $z = tx + (1-t)y$. Notice that we must have $0 < t < 1$ (since $m > 0$). The proof is exactly as in [37] in this case. Considering that the gradient of C with respect to x and y must vanish we find that $\nabla u(x) = \nabla u(y) = \nabla u(z) = p$. Then, computing the second variation of $C(x+\tau, y+\tau, t)$ along τ, we find

$$D^2u(z) \;-\; tD^2u(x) \;-\; (1-t)D^2u(y) \;\leq\; 0, \text{ so that}$$

$$a_{i,j}(p)\partial^2_{i,j}u(z) \;-\; ta_{i,j}(p)\partial^2_{i,j}u(x) \;-\; (1-t)a_{i,j}(p)\partial^2_{i,j}u(y) \;\le\; 0.$$

This yields

$$m \;=\; u(z) \;-\; tu(x) \;-\; (1-t)u(y) \;\le\; g(z) \;-\; tg(x) \;-\; (1-t)g(y) \;\le\; 0,$$

a contradiction.

2) Assume now that either x_n or y_n is unbounded. Up to a subsequence, we can assume that $|x_n| \to \infty$ as $n \to \infty$ (the case where $|y_n| \to \infty$ is symmetrical). If z_n is also unbounded, we write that $C(x_n, y_n, t_n) \le g(z_n) + \lambda c/|z_n| - tg(x_n) - (1-t)g(y_n) \le \lambda c/|z_n|$ for $|z_n|$ large enough, and in the limit we get $m \le 0$, a contradiction. Hence z_n must be bounded. We can assume that $y_n \to y$, $z_n \to z$, and it must be that $t_n \to 0$.

For any $q \in \mathbb{R}^N$ we write

$$\begin{aligned} m - C(x_n, y_n, t_n) \;&\ge\; C(x_n, y_n + q, t_n) - C(x_n, y_n, t_n) \\ &=\; u(z_n + (1-t_n)q) - (1-t_n)u(y_n + q) - (u(z_n) - (1-t_n)u(y_n)). \end{aligned}$$

Sending $n \to \infty$, we find that

$$u(z+q) - u(y+q) \;\le\; u(z) - u(y),$$

so that $\nabla u(z) = \nabla u(y)$ and $D^2u(z) \le D^2u(y)$. We deduce that

$$a_{i,j}(\nabla u(z))\partial^2_{i,j}u(z) \;\le\; a_{i,j}(\nabla u(y))\partial^2_{i,j}u(y),$$

hence $(u(z) - g(z)) - (u(y) - g(y)) \le 0$. But

$$\begin{aligned} C(x_n, y_n, t_n) \;&=\; u(z_n) - t_n u(x_n) - (1-t_n)u(y_n) \\ &\le\; u(z_n) - t_n g(x_n) - (1-t_n)u(y_n) \\ &\le\; u(z_n) - g(z_n) - (1-t_n)(u(y_n) - g(y_n)), \end{aligned}$$

and sending $n \to \infty$ we find $m \le 0$, a contradiction. Hence u must be convex in $\mathbb{R}^N$.

Notice that because of (23), $\limsup_{|x|\to\infty} u(x)/|x| = \lambda$. We deduce that u is λ–Lipschitz continuous.

Step 2 In the general case, we only know that $\limsup_{|x|\to\infty} g(x)/|x| = L < +\infty$. If $\delta > 0$, there exists $R > 0$ such that $g(x) \le (L+\delta)|x|$ for $|x| \ge R$. Let $\lambda = L + 2\delta$. For each $n \ge 0$ we let $g_n(x) = g(x) \vee (\lambda|x| - n) = \max\{g(x), (\lambda|x| - n)\}$. If $|x| \ge R$,

$$g(x) \;\le\; (L+\delta)|x| \;=\; \lambda|x| \;-\; n \;+\; n \;-\; \delta|x|$$

hence as soon as $|x| \ge (n/\delta)$, $g_n(x) = \lambda|x| - n$. Therefore by Step 1, the solution u_n of (22) with the function g_n is convex, and Lipschitz continuous with constant λ. Moreover, since for all $n \geq 0$, $g \leq g_{n+1} \leq g_n$, the comparison principle for viscosity solutions yields $u \leq u_{n+1} \leq u_n$, where u is the unique viscosity solution of (22) in the class of uniformly continuous functions with growth at most linear at infinity.

As $n \to \infty$, $g_n \to g$ (locally uniformly), and we deduce that u_n converges locally uniformly to u, which is therefore convex. We find out that this function is $(L+2\delta)$–Lipschitz for any $\delta > 0$, hence it is L–Lipschitz. This proves the Theorem in the smooth case.

Step 3 We have shown that u^ε, solution of the regularized problem, is L–Lipschitz and convex. To deduce that (up to a subsequence, using Ascoli-Arzelà's theorem) it converges locally uniformly to some convex, L–Lipschitz function u, we need some uniform bound for $u^\varepsilon(x)$, at some $x \in \mathbb{R}^N$.

To find this bound, we build a supersolution for (22), that can easily be bounded uniformly in ε. We observe that ∇F^ε is globally invertible in $\mathbb{R}^N$, with continuous (and even C^∞) inverse, since F^ε is C^∞ and $D^2 F^\varepsilon \geq \varepsilon I$. Moreover, if we introduce $(F^\varepsilon)^*$, the Legendre-Fenchel transform of F^ε, by the Legendre-Fenchel identity we have $(\nabla F^\varepsilon)^{-1} = \nabla (F^\varepsilon)^*$, hence $(F^\varepsilon)^*$ is C^∞, and $\nabla F^\varepsilon(\nabla (F^\varepsilon)^*(x)) = x$ for all $x \in \mathbb{R}^N$. In particular, $a_{i,j}(\nabla(F^\varepsilon)^*(x))\partial^2_{i,j}(F^\varepsilon)^*(x) = \operatorname{div} \nabla F^\varepsilon(\nabla(F^\varepsilon)^*(x)) = N$. Let us define the function $w^\varepsilon_c = (F^\varepsilon)^* + N + c$, $c \in \mathbb{R}$: it will therefore be a supersolution of (22) if and only if $(F^\varepsilon)^* + c \geq g$ in $\mathbb{R}^N$.

First, we find a c such that this inequality holds for all $\varepsilon \in (0,1)$. Observe that if $\varepsilon < 1$, $F^\varepsilon(p)$ is below the convex function $G(p) = \max_{|q|\leq 1} F(p-q) + |p|^2/2$ (assuming the mollifier ρ_ε has support in $B(0,\varepsilon)$). Hence $(F^\varepsilon)^* \geq G^*$. Since G is finite everywhere in $\mathbb{R}^N$, G^* must have superlinear growth, so that there exists $c \in \mathbb{R}$ with $G^* + c \geq g$ in $\mathbb{R}^N$. For such a choice of c, w^ε_c is a supersolution of (22) for all $\varepsilon < 1$, and $u^\varepsilon \leq w^\varepsilon_c$.

On the other hand, if $\varepsilon \in (0,1)$, $F^\varepsilon(p) \geq H(p) = \min_{|q|\leq 1} F(p-q)$. Hence $(F^\varepsilon)^* \leq H^*$, and $u^\varepsilon \leq w^\varepsilon_c \leq H^* + c$, and if x is in the domain of H^* (the set where H^* is finite), then it provides a uniform upper bound for $u^\varepsilon(x)$. A uniform lower bound is provided by the function g itself.

We deduce that u^ε, up to a subsequence, converges locally uniformly to a function u that is convex and L–Lipschitz. Now, for any $R > 0$ and $v \in C^1_c(B_R)$, it is easy to see that

$$\int_{B_R} F^\varepsilon(\nabla u^\varepsilon) + \frac{(u^\varepsilon - g)^2}{2}\, dx \leq \int_{B_R} F^\varepsilon(\nabla(u^\varepsilon + v)) + \frac{(u^\varepsilon + v - g)^2}{2}\, dx.$$

If we can show that $\nabla u^\varepsilon \to \nabla u$ a.e. (up to a subsequence), we will have that $F^\varepsilon(\nabla(u^\varepsilon + v)) \to F(\nabla u + v)$ a.e. in B_R (since $F^\varepsilon \to F$ locally uniformly), and by Lebesgue's theorem we will deduce that $\lim_{\varepsilon\to 0} \int_{B_R} F^\varepsilon(\nabla(u^\varepsilon + v))\, dx = \int_{B_R} F(\nabla(u+v))$. Sending ε to 0, this will yield

$$\int_{B_R} F(\nabla u) + \frac{(u-g)^2}{2}\, dx \leq \int_{B_R} F(\nabla(u+v)) + \frac{(u+v-g)^2}{2}\, dx,$$

showing that u is a convex minimizing solution of (21), hence proving the theorem. Note that, by Proposition 3.1, u is a solution of (21) in the sense of Definition 2.

It remains to study the convergence of ∇u^ε to ∇u. In fact, one shows that $\nabla u^\varepsilon \to \nabla u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ (hence any $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for $p < +\infty$). Indeed, if $\varphi \in C^1_c(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} |\nabla u^\varepsilon - \nabla u|^2 \varphi \, dx \;=\; -\int_{\mathbb{R}^N} (u^\varepsilon - u)\nabla\varphi \cdot (\nabla u^\varepsilon - \nabla u) - \int_{\mathbb{R}^N} (u^\varepsilon - u)\varphi(\Delta u^\varepsilon - \Delta u). \quad (24)$$

The two integrals on the right-hand side clearly go to zero, the first one because $|\nabla u^\varepsilon - \nabla u| \le 2L$ a.e., the second one because the distributions Δu^ε and Δu are nonnegative Radon measures in $\mathbb{R}^N$ that are uniformly bounded in the sense of Radon measures (indeed, for any $R > 0$, $\int_{B_R} |\Delta u^\varepsilon| = \int_{\partial B_R} \nabla u^\varepsilon \cdot \nu \le L\mathcal{H}^{N-1}(\partial B_R)$). This achieves the proof of Theorem 3.□

5 Implicit time discretization of the mean curvature flow

5.1 Notations and main problem

In this section and the rest of the paper, φ is a convex, even, one–homogeneous function satisfying (6), that will represent a distance in $\mathbb{R}^N$ ($d^\varphi(x,y) = \varphi(x-y)$). Given a set $E \subset \mathbb{R}^N$, the signed distance to ∂E is

$$d^\varphi_E(x) \;=\; d^\varphi(x,E) \;-\; d^\varphi(x, \mathbb{R}^N \setminus E)\,,$$

it is negative in the interior of E and positive outside of its closure ($d^\varphi(x,E) = \inf_{y\in E}\varphi(x-y)$).

The polar of φ is denoted by $\varphi^\circ(x) = \sup_{\varphi(y)\le 1} y\cdot x$. In this section and the subsequent ones, the "Wulff shape" of radius r and center x is denoted by $W(x,r) = \{y \in \mathbb{R}^N : \varphi(y-x) < R\}$, whereas $\overline{W}(x,r) = \{y \in \mathbb{R}^N : \varphi(y-x) \le r\}$.

As in Section 2.2, and following the notations in [18], we will denote by $P_\varphi(F)$ the anisotropic perimeter associated to the anisotropy φ°, given by

$$P_\varphi(F) \;=\; \int_{\mathbb{R}^N} \varphi^\circ(D\chi_F) \;=\; \int_{\partial_* F} \varphi^\circ(\nu_F(x))\, d\mathcal{H}^{N-1}(x).$$

We will make use of the corresponding anisotropic total variation (in any open set $\Omega \subseteq \mathbb{R}^N$)

$$\int_\Omega \varphi^\circ(Du) = \sup\left\{\int_\Omega u \operatorname{div}\sigma : \sigma \in C^1_c(\Omega;\mathbb{R}^N)\,, \varphi(\sigma(x)) \le 1 \;\forall x \in \Omega\right\}.$$

Given a bounded set $E \subset \mathbb{R}^N$, let us consider the equation

$$-h \operatorname{div} \partial\varphi^\circ(\nabla u) \;+\; u \;-\; d^\varphi_E \;\ni\; 0 \;\text{ in } \mathbb{R}^N\,. \quad (25)$$

We will show in this section that (25) has a solution u, globally Lipschitz, unique in the class of functions with bounded sublevel sets $[u < t] := \{x \in \mathbb{R}^N : u(x) < t\}$, that can be recovered as the limit of solutions of the same problem on increasing bounded domains (with Neumann boundary conditions). We will then define the set $T_h E = [u < 0]$ as an implicit evolution of time step h of ∂E by anisotropic mean curvature.

5.2 A variational problem for the level sets

Lemma 5.1 *Assume $v \in BV_{loc}(\mathbb{R}^N)$ is a solution of (25) such that $[v \leq s]$ are bounded for any $s \in \mathbb{R}$. Then for any $s \in \mathbb{R}$, the set $[v < s]$ is a solution of the variational problem*

$$(P_s) \qquad \min_F P_\varphi(F) \; + \; \frac{1}{h}\int_F (d_E^\varphi(x) - s)\, dx$$

Proof. If $v \in BV_{\mathrm{loc}}(\mathbb{R}^N)$ is a solution of (25), it means (cf Def. 2) that there exists a vector field z with $\varphi(z(x)) \leq 1$ a.e. in $\mathbb{R}^N$, such that $z \cdot Dv = \varphi(Dv)$ as measures (locally in $\mathbb{R}^N$), and $-\mathrm{div}\, z + u = d_E^\varphi$ in the distributional sense in $\mathbb{R}^N$. First, let us observe that $-z \cdot D\chi_{[v \leq s]} = \varphi^\circ(D\chi_{[v\leq s]})$ as measures in $\mathbb{R}^N$ for almost any $s \in R$. Indeed, since for any $k > 0$ $T_k(v) = k - \int_{-k}^k \chi_{[v\leq t]}\, dt$, by [9, Proposition 2.7], we have

$$z \cdot DT_k(v) = -\int_{-k}^k z \cdot D\chi_{[v\leq t]}\, dt, \qquad k > 0.$$

Since $\varphi^\circ(DT_k(v)) = \int_{-k}^k \varphi^\circ(D\chi_{[v\leq t]})$, and, by Remark 3.2, we have $\varphi^\circ(DT_k(v)) = z\cdot DT_k(v)$, we obtain

$$-\int_{-k}^k z \cdot D\chi_{[v\leq t]}\, dt = \int_{-k}^k \varphi^\circ(D\chi_{[v\leq t]})\, dt, \qquad k > 0,$$

and this implies our claim. Let us denote by $\mathcal{N} \subset \mathbb{R}$ the negligible set of values for which the claim is false. For $s \notin \mathcal{N}$, let $E_s := [v \leq s]$, and let $F \subseteq \mathbb{R}^N$ be a set of finite perimeter, with finite measure. Multiplying (25) by $\chi_{E_s} - \chi_F$ and integrating by parts, we obtain

$$\begin{aligned} \frac{1}{h}\int_{\mathbb{R}^N}(v - d_E^\varphi)(\chi_{E_s} - \chi_F) \;&=\; \int_{\mathbb{R}^N} \mathrm{div}\, z\,(\chi_{E_s} - \chi_F) = -\int_{\mathbb{R}^N} z\cdot D\chi_{E_s} + \int_{\mathbb{R}^N} z \cdot D\chi_F \\ &=\; P_\varphi(E_s) + \int_{\mathbb{R}^N} z\cdot D\chi_F \geq P_\varphi(E_s) - P_\varphi(F). \end{aligned}$$

Hence

$$P_\varphi(E_s) + \frac{1}{h}\int_{E_s\setminus F}(d_E^\varphi(x) - v)\, dx \;\;\leq\;\; P_\varphi(F) + \frac{1}{h}\int_{F\setminus E_s}(d_E^\varphi(x) - v)\, dx,$$

and, therefore,

$$P_\varphi(E_s) + \frac{1}{h}\int_{E_s\setminus F}(d_E^\varphi(x) - s)\, dx \;\;\leq\;\; P_\varphi(F) + \frac{1}{h}\int_{F\setminus E_s}(d_E^\varphi(x) - s)\, dx.$$

By adding $(1/h)\int_{E_s\cap F}(d_E^\varphi - s)\, dx$ to both sides of the inequality, we obtain that

$$P_\varphi(E_s) + \frac{1}{h}\int_{E_s}(d_E^\varphi(x) - s)\, dx \;\;\leq\;\; P_\varphi(F) + \frac{1}{h}\int_F (d_E^\varphi(x) - s)\, dx.$$

If $s \in \mathcal{N}$, we observe that $[v < s] = \bigcup_{s'<s, s' \notin \mathcal{N}} [v < s']$ and we deduce the thesis of the Lemma by approximation and semicontinuity of the total variation. Observe also that since for any s, $[v \le s] = \bigcap_{s'>s} [v < s']$, the set $[v \le s]$ is also a solution of (P_s).

Remark 5.2 It is simple to check that Problem (P_s) is equivalent to the variational problem

$$(P'_s) \qquad \min_F P_\varphi(F) \;+\; \frac{1}{h} \int_{F \triangle E_s} |d_E^\varphi(x) - s| \, dx$$

where $E_s = [d_E^\varphi < s]$.

5.3 A uniqueness result

We will show, as a consequence of Lemma 5.1 and of the following lemma, that (25) can have at most one solution in the class of functions with bounded sublevel sets $[u < s]$, $s \in \mathbb{R}$.

Lemma 5.3 *Let us denote by F_s a minimizer of (P_s), for any $s \in \mathbb{R}$. Then, if $s < s'$, we have $F_s \subseteq F'_s$.*

Proof. Let $s < s'$, and let us prove that $F_s \subseteq F'_s$. Indeed, one has

$$P_\varphi(F_s) \;+\; \frac{1}{h} \int_{F_s} (d_C^\varphi(x) - s) \, dx \le \; P_\varphi(F_s \cap F_{s'}) \;+\; \frac{1}{h} \int_{F_s \cap F_{s'}} (d_C^\varphi(x) - s) \, dx \,,$$

$$P_\varphi(F_{s'}) \;+\; \frac{1}{h} \int_{F_{s'}} (d_C^\varphi(x) - s') \, dx \le \; P_\varphi(F_s \cup F_{s'}) \;+\; \frac{1}{h} \int_{F_s \cup F_{s'}} (d_C^\varphi(x) - s') \, dx \,.$$

Now, summing the inequalities and using the well–known inequality

$$P_\varphi(F_s \cap F_{s'}) \;+\; P_\varphi(F_s \cup F_{s'}) \;\le\; P_\varphi(F_s) \;+\; P_\varphi(F_{s'}) \,,$$

we find that

$$\int_{F_s} (d_C^\varphi(x) - s) \, dx \;+\; \int_{F_{s'}} (d_C^\varphi(x) - s') \, dx \le \int_{F_s \cap F_{s'}} (d_C^\varphi(x) - s) \, dx \;+\; \int_{F_s \cup F_{s'}} (d_C^\varphi(x) - s') \, dx \,.$$

In other words,

$$s'(|F_s \cup F_{s'}| - |F_{s'}|) \;\le\; s(|F_s| - |F_s \cap F_{s'}|) \,,$$

that is $s'|F_s \setminus F_{s'}| \le s|F_s \setminus F_{s'}|$. Hence, if $s' > s$, we must have $|F_s \setminus F_{s'}| = 0$ so that (up to a negligible set) $F_s \subseteq F_{s'}$. □

Corollary 5.4 *If u and v are two solutions of (25), such that $[u < s]$ and $[v < s]$ are bounded for any s, then $u = v$ a.e. in $\mathbb{R}^N$.*

Proof. By Lemma 5.1, $[u < s]$ and $[v < s]$ solve (P_s) for any s. By Lemma 5.3, we find that if $s' > s$, $[u < s] \subseteq [v < s']$ and $[v < s] \subseteq [u < s']$. We easily deduce $[v < s] \subseteq [u < s]$ for any s, as well as the reverse inclusion (up to negligible sets). Hence $u = v$ a.e. in $\mathbb{R}^N$.

Remark 5.5 One also deduces that $[u < s]$ is the smallest solution of (P_s), while $[u \leq s]$ is the largest. In particular, for almost any s (but a countable set), $[u = s]$ is negligible and (P_s) has a unique solution.

Remark 5.6 One could try to deduce from Lemmas 5.1 and 5.3 an approach for building a solution to (25) (through the sets F_s solving (P_s)), hence showing existence. However, a simpler approach is proposed in the next section, that yields other important properties of this solution.

5.4 Approximation by variational problems on bounded domains

We show, here, existence of a solution u to (25), with bounded sublevels $[u < s]$, by means of an approximation by Neumann (variational) problems in increasing bounded domains.

If $E \subset \mathbb{R}^N$ is bounded and $R > 0$ is such that $E \subset\subset W(0, R)$, we define u_R as the solution of

$$\min_{u \in BV(W(0,R))} \int_{W(0,R)} \varphi^\circ(Du) + \frac{1}{2h} \int_{W(0,R)} |u(x) - d_E^\varphi(x)|^2 \, dx \tag{26}$$

We may assume, without loss of generality, that $0 \in E$. In this case, there exists $M > 0$ such that $\varphi - M \leq d_E^\varphi \leq \varphi$. By comparison (cf [19, Lemma 2.1]), one gets that $\bar{u}_{R,h} - M \leq u_R \leq \bar{u}_{R,h}$ where $\bar{u}_{R,h}$ is given by (33), in Appendix B. We deduce that if $R' > R$ is large enough, the sublevel $[u_{R'} < R + h(N-1)/R + 1]$ contains $W(0, R)$. On the other hand, $u \geq u_{R',h} - M > R + h(N-1)/R + 1$ on $\partial W(0, R' - 2\sqrt{h})$ as soon as R' is large enough, so that $[u_{R'} < R + h(N-1)/R + 1] \subset\subset W(0, R')$. By [19, Cor. A.2], we deduce that if $\Omega \supseteq W(0, R')$, and if u_Ω is the solution of (26) with $W(0, R)$ replaced with Ω, then $u_\Omega \wedge (R + h(N-1)/R + 1) = u_{R'} \wedge (R + h(N-1)/R + 1)$, and in particular, $u_\Omega|_{W(0,R)} = u_{R'}|_{W(0,R)}$. We deduce the following lemma:

Lemma 5.7 *For any $R > 0$, there exists $R_0 > R$ such that if $R' \geq R_0$ the restriction to $W(0, R)$ of the solution $u_{R'}$ of (26) (with R replaced with R') is independent of R'.*

We deduce, in particular, that as $R \to \infty$, u_R converges to a function u that solves (25) (it is obviously a "minimizing solution" in the sense of Definition 1), and with $\bar{u}_h - M \leq u \leq \bar{u}_h$

(where $\bar{u}_h$ is the limit of $\bar{u}_{R,h}$, given by (32)). By Corollary 5.4, this function u is the unique solution of (25) in the class of function with bounded sublevels $[u < s]$.

From [19, Cor. A.9], which we can invoke in the sets $[u_R < s]$ for any level s and R large enough (so that $[u_R < s] = [u < s]$), we deduce the following lemma:

Lemma 5.8 *$\varphi^\circ(\nabla u) \leq 1$ a.e. in $\mathbb{R}^N$.*

Let us mention that this lemma also easily follows by approximation with smooth problems, and using the comparison principle for viscosity solutions mentionned in Section 4.

To summarize, we have shown the following result.

Theorem 4 *For any bounded $E \subset \mathbb{R}^N$, problem (25) admits a unique solution u in the class of functions with bounded sublevels $[u < s]$. This function u is globally Lipschitz, and each sublevel $[u < s]$ is a minimizer of (P_s) or (P'_s). The function u is the limit of the solutions of (26) as $R \to \infty$. In particular, if $E \subseteq E'$, one deduces that $u' \geq u$, where u' is the solution of (25) with $d^\varphi_{E'}$ replacing d^φ_E, and $[u < s] \subseteq [u' < s]$ for any $s \in \mathbb{R}$.*

5.5 Implicit time discretization of the mean curvature flow

Given u the solution of (25) given by Theorem 4, we let $T_h E = [u < 0]$. The set $T_h E$ is a solution of (P'_0), which is the variational problem suggested by Almgren, Taylor and Wang [2] as an implicit time discretization of the mean curvature flow (see also Luckhaus and Sturzenecker [38]). From Theorem 4, we have the important property that if $E \subseteq E'$, then $T_h E \subseteq T_h E'$.

The evolution generated by T_h, as $h \to 0$, has been studied in [19], and is shown to converge, when the anisotropy φ is smooth and uniformly convex, to the generalized motion by anisotropic mean curvature (in the sense of minimal barriers or viscosity solutions, see [13, 14, 15, 42, 10, 11]). In this paper, we are interested in the particular case of convex sets.

We thus consider C a bounded, convex set, with nonempty interior. Observe that in this case the boundary of C is negligible and for our problem it is irrelevant to consider C closed or open. In this case, d^φ_C is convex (cf [39, Lem. 4.2]), so that the solution u of (25) for $E = C$ is convex too, by Theorem 3. One also deduces $u \geq d^\varphi_C$, in particular, $[u \leq 0] \subset \overline{C}$. If $\min u < 0$ (which will be true as soon as h is small enough), then one sees that $T_h C = [u < 0]$ is an open convex set with closure $[u \leq 0]$. This generalizes McCann's Theorem 4.1 in [39], which shows the same result for C a two–dimensional, balanced convex set.

The set $T_h C$ is a solution of Problem (P'_0), that is,

$$\min_F P_\varphi(F) + \frac{1}{h} \int_{F \triangle C} |d^\varphi_C(x)|\, dx \tag{27}$$

where the minimizer is taken among all finite perimeter sets $F \subset \mathbb{R}^N$.

We have $T_h C = \emptyset$ if $\min u \geq 0$. If $\min u > 0$, it is the unique solution of (27). If $T_h C \neq \emptyset$, it must be that $|[u = 0]| = 0$ and we find again that $T_h C$ is the unique solution of (27). It is only in the case $\min u = 0$ that (27) might have multiple solutions: indeed, $[u < 0] = \emptyset$ and $[u = 0]$ might provide two different solutions (here, of course, "unique" and "different" are intended in the sense of finite-perimeter sets, that is, up to a negligible set in $\mathbb{R}^N$, in particular, whenever $|[u = 0]| = 0$, then $[u < 0]$ and $[u \leq 0]$ are the same set).

We then define the discrete–in–time evolution $C_h(t) = (T_h)^{[t/h]}(C)$, for all $t > 0$. We also let $\mathcal{C}_h \subset \mathbb{R}^N \times [0, +\infty)$ be the "tube" $\{(x,t) : t \geq 0, x \in C_h(t)\}$.

We will show that as $h \to 0$, $C_h(t)$ converges to an evolution $C(t)$ which satisfies, in the case of a nonsmooth anisotropy φ, φ°, a weak form of the mean curvature motion equation. In the case of a smooth anisotropy, the fact that $C(t)$ is the mean curvature evolution starting from C already follows from the results in [2], [19], and [38] in the isotropic case. However, our construction also proves that given any smooth anisotropy, the evolution of a convex set remains convex, which did seem to be known only in dimension 2 [39].

6 Evolution of the Wulff shape.

Let us compute the evolution of the Wulff shape of radius r_0, that is, the set $\overline{W}(0, r_0) = \{\varphi \leq r_0\}$. We first compute $T_h \overline{W}(0, r_0) = T_h W(0, r_0)$.

Equation (40) in Appendix B shows that if $C = W(0, r_0)$ in (25), then u is given by

$$u(x) = \begin{cases} \varphi(x) - r_0 + \dfrac{h(N-1)}{\varphi(x)} & \text{if } \varphi(x) \geq \sqrt{h(N+1)}\,, \\ \dfrac{2N\sqrt{h}}{\sqrt{N+1}} - r_0 & \text{otherwise.} \end{cases}$$

We deduce that $T_h W(0, r_0) = \emptyset$ if $h \geq (N+1) r_0^2/(4N^2)$, and $T_h W(0, r_0) = W(0, S_h(r_0))$, with $S_h(r_0) = r_0(1 + \sqrt{1 - 4h(N-1)/r_0^2})/2$, otherwise (by convention, let us set $S_h(r_0) = 0$ when $h \geq (N+1) r_0^2/(4N^2)$). One can show that $(S_h)^{[t/h]}(r_0) \to r(t)$ uniformly in $[0, +\infty)$ as $h \to 0$ where $r(t)$ solves $\dot{r} = -(N-1)/r$, $r(0) = r_0$, that is: $r(t) = \sqrt{r_0^2 - 2(N-1)t}$ for $t \leq t_{r_0} = r_0^2/(2(N-1))$, and, by convention, $r(t) = 0$ for $t > t_{r_0}$. One also shows that for any $T > t_{r_0}$, $(S_h)^{[t/h]}(r_0) = 0$ for h small enough and $t \geq T$. One deduces the following lemma (we refer to Appendix A for the definition and properties of the Hausdorff distance and convergence).

Lemma 6.1 *Let $C(t) = \overline{W}(0, r(t))$ for $t \leq t_{r_0}$, and $\emptyset$ for t larger, and let $C_h(t) = (T_h)^{[t/h]}(W(0, r_0))$. Then, as $h \to 0$, $C_h(t)$ converges to $C(t)$ in the following senses:*

- *For each $t \in (0, t_{r_0})$, $C_h(t)$ converges to $C(t)$ in the Hausdorff sense, moreover, if $T > t_{r_0}$, then for h small enough $C_h(t) = \emptyset$ for any $t \geq T$,*

- *For each $0 \le t < t_{r_0}$, $\partial C_h(t)$ converges to $\partial C(t) = \{x : \varphi(x) = r(t)\}$ in the Hausdorff sense,*
- *The "tube" $\mathcal{C}_h = \{(x,t) : x \in C_h(t)\}$ converges in the Hausdorff sense to $\{(x,t) : x \in C(t)\} = \{(x,t) : 0 \le t \le t_{r_0}, \varphi(x) \le r(t)\}$, in $\mathbb{R}^N \times [0,+\infty)$*
- *The boundary of $\mathcal{C}_h$, $\partial\mathcal{C}_h$, converges to the set $\{(x,t) : 0 \le t \le t_{r_0}, \varphi(x) = r(t)\}$ in $\mathbb{R}^N \times [0,+\infty)$.*

Remark 6.2 These properties also yield that $C_h(t)^c$ converges to $\overline{C(t)^c}$ in the Hausdorff sense, and, as well, that the complements of $\mathcal{C}_h$ converge to $\{(x,t) : 0 \le t \le t_{r_0}, \varphi(x) \ge r(t)\}$.

Remark 6.3 In the same way, for any $x \in \mathbb{R}^N$, $r_0 > 0$, $t \ge 0$, one shows similar convergence properties of $T_h^{[s/h]-[t/h]}(W(x,r_0))$ in $\mathbb{R}^N \times [t,+\infty)$, to the tube given by $W(x, \sqrt{r_0^2 - 2(N-1)(s-t)})$ for $t \le s \le t + r_0^2/(2(N-1))$ and $\emptyset$ for $s \ge t + r_0^2/(2(N-1))$.

7 Evolution of a general convex set

Let C_0 be an initial bounded convex set, with nonempty interior, and let, for $h > 0$ small enough, $C_h(t) = (T_h)^{[t/h]}(C_0)$. Let also $\mathcal{C}_h \subset \mathbb{R}^N \times [0,+\infty)$ be the tube $\{(x,t) : t \ge 0, x \in C_h(t)\}$.

Up to a subsequence (that we will still denote by $(h)_{h>0}$ throughout the sequel), we may assume that $\mathcal{C}_h \to \mathcal{C}^*$ in the Hausdorff sense as $h \to 0$, and, as well, that $(\mathcal{C}_h)^c \to (\mathcal{C}_*)^c$ (see Appendix A). One has $\mathcal{C}_* \subset \mathcal{C}^*$ and $\mathcal{C}^* \setminus \mathcal{C}_*$ is the Hausdorff limit of $\partial\mathcal{C}_h$.

We denote by $C^*(t)$ (resp., $C_*(t)$) the section $\{x : (x,t) \in \mathcal{C}^*\}$ (resp., $\{x : (x,t) \in \mathcal{C}_*\}$).

Let us observe that, from the existence of some $R > 0$ with $C \subset W(0,R)$ and the convergence properties established in the previous section, $C^*(t)$ is empty for $t > t_R$.

From the properties of $C_h(t)$, it is straightforward that both $C^*(t)$ and $C_*(t)$ are convex for any t, and, as well, that if $t < s$, $C^*(t) \supseteq C^*(s)$ and $C_*(t) \supseteq C_*(s)$. We call t_C the extinction time for $\mathcal{C}_*$, that is, $t_C = \min\{t \ge 0 : C_*(t) = \emptyset\}$ (observe that $C_*(t) \ne \emptyset$ for any $t < t_C$). Similarly, let t_C^* be the extinction time of $\mathcal{C}^*$, that is, $\max\{t \ge 0 : C^*(t) \ne \emptyset\}$. It is also clear (since $\mathcal{C}_* \subset \mathcal{C}^*$) that $t_C \le t_C^*$, and that $t_C^* = \lim_{h\to 0} t_h$ where t_h is the extinction time of $\mathcal{C}_h$ ($t_h = \min\{t \ge 0 : C_h(t) = \emptyset\}$). It is very likely that $t_C^* = t_C$, although we were not able to prove it.

Let us first show the following properties for the tubes $\mathcal{C}_*$ and $\mathcal{C}^*$ and their sections:

(i) $\displaystyle\bigcup_{t'>t} C_*(t') = C_*(t)$ for any $t \ge 0$,

(ii) $\operatorname{int}(\bigcap_{t'<t} C_*(t')) = C_*(t)$ for any $t > 0$,

(iii) $\operatorname{int}(C^*(t)) = C_*(t)$ for any $t > 0$, and $C_*(0) = \operatorname{int}(C_0)$, $C^*(0) = \overline{C_0}$.

To show (i) we first observe that since C_* is nonincreasing in time, $C_*(t) \supseteq \cup_{t'>t} C_*(t')$. On the other hand, if $x \in C_*(t)$, then there exists $\rho > 0$ such that $W(x,\rho) \times \{t\} \subset\subset C_*$, hence is contained in C_h for h small enough. Then, $T_h^{[s/h]-[t/h]} W(x,\rho) \subset C_h(s)$ for $s \geq t$, or,

$$\left\{(y,s) \in \mathbb{R}^N \times [t,+\infty) \,:\, y \in T_h^{[s/h]-[t/h]} W(x,\rho)\right\}^c \supset (C_h)^c.$$

Passing to the Hausdorff limit and using Lemma 6.1 and Remark 6.3, we see that for any $s \in [t, t+\rho^2/(2(N-1)))$, $W(x, \sqrt{\rho^2 - 2(N-1)(s-t)}) \subset C_*(s)$. In particular, $x \in C_*(s)$ for s close enough to t, $s > t$. This shows (i).

The proof of (ii) is similar. We fix $t > 0$. First, since C_* is nonincreasing in time, $C_*(t) \subset \operatorname{int}(\cap_{t'<t} C_*(t'))$. Now, if $x \in \operatorname{int}(\cap_{t'<t} C_*(t'))$ there exists $\rho > 0$ such that $W(x,\rho) \subset\subset C_*(t')$ for any $t' < t$. Repeating the proof of (i), we find that $W(x, \sqrt{\rho^2 - 2(N-1)(s-t')}) \subset C_*(s)$ for any $s \in [t', t'+\rho^2/(2(N-1)))$, and sending t' to t, that for $t \leq s \leq t+\rho^2/(2(N-1))$, $W(x, \sqrt{\rho^2 - 2(N-1)(s-t)}) \subset C_*(s)$. In particular, $W(x,\rho) \subset C_*(t)$, showing our claim.

Let us now prove (iii). Let $t > 0$. If $x \in \operatorname{int}(C^*(t))$ then, by convexity, we can choose $N+1$ points $\xi^1, \ldots, \xi^{N+1} \in C^*(t)$ and $\rho > 0$ such that $W(x,\rho)$ is contained in the convex envelope of the $(\xi^i)_{i=1}^{N+1}$. Then, there exist sequences $(\xi_h^i, t_h^i)_{h>0}$ for each i, with $\xi_h^i \in C_h(t_h^i)$, and such that $t_h^i \to t$ and $\xi_h^i \to \xi^i$ as $h \to 0$. If $t' < t$, then for h small enough, $t' \leq t_h^i$ so that $\xi_h^i \in C_h(t')$. By convexity, the simplex with vertices ξ_h^i is also in $C_h(t')$, and if h is small enough it will contain $W(x,\rho/2)$. Reproducing the arguments in the proofs of (i) and (ii), we find that $W(x, \sqrt{\rho^2/4 - 2(N-1)(s-t')}) \subset C_*(s)$ for $s > t'$, s close to t'. Sending t' to t we find in particular that $W(x,\rho/2) \subset C_*(t)$, hence $\operatorname{int}(C^*(t)) \subset C_*(t)$. Since $C_*(t) \subset C^*(t)$, we find that $\operatorname{int}(C^*(t)) = C_*(t)$.

Eventually, we see that if $x \in \operatorname{int} C_0$, the proof of (i) can be reproduced to show that $x \in C_*(0)$, now, we find $\operatorname{int}(C_0) \subseteq C_*(0) \subset C^*(0) \subseteq \overline{C_0}$ (since $\overline{C_0}$ contains $C_h(t)$ for all h and t). Since $\operatorname{int}(C_0)$ is the interior of $\overline{C_0}$, and $\overline{C_0}$ the closure of $\operatorname{int}(C_0)$, it yields the two last statements of (iii).

Point (iii) shows that $\operatorname{int}(C^*(t)) = C_*(t)$ for all $t \geq 0$. In particular, $\operatorname{int}(C^*(t))$ is empty if $t \geq t_C$. If $t < t_C$, we deduce also that $\overline{C_*(t)} = C^*(t)$. Fix $t \geq 0$ and consider $A \subset B$ such that A^c is the Hausdorff limit of a converging subsequence of $((C_h(t))^c)_{h>0}$, while B is the limit of $C_h(t)$ along the same subsequence. One has $C_*(t) \subseteq A \subset B \subseteq C^*(t)$. Hence, if $t < t_C$, we deduce that $A = C_*(t)$ and $B = C^*(t)$ (showing in particular that the whole sequences $((C_h(t))^c)_{h>0}$ and $(C_h(t))_{h>0}$ converge to A^c and B). Notice that the convergence of $C_h(t)$ to $C^*(t) = \overline{C_*(t)}$ and of $(C_h(t))^c$ to $(C_*(t))^c$ is equivalent to the convergence of

$\partial C_h(t)$ to $\partial C^*(t)$ (see Section A.5). If $t \geq t_R$, we find that $A = \emptyset = C_*(t)$. In particular, we have shown the following lemma:

Lemma 7.1 *If $t \in [0, t_C)$, then $C_h(t) \to C^*(t)$ in the Hausdorff sense while $(C_h(t))^c \to (C_*(t))^c$, and one has $C^*(t) = \overline{C_*(t)}$ while $C_*(t)$ is the interior of $C^*(t)$. In particular $\partial C_h(t)$ goes to $\partial C^*(t)$.*

We observe that in particular (*cf* Section A.5), $C^*(t) \setminus C_*(t) = \partial C^*(t)$ is Lebesgue–negligible for any $t < t_C$, whereas if $t \geq t_C$, then $C^*(t)$ has empty interior (from point (iii)). As a consequence, also $C^* \setminus C_*$ is negligible in $\mathbb{R}^N \times [0, +\infty)$.

Lemma 7.2 *The function $t \mapsto \partial C^*(t)$ is Hausdorff–continuous on $[0, t_C]$. Moreover, for any $T < t_C$, the convergence of $\partial C_h(t)$ to $\partial C_*(t) = \partial C^*(t)$ in the Hausdorff distance is uniform on $[0, T]$.*

Proof. We refer to Section A.6 in the appendix. From property (i) above, we get that as long as $t < t_C$, $\partial C^*(t') \to \partial C^*(t)$ if $t' \to t$, $t' > t$, On the other hand, the semicontinuity property $C^*(t) = \bigcap_{t'<t} C^*(t')$ is clearly true for any $t > 0$. Indeed, if $x \in \bigcap_{t'<t} C^*(t')$ it means that $(x, t') \in C^*$ for any $t' < t$, but C^* is closed, so that $(x, t) \in C^*$ hence $x \in C^*(t)$. Conversely if $x \in C^*(t)$, then $(x, t) = \lim_{h \to 0}(x_h, t_h)$ for some sequence $(x_h, t_h) \in C_h$, but as if $t' < t$, $t_h \geq t'$ for h small enough, one deduces $x \in C^*(t')$, hence $x \in \bigcap_{t'<t} C^*(t')$. We deduce that for any $t > 0$, $\partial C^*(t') \to \partial C^*(t)$ if $t' \to t$, $t' < t$.

Now let us consider the convergence of $\partial C_h(t)$ to $\partial C^*(t)$. As mentionned in Section A.5, the fact that for any $t < t_R$, $\partial C_h(t) \to \partial C^*(t)$ is equivalent to the uniform convergence in $\mathbb{R}^N$ of the distance functions $d_h(x, t) = d^{\varphi}_{C_h(t)}(x)$ to $d(x, t) = d^{\varphi}_{C^*(t)}(x)$. To get uniform convergence in $t \in [0, T]$, $T < t_C$, we must show that $\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^N} |d_h(x, t) - d(x, t)| \to 0$ as $h \to 0$. But, by the previous paragraph, $t \mapsto d(x, t)$ is continuous, hence uniformly continuous, as a function from $[0, T]$ to $C(\mathbb{R}^N)$ (with uniform convergence). Moreover, we know that both d_h and d are nondecreasing in time. Let us fix $\varepsilon > 0$ and find $t_0 = 0 < t_1 < \cdots < t_{k-1} < t_k = T$ such that $0 \leq d(x, t_i) - d(x, t_{i-1}) \leq \varepsilon$ for any $i = 1, \ldots, k$ and any $x \in \mathbb{R}^N$. If h is small enough, then $|d_h(x, t_i) - d(x, t_i)| \leq \varepsilon$ for any $i = 0, \ldots, k$ and any $x \in \mathbb{R}^N$. For such h, $t \in [0, T]$, $x \in \mathbb{R}^N$, if i is an index such that $t_i \leq t \leq t_{i+1}$, we have

$$d_h(x, t) - d(x, t) \leq d_h(x, t_i) - d(x, t_{i+1}) \leq d_h(x, t_i) - d(x, t_i) + \varepsilon \leq 2\varepsilon,$$

$$d_h(x, t) - d(x, t) \geq d_h(x, t_{i+1}) - d(x, t_i) \geq d_h(x, t_{i+1}) - d(x, t_{i+1}) - \varepsilon \geq -2\varepsilon,$$

which proves the desired property and achieves the proof of the lemma. □

If φ, φ° are smooth, we deduce that on $[0, t_C)$, the limit evolution $\partial C^*(t)$ is the generalized anisotropic mean curvature motion starting from C_0, by [2, Thm 6.2]. Here "generalized" is intended in the sense of viscosity solutions, see for instance [11]. In particular, we deduce that for any smooth anisotropy, the mean curvature motion starting from a convex set remains convex at all time. Notice that t_C must be the extinction time for the curvature flow, since the limit of $C_*(t)$ as $t \to t_C$ has empty interior.

In all cases, including the nonsmooth or "crystalline" case, we can derive a weak form of the Mean Curvature Flow equation:

Theorem 5 *There exists $z \in L^\infty(\mathbb{R}^N \times (0, t_C))$ with $z \in \partial\varphi^\circ(\nabla d)$ a.e. in $\mathbb{R}^N \times (0, t_C)$, such that*

$$-\operatorname{div} z(x,t) \ + \ \frac{\partial d}{\partial t} \ \geq \ 0 \tag{28}$$

as measures out of $C^* \cap \mathbb{R}^N \times (0, t_C)$, and

$$-\operatorname{div} z(x,t) \ + \ \frac{\partial d}{\partial t} \ \leq \ 0 \tag{29}$$

as measure in C_*.

Proof. Let t_h be the extinction time of C_h, we recall $t_h \to t_C^* \geq t_C$. For $h > 0$ and $t \in [0, t_h)$ we let $d_h(x,t) = d^\varphi_{C_h(t)}(x)$. We also denote by $u_h(t,x)$ the solution of (25) for the convex $C_h(t)$. Since $C_h(t+h) = \{u_h(\cdot, t) < 0\}$ and (by Lemma (5.8)), $\varphi^\circ(\nabla u_h(x,t)) \leq 1$, we can show that $u_h(x,t) \leq d_h(x, t+h)$ for any $(x,t) \notin C_h$ whereas $u_h(x,t) \geq d_h(x, t+h)$ if $(x,t) \in C_h$. Let us show for instance the first assertion: for $x \notin C_h(t)$, there exists $x_0 \in \partial C_h(t)$ with $\varphi(x - x_0) = d_h(x, t+h)$. Since $u_h(x_0) = 0$, one has

$$\begin{aligned} u_h(x) \ &= \ \int_0^1 \nabla u_h(x_0 + s(x - x_0)) \cdot (x - x_0)\, ds \\ &= \ \varphi(x - x_0) \int_0^1 \nabla u_h(x_0 + s(x - x_0)) \cdot \frac{x - x_0}{\varphi(x - x_0)}\, ds \end{aligned}$$

Since $\varphi((x - x_0)/\varphi(x - x_0)) = 1$, the scalar product in the integral is less than $\varphi^\circ(\nabla u_h(x_0 + s(x - x_0))) \leq 1$. Hence $u_h(x) \leq \varphi(x - x_0) = d_h(x, t+h)$. The proof of the second assertion is identical: if $x \in C_h(t)$, there exists $x_0 \in \partial C_h(t)$ with $-\varphi(x_0 - x) = d_h(x, t+h)$, and one gets in the same way $-u_h(x,t) = u_h(x_0, t) - u_h(x,t) \leq \varphi(x_0 - x) = -d_h(x, t+h)$.

Therefore, for each $t < t_h$, there exists a field $z_h(x,t) \in \partial\varphi^\circ(\nabla u_h(x, t+h))$ (piecewise constant in t) such that

$$\begin{cases} -\operatorname{div} z_h(x,t) \ + \ \dfrac{d_h(x,t+h) - d_h(x,t)}{h} \ \geq \ 0 & \text{for } (x,t) \notin C_h\,, \\ -\operatorname{div} z_h(x,t) \ + \ \dfrac{d_h(x,t+h) - d_h(x,t)}{h} \ \leq \ 0 & \text{for } (x,t) \in C_h\,. \end{cases} \tag{30}$$

There exists $z \in L^\infty(\mathbb{R}^N \times (0,t_C);\mathbb{R}^N)$ such that, up to a subsequence, $z_h \to z$ weakly$-*$ (here z_h, which strictly speaking is defined only up to $t = t_h$, can be arbitrarily extended, for instance by 0, after t_h if $t_h < t_C$). Assume we have shown that $\nabla u_h(x,t+h) \to \nabla d(x,t)$ in $L^2_{\rm loc}(\mathbb{R}^N \times (0,t_C))$. If $p(x,t) \in L^2_{\rm loc}(\mathbb{R}^N \times (0,t_C);\mathbb{R}^N)$ is an arbitrary vector field and $\psi \in C_c^\infty(\mathbb{R}^N \times (0,t_C))$ a nonnegative test function, one has for any h small enough

$$\begin{aligned}\int_0^{t_C}\int_{\mathbb{R}^N} \varphi^\circ(p(x,t))\psi(x,t)\,dxdt \;\geq\; &\int_0^{t_C}\int_{\mathbb{R}^N} \varphi^\circ(\nabla u_h(x,t+h))\psi(x,t)\,dxdt \\ &+ \int_0^{t_C}\int_{\mathbb{R}^N} \left(z_h(x,t)\cdot(p(x,t) - \nabla u_h(x,t+h))\right)\psi(x,t)\,dxdt.\end{aligned}$$

Sending $h \to 0$, we find

$$\begin{aligned}\int_0^{t_C}\int_{\mathbb{R}^N} \varphi^\circ(p(x,t))\psi(x,t)\,dxdt \;\geq\; &\int_0^{t_C}\int_{\mathbb{R}^N} \varphi^\circ(\nabla d(x,t))\psi(x,t)\,dxdt \\ &+ \int_0^{t_C}\int_{\mathbb{R}^N} \left(z(x,t)\cdot(p(x,t) - \nabla d(x,t))\right)\psi(x,t)\,dxdt.\end{aligned}$$

We easily deduce that a.e. in $\mathbb{R}^N \times (0,t_C)$, $z \in \partial\varphi^\circ(\nabla d)$. Now, let us choose a nonnegative test function $\psi \in C_c^\infty(\mathbb{R}^N \times (0,t_C))$ such that the support of ψ is out of C^*, then for h small enough $\operatorname{supp}\psi \subset (C_h)^c$, and integrating against the first equation in (30), integrating by parts and passing to the limit, we find (28). We show (29) in the same way. We have checked the inequalities (28-29) in the distributional sense, however, since both $\partial d/\partial t$ and $\operatorname{div} z$ are nonnegative (see Remark 7.4), they also are Radon measures and (28-29) must be true in the sense of measures, in their respective domains.

It remains to check that $\nabla u_h(x,t+h) \to \nabla d(x,t)$ in $L^2_{\rm loc}(\mathbb{R}^N \times (0,t_C))$. Let us consider equation (25): if we multiply it by $(u - d_C^\varphi)e^{-|x|}$, and take the integral over $\mathbb{R}^N$, we find that

$$h\int_{\mathbb{R}^N} z(x)\cdot\nabla((u(x) - d_C^\varphi(x))e^{-|x|})\,dx \;+\; \int_{\mathbb{R}^N} |u(x) - d_C^\varphi(x)|^2 e^{-|x|}\,dx \;=\; 0.$$

Hence (using Lemma 5.8),

$$\int_{\mathbb{R}^N} |u(x) - d_C^\varphi(x)|^2 e^{-|x|}\,dx \;\leq\; 2h\int_{\mathbb{R}^N} e^{-|x|}\,dx \,+\, Ch\int_{\mathbb{R}^N} |u(x) - d_C^\varphi(x)| e^{-|x|}$$

where $C = \max_{\varphi(z)\leq 1}|z|$. Using Cauchy–Schwarz inequality, we deduce

$$\int_{\mathbb{R}^N} |u(x) - d_C^\varphi(x)|^2 e^{-|x|}\,dx \;\leq\; C'h\left(2 + C\left(\int_{\mathbb{R}^N} |u(x) - d_C^\varphi(x)|^2 e^{-|x|}\,dx\right)^{\frac{1}{2}}\right)$$

where $C' = N\omega_N\Gamma(N)$. We deduce that $\int_{\mathbb{R}^N} |u(x) - d_C^\varphi(x)|^2 e^{-|x|}\,dx = O(h)$. In particular, for any $t < t_h$, $\int_{\mathbb{R}^N} |u_h(x,t+h) - d_h(x,t)|^2 e^{-|x|}\,dx \to 0$ as $h \to 0$ and since $\varphi^\circ(\nabla u_h) \leq 1$ a.e., we deduce that $u_h(x,t+h)$ goes locally uniformly (in x) to $d(x,t)$ as $h \to 0$ (since d

is the limit of d_h). Using the same argument as in the last paragraph of Section 4 (based on the integration by parts (24)) we deduce that $\nabla u_h(x, t+h)$ goes to $\nabla d(x,t)$, first in $L^2_{\rm loc}(\mathbb{R}^N)$ for each $t \in [0, t_C)$, and by Lebsegue's dominated convergence theorem, also in $L^2_{\rm loc}(\mathbb{R}^N \times (0, t_C))$. Notice that using the same argument as in the proof of Lemma 7.2, we deduce (since also u_h is nondecreasing in t) that $u_h(x, t+h)$ goes to $d(x,t)$ locally uniformly in $\mathbb{R}^N \times [0, t_C)$. □

Remark 7.3 In fact, one shows similarly that $d_h(x,t)$ and $u_h(x, t+h)$ converge to d for $t \in [t_C, t_C^*)$ at least at all points where d is continuous (in time, a careful analysis shows then that it is the case except at an at most countable number of points). One deduces that (28) holds up to t_C^*. We expect this inequation can be used to show that $t_C^* = t_C$ (since $C^*(t)$ is $(N-1)$–dimensional for $t_C \le t \le t_C^*$).

Remark 7.4 Since for all x and $t \le t_h$, one has $\operatorname{div} z_h(x,t) \ge 0$ (indeed $\operatorname{div} z_h$ is the (Lipschitz) function $(u_h - d_h)/h$, which is known to be nonnegative), in the limit one also deduces $\operatorname{div} z \ge 0$, which is not obvious out of C^*. On the other hand, one also clearly has $\partial d/\partial t \ge 0$.

Remark 7.5 Other properties can be shown about the measure $\partial d/\partial t$, for instance, it is locally L^∞ inside C_* since one may show that $\partial d/\partial t \le -(N-1)/d$ where $d < 0$. We conjecture that (28) and (29) yield that $\partial C_*(t)$ is a convex flow by crystalline curvature in the classical sense. This is the subject of future studies.

A Hausdorff distance

A.1 Definition

In this section, we recall some properties of the Hausdorff distance and convergence. The Hausdorff distance between two sets A and B in $\mathbb{R}^N$ is defined as

$$d_{\mathcal{H}}(A,B) \;=\; \max\left\{\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\right\}$$

where the distance of a point x to a set E is defined, as usual, by $\operatorname{dist}(x,E) = \inf_{y\in E}|x-y|$. Equivalently, one has

$$d_{\mathcal{H}}(A,B) \;=\; \inf\left\{t>0 \,:\, X \subseteq Y^t \text{ and } Y \subseteq X^t\right\},$$

where E^t is the set $\{x \,:\, \operatorname{dist}(x,E) \le t\}$.

Let us show that

$$d_{\mathcal{H}}(A,B) \;=\; \sup_{x\in\mathbb{R}^N} |\operatorname{dist}(x,A) - \operatorname{dist}(x,B)|. \qquad (31)$$

Since by definition, $d_{\mathcal{H}}(A,B) = \sup_{x\in A\cup B} |\mathrm{dist}\,(x,A) - \mathrm{dist}\,(x,B)|$, we need to show that for any $x \notin A\cup B$, $|\mathrm{dist}\,(x,A) - \mathrm{dist}\,(x,B)| \le d_{\mathcal{H}}(A,B)$. Let $t \ge d_{\mathcal{H}}(A,B)$. Given $x \notin A \cup B$, let $a_n \in A$ such that $\mathrm{dist}\,(x,A) = \lim_{n\to\infty} |x - a_n|$. For all n, since $a_n \in A \subset B^t$, there exists $b_n \in B$ with $|a_n - b_n| \le t + 1/n$. Then, $\mathrm{dist}\,(x,B) \le |x - b_n| \le |x - a_n| + t + 1/n$. Passing to the limit, we find that $\mathrm{dist}\,(x,B) - \mathrm{dist}\,(x,A) \le t$. Interverting A and B we find that $|\mathrm{dist}\,(x,A) - \mathrm{dist}\,(x,B)| \le t$, hence $|\mathrm{dist}\,(x,A) - \mathrm{dist}\,(x,B)| \le d_{\mathcal{H}}(A,B)$ for any $x \notin A\cup B$. This shows the identity (31).

We say that the sequence of sets $(A_n)_{n\ge1}$ converges to A in the Hausdorff sense if and only if $d_{\mathcal{H}}(A_n, A) \to 0$ and A is closed (this assumption being added in order for the limit to be unique, since for any $A, B \subset \mathbb{R}^N$, $d_{\mathcal{H}}(A,B) = d_{\mathcal{H}}(\overline{A}, \overline{B})$). By (31), it means that the distance to A_n converges uniformly in $\mathbb{R}^N$ to the distance to A.

A.2 Characterization of limits

Let $(A_n)_{n\ge1}$ be a uniformly bounded sequence of sets, and let A be a closed set. Then it is well known that A_n goes to A in the Hausdorff sense as n goes to infinity if and only if

(i) Given any sequence $(x_n)_{n\in N}$ with $x_n \in A_n$, if x is the limit of a subsequence of $(x_n)_{n\ge1}$, then $x \in A$;

(ii) For every $x \in A$, there exists a sequence of points $x_n \in A_n$ such that $\lim_{n\to\infty} x_n = x$.

A.3 Sets with bounded complement

Assume $(A_n)_{n\ge1}$ is a sequence of unbounded sets such that their *complements* A_n^c are uniformly bounded in $\mathbb{R}^N$, and $A \subset \mathbb{R}^N$: then again, the sets A_n converge to A if and only if (i) and (ii) hold. Indeed, if R is large enough so that $A_n^c \subset B(0, R-1)$ for any n, then A_n goes to A if and only if $\overline{B}(0,R)^c \subset A$ and the bounded sets $\overline{B}(0,R) \cap A_n$ converge to $\overline{B}(0,R) \cap A$: then we can use the characterization given in A.2.

A.4 Compactness

Let $(A_n)_{n\ge1}$ be a sequence of sets in $\mathbb{R}^N$, such that either $(A_n)_{n\ge1}$, or their complements $(A_n^c)_{n\ge1}$, are uniformly bounded. Then, there exists a closed set $A \subset \mathbb{R}^N$ such that, up to a subsequence, $A_n \to A$ in the Hausdorff sense.

For each $n \ge 1$, let $d_n(x) = \mathrm{dist}\,(x, A_n)$. Since the functions d_n are (locally) uniformly bounded and equicontinuous (being all 1–Lipschitz) then, up to a subsequence (still denoted by $(d_n)_{n\ge1}$), they converge locally uniformly to some function d. Let $A = \{d = 0\}$. Then, if $x_n \in A_n$ for all n and $x_{n_k} \to x$ as $k \to \infty$, since the subsequence (x_{n_k}) is bounded one has $d_{n_k}(x_{n_k}) \to d(x)$ as $k \to \infty$, so that $d(x) = 0$ (since $d_n(x_n) = 0$ for all n) and $x \in A$.

Let now $x \in A$: then $d_n(x) \to 0$ as $n \to \infty$. Hence, for any n, there exists $x_n \in A_n$, with $\lim_{n\to\infty} |x_n - x| = 0$. We have checked both points (i) and (ii) of paragraph A.2. Hence A is the Hausdorff limit of the sets A_n. By (31), we deduce that $d(x) = \operatorname{dist}(x, A)$, and that $\lim_{n\to\infty} \sup_{x\in\mathbb{R}^N} |d_n(x) - d(x)| = 0$.

To summarize, for such sets, $A_n \to A$ as $n \to \infty$ if and only if $\operatorname{dist}(x, A_n) \to \operatorname{dist}(x, A)$ locally uniformly, if and only if $\operatorname{dist}(x, A_n) \to \operatorname{dist}(x, A)$ uniformly in $\mathbb{R}^N$.

A.5 Signed distance function

For any set $E \subset \mathbb{R}^N$ let $d_E(x) = \operatorname{dist}(x, E) - \operatorname{dist}(x, E^c)$ be the signed distance to ∂E. Let $(A_n)_{n\geq 1}$ be a sequence of uniformly bounded sets in $\mathbb{R}^N$. Up to a subsequence, d_{A_n} converges to some Lipschitz function d (locally uniformly, and, in fact, uniformly by the previous results). From the paragraph above, we have that $A^* = \{d \leq 0\}$ is the Hausdorff limit of the sets A_n, whereas $A_* = \{d < 0\} \subset A^*$ is the complement of the Hausdorff limit of the sets A_n^c. Also, since $|d_{A_n}(x)| = \operatorname{dist}(x, \partial A_n)$, $\{d = 0\} = A^* \setminus A_*$ is the limit of the boundaries ∂A_n, and $|d|$ is the distance to $A^* \setminus A_*$.

In particular, if $A_* = \operatorname{int}(A^*)$, then the boundaries ∂A_n converge in the Hausdorff sense to $\partial A^* = A^* \setminus A_*$, and $d = d_{A^*}$.

If A_n is convex for all n, then d_{A_n} is convex, so that also d is convex, and A_*, A^* are also convex. One has $|\nabla d| = 1$ a.e. in $\mathbb{R}^N$, since one can show that $\nabla d_{A_n} \to \nabla d$ in $L^2_{loc}(\mathbb{R}^N)$ (see (24) and the last paragraph of Section 4). In particular, we deduce $|\{d = 0\}| = |A^* \setminus A_*| = 0$. One must have $\min d \leq 0$ (since $\min d_{A_n} \leq 0$ for all n). If $\min d < 0$, then $A_* = \operatorname{int}(A^*)$, $A^* = \overline{A_*}$, and $d = d_{A^*} = d_{A_*}$, in this case, the boundaries ∂A_n converge to $\partial A^* = \partial A_*$. If $\min d = 0$, A^* has empty interior, since it is negligible, and, again, $\partial A_n \to \partial A^* = A^*$.

Let us observe that all that has been said about the distance functions remains valid if the Euclidean distance $|x - y|$ is replaced with some arbitrary anisotropic distance of the form $\varphi(x - y)$, with φ an even, convex, one–homogeneous function in $\mathbb{R}^N$, equivalent to the Euclidean norm.

A.6 Nonincreasing sequences of sets

Let $(A_n)_{n\geq 1}$ be a sequence of closed sets in $\mathbb{R}^N$, such that either $(A_n)_{n\geq 1}$, or their complements $(A_n^c)_{n\geq 1}$, are uniformly bounded. Assume $A_n \supseteq A_{n+1}$ for any n and $A = \bigcap_{n\geq 1} A_n$. Then, $A_n \to A$ in the Hausdorff sense, and $(A_n)^c \to \overline{A^c}$, as $n \to \infty$. In particular, one deduces that ∂A_n goes to ∂A (and $d_{A_n} \to d_A$).

Let us show this: first, if $x \in A$, then $x \in A_n$ for all n hence it is a limit of a sequence (x_n) with $x_n \in A_n$ for all n. Conversely, if $x_n \in A_n$ and $x_{n_k} \to x$, assume that $x \notin A$. Then

$x \notin A_n$ for some $n \geq 1$, hence there exists $\rho > 0$ such that $B(x,\rho) \cap A_n = \emptyset$. Now, if k is large enough $x_{n_k} \in B(x,\rho)$ and $n_k \geq n$, a contradiction since $A_{n_k} \subset A_n$. This shows that A is the Hausdorff limit of $(A_n)_{n\geq 1}$. Let us show that $\overline{A^c}$ is the limit of the complements A_n^c (or of their closure, which is equivalent). If $x_n \notin A_n$ for all n, since $\bigcup_{n=1}^\infty A_n^c = A^c$, then any cluster point of $\{x_n : n \geq 1\}$ is in $\overline{A^c}$. Conversely, if $x \in \overline{A^c}$, there exists $y_k \in A^c$ with $y_k \to x$ as $k \to \infty$. Now, for any k, there exists $n(k)$ such that $y_k \in A^c_{n(k)}$. By induction we may build the sequence $n(k)$ in order to always have $n(k+1) > n(k)$. Then, we define x_n as follows: we let $x_n = y_k$ whenever $n(k) \leq n < n(k+1)$. We have $x_n \in A_n^c$ for all n, and $x_n \to x$.

B Explicit solution for $d = \varphi$.

In this section we show how to compute the solution of

$$-h \operatorname{div} \partial\varphi^\circ(Du) \ + \ u \ = \ \varphi \tag{32}$$

in $\mathbb{R}^N$. We also consider the problem

$$\min_{u\in BV(W(0,R))} \int_{W(0,R)} |Du| \ + \int_{W(0,R)} \frac{(u(x)-\varphi(x))^2}{2h}\, dx\,, \tag{33}$$

for $R > 0$ finite. By Theorem 3, equation (32) has a convex solution $\bar u_h$ (larger than φ), whereas by Lemma 5.4, it is unique in the class of functions with bounded level sets $[u < s]$, $s \in \mathbb{R}$. We will show that the solutions $\bar u_{R,h}$ of (33) coincide with $\bar u_h$, in a Wulff shape $W(0,\rho(R,h))$ for some $\rho(R,h)$ close to R ($\rho(R,h) \simeq R - \sqrt{2h}$ as $R \to \infty$ and $h \to 0$). In particular, we have $\bar u_{R,h} \to \bar u_h$ as $R \to \infty$.

Let us first consider Problem (33). If v is Lipschitz,

$$\int_{W(0,R)} \varphi^\circ(Dv) = \int_{W(0,R)} \varphi^\circ(\nabla v(x))\, dx = \int_0^R \int_{\{\varphi(x)=s\}} \varphi^\circ(\nabla v(x))\, \frac{d\mathcal{H}^{N-1}(x)}{|\nabla\varphi(x)|}\, ds,$$

hence

$$\int_{W(0,R)} \varphi^\circ(Dv) \ = \int_{\{\varphi(\xi)=1\}} \frac{d\mathcal{H}^{N-1}(\xi)}{|\nabla\varphi(\xi)|} \int_0^R \varphi^\circ(\nabla v(s\xi))\, ds.$$

As $\varphi^\circ(\nabla v) \geq |\nabla v \cdot \xi|$ for any ξ with $\varphi(\xi) \leq 1$ (here we use the fact φ° is even, otherwise one just has $\varphi^\circ(\nabla v) \geq (\nabla v \cdot \xi)^+$, which probably yields the same result), we find

$$\int_{W(0,R)} \varphi^\circ(Dv) \ \geq \int_{\{\varphi(\xi)=1\}} \frac{d\mathcal{H}^{N-1}(\xi)}{|\nabla\varphi(\xi)|} \int_0^R |\nabla v(s\xi)\cdot\xi|\, s^{N-1}\, ds.$$

Denoting by $v_\xi(s)$ the function $s \mapsto v(s\xi)$, we get

$$\int_{W(0,R)} \varphi^\circ(Dv) \ \geq \int_{\{\varphi(\xi)=1\}} \frac{d\mathcal{H}^{N-1}(\xi)}{|\nabla\varphi(\xi)|} \int_0^R |v_\xi'(s)| s^{N-1}\, ds,$$

with equality whenever v is of the form $f \circ \varphi$.

On the other hand,

$$\int_{W(0,R)} \frac{\left(v(x) - d_C^\varphi(x)\right)^2}{2h}\, dx = \int_{\{\varphi(\xi)=1\}} \frac{d\mathcal{H}^{N-1}(\xi)}{|\nabla\varphi(\xi)|} \int_0^R \frac{\left(v_\xi(s) - s\right)^2}{2h} s^{N-1}\, ds\,.$$

We see that, for v a Lipschitz function,

$$\int_{W(0,R)} \varphi^\circ(Dv) + \int_{W(0,R)} \frac{\left(v(x) - d_C^\varphi(x)\right)}{2h}\, dx$$
$$\geq \int_{\{\varphi(\xi)=1\}} \frac{d\mathcal{H}^{N-1}(\xi)}{|\nabla\varphi(\xi)|} \int_0^R \left(|v_\xi'(s)| + \frac{\left(v_\xi(s) - s\right)^2}{2h} \right) s^{N-1}\, ds\,. \quad (34)$$

This inequality is easily extended by approximation to any $v \in BV(W(0,R))$. We deduce that the unique minimizer $\bar{u}_{R,h}$ of (33) is of the form $f_{R,h}(\varphi(x))$, with $f_{R,h}$ minimizing (over f) the integral

$$\int_0^R \left(|f'(s)| + \frac{\left(f(s) - s\right)^2}{2h} \right) s^{N-1}\, ds$$

that appears in (34). In particular, $f_{R,h}(|x|)$ is the solution of (33) in the isotropic case $\varphi = \varphi^\circ = |\cdot|$. (In fact, by uniqueness and symmetry it was already clear that this solution is radial.)

In the isotropic case, the Euler equation for (33) is

$$-h \operatorname{div} z \ + \ \bar{u}_{R,h} \ = \ |x| \quad (35)$$

in $B(0,R)$, where $|z| \leq 1$ and $z \cdot D\bar{u}_{R,h} = |D\bar{u}_{R,h}|$, and, at the boundary $\partial B(0,R)$, $z \cdot \nu = 0$. Simple scaling arguments show that $\bar{u}_{R,h}(x) = R\bar{u}_{1,h/R^2}(x/R)$, hence it is enough to consider the case $R = 1$.

By classical rearrangement arguments, one sees that $\bar{u}_{1,h}$ is radially nondecreasing (in other words, $f_{1,h}$ is nondecreasing). Hence there are essentially two situations. Either $\bar{u}_{1,h}$ is radially increasing, and $z = x/|x|$: in this case, the Euler equation shows that $\bar{u}_{1,h}(x) = |x| + h(N-1)/|x|$. Otherwise $\bar{u}_{1,h}$ is constant.

There exist, hence, two radii α and β such that $\bar{u}_{1,h}(x) = |x| + h(N-1)/|x|$ if $\alpha \leq |x| \leq \beta$, $\bar{u}_{1,h} = c_\alpha = \alpha + h(N-1)/\alpha$ if $|x| \leq \alpha$, and $\bar{u}_{1,h} = c_\beta = \beta + h(N-1)/\beta$ if $\beta \leq |x| \leq 1$. When $x \in [\alpha, \beta]$, the field z in the Euler equation is $x/|x|$, otherwise, it is of the form $\sigma(|x|)x/|x|$ with $|\sigma| \leq 1$. Then, the Euler equation yields $\sigma'(r) + (N-1)\sigma(r)/r = (c - r)/h$ with $c = c_\alpha$ if $r \leq \alpha$ and $c = c_\beta$ if $r \geq \beta$. We deduce that

$$\sigma(r) \ = \ -\frac{r^2}{h(N+1)} + c\frac{r}{hN} + \frac{d}{r^{N-1}}, \quad (36)$$

where $c = c_\alpha, c_\beta$ (depending on r) and the second constant $d = d_\alpha$ if $r \leq \alpha$, $d = d_\beta$ if $r \geq \beta$ is to be determined. If $r \leq \alpha$, the constraint $|\sigma(r)| \leq 1$ yields $d_\alpha = 0$, and α is given by the radial continuity of σ which yields $\sigma(\alpha) = 1$. One shows that it implies $\alpha = \sqrt{h(N+1)}$, and $c_\alpha = 2N\sqrt{h}/\sqrt{N+1}$. We deduce

$$z(x) \;=\; \left(1 - \left(\frac{|x|}{h\sqrt{N+1}} - 1\right)^2\right)\frac{x}{|x|}$$

if $|x| \leq \sqrt{h(N+1)}$.

It remains to find β, d_β, for $|x| \geq \beta$. The Neumann boundary condition for z shows that $\sigma(1) = 0$, whereas by continuity $\sigma(\beta) = 1$. Hence

$$d_\beta \;=\; \frac{1}{h(N+1)} - \frac{c_\beta}{hN} \quad \text{and} \quad -\frac{\beta^2}{h(N+1)} + c_\beta\frac{\beta}{hN} + \frac{d_\beta}{\beta^{N-1}} \;=\; 1\,.$$

We deduce the following equation for β (remember $c_\beta = \beta + h(N-1)/\beta$):

$$h\beta^{N-1} \;+\; \frac{\beta^{N+1}-1}{N+1} \;-\; \left(\beta + \frac{h(N-1)}{\beta}\right)\frac{\beta^N - 1}{N} \;=\; 0,$$

which can be written in polynomial form as

$$P(\beta) := -\frac{\beta^{N+2}}{N(N+1)} + \frac{h}{N}\beta^N + \frac{\beta^2}{N} - \frac{\beta}{N+1} + \frac{h(N-1)}{N} = 0. \tag{37}$$

Notice that in the sequence of the coefficients of P, $\left(\frac{h(N-1)}{N}, -\frac{1}{N+1}, \frac{1}{N}, \frac{h}{N}, -\frac{1}{N(N+1)}\right)$, the number of consecutive sign changes is 3. By Sturm's Theorem [22, p. 69] the number of roots of (37) in $(0, +\infty)$ is at most 3. Since, by Descartes' Rule [22, p. 69], the number of sign changes of the above sequence minus the number of roots of (37) is an even number, the number of roots of P must be either 1 or 3. Observe that $P(0) > 0$, $P(\alpha) = P(\sqrt{h(N+1)}) = 2h - \sqrt{\frac{h}{N+1}} < 0$ when $2\sqrt{h(N+1)} < 1$. Thus, for h small enough, there is a root of P in $(0, \alpha)$. On the other hand, since $P(1) = \frac{h}{N}$ and $P(+\infty) = -\infty$, there is a second root of P in $(1, \infty)$. Let us compute the third root which is in $(\alpha, 1)$. Indeed, one deduces

$$h \;=\; \frac{\frac{N}{N+1} + \frac{\beta^{N+1}}{N+1} - \beta}{\beta^{N-1} + \frac{N-1}{\beta}}\,.$$

Writing $\beta = 1 - \delta$, with δ close to 0 as $h \to 0$, we find

$$\sqrt{h} \;=\; \frac{\delta}{\sqrt{2}}\left(1 - \frac{N-1}{6}\delta + o(\delta)\right),$$

and the local inversion theorem hence yields that

$$\beta \;=\; 1 - \sqrt{2h} - \frac{N-1}{3}h + o(h)$$

as $h \to 0$. Equation (37) has no other solution in the interval (α, β).

Thus, the vector field $z(x) = \frac{x}{|x|}$ when $\alpha \leq |x| \leq \beta$ and $z(x) = \sigma(|x|)\frac{x}{|x|}$ when $0 \leq |x| \leq \alpha$ or $\beta \leq |x| \leq 1$, and $\sigma(|x|)$ being of the form (36) with corresponding constants c_α, d_α and

c_β, d_β depending on the region, satisfies $|z| \leq 1$, and together with $\bar{u}_{1,h}$ is a solution of (35) in $\mathcal{D}'(B(0,1))$. Moreover, since $\sigma(1) = 0$, we have $z \cdot \nu = 0$. Finally, one easily proves that $z \cdot D\bar{u}_{1,h} = |D\bar{u}_{1,h}|$. Thus, $\bar{u}_{1,h}$ is the solution of (35).

In general, we deduce that for some $\rho(R,h)$ satisfying

$$\rho(R,h) \; = \; R - \sqrt{2h} - \frac{N-1}{3}\frac{h}{R} + o\left(\frac{h}{R}\right) \tag{38}$$

the solution of (33) is given by

$$\bar{u}_{R,h}(x) \; = \begin{cases} \sqrt{h}\dfrac{2N}{\sqrt{N+1}} & \text{if } \varphi(x) \leq \sqrt{h(N+1)}\,, \\ \varphi(x) \, + \, h\dfrac{N-1}{\varphi(x)} & \text{if } \sqrt{h(N+1)} \leq \varphi(x) \leq \rho(R,h)\,, \\ \rho(R,h) \, + \, h\dfrac{N-1}{\rho(R,h)} & \text{if } \rho(R,h) \leq \varphi(x) \leq R\,. \end{cases} \tag{39}$$

In particular, this solution is independent of R in $W(0,R')$ as soon as $\rho(R,h) \geq R'$, which is always true for R large enough. As $R \to \infty$, it hence converges to a solution $\bar{u}_h$ of (32), given by

$$\bar{u}_h(x) \; = \begin{cases} \sqrt{h}\dfrac{2N}{\sqrt{N+1}} & \text{if } \varphi(x) \leq \sqrt{h(N+1)}\,, \\ \varphi(x) \, + \, h\dfrac{N-1}{\varphi(x)} & \text{if } \varphi(x) \geq \sqrt{h(N+1)}\,. \end{cases} \tag{40}$$

C A comparison principle

Let $\alpha \geq 2$, $T_k(r) := \max(\min(r,k), -k)$, $T_k^+(r) = \max(T_k(r), 0)$ $(k \geq 0)$ and let $j_\alpha = rT_k^+(r)^{\alpha-1}$.

Theorem 6 *Let $u, \overline{u} \in L^1_{\text{loc}}(\mathbb{R}^N)$ be two solutions of (13) corresponding to the right hand sides $g, \overline{g} \in L^\alpha_{\text{loc}}(\mathbb{R}^N)$, $\alpha \geq 2$, respectively. Then*

$$\Big(\int_{\mathbb{R}^N} j_\alpha(u - \overline{u})\varphi^\alpha\Big)^{1/\alpha} \leq \Big(\int_{\mathbb{R}^N} (g - \overline{g})^{+\alpha}\varphi^\alpha\Big)^{1/\alpha} + 2M\Big(\int_{\mathbb{R}^N} |\nabla\varphi|^\alpha\Big)^{1/\alpha} \tag{41}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$.

Proof. Let $p(r) := \alpha T_k^+(r)^{\alpha-1}$, $p^*(r) := j^{*\prime}(r) = -p(-r)$. Let $z, \overline{z} \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ with $(z, Dp(u)) = F(Dp(u))$, $(\overline{z}, Dp(\overline{u})) = F(Dp(\overline{u}))$, and such that

$$u - \text{div } z = g \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{42}$$

$$\overline{u} - \text{div } \overline{z} = \overline{g} \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{43}$$

Let $l_1, l_2 \in \mathbb{R}$, $r, \overline{r} \in \mathbb{R}^N$. Multiplying (42) by $p(u-l_1)$ and (43) by $p(\overline{u}-l_2)$, adding and substracting $\int_{\mathbb{R}^N} r \cdot \nabla_x \eta \, p(u-l_1)$, and $\int_{\mathbb{R}^N} \overline{r} \cdot \nabla_x \eta \, p(\overline{u}-l_2)$, respectively, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} up(u-l_1)\eta + \int_{\mathbb{R}^N} \eta \, z \cdot D_x p(u-l_1) + \int_{\mathbb{R}^N} (z-r) \cdot \nabla_x \eta \, p(u-l_1) \\ &+ \int_{\mathbb{R}^N} r \cdot \nabla_x \eta \, p(u-l_1) = \int_{\mathbb{R}^N} gp(u-l_1)\eta, \end{aligned} \tag{44}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N} \overline{u}p^*(\overline{u}-l_2)\eta + \int_{\mathbb{R}^N} \eta \, \overline{z} \cdot D_y p^*(\overline{u}-l_2) + \int_{\mathbb{R}^N} (\overline{z}-\overline{r}) \cdot \nabla_y \eta \, p^*(\overline{u}-l_2) \\ &+ \int_{\mathbb{R}^N} \overline{r} \cdot \nabla_y \eta \, p^*(\overline{u}-l_2) = \int_{\mathbb{R}^N} \overline{g}p^*(\overline{u}-l_2)\eta, \end{aligned} \tag{45}$$

for all $\eta \in C_0^\infty(\mathbb{R}^N)$.

We choose two different variables x, y and consider u, z, g as functions of x and $\overline{u}$, $\overline{z}$, $\overline{g}$ as functions of y. Let $0 \leq \psi \in C_0^\infty(\mathbb{R}^N)$, and (ρ_n) a standard sequence of mollifiers in $\mathbb{R}^N$. Define

$$\eta_n(x,y) := \rho_n(x-y)\psi\left(\frac{x+y}{2}\right) \geq 0.$$

Note that for n sufficiently large,

$$x \mapsto \eta_n(x,y) \in C_0^\infty\left(\mathbb{R}^N\right) \qquad \forall\, y \in \mathbb{R}^N,$$

$$y \mapsto \eta_n(x,y) \in C_0^\infty\left(\mathbb{R}^N\right) \qquad \forall\, x \in \mathbb{R}^N.$$

Hence, for y fixed, if we take $l_1 = \overline{u}(y)$ and $r = \overline{z}(y)$ in (44), we get

$$\begin{aligned} &\int_{\mathbb{R}^N} up(u-\overline{u}(y))\eta_n + \int_{\mathbb{R}^N} \eta_n \, z \cdot D_x p(u-\overline{u}(y)) \\ &+ \int_{\mathbb{R}^N} (z-\overline{z}(y)) \cdot \nabla_x \eta_n \, p(u-\overline{u}(y)) \\ &+ \int_{\mathbb{R}^N} \overline{z}(y) \cdot \nabla_x \eta_n \, p(u-\overline{u}(y)) = \int_{\mathbb{R}^N} gp(u-\overline{u}(y))\eta_n \end{aligned} \tag{46}$$

Similarly, for x fixed, if we take $l_2 = u(x)$ and $\overline{r} = z(x)$ in (45), we get

$$\begin{aligned} &\int_{\mathbb{R}^N} \overline{u}p^*(\overline{u}-u(x))\eta_n + \int_{\mathbb{R}^N} \eta_n \, \overline{z} \cdot D_y p^*(\overline{u}-u(x)) \\ &+ \int_{\mathbb{R}^N} (\overline{z}-z(x)) \cdot \nabla_y \eta_n \, p^*(\overline{u}-u(x)) \\ &+ \int_{\mathbb{R}^N} z(x) \cdot \nabla_y \eta_n \, p^*(\overline{u}-u(x)) = \int_{\mathbb{R}^N} \overline{g}p^*(u-\overline{u}(y))\eta_n. \end{aligned} \tag{47}$$

Now, since $p^*(r) = -p(-r)$, we can rewrite (47) as

$$\begin{aligned}
&-\int_{\mathbb{R}^N} \overline{u}p(u(x)-\overline{u})\eta_n - \int_{\mathbb{R}^N} \eta_n\,\overline{z}\cdot D_y p(u(x)-\overline{u}) \\
&+\int_{\mathbb{R}^N} (z(x)-\overline{z})\cdot\nabla_y\eta_n\; p(u(x)-\overline{u}) \qquad (48)\\
&-\int_{\mathbb{R}^N} z(x)\cdot\nabla_y\eta_n\, p(u(x)-\overline{u}) = -\int_{\mathbb{R}^N} \overline{g}p(u-\overline{u})\eta_n.
\end{aligned}$$

Integrating (46) with respect to y and (48) with respect to x and taking the sum yields

$$\begin{aligned}
&\int_{\mathbb{R}^N\times\mathbb{R}^N} (u-\overline{u})p(u-\overline{u})\eta_n \\
&+\int_{\mathbb{R}^N\times\mathbb{R}^N} \eta_n\, z\cdot D_x p(u-\overline{u}) - \int_{\mathbb{R}^N\times\mathbb{R}^N} \eta_n\,\overline{z}\cdot D_y p(u-\overline{u}) \\
&+\int_{\mathbb{R}^N\times\mathbb{R}^N} \big(z-\overline{z}\big)\cdot\big(\nabla_x\eta_n+\nabla_y\eta_n\big)\, p(u-\overline{u}) \qquad (49)\\
&+\int_{\mathbb{R}^N\times\mathbb{R}^N} \overline{z}\cdot\nabla_x\eta_n\, p(u-\overline{u}) - \int_{\mathbb{R}^N\times\mathbb{R}^N} z\cdot\nabla_y\eta_n\, p(u-\overline{u}) \\
&= \textstyle\int_{\mathbb{R}^N\times\mathbb{R}^N} (g-\overline{g})p(u-\overline{u})\eta_n.
\end{aligned}$$

Let us observe that

$$\begin{aligned}
&\int_{\mathbb{R}^N\times\mathbb{R}^N} \eta_n\, z\cdot D_x p(u-\overline{u}) + \int_{\mathbb{R}^N\times\mathbb{R}^N} \overline{z}\cdot\nabla_x\eta_n\, p(u-\overline{u}) \\
=\; &\int_{\mathbb{R}^N\times\mathbb{R}^N} (F(Dp(u-\overline{u})) - \overline{z}\cdot D_x p(u-\overline{u}))\eta_n \geq 0
\end{aligned}$$

Similarly, we have

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \eta_n\,\overline{z}\cdot D_y p(u-\overline{u}) - \int_{\mathbb{R}^N\times\mathbb{R}^N} z\cdot\nabla_y\eta_n\, p(u-\overline{u}) \geq 0$$

Taking into account these last two estimates we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N\times\mathbb{R}^N} (u-\overline{u})p(u-\overline{u})\eta_n + \int_{\mathbb{R}^N\times\mathbb{R}^N} \big(z-\overline{z}\big)\cdot\big(\nabla_x\eta_n+\nabla_y\eta_n\big)\, p(u-\overline{u}) \\
\leq\; &\int_{\mathbb{R}^N\times\mathbb{R}^N} (g-\overline{g})p(u-\overline{u})\eta_n.
\end{aligned}$$

Since

$$\nabla_x\eta_n + \nabla_y\eta_n = \rho_n(x-y)\nabla\psi\Big(\frac{x+y}{2}\Big)$$

letting $n \to \infty$ in the above inequality we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} (u-\overline{u})p(u-\overline{u})\psi &\leq \int_{\mathbb{R}^N} (g-\overline{g})p(u-\overline{u})\psi \\ &+ \int_{\mathbb{R}^N} \left(\overline{z}-z\right)\cdot\nabla\psi\, p(u-\overline{u}).\end{aligned}$$

Let us replace ψ by ψ^α, $\psi \in C_0^\infty(\mathbb{R}^N)$, $\psi \geq 0$. Let $M>0$ be such that $\|z\|_\infty \leq M$, $\|\overline{z}\|_\infty \leq M$. Since $p(r) = \alpha T_k^+(r)^{\alpha-1}$, afetr dividing by α we may write the above inequality as

$$\begin{aligned}\int_{\mathbb{R}^N} j_\alpha(u-\overline{u})\psi^\alpha &\leq \int_{\mathbb{R}^N} (g-\overline{g})T_k^+(u-\overline{u})^{\alpha-1}\psi^\alpha \\ &+ 2M\int_{\mathbb{R}^N} \psi^{\alpha-1}|\nabla\psi|\, T_k^+(u-\overline{u})^{\alpha-1}.\end{aligned}$$

Since

$$\begin{aligned}&\int_{\mathbb{R}^N} (g-\overline{g})T_k^+(u-\overline{u})^{\alpha-1}\psi^\alpha \leq \int_{\mathbb{R}^N} (g-\overline{g})^+T_k^+(u-\overline{u})^{\alpha-1}\psi^\alpha \\ \leq\ &\Big(\int_{\mathbb{R}^N} T_k^+(u-\overline{u})^\alpha\psi^\alpha\Big)^{(\alpha-1)/\alpha}\Big(\int_{\mathbb{R}^N} (g-\overline{g})^{+\alpha}\psi^\alpha\Big)^{1/\alpha},\end{aligned}$$

$$\int_{\mathbb{R}^N} \psi^{\alpha-1}|\nabla\psi|\, T_k^+(u-\overline{u})^{\alpha-1} \leq \Big(\int_{\mathbb{R}^N} T_k^+(u-\overline{u})^\alpha\psi^\alpha\Big)^{(\alpha-1)/\alpha}\Big(\int_{\mathbb{R}^N} |\nabla\psi|^\alpha\Big)^{1/\alpha},$$

and $T_k^+(r)^\alpha \leq j_\alpha(r)$ for all $r \in \mathbb{R}$, we obtain

$$\Big(\int_{\mathbb{R}^N} j_\alpha(u-\overline{u})\psi^\alpha\Big)^{1/\alpha} \leq \Big(\int_{\mathbb{R}^N} (g-\overline{g})^{+\alpha}\psi^\alpha\Big)^{1/\alpha} + 2M\Big(\int_{\mathbb{R}^N} |\nabla\psi|^\alpha\Big)^{1/\alpha}.$$

□

Remark C.1 For the proof we only need the inequality $\leq$ both in (44) and (45). This permits to extend the above result. Indeed, we could have defined the notion of solution saying that there exists $z \in L^\infty(\mathbb{R}^N;\mathbb{R}^N)$ with $\operatorname{div} z \in L^2_{\rm loc}(\mathbb{R}^N)$, such that $u - \operatorname{div} z = g$ in $\mathcal{D}'(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} up(u)\eta + \int_{\mathbb{R}^N} \eta F(D_x p(u)) + \int_{\mathbb{R}^N} z\cdot\nabla_x\eta\ p(u) \leq \int_{\mathbb{R}^N} gp(u)\eta, \tag{50}$$

for any $\eta \in C_0^\infty(\mathbb{R}^N)$, $\eta \geq 0$, and any $p \in \mathcal{P}$.

The following corollary proves Theorem 2.

Corollary C.2 *Let $u, \overline{u} \in L^1_{\rm loc}(\mathbb{R}^N)$ be two solutions of (13) corresponding to the right hand sides $g, \overline{g} \in L^\alpha_{\rm loc}(\mathbb{R}^N)$, respectively. Assume that $\alpha > \max(N,2)$. If $g \leq \overline{g}$, then $u \leq \overline{u}$.*

Proof. Let $\psi \in C_0^\infty$, $0 \le \psi \le 1$, $\psi(x) = 1$ if $x \in B(0,1)$, $\psi(x) = 0$ outside $B(0,2)$. Setting $\psi_n(x) := \psi(\frac{x}{n})$ instead of $\psi(x)$ we get

$$\begin{aligned}\Big(\int_{\mathbb{R}^N} j_\alpha(u-\overline{u})\psi_n^\alpha\Big)^{1/\alpha} &\le \Big(\int_{\mathbb{R}^N} (g-\overline{g})^{+\alpha}\psi_n^\alpha\Big)^{1/\alpha} + 2M\Big(\int_{\mathbb{R}^N} |\nabla\psi_n|^\alpha\Big)^{1/\alpha} \\ &\le 2M\Big(\int_{\mathbb{R}^N} |\nabla\psi_n|^\alpha\Big)^{1/\alpha}.\end{aligned}$$

Now, observe that

$$\int_{\mathbb{R}^N} |\nabla\psi_n|^\alpha = n^{-\alpha}\int_{\mathbb{R}^N} |\nabla\psi(\frac{x}{n})|^\alpha = n^{N-\alpha}\int_{\mathbb{R}^N} |\nabla\psi(x)|^\alpha \to 0$$

as $n \to \infty$. Thus, letting $n \to \infty$, we obtain

$$\int_{\mathbb{R}^N} j_\alpha(u-\overline{u})\,dx \le 0,$$

and, we conclude that $u \le \overline{u}$. □

D Evolutions with different mobilities

In this brief appendix we explain how to adapt our results whenever one considers the anisotropic curvature motion with a mobility different from $\varphi^\circ(\nu)$. The general motion of the interface we consider is governed by the formal equation

$$V(x) \;=\; \psi^\circ(\nu(x))\kappa_\varphi(x)$$

where V is the (Euclidean) normal speed, κ_φ is the anisotropic φ–curvature (that is, $\operatorname{div}\partial\varphi^\circ(\nabla d(x))$), and ψ° is a convex, one–homogeneous function that may be different from φ°. ψ° is the polar of some ψ (that is, $\psi^\circ(\xi) = \sup_{\psi(\eta)\le 1}\eta\cdot\xi$) that has to satisfy, for some constants $c_2 > c_1 > 0$,

$$c_1\varphi \;\le\; \psi \;\le\; c_2\varphi\,.$$

In this case, all the results in this paper hold, replacing each time the distance d^φ with d^ψ, except the results in Section B that concern the explicit evolution of the Wulff shape. However, inner and outer estimates can be easily computed, as follows.

Consider the set $W(0,r) = [\varphi \le r]$, $r > 0$, and the associated problem

$$-h\operatorname{div}\partial\varphi^\circ(\nabla u) \;+\; u \;-\; d^\psi_{W(0,r)} \;\ni\; 0 \;\text{ in } \mathbb{R}^N\,. \tag{51}$$

One easily checks that for any $x \in \mathbb{R}^N$,

$$c_1(\varphi(x)-r)\wedge c_2(\varphi(x)-r) \;\le\; d^\psi_{W(0,r)}(x) \;\le\; c_1(\varphi(x)-r)\vee c_2(\varphi(x)-r)\,.$$

Notice that $c_1(\varphi-r)\wedge c_2(\varphi-r) = c_1(\varphi-r)^+ - c_2(\varphi-r)^-$ and $c_1(\varphi-r)\vee c_2(\varphi-r) = c_2(\varphi-r)^+ - c_1(\varphi-r)^-$. One shows, following the approaches in Section 5 and Appendix B, that if $g(x) = a(\varphi(x) - r)$ for $\varphi(x) \le r$ and $g(x) = b(\varphi(x) - r)$ for $\varphi(x) \ge r$, then, the solution w of

$$-h \operatorname{div} \partial\varphi^\circ(\nabla w) \; + \; w \; - \; g \; \ni \; 0 \text{ in } \mathbb{R}^N$$

is given by

$$w(x) \;=\; \begin{cases} g(x) \;+\; h\dfrac{N-1}{\varphi(x)} & \text{if } \varphi(x) \ge \sqrt{\frac{h}{a}(N+1)}\,, \\ \dfrac{2N\sqrt{ah}}{\sqrt{N+1}} \;-\; ar & \text{otherwise,} \end{cases}$$

as soon as $h \le ar^2/(N+1)$. Hence, if u is the solution of (51), the set $T_h(W(0,r)) = [u < 0]$ satisfies, as soon as h is small enough,

$$W(0, S_{h/c_1}(r)) \;\subseteq T_h(W(0,r)) \;\subseteq W(0, S_{h/c_2}(r))$$

where as in Section 6, $S_h(r) = r(1 + \sqrt{1 - 4h(N-1)/r^2})/2$. We deduce that any limit, as h goes to zero, of the discrete evolution $T_h^{[t/h]}(W(0,r_0))$, is between the evolutions $W(0, \sqrt{r_0^2 - 2(N-1)t/c_1})$ and $W(0, \sqrt{r_0^2 - 2(N-1)t/c_2})$. This allows to extend all the proofs in Section 7 to the case of a general mobility ψ°.

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