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**Abstract**

A global existence theorem is proved for the Landau-Lifshitz-Gilbert equations with biquadratic exchange coupling energy. The main difficulty relies in the cubic nonlinear Neumann boundary condition satisfied by the magnetisation at the interfaces. We use several regularization procedures to obtain global weak solutions with finite energy to the problem.

**Key words.** Ferromagnets, biquadratic exchange coupling .
**1991 AMS subject classifications.** 73R05, 73K05, 47J35, 34G20, 35L10,35K05,46E35.

## 1 Biquadratic exchange coupling energy

We are dealing with a three layers of material constitued by two ferromagnets separeted by a nonmagnetic spacer. The adjacent interfaces of the ferromagnets are coupled via the so called biquadratic exchange coupling energy see [5], [6], [7] for example. The case of bilinear exchange coupling energy was considered in [4] and [13].

In the following, the domain occupied by the ferromagnetic material is denoted by $\Omega = \Omega^+ \cup \Omega^-$ where $\Omega^+ = \widehat{\Omega} \times (h, l)$, $\Omega^- = \widehat{\Omega} \times (-l, -h)$ with $0 < h < l$ and the nonmagnetic spacer occupies the domain $\Omega^0 = \widehat{\Omega} \times (-h, h)$. We denote by $\Gamma^\pm = \widehat{\Omega} \times \{z = \pm h\}$ the adjacent interfaces of the ferromagnetic material. The generic point of $\Omega$ will be denoted by $x = (\widehat{x}, z)$.

Let $M(t, x) \in S^2$ be the magnetization of $\Omega$ at the time $t \geq 0$ in the position $x$ where $S^2$ denotes the unit sphere of $\mathbb{R}^3$. The biquadratic interlayer exchange coupling energy acting between the interfaces $\Gamma^+$ and $\Gamma^-$ takes the form

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$$\mathcal{E}_{bq}(M) = K_{bq} \int_{\widehat{\Omega}} (1 - (M^+ \cdot M^-)^2) d\widehat{x} \tag{1}$$

where $M^{\pm}(t, \widehat{x}) = M(t, \widehat{x}, \pm h)$ and $K_{bq} > 0$ is a physical constant . Since, by using the saturation condition $|M^{\pm}|^2 = 1$, we have $1-(M^+ \cdot M^-)^2 = \frac{1}{4}|M^+ - M^-|^2 |M^+ + M^-|^2$. Hence $\mathcal{E}_{bq}(M) = 0$ if and only if t $M^+ = M^-$ or $M^+ = -M^-$.

Let us precise the model we shall discuss. In the ferromagnetic domain $\Omega$ the magnetization $M(t, x) \in S^2$ satisfies the Landau-Lifshitz-Gilbert (LLG) equations in $\mathbb{R}^+ \times \Omega$

$$\begin{cases} \partial_t M - \alpha M \times \partial_t M = -(1 + \alpha^2) M \times \mathcal{H}(M) \\ M(0, x) = M_0(x), \ |M_0(x)|^2 = 1 \text{ a.e.} \end{cases} \tag{2}$$

The effective magnetic field $\mathcal{H}(M)$ is given by

$$\mathcal{H}(M) = A\Delta M - \nabla_M \psi(M) + \nabla\varphi \tag{3}$$

where $A > 0$ is a fixed constant called the anisotropy exchange constant, $\nabla\varphi$ is the demagnetizing field given by the stray equation (or magnetostatic equation)

$$\nabla \cdot (\nabla\varphi + \chi(\Omega) M) = 0 \ \text{ in } \ \mathbb{R}^+ \times \mathbb{R}^3 \tag{4}$$

$\chi(\Omega)$ being the characteristic function of $\Omega$ and $\nabla_M \psi(M)$ is the volume anisotropy field associated with a regular function $\psi \in C^2(\mathbb{R}^3)$ satisfying $\psi(X) \geq 0$ and $|D^2\psi(X)| \leq C$ for all vector $X \in \mathbb{R}^3$. The magnetization $M(t, x)$ satisfies the saturation condition

$$|M(t, x)|^2 = 1 \ \text{ a.e in } \ \mathbb{R}^+ \times \Omega. \tag{5}$$

The boundary condition satisfied by $M$ on $\mathbb{R}^+ \times (\partial\Omega \setminus (\Gamma^+ \cup \Gamma^-))$ is given by

$$M \times A \frac{\partial M}{\partial n} = 0 \tag{6}$$

while on $\mathbb{R}^+ \times \Gamma^{\pm}$ the biquadratic exchange coupling energy gives (by using the second form of the local biquadratic energy) the boundary condition

$$M^{\pm} \times (\mp A(\frac{\partial M}{\partial z})^{\pm} + K_{bq}(|M^{\pm}|^2 M^{\pm} + |M^{\mp}|^2 M^{\pm} - 2(M^+ \cdot M^-) M^{\mp})) = 0. \tag{7}$$

In this paper we shall discuss the global existence of weak solutions with finite energy of problem (2)-(4)-(5)-(6)-(7).

We use the following notations. $\mathbb{L}^2(Q)$ is the vectorial Lebesgue space $(L^2(Q))^3$ with norm and scalar product denoted respectively by $|\cdot|$ and $(\cdot\ ;\ \cdot)$. The Hilbert space $\mathbb{H}^1(Q)$ is the usual Sobolev space $(H^1(Q))^3$ where $Q$ is an open and regular set of $\mathbb{R}^n$. If $Q$ is a bounded domain of $\mathbb{R}^n$, $|Q|$ will denote its Lebegue measure. At the end let us announce that in the sequel $C$ will reprent various positive constants which are independent upon the different parameters.

## 2 The approximated models



Without loss of generality we set $\alpha = 1$ and $A = 1$. The saturation condition $|M|^2 = 1$ allows to write $\partial_t M = -M \times (M \times \partial_t M)$. Substituting into the Landau-Lifshitz-Gilbert equation we obtain

$$M \times (-\frac{M}{1+|M|} \times \partial_t M - \frac{1}{2}\partial_t M + \Delta M - \nabla_M \psi(M) + \nabla\varphi + pM) = 0 \tag{8}$$

where $p$ is an arbitrarly scalar function depending eventually of $|M|$.

The idea is to obtain the solutions of our problem as a limit of approximated solutions of an intermediary problem which penalizes the saturation condition. The scalar function $p$ plays the role of the penalization operator.

Let $\nu > 0$ and $\eta > 0$ be two fixed small parameters. We introduce the vector function $U$ satisfying in $Q = (0,T) \times \Omega$ with $0 < T < \infty$, the intermediary problem (see [3])

$$\begin{cases} \frac{1}{2}\partial_t U - \Delta U = F_\nu(U) + G(U, \partial_t U) \ \text{ in } \ Q \\ U(0) = M_0, \ |M_0(x)|^2 = 1 \ \text{ in } \ \Omega \\ \dfrac{\partial U}{\partial n} = 0 \ \text{ on } \ (0,T) \times (\partial\Omega \setminus (\Gamma^+ \cup \Gamma^-)) \\ \mp(\dfrac{\partial U}{\partial z})^\pm + K_{bq}(B_\nu^\pm(U^+, U^-) + R_\eta(U^\pm)) = 0 \ \text{ on } \ (0,T) \times \Gamma^\pm \end{cases} \tag{9}$$

with the stray equation

$$\nabla \cdot (\nabla\varphi + \chi(\Omega)U) = 0 \ \text{ in } \ Q^\infty = (0,T) \times \mathbb{R}^3. \tag{10}$$

We set

$$G(U, \partial_t U) = -\frac{U}{1+|U|} \times \partial_t U \tag{11}$$

and

$$F_\nu(U) = -\nabla_U \psi(U) + \mathbb{D}(U) - \frac{1}{\nu}\nabla_U(\gamma(|U|)) \tag{12}$$

where

$$\gamma(y) = \frac{1}{2}(\sqrt{2} - \sqrt{1+y^2})^2 \ , \ y \in \mathbb{R} \tag{13}$$

and $\mathbb{D}$ is the linear opertor $U \mapsto \nabla\varphi$ where $\varphi$ satisfies (10). The boundary operators $B_\nu^\pm(U^+, U^-)$ , $R_\eta(U^\pm)$ are defined by

$$\begin{cases} B_\nu^\pm(U^+, U^-) = \nabla_{U^\pm}(\Phi_\nu(U^+, U^-)) \\ \Phi_\nu(U^+, U^-) = \frac{1}{4\nu}\log(1 + \nu|U^+ + U^-|^2|U^+ - U^-|^2) \\ R_\eta(U^\pm) = \nabla_{U^\pm}\Theta_\eta(U^\pm) \\ \Theta_\eta(U^\pm) = \frac{1}{2\eta}(s(U^\pm) - \log(1 + s(U^\pm))), \ s(U^\pm) = max(|U^\pm|^2 - 1, 0) \end{cases} \tag{14}$$

which is to say

$$\begin{cases} B_\nu^\pm(U^+, U^-) = \dfrac{|U^\pm|^2 U^\pm + |U^\mp|^2 U^\pm - 2(U^+ \cdot U^-)U^\mp}{1 + \nu|U^+ + U^-|^2|U^+ - U^-|^2} \\ R_\eta(U^\pm) = \dfrac{1}{\eta}\,\dfrac{s(U^\pm)U^\pm}{1 + s(U^\pm)}. \end{cases} \tag{15}$$

Notice that $\Theta_\eta(U^\pm)=0$ if and only if $|U^\pm|\le 1$ as well as $\gamma(|U|)=0$ if and only if $|U|=1$ and if we multiply the equation (10) by $\varphi$ and integrate on $\mathbb{R}^3$, we get

$$\int_\Omega \mathbb{D}(U).U dx = \int_\Omega \nabla\varphi.U dx = -|\nabla\varphi|^2_{\mathbb{L}^2(\mathbb{R}^3)} \tag{16}$$

which leads to the estimate

$$|\mathbb{D}(U)|_{\mathbb{L}^2(\mathbb{R}^3)} \le |U|_{\mathbb{L}^2(\Omega)}. \tag{17}$$

Let $W$ be a regular test function. Multiplying the intermediary problem (9) by $U\times W$ and observing that $\nabla_U(\gamma(|U|))\cdot U\times W=0$ then integrating by parts we get the same weak formulation as for problem (2)-(4)-(5)-(6)-(7). Hence the intermediary problem appears as an approximation to our problem.

In order to solve problem (9) we introduce the change of unknown function

$$U=e^{kt}(V_\nu^\eta+M_0) \tag{18}$$

for $k>0$ fixed which will be choosen later. Hence $V_\nu^\eta$ satisfies the intermediary problem

$$\begin{cases} \dfrac{1}{2}\partial_t V_\nu^\eta - \Delta V_\nu^\eta + \dfrac{k}{2}V_\nu^\eta = F_\nu(t,V_\nu^\eta) + G_\nu(t,V_\nu^\eta,\partial_t V_\nu^\eta) + H_{M_0} \ \text{ in } \ Q \\ V_\nu^\eta(0)=0 \ \text{ in } \ \Omega \\ \dfrac{\partial(V_\nu^\eta+M_0)}{\partial n} = 0 \ \text{ on } \ (0,T)\times(\partial\Omega\setminus(\Gamma^+\cup\Gamma^-)) \\ \mp\Big(\dfrac{\partial(V_\nu^\eta+M_0)}{\partial z}\Big)^\pm + K_{bq}B_\nu^\pm(t,V_\nu^{\eta,+},V_\nu^{\eta,-}) \\ +K_{bq}R_\eta(t,V_\nu^{\eta,\pm}) = 0 \ \text{ on } \ (0,T)\times\Gamma^\pm \end{cases} \tag{19}$$

with

$$\begin{cases} F_\nu(t,V) = e^{-kt}F_\nu(e^{kt}(V+M_0)) \\ B_\nu^\pm(t,V^+,V^-) = e^{-kt}B_\nu^\pm(e^{kt}(V^+ + M_0^+), e^{kt}(V^- + M_0^-)) \\ R_\eta(t,V^\pm) = e^{-kt}R_\eta(e^{kt}(V^\pm + M_0^\pm)) \\ G(t,V,\partial_t V) = \frac{e^{kt}(V+M_0)}{1+e^{kt}|V+M_0|}\times\partial_t V \\ H_{M_0} = \Delta M_0 - \frac{k}{2}M_0 \end{cases} \tag{20}$$

Problem (19) looks like a heat equation with a nonlinear Neumann boundary condition on the interfaces except that the force $G$ depends of $\partial_t V_\nu^\eta$. To avoid this difficulty we introduce the following elliptic regularization of (19) where $\varepsilon>0$ is a fixed small parameter (see [8], [9] for example)

$$\begin{cases} -\varepsilon^2 \partial_t^2 V_{\nu,\eta}^\varepsilon - \Delta V_{\nu,\eta}^\varepsilon + \frac{1}{2}\partial_t V_{\nu,\eta}^\varepsilon + \frac{k}{2} V_{\nu,\eta}^\varepsilon = \\ F_\nu(t, V_{\nu,\eta}^\varepsilon) + G_\nu(t, V_{\nu,\eta}^\varepsilon, \partial_t V_{\nu,\eta}^\varepsilon) + H_{M_0} \text{ in } \ Q \\ V_{\nu,\eta}^\varepsilon(0) = 0, \ \partial_t V_{\nu,\eta}^\varepsilon(T) = 0 \text{ in } \ \Omega \\ \dfrac{\partial (V_{\nu,\eta}^\varepsilon + M_0)}{\partial n} = 0 \ \text{ on } \ (0,T) \times (\partial\Omega \setminus (\Gamma^+ \cup \Gamma^-)) \\ \mp \Big( \dfrac{\partial (V_{\nu,\eta}^\varepsilon + M_0)}{\partial z} \Big)^\pm + K_{bq} B_\nu^\pm (t, V_{\nu,\eta}^{\varepsilon,+}, V_{\nu,\eta}^{\varepsilon,-}) \\ + K_{bq} R_\eta (t, V_{\nu,\eta}^{\varepsilon,\pm}) = 0 \ \text{ on } \ (0,T) \times \Gamma^\pm \end{cases} \tag{21}$$

Let us recall the role of the parameters $\nu$, $\eta$, $k$ and $\varepsilon$. The parameter $\varepsilon$ is the elliptic regularization parameter while $k$ allows to obtain the $L^2$-norm of the solution. The parameter $\nu$ is used to penalize the saturation condition while $\eta$ allows to show that the traces of the solutions on the interfaces are in the ball of radius 1. This condition is essential in passing to the limit in the nonlinear Neumann boundary condition.

# 3 Solving the regularized problem (21)

Let us introduce the weak formulation of problem (21). We set $\mathbb{V} = \{V \in \mathbb{H}^1(Q); V(0) = 0\}$ endowed with the usual norm of $\mathbb{H}^1(Q)$ and define on $\mathbb{V} \times \mathbb{V}$ the bilinear form

$$a_{\varepsilon,k}(V, W) = \varepsilon^2 (\partial_t V; \partial_t W) + (\nabla V; \nabla W) + \frac{1}{2}(\partial_t V; W) + \frac{k}{2}(V; W) \tag{22}$$

where $(\cdot;\cdot)$ denotes the scalar product of the Hilbert space $\mathbb{H} = \mathbb{L}^2(Q)$. We also define

$$\begin{cases} b_\nu(V; W) = K_{bq} \int_{\widehat{Q}} \widetilde{B}_\nu(t, \widetilde{V}) \cdot \widetilde{W} d\widehat{x} dt \\ r_\eta(V; W) = K_{bq} \int_{\widehat{Q}} \widetilde{R}_\eta(t, \widetilde{V}) \cdot \widetilde{W} d\widehat{x} dt \end{cases} \tag{23}$$

where we set $\widehat{Q} = (0,T) \times \widehat{\Omega}$ and

$$\widetilde{V} = (V^+, V^-), \ \widetilde{B}_\nu = (B_\nu^+, B_\nu^-), \ \widetilde{R}_\eta(t, \widetilde{V}) = (R_\eta(t, V^+), R_\eta(t, V^-)). \tag{24}$$

The weak formulation of the problem (21) becomes: for all $W \in \mathbb{V}$

$$(a_{\varepsilon,k} + b_\nu + r_\eta)(V_{\nu,\eta}^\varepsilon, W) = (F_\nu(t, V_{\nu,\eta}^\varepsilon); W) + (G(t, V_{\nu,\eta}^\varepsilon, \partial_t V_{\nu,\eta}^\varepsilon); W) + L_{M_0}(W) \tag{25}$$

with

$$L_{M_0}(W) = -(\nabla M_0; \nabla W) - \frac{k}{2}(M_0; W). \tag{26}$$

Hereafter we will precise some useful properties satisfied by the operators intervening in the weak formulation (25), $\langle \cdot;\cdot \rangle$ will denote the duality product between $\mathbb{V}$ and $\mathbb{V}'$.

**Lemma 1** *The linear operator $\mathcal{A}_{\varepsilon,k}$ defined from $\mathbb{V}$ into $\mathbb{V}'$ by setting*

$$\langle \mathcal{A}_{\varepsilon,k}(V); W \rangle = a_{\varepsilon,k}(V, W) \tag{27}$$

*is monotone continuous and coecive.*

**Lemma 2** *The mapping $X \in \mathbb{R}^3 \mapsto R_\eta(X)$ is lipschitz continuous and monotone. Therefore the operator $\mathcal{R}_\eta$ defined on $\mathbb{V}$ by*

$$\langle \mathcal{R}_\eta(U); V\rangle = r_\eta(U, V) \tag{28}$$

*is monotone and satisfies the lipschitz property*

$$|\mathcal{R}_{\nu,\eta}(U) - \mathcal{R}_\eta(V)| \le 2\eta^{-1}(|U^+ - V^+|_{L^2(\widehat{Q})} + |U^- - V^-|_{L^2(\widehat{Q})}) \tag{29}$$

*for all $U, V \in \mathbb{V}$.*

*Proof.* Indeed computing the gradient of $R_\eta(t, .)$, we get easily

$$|\nabla_X R_\eta(X)| \le \frac{2}{\eta} \tag{30}$$

which lead to (29). Let us verify the monotonicity. For $X, Y \in \mathbb{R}^3$, we have $\eta(R_\eta(X) - R_\eta(Y)) \cdot (X - Y) = \frac{s(X)}{1+s(X)}(|X|^2 - X \cdot Y) + \frac{s(Y)}{1+s(Y)}(|Y|^2 - X \cdot Y)$ then $\eta(R_\eta(X) - R_\eta(Y)) \cdot (X - Y) \ge (|X| - |Y|)(\frac{s(X)}{1+s(X)}|X| - \frac{s(Y)}{1+s(Y)}|Y|)$. Writing $\frac{s(X)}{1+s(X)}|X| - \frac{s(Y)}{1+s(Y)}|Y| = (|X| - |Y|)\frac{s(X)}{1+s(X)} + |Y|(\frac{s(X)}{1+s(X)} - \frac{s(Y)}{1+s(Y)})$ and using the inequality $(|X| - |Y|)(\frac{s(X)}{1+s(X)} - \frac{s(Y)}{1+s(Y)}) \ge 0$ which holds for all $X, Y \in \mathbb{R}^3$, we get

$$\eta(R_\eta(X) - R_\eta(Y)) \cdot (X - Y) \ge 0 \tag{31}$$

for all $X, Y \in \mathbb{R}^3$. Thus, the proof is complete.□

**Lemma 3** *The nonlinear operator $\mathcal{F}_\nu$ defined for all $U, V \in \mathbb{H}$ by*

$$(\mathcal{F}_\nu(V); V) = \int_Q F_\nu(t, V) \cdot V dxdt \tag{32}$$

*satisfies the lipschitz property*

$$|\mathcal{F}_\nu(U) - \mathcal{F}_\nu(V)|_{\mathbb{L}^2(Q)} \le C(1 + \nu^{-1})|U - V|_{\mathbb{L}^2(Q)} \tag{33}$$

*for all $U, V \in \mathbb{H}$ with $C > 0$ is a a constant whis is independent of the parameter $\nu$.*

*Proof.* Since $\nabla_U \psi(U)$ and $\nabla\gamma(|U|)$ are lipschitzian then using property (17) of the operator $\mathbb{D}$, we get the result.

Summarizing all these results, we obtain

**Lemma 4** *For all $\varepsilon$, $\nu$, $\eta > 0$, the operator $\mathcal{L}^\varepsilon_{\nu,\eta} = \mathcal{A}_{\varepsilon,k} + \mathcal{R}_\eta - \mathcal{F}_\nu$ is hemicontinuous, that is the mapping $s \in \mathbb{R} \mapsto \mathcal{L}^\varepsilon_{\nu,\eta}(U + sV)$ is continuous for all $U, V \in \mathbb{V}$. Moreover there exists a constant $k_\nu$ wich depends of $\nu$ such that for $k > k_\nu$ the operator $\mathcal{L}^\varepsilon_{\nu,\eta}$ is strictly monotone and satisfies the coercivity property*

$$\frac{\langle \mathcal{L}^\varepsilon_{\nu,\eta}(U); U\rangle}{|U|_{\mathbb{V}}} \to +\infty, \quad \textit{if} \;\; |U|_{\mathbb{V}} \to +\infty. \tag{34}$$

*Proof.* For all $U, V \in \mathbb{V}$ it holds

$$\left\{ \begin{array}{l} \langle \mathcal{L}^{\varepsilon}_{\nu,\eta}(U) - \mathcal{L}^{\varepsilon}_{\nu,\eta}(V); U - V \rangle \geq \varepsilon^2 |\partial_t U - \partial_t V|^2_{L^2(Q)} + |\nabla U - \nabla V|^2_{L^2(Q)} \\ +(k - k_\nu)|U - V|^2_{L^2(Q)} + \frac{1}{4}|U(T) - V(T)|^2_{L^2(\Omega)} \end{array} \right. \tag{35}$$

with a positive constant $k_\nu$. Indeed according to the monotonicity of $\mathcal{R}_\eta$ and the lipschitz property of $\mathcal{F}_\nu$, we have

$$\left\{ \begin{array}{l} \langle \mathcal{L}^{\varepsilon}_{\nu,\eta}(U) - \mathcal{L}^{\varepsilon}_{\nu,\eta}(V); U - V \rangle \geq \varepsilon^2 |\partial_t U - \partial_t V|^2_{L^2(Q)} + |\nabla U - \nabla V|^2_{L^2(Q)} \\ +(\frac{k}{2} - C(1 + \nu^{-1}))|U - V|^2_{L^2(Q)} + \frac{1}{4}|U(T) - V(T)|^2_{L^2(\Omega)} \end{array} \right. \tag{36}$$

so choosing $k > 2C(1 + \nu^{-1}) + 1$, we get the result.□

Let us introduce the operators $\mathcal{B}_\nu$ and $\mathcal{G}$ defined on $\mathbb{V}$ respectively by

$$\langle \mathcal{B}_\nu(U), V \rangle = b_\nu(U, V) \tag{37}$$

$$(\mathcal{G}(U); V) = -\int_Q G(t, U, \partial_t U) \cdot V dxdt. \tag{38}$$

Unfortunately these operators are not lipschitz perturbations of the monotone operator $\mathcal{L}^{\varepsilon}_{\nu,\eta}$. Consequently we will consider an alternative argument involving operators of type $M$. Thereby we begin by recalling some general results about such operators (see [12] for example).

**Definition 1** *([12]) Let $\mathcal{A}$ be an operator defined on a reflexive Banach space $\mathbb{F}$ into $\mathbb{F}'$. We will say that $\mathcal{A}$ is of type $M$ (respectively $M_0$) if it is continuous on finite dimensional subset $F \subset \mathbb{F}$ and if $U_i$ is some filter on a compact set $K \subset \mathbb{F}$ such that $U_i \to U$ in $\mathbb{F}$, $\mathcal{A}(U_i) \rightharpoonup V$ in $\mathbb{F}'$ weak-$\star$, $\limsup(\mathcal{A}(U_i); U_i) \leq (V; U)$ (respectively $(\mathcal{A}(U_i); U_i) \to (V; U)$) then we have $\mathcal{A}(U) = V$.*

In particular we get

**Lemma 5** *([12]) Let $\mathcal{A}$ , $\mathcal{B} : \mathbb{F} \to \mathbb{F}'$. We have*

- *If $\mathcal{A}$ is monotone hemicontinuous then it is of type $M$*
- *If $\mathcal{A}$ is of type $M$ then $\mathcal{A}$ is of type $M_0$*
- *If $\mathcal{A}$ is of type $M_0$ then so is $-\mathcal{A}$*
- *If $\mathcal{B}$ is continuous from $\mathbb{F}$ weak into $\mathbb{F}'$ strong and if $\mathcal{A}$ is of type $M_0$ then $\mathcal{A} + \mathcal{B}$ is of type $M_0$.*
- *If $\mathcal{A}$ is of type $M$ and $\mathcal{B}$ is bounded (mapping bounded sets into bounded sets) weakly continuous (that is from $\mathbb{F}$ weak into $\mathbb{F}'$ weak-$\star$) and the mapping $U \mapsto (\mathcal{B}(U); U)$ is weakly lower semi-continuous (that is if we have $U_i \rightharpoonup U$ then $\limsup(\mathcal{B}(U_i); U_i) \geq (\mathcal{B}(U); U)$), then $\mathcal{A} + \mathcal{B}$ is of type $M$.*

**Lemma 6** *([12]) If $\mathcal{A}$ is of type $M_0$ and coercive then $\mathcal{A}$ is surjective.*

We will employ these results to solve the problem (21). First we notice that the operator $\mathcal{L}^{\varepsilon}_{\nu,\eta}$ defined in lemma 4 is of type $M$. Moreover it holds that

**Lemma 7** *The nonlinear operator $\mathcal{B}_\nu$ is bounded and weakly continuous on $\mathbb{V}$. Moreover the mapping $V \mapsto \langle \mathcal{B}_\nu(V), V\rangle$ is weakly continuous on $\mathbb{V}$.*

*Proof.* For all $V \in \mathbb{V}$, we have

$$B^{\pm}_{\nu}(V^+, V^-) = \frac{1}{2}\,\frac{|V^+ - V^-|^2(V^+ + V^-) \pm |V^+ + V^-|^2(V^+ - V^-)}{1 + \nu|V^+ + V^-|^2|V^+ - V^-|^2} \tag{39}$$

so

$$|B^{\pm}_{\nu}(V^+, V^-)| \leq (4\nu)^{-1/2}(|V^+| + |V^-|) \tag{40}$$

and hence $\mathcal{B}_\nu$ maps bounded sets into bounded sets. Let $(V_n)_n$ be a sequence of $\mathbb{V}$ such that $V_n \rightharpoonup V$ weakly in $\mathbb{V}$ then $V_n^{\pm} \rightharpoonup V^{\pm}$ weakly in $\mathbb{H}^{1/2}(\widehat{Q})$ and then strongly in $\mathbb{L}^2(\widehat{Q})$. Using the Lebesgue dominated convergence theorem, we get that for all $W \in \mathbb{V}$

$$\frac{|V_n^+ \mp V_n^-|(V_n^+ \pm V_n^-)}{1 + \nu|V_n^+ + V_n^-|^2|V_n^+ - V_n^-|^2} \cdot W^{\pm} \to \frac{|V^+ \mp V^-|(V^+ \pm V^-)}{1 + \nu|V^+ + V^-|^2|V^+ - V^-|^2} \cdot W^{\pm}$$

strongly in $L^2(\widehat{Q})$. Therefore $(B^{\pm}_{\nu}(V_n^+, V_n^-); W^{\pm}) \to (B^{\pm}_{\nu}(V^+, V^-); W^{\pm})$ for all $W \in \mathbb{V}$. Similarly we get

$$(B^{\pm}_{\nu}(t, V_n^+, V_n^-); W^{\pm}) \to (B^{\pm}_{\nu}(t, V^+, V^-); W^{\pm})\,,\ \forall W \in \mathbb{V}$$

so $\mathcal{B}_\nu$ is weakly continuous from $\mathbb{V}$ to $\mathbb{V}'$. For $V \in \mathbb{V}$, if we set $U = e^{kt}(V + M_0)$ and $\widetilde{U} = (U^+, U^-)$, we can write

$$\langle \mathcal{B}_\nu(V); V\rangle = (e^{-2kt}\widetilde{B}_\nu(\widetilde{U}); \widetilde{U}) - (e^{-kt}\widetilde{B}_\nu(\widetilde{U}); \widetilde{M_0}) \tag{41}$$

with

$$(e^{-2kt}\widetilde{B}_\nu(\widetilde{U}); \widetilde{U}) = e^{-2kt}\frac{|U^+ + U^-|^2|U^+ - U^-|^2}{1 + \nu|U^+ + U^-|^2|U^+ - U^-|^2}. \tag{42}$$

Therefore if $V_n \rightharpoonup V$ weakly in $\mathbb{V}$, the weak continuity of $\mathcal{B}_\nu$ leads to the convergence of $(e^{-kt}\widetilde{B}_\nu(\widetilde{U}_n); \widetilde{M_0})$ towards $(e^{-kt}\widetilde{B}_\nu(\widetilde{U}); \widetilde{M_0})$ and in view of (42) and the Lebesgue dominated convergence theorem we obtain $(e^{-2kt}\widetilde{B}_\nu(\widetilde{U}_n); \widetilde{U}_n) \to (e^{-2kt}\widetilde{B}_\nu(\widetilde{U}); \widetilde{U})$. So the proof of the lemma is complete.□

**Lemma 8** *The operator $\mathcal{G}$ satisfies the same properties as $\mathcal{B}_\nu$.*

*Proof.* First $\mathcal{G}$ is bounded from $\mathbb{V}$ into $\mathbb{V}'$ because we have $|\mathcal{G}(V)| \leq |\partial_t V|$, for all V in $\mathbb{V}$. Assume $V_n \rightharpoonup V$ weakly in $\mathbb{V}$ then $V_n \to V$ strongly in $\mathbb{H}$ and $\partial_t V_n \rightharpoonup \partial_t V$ weakly in $\mathbb{H}$. Since $|\frac{e^{kt}(V_n+M_0)}{1+e^{kt}|V_n+M_0|}| \leq 1$ it follows that $\frac{e^{kt}(V_n+M_0)\cdot W}{1+e^{kt}|V_n+M_0|} \to \frac{e^{kt}(V+M_0)\cdot W}{1+e^{kt}|V+M_0|}$ strongly in $\mathbb{H}$ for all $W \in \mathbb{H}$ and then $\mathcal{G}(V_n) \rightharpoonup \mathcal{G}(V)$ weakly in $\mathbb{H}$ and finally weakly-$\star$ in $\mathbb{V}'$. Moreover since $(\mathcal{G}(V_n); V_n) = -(\mathcal{G}(V_n); M_0)$ thus from what preceed we deduce that $(\mathcal{G}(V_n); M_0) \to (\mathcal{G}(V); M_0) = -(\mathcal{G}(V); V)$.□

Now we are able to establish the following existence theorem for problem (21)

**Theorem 1** *Let $M_0 \in H^2(\Omega)$ be such that $|M_0(x)|^2 = 1$ in $\overline{\Omega}$, $\frac{\partial M_0}{\partial n} = 0$ on $\partial\Omega$, $M_0^+ = \pm M_0^-$ on $\widehat{\Omega}$ and let $\varepsilon$, $\nu$, $\eta > 0$ be fixed. Then there exists $k_0 > 0$ depending upon $\nu$ such that for $k > k_0$, the problem (21) admits a solution $V^\varepsilon_{\nu,\eta} \in \mathbb{V}$.*

*Proof.* We apply the result of lemma 6. In view of (41), it holds that $\langle \mathcal{B}_\nu(V); V\rangle \geq -(4\nu)^{-1/2}(|\widetilde{V}| + |\widetilde{M_0}|)|\widetilde{M_0}|$, $\forall V \in \mathbb{V}$. Since $\mathcal{G}$ satisfies $|(\mathcal{G}(V); V)| \leq \int_Q |\partial_t V| dx$ and $\mathcal{L}^\varepsilon_{\nu,\eta}$ is coercive for $k > k_0$, then so is $\mathcal{L}^\varepsilon_{\nu,\eta} + \mathcal{B}_\nu + \mathcal{G}$. As $\mathcal{B}_\nu + \mathcal{G}$ verifies the fifth condition quoted in lemma 5 then $\mathcal{L}^\varepsilon_{\nu,\eta} + \mathcal{B}_\nu + \mathcal{G}$ is of type $M$ and since it is coercive for $k > k_0$ then it is surjective from $\mathbb{V}$ into $\mathbb{V}'$. Therefore there exists $V^\varepsilon_{\nu,\eta} \in \mathbb{V}$ solving the equation

$$-\varepsilon^2 \partial_t^2 V^\varepsilon_{\nu,\eta} - \Delta V^\varepsilon_{\nu,\eta} + \frac{1}{2}\partial_t V^\varepsilon_{\nu,\eta} + \frac{k}{2} V^\varepsilon_{\nu,\eta} = F_\nu(t, V^\varepsilon_{\nu,\eta}) + G_\nu(t, V^\varepsilon_{\nu,\eta}, \partial_t V^\varepsilon_{\nu,\eta}) + H_{M_0} \quad (43)$$

in $\mathbb{V}'$. Hence $V^\varepsilon_{\nu,\eta}(0) = 0$ and solves problem (21) in the weak sense. Moreover since $V^\varepsilon_{\nu,\eta}$ satisfies in the sense of distributions $-\varepsilon^2 \partial_t^2 V^\varepsilon_{\nu,\eta} - \Delta V^\varepsilon_{\nu,\eta} + \frac{k}{2} V^\varepsilon_{\nu,\eta} \in L^2(Q)$ with the traces $\partial V^\varepsilon_{\nu,\eta}/\partial N \in H^{1/2}(\partial Q \setminus (\{0\} \times \Omega))$ and $V^\varepsilon_{\nu,\eta}(0) = 0$ on $\Omega$ (here $N$ denotes the outward unit normal to $\partial Q$) then using the classical result of the regularity of solutions of elliptic mixed problems we deduce that

$$V^\varepsilon_{\nu,\eta} \in \mathbb{H}^2(Q). \quad (44)$$

Indeed in view of the hypotheses given on $M_0$, we can transform our problem into a problem with zero boundary conditions according to the trace theorem [2]. Then the geometry of the domain $(0, T) \times \Omega^\pm$ leads by the well known reflection argument to the $H^2$-regularity of the solution.

**Remark 1** *Notice that the coerciveness property of $\mathcal{L}^\varepsilon_{\nu,\eta} + \mathcal{B}_\nu + \mathcal{G}$ does not lead to uniform bounds of solutions $V^\varepsilon_{\nu,\eta}$ with respect to $\varepsilon$. The regularity $\mathbb{H}^2$ of the solutions $V^\varepsilon_{\nu,\eta}$ will be relevant to establish uniform estimates.*

# 4 Convergence as $\varepsilon \to 0$

We multiply (43) by $e^{-2kt}\partial_t V^\varepsilon_{\nu,\eta}$ and integrate on $Q$. To simplify notations, let us temporarily write $V^\varepsilon_{\nu,\eta} = V$, $U = e^{kt}(V + M_0)$, $\widetilde{U} = (U^+, U^-)$, $\widetilde{\Theta}_\eta(\widetilde{U}) = \Theta_\eta(U^+) + \Theta_\eta(U^-)$. First notice that

$$(G(t, V, \partial_t V); e^{-2kt}\partial_t V) = 0 \quad (45)$$

and using integrations by parts, we get successively

$$(-\varepsilon^2 \partial_t^2 V + \frac{1}{2}\partial_t V; e^{-2kt}\partial_t V) = \frac{\varepsilon^2}{2}|\partial_t V(0)|^2 + \frac{1}{2}(1 - 2\varepsilon^2 k)\int_0^T e^{-2kt}|\partial_t V|^2 dt \quad (46)$$

$$\frac{k}{2}(V + M_0; e^{-2kt}\partial_t V) = \frac{k}{4}e^{-4kT}|U(T)|^2 - \frac{k}{4}|M_0|^2 + \frac{k^2}{2}\int_0^T e^{-4kt}|U|^2 dt. \quad (47)$$

Then since $\Phi_\nu(M_0^+, M_0^-) = 0$, $\Theta_\eta(M_0^\pm) = 0$, we have

$$(-\Delta(V+M_0);e^{-2kt}\partial_t V)=\frac{1}{2}e^{-4kT}|\nabla U(T)|^2+K_{bq}e^{-4kT}\int_{\widehat{\Omega}}(\Phi_\nu+\widetilde{\Theta}_\eta)(\widetilde{U}(T))d\hat{x}$$
$$-\frac{1}{2}|\nabla M_0|^2+k\int_0^T e^{-4kt}|\nabla U|^2dt+4kK_{bq}\int_{\widehat{Q}}e^{-4kt}(\Phi_\nu+\widetilde{\Theta}_\eta)(\widetilde{U})d\hat{x}dt$$
$$-kK_{bq}\int_{\widehat{Q}}e^{-4kt}(\widetilde{B}_\nu+\widetilde{R}_\eta)(\widetilde{U})\cdot\widetilde{U}d\hat{x}dt.$$

Let us set for $U\in\mathbb{V}$

$$\tau_\eta(U^\pm)=4\Theta_\eta(U^\pm)-\frac{1}{\eta}\frac{s(U^\pm)}{1+s(U^\pm)}\ ,\ \widetilde{\tau}_\eta(\widetilde{U})=\tau_\eta(U^+)+\tau_\eta(U^-)\tag{48}$$

with $s(U^\pm)=max(|U^\pm|^2-1,0)$. Hence $\tau_\eta(U^\pm)\geq 0$ and we have $\int_{\widehat{Q}}e^{-4kt}(4\widetilde{\Theta}_\eta(\widetilde{U})-\widetilde{R}_\eta(\widetilde{U})\cdot\widetilde{U})d\hat{x}dt=\int_{\widehat{Q}}e^{-4kt}\widetilde{\tau}_\eta(\widetilde{U})d\hat{x}dt-\frac{1}{\eta}\int_{\widehat{Q}}e^{-4kt}(\frac{s(U^+)}{1+s(U^+)}+\frac{s(U^-)}{1+s(U^-)})d\hat{x}dt$ so

$$\int_{\widehat{Q}}e^{-4kt}(4\widetilde{\Theta}_\eta(\widetilde{U})-\widetilde{R}_\eta(\widetilde{U})\cdot\widetilde{U})d\hat{x}dt\geq\int_{\widehat{Q}}e^{-4kt}\widetilde{\tau}_\eta(\widetilde{U})d\hat{x}dt-\frac{1}{2k\eta}|\widehat{\Omega}|.\tag{49}$$

Furthermore in view of (42), we have

$$k\int_{\widehat{Q}}e^{-4kt}\widetilde{B}_\nu(\widetilde{U})\cdot\widetilde{U}d\hat{x}dt\leq(4\nu)^{-1}|\widehat{\Omega}|\tag{50}$$

which together with (49) give

$$\begin{aligned}&(-\Delta(V+M_0);e^{-2kt}\partial_t V)\geq k\int_0^T e^{-4kt}|\nabla U|^2dt+\\&kK_{bq}\int_{\widehat{Q}}e^{-4kt}(4\Phi_\nu+\widetilde{\tau}_\eta)(\widetilde{U})d\hat{x}dt-\frac{1}{2}|\nabla M_0|^2-(\frac{1}{4\nu}+\frac{1}{2\eta})|\widehat{\Omega}|.\end{aligned}\tag{51}$$

Since $\gamma(|M_0|)=0$, we have

$$(-F_\nu(t,V);e^{-2kt}\partial_t V)=e^{-4kT}\int_\Omega(\psi(U(T))+\nu^{-1}\gamma(|U(T)|))dx-\int_\Omega\psi(M_0)dx$$
$$+4k\int_Q e^{-4kt}(\psi(U)+\nu^{-1}\gamma(|U|))dxdt+\frac{1}{2}e^{-4kT}\int_{\mathbb{R}^3}|\mathbb{D}(U(T))|^2dx$$
$$-\frac{1}{2}\int_{\mathbb{R}^3}|\mathbb{D}(M_0)|^2dx+2k\int_{Q\infty}e^{-4kt}|\mathbb{D}(U)|^2dxdt+k(F_\nu(U);e^{-4kt}U).$$

Writing $(\nabla_U\psi(U);e^{-4kt}U)=(\nabla_U\psi(U)-\nabla_U\psi(0);e^{-4kt}U)+(\nabla_U\psi(0);e^{-4kt}U)$ and using the lipschitz property of $\nabla_U\psi$, we get

$$|(\nabla_U\psi(U);e^{-4kt}U)|\leq C\int_0^T e^{-4kt}(|U|^2+|U|)dt\tag{52}$$

with a constant $C>0$. Hence since $|\nabla_U\gamma(|U|)|\leq C|U|$ then using (17), we obtain

$$|(F_\nu(U);e^{-4kt}U)|\leq C(1+\nu^{-1})\int_0^T e^{-4kt}|U|^2dt+C\tag{53}$$

which leads to

$$(-F_\nu(t,V); e^{-2kt}\partial_t V) \geq 4k\int_Q e^{-4kt}(\psi(U)+\nu^{-1}\gamma(|U|))dxdt - \int_\Omega \psi(M_0)dx$$
$$+2k\int_{Q^\infty} e^{-4kt}|\mathbb{D}(U)|^2dxdt - \frac{1}{2}\int_{\mathbb{R}^3}|\mathbb{D}(M_0)|^2dx - k_\nu\int_Q e^{-4kt}|U|^2dxdt - kC \tag{54}$$

with $k_\nu = C(1+\nu^{-1})$. Combining the results obtained in (45), (46), (47), (51) and (54), we get

$$\frac{1}{2}(1-2k\varepsilon^2)\int_0^T e^{-2kt}|\partial_t V|^2dt + k\int_0^T e^{-4kt}|\nabla U|^2dt + k(\frac{k}{2}-k_\nu)\int_0^T e^{-4kt}|U|^2dt$$
$$+kK_{bq}\int_{\widehat{Q}} e^{-4kt}(4\Phi_\nu+\widetilde{\tau}_\eta)(\widetilde{U})d\hat{x}dt + 4k\int_Q e^{-4kt}(\psi(U)+\nu^{-1}\gamma(|U|))dxdt \tag{55}$$
$$+2k\int_{Q^\infty} e^{-4kt}|\mathbb{D}(U)|^2dxdt \leq C(k,\nu,\eta,M_0)$$

where

$$\begin{cases} C(k,\nu,\eta,M_0) = \dfrac{k}{4}|M_0|^2 + \dfrac{1}{2}|\nabla M_0|^2 + \displaystyle\int_\Omega \psi(M_0)dx \\ +\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^3}|\mathbb{D}(M_0)|^2dx + kC + (\dfrac{1}{4\nu}+\dfrac{1}{2\eta})|\widehat{\Omega}|. \end{cases} \tag{56}$$

Therefore $\nu$ and $\eta$ being fixed, for $k > 2k_\nu + 1$ and $\varepsilon < (4k)^{-1/2}$, we get in particular

**Lemma 9** *For every $\nu > 0$ and $\eta > 0$, there exists $k_0 > 0$, $\varepsilon_0 > 0$ and a constant $C > 0$ with is independent of $\varepsilon$ such that for $k > k_0, \varepsilon < \varepsilon_0$, the solutions $V^\varepsilon_{\nu,\eta}$ of the problem (21) satisfy*

$$\|V^\varepsilon_{\nu,\eta}\|_{\mathbb{V}} + |\mathbb{D}(V^\varepsilon_{\nu,\eta})|_{L^2(Q^\infty)} \leq C. \tag{57}$$

Now we are able to pass to the limit in problem (21) when $\varepsilon \to 0$. The bounds given in lemma 9 imply that there exists a subsequence still denoted $V^\varepsilon_{\nu,\eta}$ and $V^\eta_\nu \in \mathbb{V}$ such that

$$\begin{cases} V^\varepsilon_{\nu,\eta} \rightharpoonup V^\eta_\nu \ \text{ in } \mathbb{H}^1(Q) \ \text{ weak} \\ V^\varepsilon_{\nu,\eta} \to V^\eta_\nu \ \text{ in } \mathbb{L}^2(Q) \text{ strong} \\ \partial_t V^\varepsilon_{\nu,\eta} \rightharpoonup \partial_t V^\eta_\nu \ \text{ in } \mathbb{L}^2(Q) \ \text{ weak} \\ V^{\varepsilon,\pm}_{\nu,\eta} \rightharpoonup V^{\eta,\pm}_\nu \ \text{ in } H^{1/2}(\widehat{Q}) \ \text{ weak.} \end{cases} \tag{58}$$

Since $\mathbb{D}$ is linear and $\nabla_U\psi$ and $\nabla_U\gamma$ are lipschitzian, we have

$$\begin{cases} \mathbb{D}(V^\varepsilon_{\nu,\eta}) \to \mathbb{D}(V^\eta_\nu) \ \text{ in } \mathbb{L}^2(Q) \text{ strong} \\ \nabla_U\psi(V^\varepsilon_{\nu,\eta}) \to \nabla_U\psi(V^\eta_\nu) \ \text{ in } \mathbb{L}^2(Q) \text{ strong} \\ \nabla_U\gamma(|V^\varepsilon_{\nu,\eta}|) \to \nabla_U\gamma(|V^\eta_\nu|) \ \text{ in } \mathbb{L}^2(Q) \text{ strong.} \end{cases} \tag{59}$$

For the boundary terms, we get the following

**Lemma 10** *It holds that*

$$\left\{\begin{array}{l} V_{\nu,\eta}^{\varepsilon,\pm} \to V_\nu^{\eta,\pm} \quad in \quad L^2(\widehat{Q}) \quad strong \\ B_\nu^\pm(V_{\nu,\eta}^{\varepsilon,+}, V_{\nu,\eta}^{\varepsilon,-}) \rightharpoonup B_\nu^\pm(V_\nu^{\eta,+}, V_\nu^{\eta,-}) \quad in \quad L^2(\widehat{Q}) \quad weak \\ R_\eta(V_{\nu,\eta}^{\varepsilon,\pm}) \to R_\eta(V_\nu^{\eta,\pm}) \ in \ \ L^2(\widehat{Q}) \quad strong\ . \end{array}\right. \tag{60}$$

*Proof.* We get the strong convergence of the traces thanks to the compactness of the continuous imbedding $H^{1/2}(\widehat{Q}) \subset \mathbb{L}^2(\widehat{Q})$. Next the lipschitz property of $R_\eta$ leads to the the strong convergence of $R_\eta(V_{\nu,\eta}^{\varepsilon,\pm})$ while the weak convergence of $B_\nu^\pm(V_{\nu,\eta}^{\varepsilon,+}, V_{\nu,\eta}^{\varepsilon,-})$ is obtained proceeding as in the proof of lemma 7.□

Passing to the limit in problem (21), when $\varepsilon \to 0$ with $\nu$ and $\eta$ fixed, we get

**Theorem 2** *Assume that $M_0$ satifies the hypotheses of theorem 1 and let $\nu, \eta > 0$ be fixed. Then for any $k > k_0$ and $T > 0$ (19) admits a solution $V_\nu^\eta \in \mathbb{V}$ obtained as the limit of the sequence $(V_{\nu,\eta}^\varepsilon)_\varepsilon$ when $\varepsilon \to 0$.*

# 5 Convergence for $\eta \to 0$ and $\nu \to 0$

Let $V_\nu^\eta$ be the solution of (19) provided by theorem 2. We set

$$U_\nu^\eta = e^{kt}(V_\nu^\eta + M_0) \tag{61}$$

then $U_\nu^\eta \in \mathbb{H}^1(Q)$ and satisfies the intermediary problem (9). It follows that $\Delta U_\nu^\eta \in L^2(0,T;\mathbb{L}^2(\Omega))$, $\partial U_\nu^\eta/\partial n \in L^2(0,T;\mathbb{H}^{1/2}(\partial\Omega))$. Hence using the regularity property satisfied by the solution of a Laplace's equation we deduce that

$$U_\nu^\eta \in L^2(0,T;\mathbb{H}^2(\Omega)) \cap \mathbb{H}^1(Q). \tag{62}$$

We have the following energy bound

**Lemma 11** *$U_\nu^\eta$ satisfies for $t \in (0,T)$ the energy inequality*

$$\mathcal{E}_\nu^\eta(U_\nu^\eta(t)) + \frac{1}{2}\int_0^t |\partial_t U_\nu^\eta(s)|^2_{\mathbb{L}^2(\Omega)} ds \le \mathcal{E}(M_0) \tag{63}$$

*where*

$$\left\{\begin{array}{l} \mathcal{E}_\nu^\eta(V) = |\nabla V|^2_{\mathbb{L}^2(\Omega)} + |\nabla\varphi|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \displaystyle\int_\Omega (\psi(V) + \nu^{-1}\gamma(|V|))dx \\ + \displaystyle\int_{\widehat{\Omega}} (\Phi_\nu(V^+, V^-) + \Theta_\eta(V^+) + \Theta_\eta(V^-))d\widehat{x} \end{array}\right. \tag{64}$$

$$\mathcal{E}(M_0) = |\nabla M_0|^2_{\mathbb{L}^2(\Omega)} + |\nabla\varphi_0|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \int_\Omega \psi(M_0)dx \tag{65}$$

*and $\nabla\varphi = \mathbb{D}(V)$, $\nabla\varphi_0 = \mathbb{D}(M_0)$.*

*Moreover there exists $\nu_0 > 0$ and $C > 0$ which depends only upon the initial data $M_0$ such that for $0 < \nu < \nu_0$ and $\eta > 0$, we have*

$$|U_\nu^\eta(t)|_{\mathbb{L}^2(\Omega)} \le C,\ \ t \in (0,T). \tag{66}$$

*Proof.* Recall that $\gamma(|M_0|) = 0$, $\Theta_\eta(M_0^\pm) = 0$ and $\Phi_\nu(M_0^+, M_0^-) = 0$ so if we multiply the equation (9) by $\partial_t U_\nu^\eta$ and integrate on $(0,t)\times\Omega$ we get the result stated in (63). Therefore since the function $\gamma$ satisfies the inequality $s^2 \leq 4\gamma(s) + 3$, we obtain $|U_\nu^\eta(t)|_{\mathbb{L}^2(\Omega)} \leq 4\nu\mathcal{E}(M_0) + 3$ wich leads to (66).□

Now we pass to the limit in (9) for $\eta \to 0$ and $\nu$ fixed. There exists a subsequence also denoted $U_\nu^\eta$ and $U_\nu \in L^\infty(0,T;\mathbb{H}^1(\Omega)) \cap \mathbb{H}^1(Q)$ such that when $\eta \to 0$ we have

$$\begin{cases} U_\nu^\eta \rightharpoonup U_\nu \;\text{ in }\; L^\infty(0,T;H^1(\Omega)) \;\text{ weak } -\star, \\ U_\nu^\eta \to U_\nu \;\text{ in }\; \mathbb{L}^2(Q) \;\text{ strong} \\ \partial_t U_\nu^\eta \rightharpoonup \partial_t U_\nu \;\text{ in }\; \mathbb{L}^2(Q) \;\text{ weak} \\ U_\nu^{\eta,\pm} \to U_\nu^\pm \;\text{ in }\; \mathbb{L}^2(\widehat{Q}) \;\text{ strong} \end{cases} \tag{67}$$

Moreover we have

$$\begin{cases} \mathbb{D}(U_\nu^\eta) \to \mathbb{D}(U_\nu) \;\text{ in }\; \mathbb{L}^2(Q) \;\text{ strong} \\ \nabla_U\psi(U_\nu^\eta) \to \nabla_U\psi(U_\nu) \;\text{ in } \mathbb{L}^2(Q) \text{ strong} \\ \nabla_U\gamma(|U_\nu^\eta|) \to \nabla_U\gamma(|U_\nu|) \;\text{ in } \mathbb{L}^2(Q) \text{ strong} \end{cases} \tag{68}$$

and we get

**Lemma 12** *When $\eta \to 0$, it holds the following convergences*

$$\begin{cases} B_\nu^\pm(U_\nu^{\eta,+}, U_\nu^{\eta,-}) \to B_\nu^\pm(U_\nu^+, U_\nu^-) \;\text{ in } \mathbb{L}^2(\widehat{Q}) \text{ strong} \\ \Theta_\eta(U_\nu^{\eta,\pm}) \to 0 \;\text{ in } L^\infty(0,T;L^1(\widehat{\Omega})) \text{ strong.} \end{cases} \tag{69}$$

*Moreover $U_\nu$ is such that*

$$|U_\nu^\pm(t,x)|^2 \leq 1 \text{ a.e. } (t,x) \in \widehat{Q} \tag{70}$$

*and satisfies the energy inequality*

$$\mathcal{E}^\nu(U_\nu(t)) + \frac{1}{2}\int_0^t |\partial_t U_\nu(s)|^2_{L^2(\Omega)} ds \leq \mathcal{E}(M_0) \tag{71}$$

*where*

$$\begin{cases} \mathcal{E}^\nu(U_\nu(t)) = |\nabla U_\nu(t)|^2_{\mathbb{L}^2(\Omega)} + |\mathbb{D}(U_\nu(t))|^2_{L^2(\mathbb{R}^3)} + \displaystyle\int_\Omega \psi(U_\nu(t))dx \\ +\nu^{-1}\displaystyle\int_\Omega \gamma(|U_\nu(t)|)dx + \int_{\widehat{\Omega}} \Phi_\nu(U_\nu^+(t), U_\nu^-(t))d\widehat{x}. \end{cases} \tag{72}$$

*Proof.* First we get $B_\nu^\pm(U_\nu^{\eta,+}, U_\nu^{\eta,-}) \rightharpoonup B_\nu^\pm(U_\nu^+, U_\nu^-)$ in $\mathbb{L}^2(\widehat{Q})$ weak as in the previous section. Following the proof of lemma 7, we obtain

$$\frac{|U_\nu^{\eta,+} \mp U_\nu^{\eta,-}|(U_\nu^{\eta,+} \pm U_\nu^{\eta,-})}{1 + \nu|U_\nu^{\eta,+} + U_\nu^{\eta,-}|^2|U_\nu^{\eta,+} - U_\nu^{\eta,-}|^2} \to \frac{|U_\nu^+ \mp U_\nu^-|(U_\nu^+ \pm U_\nu^-)}{1 + \nu|U_\nu^+ + U_\nu^-|^2|U_\nu^+ - U_\nu^-|^2}$$

strongly in $L^2(0,T;\mathbb{L}^4(\widehat{\Omega}))$ thanks to the Lebesgue dominated convergence theorem. According to continuous imbedding $H^{1/2}(\widehat{\Omega}) \subset L^4(\widehat{\Omega})$ (see [1]), $U_\nu^{\eta,\pm}$ is bounded in

$L^\infty(0,T;\mathbb{L}^4(\widehat{\Omega}))$ so we obtain the strong convergence of $B_\nu^\pm(U_\nu^{\eta,+},U_\nu^{\eta,-})$ stated in the lemma. The strong convergence of the traces leads also to

$$\Theta_\eta(U_\nu^{\eta,\pm}) \to s(U_\nu^\pm) \ \text{ a.e. in } \widehat{Q} \tag{73}$$

but since $|\Theta_\eta(U_\nu^{\eta,\pm})|_{L^\infty(0,T;L^1(\widehat{\Omega}))} \leq 2\eta\mathcal{E}(M_0)$, we get

$$\Theta_\eta(U_\nu^{\eta,\pm}) \to 0 \text{ strongly in } L^\infty(0,T;L^1(\widehat{\Omega})).$$

Combining these results we conclude that $s(U_\nu^\pm)=0$ a.e. in $\widehat{Q}$ which leads to (70). (71) is obtained by taking the limit in (63) when $\eta\to 0$.□

Let $W\in\mathcal{D}(\overline{Q})$ be a test function. We multiply the equation (9) by $U_\nu^\eta\times W$ and integrate by parts. Observing that $\nabla_U\gamma(|U_\nu^\eta|)\cdot U_\nu^{\eta,\pm}\times W^\pm=0$ and $R_\eta(U_\nu^{\eta,\pm})\cdot U_\nu^{\eta,\pm}\times W^\pm=0$, we get the weak formulation of (9)

$$\left\{\begin{array}{l} \displaystyle\frac{1}{2}\int_Q \partial_t U_\nu^\eta\cdot U_\nu^\eta\times W\ dxdt+\int_Q\frac{U_\nu^\eta}{1+|U_\nu^\eta|}\times\partial_t U_\nu^\eta\cdot U_\nu^\eta\times W\ dxdt\\ \displaystyle+\int_Q\nabla U_\nu^\eta\cdot U_\nu^\eta\times\nabla W\ dxdt-\int_Q(\nabla\varphi_\nu^\eta-\nabla_U\psi(U_\nu^\eta))\cdot U_\nu^\eta\times W\ dxdt\\ \displaystyle=\int_{\widehat{Q}}(B_\nu^+(U_\nu^{\eta,+},U_\nu^{\eta,-})\cdot U_\nu^{\eta,+}\times W^+ + B_\nu^-(U_\nu^{\eta,+},U_\nu^{\eta,-})\cdot U_\nu^{\eta,-}\times W^-)d\widehat{x}dt.\end{array}\right. \tag{74}$$

Using the strong convergence of $U_\nu^\eta$ and the weak convergence of $\partial_t U^\nu$ in $\mathbb{L}^2(Q)$, we can pass to the limit in each volume integral. Moreover the strong convergence in $\mathbb{L}^2(\widehat{Q})$ of the traces $U_\nu^{\eta,\pm}$ and $B_\nu^\pm(U_\nu^{\eta,+},U_\nu^{\eta,-})$ allow to pass to the limit in the boundary terms. Hence the limit $U_\nu$ satisfies the weak formulation

$$\left\{\begin{array}{l} \displaystyle\frac{1}{2}\int_Q \partial_t U_\nu\cdot U_\nu\times W dxdt+\int_Q\frac{U_\nu}{1+|U_\nu|}\times\partial_t U_\nu\cdot U_\nu\times W dxdt\\ \displaystyle+\int_Q\nabla U_\nu\cdot U_\nu\times\nabla W dxdt-\int_Q(\nabla\varphi_\nu-\nabla_U\psi(U_\nu))\cdot U_\nu\times W dxdt\\ \displaystyle=\int_{\widehat{Q}}(B_\nu^+(U_\nu^+,U_\nu^-)\cdot U_\nu^+\times W^+ + B_\nu^-(U_\nu^+,U_\nu^-)\cdot U_\nu^-\times W^-)d\widehat{x}dt.\end{array}\right. \tag{75}$$

Now we are able to prove our main theorem

**Theorem 3** *Let $M_0\in H^2(\Omega)$ be such that $|M_0(x)|=1$ in $\overline{\Omega}$, $\frac{\partial M_0}{\partial n}=0$ on $\partial\Omega$, $M_0^+=\pm M_0^-$ on $\widehat{\Omega}$. There exists a solution of the Landau-Lifshitz equation with biquadratic interlayer exchange coupling satisfying $M\in L^\infty(\mathbb{R}^+;\mathbb{H}^1(\Omega))$, $\partial_t M\in L^2_{loc}(\mathbb{R}^+;\mathbb{L}^2(\Omega))$, $|M(t,x)|^2=1$ a.e in $\mathbb{R}^+\times\Omega$, $\nabla\varphi\in L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^3))$ and the energy inequality*

$$\mathcal{E}(M(t))+\frac{1}{2}\int_0^t|\partial_t M(s)|^2_{\mathbb{L}^2(\Omega)}ds\leq\mathcal{E}(M_0) \tag{76}$$

*where*

$$\mathcal{E}(M)=|\nabla M|^2_{\mathbb{L}^2(\Omega)}+|\nabla\varphi|^2_{\mathbb{L}^2(\mathbb{R}^3)}+\int_\Omega\psi(M)dx+\int_{\widehat{\Omega}}\Phi(M^+,M^-)d\widehat{x} \tag{77}$$

*with* $\Phi(M^+,M^-) = 1-(M^+\cdot M^-)^2$, $\nabla\varphi = \mathbb{D}(M)$, $\varphi_0 = \mathbb{D}(M_0)$ *and the initial energy is given by*

$$\mathcal{E}(M_0) = |\nabla M_0|^2_{\mathbb{L}^2(\Omega)} + |\nabla\varphi_0|^2_{L^2(\mathbb{R}^3)} + \int_\Omega \psi(M_0)dx \tag{78}$$

*Proof.* Let $U_\nu$ the limit of $U_\nu^\eta$ when $\eta\to 0$. Hence $U_\nu$ satisfies the energy inequality (71) and the trace estimate $|U_\nu^\pm(t,\widehat{x})|^2 \le 1$ a.e. in $\mathbb{R}^+\times\widehat{\Omega}$. Clearly the energy inequality implies the following convergence for a subsequence $U_\nu$

$$\begin{cases} U_\nu \rightharpoonup M \ \text{ in } \ L^\infty(\mathbb{R}^+;H^1(\Omega)) \ \text{ weak} -\star, \\ \partial_t U_\nu \rightharpoonup \partial_t M \ \text{ in } \ L^2_{loc}(\mathbb{R}^+;L^2(\Omega)) \ \text{ weak,} \\ U_\nu \to M \ \text{ in } \ L^2_{loc}(\mathbb{R}^+;L^2(\Omega)) \ \text{ strong} \\ U_\nu^\pm \rightharpoonup M^\pm \ \text{ in } \ L^\infty(\mathbb{R}^+;\mathbb{H}^{1/2}(\widehat{\Omega})) \ \text{ weak} -\star \\ U_\nu^\pm \to M^\pm \ \text{ in } \ L^2_{loc}(\mathbb{R}^+;L^2(\widehat{\Omega})) \ \text{ strong} \end{cases} \tag{79}$$

for some $M\in L^\infty(\mathbb{R}^+;\mathbb{H}^1(\Omega))\cap\mathbb{H}^1(Q)$. Moreover we have

$$\begin{cases} \nabla\varphi_\nu \rightharpoonup \nabla\varphi \ \text{ in } \ L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^3)) \ \text{ weak} -\star \\ \nabla\varphi_\nu \to \nabla\varphi \ \text{ in } \ L^2_{loc}(\mathbb{R}^+;L^2(\mathbb{R}^3)) \ \text{ strong} \\ \nabla_M\psi(U_\nu) \to \nabla_M\psi(M) \ \text{ in } \ L^2_{loc}(\mathbb{R}^+;L^2(\Omega)) \ \text{ strong} \end{cases} \tag{80}$$

Since $|\gamma(|U_\nu|)|_{L^\infty(0,T;L^1(\Omega))} \le \nu\mathcal{E}(M_0)$, we have

$$\gamma(|U_\nu|) \to 0 \text{ strongly in } \ L^\infty(\mathbb{R}^+;L^1(\Omega))$$

and so a.e. in $\mathbb{R}^+\times\Omega$. Hence combining this result with the strong convergence of $U_\nu$ we get $\gamma(|M|)=0$ a.e. in $\mathbb{R}^+\times\Omega$ that is $|M(t,x)|^2 = 1$ a.e. in $\mathbb{R}^+\times\Omega$. Now we are interested by the convergence of the boundary terms $B_\nu^\pm(U_\nu^+,U_\nu^-)\cdot U_\nu^\pm\times W^\pm$ of the weak formulation (75). Since $U_\nu^\pm$ is bounded in $\mathbb{L}^\infty(\mathbb{R}^+\times\widehat{\Omega})$ so is $B_\nu^\pm(U_\nu^+,U_\nu^-)$ so thanks to the Lebesgue dominated convergence theorem, we get the strong convergence

$$B_\nu^\pm(U_\nu^+,U_\nu^-) \to \frac{1}{2}|M^+-M^-|^2(M^++M^-) \pm \frac{1}{2}|M^++M^-|^2(M^+-M^-) \tag{81}$$

in $L^2_{loc}(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))$ then

$$B_\nu^\pm(U_\nu^+,U_\nu^-)\cdot U_\nu^\pm\times W^\pm \to \frac{1}{2}(|M^+-M^-|^2(M^++M^-) \pm |M^++M^-|^2(M^+-M^-))\cdot M\times W^\pm \tag{82}$$

for all $W\in\mathcal{D}(\mathbb{R}^+\times\overline{\Omega})$. Hence $M$ satisfies the weak formulation

$$\begin{cases} \frac{1}{2}\int_Q \partial_t M\cdot M\times G \, dxdt + \frac{1}{2}\int_Q \ M\times\partial_t M\cdot M\times G \, dxdt \\ +\int_Q \nabla M\cdot M\times\nabla G \, dxdt - \int_Q (\nabla\varphi - \nabla_M\psi(M))\cdot M\times G \, dxdt = \\ \int_{\widehat{Q}} (B^+(M^+,M^-)\cdot M^+\times G^+ + B^-(M^+,M^-)\cdot M^-\times G^-) \, d\widehat{x}dt \end{cases} \tag{83}$$

where $B^\pm(M^+,M^-) = -2(M^+\cdot M^-)M^\pm$. This shows that $M$ is a global weak solution of LLG equations. The energy estimate satisfied by $M$ follows from the one satisfied by $U_\nu$ by passing to the limit when $\nu\to 0$. The proof of the theorem is complete. □

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