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Modeling photonic crystal fibers

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Abstract

We study guidance of electromagnetic waves in photonic fibers. Both transverse magnetic and transverse electric polarizations are investigated. We caracterize guided modes in the fiber as eigenfunctions of compact integral operators and prove their exponential decay in the cladding. Then, we prove the possiblity of opening gaps in the spectrum of the background spectrum making it possible to guide electromagnetic waves with suitable cores.

1 Introduction

Optical fibers are today finding wide use in areas covering telecommunications, sensor technologies, spectroscopy, and medicine [7].

Ordinary optical fibers guide light by total internal reflection, which relies on the refractive index of the central core being greater than that of the surrounding cladding. This physical mechanism has been known and exploited technologically for many years. However, within the past decade the research in new purpose-built materials has opened up the possibilities of localizing and controlling light in cavities and waveguides by a new physical mechanism, namely the photonic band gap effect (PBG).

The PBG effect may be achieved in periodically structured materials having a periodicity on the scale of the optical wavelength. Such periodic structures are usually referred to as photonic crystals, or photonic band gap structures. By appropriate choice of crystal structure, the dimensions of the periodic lattice, and the properties of the component materials, propagation of electromagnetic waves in certain frequency bands (the photonic band gaps) may be forbidden within the crystal [40].

In [27], Knight and colleagues describe a fundamentally different type of optical fiber, one that has a core with a lower refractive index than the cladding and so rules out the possibility of internal reflection. Instead, light is guided by a mechanism which allows it to be piped through air.

The core of the new fiber is essentially a defect surrounded by a periodic array of air holes running along the entire length of the fiber. The defect acts like the core of an optical fiber. Light, which is expelled from the periodic structure surrounding the core, can only propagate along it. The new fiber operates truly by the photonic band gap effect. We refer to such a structure as a photonic crystal fiber (PCF).

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In this paper we model the propagation of electromagnetic waves in photonic crystal fibers. We give a mathematical framework for understanding their very unusual properties compared with the conventional fibers, attributed to an operation of the well-known mechanism of total reflection, and develop theoretical tools for the modeling of these photonic crystal fibers. We show the conditions under which the guided mode exist, and the nature of such modes. We study their dispersion properties and verify the exponential confinement of guided modes. In particular, we show that there exists a discrete set of these modes parameterized by a wave-number parameter.

The paper is outlined as follows. Sections 2 and 3 state the photonic fiber problem and give the main equations governing electromagnetic propagation in the fiber. In sections 4 and 5 we give some general results from the Floquet theory and PDE's with periodic coefficients. We then formulate the guidance of electromagnetic waves with integral equations in section 6 and we study the corresponding operators in section 7. A caracterization of guided modes is given in section 8. In section 9 we present an asymptotical case under which gaps open in the spectrum of the background medium making it possible to create defect modes. Of course, this is not necessary, (we can get gaps in the background spectrum without resorting to such asymptotics), however, it is very useful, since it makes it possible to guide waves with much more general structures. Finally, in section 11, We illustrate the main findings of the investigation in numerical examples.

2 Problem statement

We consider a 2-D photonic crystal, that is a medium characterized by a dielectric permittivity being periodic in two normal directions and invariant in the third normal direction. More precisely, the dielectric permittivity is given by a L^∞ and away from 0 measurable function $\epsilon_{\rm p}(x)$. This means that there exist ε_- and ε_+ positive constants such that:

$$0 < \varepsilon_- \le \epsilon_{\rm p}(x) \le \varepsilon_+ < \infty \ , \ \text{a.e. } x \in \mathbb{R}^2 \ . \tag{2.1}$$

The bounds $\varepsilon-$ and ε_+ are supposed to be reached. The function $\epsilon_{\rm p}$ is assumed to be independent of x_3 and unit-periodic in the $x_3 = 0$ plane:

$$\epsilon_{\rm p}(x_1+1, x_2) = \epsilon_{\rm p}(x_1, x_2), \quad \epsilon_{\rm p}(x_1, x_2+1) = \epsilon_{\rm p}(x_1, x_2) \ . \tag{2.2}$$

To this perfect 2-D photonic crystal, we introduce a line defect which is represented by a perturbation to the dielectric function $(\delta\varepsilon)(x_1, x_2)$. The perturbation is confined to the domain Ω:

$$(\delta\varepsilon)(x_1, x_2) = 0, \quad x \in \Omega^c \ .$$

Then the medium with defect has the dielectric function

$$\epsilon_{\rm p}(x_1, x_2) = \varepsilon(x_1, x_2) + (\delta\varepsilon)(x_1, x_2) \ . \tag{2.3}$$

Our goal is to find the guided modes in this structure, *i.e.*, frequencies for which there exist solutions to the time-harmonic Maxwell equations that are propagating along the defect and which energy is confined to the defect area.

3 Maxwell equations

The electromagnetic fields (E, H) satisfy the following time-harmonic Maxwell equations:

$$\left\{\begin{array}{lcl} \nabla \times H & = & -i\omega\varepsilon(x)E \;, \\ \nabla \times E & = & i\omega H \;. \end{array}\right. \tag{3.4}$$

However, this system can be studied from two scalar equations. Actually, the geometry of the medium and its dielectric function are independent of the third space coordinate x_3. Since we are looking for guided waves along the third direction, we take E and H with fixed exponential variation in the coordinate x_3 of the form $e^{i\beta x_3}$. This means that the electromagnetic field have the expression $(Ee^{i\beta x_3}, He^{i\beta x_3})$, where E and H depend only on (x_1, x_2). As we can see that, under such assumption, the curl operator reduces to

$$\nabla \times (He^{i\beta x_3}) = e^{i\beta x_3} \begin{pmatrix} \dfrac{\partial H_3}{\partial x_2} - i\beta H_2 \\ i\beta H_1 - \dfrac{\partial H_3}{\partial x_1} \\ \dfrac{\partial H_2}{\partial x_1} - \dfrac{\partial H_1}{\partial x_2} \end{pmatrix} . \tag{3.5}$$

Consequently, the harmonic Maxwell system is now decoupled in two independent subsystems. The solutions to the first one:

$$\left\{\begin{array}{lcl} i\omega\left(\varepsilon(x) - \dfrac{\beta^2}{\omega^2}\right) E_1 + \dfrac{\partial H_3}{\partial x_2} & = & 0 \;, \\ i\omega\left(\varepsilon(x) - \dfrac{\beta^2}{\omega^2}\right) E_2 - \dfrac{\partial H_3}{\partial x_1} & = & 0 \;, \\ -i\omega H_3 + \dfrac{\partial E_2}{\partial x_1} - \dfrac{\partial E_1}{\partial x_2} & = & 0 \;, \end{array}\right. \tag{3.6}$$

are called transverse electric (TE) and have the property $E_3 \equiv 0$. The solutions to the second one:

$$\left\{\begin{array}{lcl} i\omega\varepsilon(x)E_3 + \dfrac{\partial H_2}{\partial x_1} - \dfrac{\partial H_1}{\partial x_2} & = & 0 \;, \\ -i\omega\left(1 - \dfrac{\beta^2}{\omega^2\varepsilon(x)}\right) H_1 + \dfrac{\partial E_3}{\partial x_2} & = & 0 \;, \\ -i\omega\left(1 - \dfrac{\beta^2}{\omega^2\varepsilon(x)}\right) H_2 - \dfrac{\partial E_3}{\partial x_1} & = & 0 \;, \end{array}\right. \tag{3.7}$$

are called transverse magnetic (TM) and have the property $H_3 \equiv 0$. In both cases, solutions can be computed from a unique scalar function (resp. H_3 or E_3) which satisfies one of the following equations:

$$\left\{\begin{array}{lcl} \nabla \cdot \dfrac{1}{\varepsilon(x) - \frac{\beta^2}{\omega^2}} \nabla H_3 + \omega^2 H_3 & = & 0 \;, \\ \nabla \cdot \dfrac{\varepsilon(x)}{\varepsilon(x) - \frac{\beta^2}{\omega^2}} \nabla E_3 + \omega^2 \varepsilon(x) E_3 & = & 0 \;. \end{array}\right. \tag{3.8}$$

The problem consists then in finding $(\omega^2, \beta^2, u) \in \mathbb{R}^+ \times \mathbb{R}^+ \times L^2(\mathbb{R}^2)$ such that $\beta^2 < \omega^2 \varepsilon_-$ and u is solution of

$$\nabla \cdot \frac{1}{\varepsilon(x) - \frac{\beta^2}{\omega^2}} \nabla u + \omega^2 u = 0 \;, \tag{3.9}$$

in the TE case and of

$$\nabla \cdot \frac{\varepsilon(x)}{\varepsilon(x) - \frac{\beta^2}{\omega^2}} \nabla u + \omega^2 \varepsilon(x) u = 0 \ , \tag{3.10}$$

in the TM case.

4 Periodic operators and Floquet theory

As we look into the equations (3.9) and (3.10), we notice that they are partial differential equations with almost periodic coefficients. More precisely, these are spectral problems of partial differential operators with coefficients that are compactly supported perturbations of periodic functions. Let us then consider the periodic operators.

For $\alpha^2 < \varepsilon_-$, we define the unbounded operators $A^{\rm p}_\alpha$ and $B^{\rm p}_\alpha$ as

$$\begin{array}{rcl} A^{\rm p}_\alpha : L^2(\mathbb{R}^2) & \to & L^2(\mathbb{R}^2) \\ u & \mapsto & A^{\rm p}_\alpha u = -\nabla \cdot \dfrac{1}{\epsilon_{\rm p}(x) - \alpha^2} \nabla u \ , \end{array} \tag{4.11}$$

and

$$\begin{array}{rcl} B^{\rm p}_\alpha : L^2(\mathbb{R}^2) & \to & L^2(\mathbb{R}^2) \\ u & \mapsto & B^{\rm p}_\alpha u = -\dfrac{1}{\epsilon_{\rm p}(x)} \nabla \cdot \dfrac{\epsilon_{\rm p}(x)}{\epsilon_{\rm p}(x) - \alpha^2} \nabla u \ . \end{array} \tag{4.12}$$

These are self-adjoint partial differential operators with periodic coefficients. The self-adjointness character of $B^{\rm p}_\alpha$ is seen in the weighted Sobolev space $L^2(\mathbb{R}^2, \epsilon_{\rm p}(x)dx)$.

Let us start by considering $A^{\rm p}_\alpha$. This is an acoustic operator. It is also the operator governing the propagation of TE-polarized electromagnetic waves in a 2-D medium with a virtual dielectric permittivity $\epsilon_{\rm p} - \alpha^2$. Its spectrum has a band structure depending on the parameter α and it is well known that such operators can have band gaps, *i.e.*, intervals of values of ω that do not belong to the spectrum of $A^{\rm p}_\alpha$ and so propagating waves at frequencies ω can not exist in the virtual 2-D photonic crystal with dielectric permittivity $\epsilon_{\rm p} - \alpha^2$.

The case of $B^{\rm p}_\alpha$ is slightly different. This is not an operator governing the propagation of TM-polarized in some dielectric medium since it is not a Helmholtz operator. However, it still has band-structure spectrum. Actually, since it is elliptic and self-adjoint, when applying the Floquet transform we find a collection of operators defined on the unit cell, depending continuously on the dual variable and with point spectrum in the positive half-real axis accumulating at infinity. Then it is clear that the spectrum of $B^{\rm p}_\alpha$ has a band-structure. The other question that we can ask is can it have gaps?

The answer is yes. First, we notice that when $\alpha = 0$, $B^{\rm p}_0$ is a Helmholtz operator. It is well known that for suitable periodic dielectric function $\epsilon_{\rm p}$, the Helmholtz operator

$$\begin{array}{rcl} B^{\rm p}_0 : L^2(\mathbb{R}^2) & \to & L^2(\mathbb{R}^2) \\ u & \mapsto & B^{\rm p}_0 u = -\dfrac{1}{\epsilon_{\rm p}(x)} \Delta u \ , \end{array} \tag{4.13}$$

can exhibit band gaps. It remains to prove the continuous dependence of the spectrum of $B^{\rm p}_\alpha$ on α to conclude that, at least for α close to 0, $B^{\rm p}_\alpha$ has gaps in its spectrum. The continuity can be seen with the dependence on α of the point spectrum of the Floquet transformed operators.

In what follows, we suppose that ϵ_{p} is such that A^{p}_{α} or B^{p}_{α} (depending on which polarization is considered) has a gap for α^2 belonging to a non-empty open subset of $(0, \varepsilon_-)$.

5 The Green's kernel

Here we define the Green's kernel for the operators A^{p}_{α} and B^{p}_{α} when they have gaps.

5.1 The TE polarization

Let us suppose that for some $\alpha \in (0, \varepsilon_-)$, the operator A^{p}_{α} has gaps in its spectrum. We denote by Σ_α the spectrum of A^{p}_{α}. Let ω^2 be in $\mathbb{R}^+ \setminus \Sigma_\alpha$. We can then define the Green's kernel $G_\alpha(\omega^2; x, y)$ as the solution to

$$\nabla \cdot \frac{1}{\epsilon_{\mathrm{p}}(x) - \alpha^2} \nabla G_\alpha(\omega^2; x, y) + \omega^2 G_\alpha(\omega^2; x, y) = \delta(x - y) \ . \tag{5.14}$$

One of the main properties of the Green's kernel is stated in the following lemma.

Lemma 5.1 *There exist positive constants C_1 and C_2 depending on α and ω_0^2 such that for any $\omega^2 \in]0, \omega_0^2[\setminus \Sigma_\alpha$:*

$$|G_\alpha(\omega^2; x, y)| \leq C_1 e^{C_2 \, dist(\omega^2, \Sigma_\alpha)|x-y|} \ , \quad |x - y| \to +\infty \ . \tag{5.15}$$

This explains why an incident wave with frequency lying in the gap is reflected by the photonic crystal and decays exponentially inside it. It also gives a justification to the exponential localization of modes created by adding a compactly supported defect in the crystal.

The exponential decay is obtained by using a Combes-Thomas [10] argument to get the appropriate estimates on the resolvent. It is known however that the radius of localization

$$\frac{1}{C_2 \mathrm{dist}(\omega^2, \Sigma_\alpha)}$$

is not optimal close to the spectrum. More precisely, let $]a, b[$ be a gap of A^{p}_{α}, *i.e.*,

$$]a, b[\cap \Sigma_\alpha = \emptyset \quad \text{and} \quad a, b \in \Sigma_\alpha \ ,$$

then it has been proved that we have a decay estimate of the form:

$$e^{-C\sqrt{|\omega^2 - a||\omega^2 - b|}|x-y|} \ .$$

This is obtained by a general operator-theoretic approach. The main idea consists in using the Paley-Wiener theorems for the Floquet transform and the exponential decay of functions for which the Floquet transform has analytic dependence on the dual variable in a neighborhood of the real axis.

Another property of the Green's function is its weak singularity when $x = y$.

Lemma 5.2 *Let D be a bounded domain in which $\epsilon_{\rm p}$ is constant. Then the function*

$$G_\alpha(\omega^2;x,y) - \frac{\epsilon_{\rm p}-\alpha^2}{2\pi}\log|x-y| \tag{5.16}$$

is continuous for x,y in D when $|x-y|\to 0$.

Proof. We recall that

$$\Delta\left(\frac{1}{2\pi}\log|x-y|\right) = \delta(x-y)\ . \tag{5.17}$$

Let us define K by

$$K(x,y) = G_\alpha(\omega^2;x,y) - \frac{\epsilon_{\rm p}-\alpha^2}{2\pi}\log|x-y|\ .$$

We remark that K satisfies the following Helmholtz equation:

$$\Delta K(x,y) + \omega^2(\epsilon_{\rm p}-\alpha^2)K(x,y) = -\omega^2\frac{(\epsilon_{\rm p}-\alpha^2)^2}{2\pi}\log|x-y|\ . \tag{5.18}$$

Since $\log|x-y|$ is L^2_y-integrable, we deduce that K, considered as a function of y for a fixed x, is in $H^2(D)$ and is continuous when $|x-y|\to 0$. □

5.2 The TM polarization

Again, the case of the TM polarization is not exactly similar to the TE polarization. Since the operator $B^{\rm p}_\alpha$ is not a Helmholtz operator, the Green's kernel is different form the one of $B^{\rm p}_0$. We use the same notations as done in the previous section calling Σ_α the spectrum of $B^{\rm p}_\alpha$ and $G_\alpha(\omega^2;x,y)$ the solution to

$$\nabla\cdot\frac{\epsilon_{\rm p}(x)}{\epsilon_{\rm p}(x)-\alpha^2}\nabla G_\alpha(\omega^2;x,y) + \omega^2\epsilon_{\rm p}(x)G_\alpha(\omega^2;x,y) = \delta(x-y)\ . \tag{5.19}$$

Nevertheless, the analogous results to the ones cited in the previous section hold. Actually, Lemma 5.15 relies on a Combes-Thomas argument [10] that can still be used. We have just to modify the duality in $L^2(\mathbb{R}^2)$ defining it as

$$\langle u,v\rangle = \int_{\mathbb{R}^2} u(x)\,\overline{v}(x)\,\epsilon_{\rm p}(x)\,dx\ .$$

The analogous result to the one in Lemma 5.2 is that the function

$$G_\alpha(\omega^2;x,y) - \frac{\epsilon_{\rm p}-\alpha^2}{2\pi\epsilon_{\rm p}}\log|x-y| \tag{5.20}$$

is continuous for x,y in D when $|x-y|\to 0$.

6 An integral formulation of the photonic fiber problem

Now we introduce a compactly supported perturbation to the dielectric function of the medium which is transformed into $\varepsilon(x)$ defined in (2.3) and we look for guided modes $(\omega^2,\beta^2,u)\in\mathbb{R}^+\times\mathbb{R}^+\times L^2(\mathbb{R}^2)$ solutions of (3.9) or (3.10).

6.1 The TE polarization

We consider here the TE polarization. Suppose that we have a guided mode. This clearly means that u is an eigenfunction of $A_{\frac{\beta}{\omega}}$ for the eigenvalue ω^2 where A_α is the operator defined for $\alpha^2 < \varepsilon_-$ as

$$\begin{aligned} A_\alpha : L^2(\mathbb{R}^2) \quad &\to \quad L^2(\mathbb{R}^2) \\ u \quad &\mapsto \quad A_\alpha u = -\nabla \cdot \frac{1}{\varepsilon(x) - \alpha^2} \nabla u \, . \end{aligned} \tag{6.21}$$

It is then interesting to look for the spectral properties of the operator A_α and for a practical characterization of its eigenvalues when they exist. The following proposition is a consequence of a classical result in spectral theory.

Proposition 6.1 *For any $\alpha \in (0, \varepsilon_-)$, the operators A^{p}_α and A_α have the same essential spectrum.*

This is a consequence of the Weyl's theorem since it can be proved that $A_\alpha - A^{\mathrm{p}}_\alpha$ is a relatively compact perturbation of A^{p}_α.

Then the spectrum of A_α lying in the gaps of A^{p}_α will consist in eigenvalues of finite multiplicity that can accumulate only at the edges of the gaps. Moreover, A_α has the same continuous spectrum as A^{p}_α. An interesting question is: what about the existence of eigenvalues of A_α in the continuous spectrum? There is no result for the moment answering whether such eigenvalues can appear or not. In the case that such eigenvalues exist, the behaviour of the corresponding eigenfunctions is not obvious. On one hand, they should be localized due to the local character of the perturbation and on the other hand, it has enough energy to propagate along the medium.

We suppose here that the guided mode we consider is such that $\omega^2 \notin \Sigma_{\frac{\beta}{\omega}}$.

Recalling that ϵ_{p} and ε are piecewise constant, we define the finite partition $(D_i)_{i\in I}$ of Ω as the disjoint subdomains of Ω in which ϵ_{p}, ε and thus $(\delta\varepsilon)$ are constant. We also define $\Pi = \cup_{i\in I}\partial D_i$. We suppose that the curves $\overline{D_i} \cap \overline{D_j}$ and $\overline{D_i} \cap \Omega^c$ are smooth.

The following proposition holds.

Proposition 6.2 *The guided modes $(\omega^2, \beta^2, u) \in \mathbb{R}^+ \times \mathbb{R}^+ \times L^2(\mathbb{R}^2)$ satisfying $\omega^2 \notin \Sigma_{\frac{\beta}{\omega}}$ are exactly the functions u satisfying*

$$\left[\frac{1}{\varepsilon - \frac{\beta^2}{\omega^2}} \partial_\nu u \right] = 0 \, ,$$

and are solutions of the following integral equation:

$$\begin{aligned} u(x) \quad = \quad & \omega^2 \int_\Omega \frac{(\delta\varepsilon)}{(\epsilon_{\mathrm{p}} - \frac{\beta^2}{\omega^2})} G_{\frac{\beta}{\omega}}(\omega^2; x, y) u(y) \, dy \\ & + \int_\Pi G_{\frac{\beta}{\omega}}(\omega^2; x, y) \left[\frac{(\delta\varepsilon)}{(\epsilon_{\mathrm{p}} - \frac{\beta^2}{\omega^2})} \right] \frac{1}{(\varepsilon - \frac{\beta^2}{\omega^2})} \partial_\nu u(y) \, dl_y \, , \end{aligned} \tag{6.22}$$

where $\partial_\nu u$ is the normal derivative of u on Π and $[f]$ represents the jump of f across Π in the ν direction.

Proof. Suppose that u satisfies the conditions above. It is clear then that u satisfies (3.9) in Ω^c. Now let us consider u in a domain D_i. Since ϵ_{p}, ε and $(\delta\varepsilon)$ are constant in D_i, we have:

$$\left(\frac{1}{\varepsilon(x)-\frac{\beta^2}{\omega^2}}\Delta_x+\omega^2\right)G_{\frac{\beta}{\omega}}(\omega^2;x,y) = \frac{\epsilon_{\mathrm{p}}(x)-\frac{\beta^2}{\omega^2}}{\varepsilon(x)-\frac{\beta^2}{\omega^2}}\delta(x-y) -\omega^2\frac{(\delta\varepsilon)(x)}{\varepsilon(x)-\frac{\beta^2}{\omega^2}}G_{\frac{\beta}{\omega}}(\omega^2;x,y)\ , \tag{6.23}$$

for any $x\in\Omega\setminus\Pi$ and any $y\in\mathbb{R}^2$. It follows that for any $i\in I$ and any $x\in D_i$, we have:

$$\begin{aligned}\left(\nabla\cdot\frac{1}{\varepsilon(x)-\frac{\beta^2}{\omega^2}}\nabla+\omega^2\right)u(x) &= \omega^2\frac{(\delta\varepsilon)(x)}{\varepsilon(x)-\frac{\beta^2}{\omega^2}}u(x)\\ &\quad-\omega^4\frac{(\delta\varepsilon)(x)}{\varepsilon(x)-\frac{\beta^2}{\omega^2}}\int_\Omega\frac{(\delta\varepsilon)}{(\epsilon_{\mathrm{p}}-\frac{\beta^2}{\omega^2})}G_{\frac{\beta}{\omega}}(\omega^2;x,y)u(y)\,dy\\ &\quad-\omega^2\frac{(\delta\varepsilon)(x)}{\varepsilon(x)-\frac{\beta^2}{\omega^2}}\int_\Pi G_{\frac{\beta}{\omega}}(\omega^2;x,y)\left[\frac{(\delta\varepsilon)}{(\epsilon_{\mathrm{p}}-\frac{\beta^2}{\omega^2})}\right]\frac{1}{(\varepsilon-\frac{\beta^2}{\omega^2})}\partial_\nu u(y)\,dl_y\\ &=0\ .\end{aligned}$$

Then u solves equation (3.9) in $\mathbb{R}^2\setminus\Pi$. Recalling the jump relation it satisfies, we conclude that u solves (3.9) in $\mathbb{R}^2$.

Conversely, let us suppose that u solves equation (3.9). Then u satisfies the jump relation

$$\left[\frac{1}{(\varepsilon-\frac{\beta^2}{\omega^2})}\partial_\nu u\right]=0$$

on Π.

Moreover, we have:

$$\begin{aligned}u(x) &= \int_{\mathbb{R}^2}\left(\nabla\cdot\frac{1}{\epsilon_{\mathrm{p}}(y)-\frac{\beta^2}{\omega^2}}\nabla+\omega^2\right)G_{\frac{\beta}{\omega}}(\omega^2;x,y)u(y)\,dy\\ &= -\int_{\mathbb{R}^2}\frac{1}{\epsilon_{\mathrm{p}}(y)-\frac{\beta^2}{\omega^2}}\nabla G_{\frac{\beta}{\omega}}(\omega^2;x,y)\cdot\nabla u(y)\,dy\\ &\quad+\omega^2\int_{\mathbb{R}^2}G_{\frac{\beta}{\omega}}(\omega^2;x,y)u(y)\,dy\\ &= \int_\Omega\frac{(\delta\varepsilon)(y)}{(\epsilon_{\mathrm{p}}(y)-\frac{\beta^2}{\omega^2})(\varepsilon(y)-\frac{\beta^2}{\omega^2})}\nabla G_{\frac{\beta}{\omega}}(\omega^2;x,y)\cdot\nabla u(y)\,dy\\ &\quad+\int_{\mathbb{R}^2}G_{\frac{\beta}{\omega}}(\omega^2;x,y)\nabla\cdot\frac{1}{\varepsilon(y)-\frac{\beta^2}{\omega^2}}\nabla u(y)\,dy\\ &\quad+\omega^2\int_{\mathbb{R}^2}G_{\frac{\beta}{\omega}}(\omega^2;x,y)u(y)\,dy\\ &= \sum_{i\in I}\int_{D_i}\frac{(\delta\varepsilon)(y)}{(\epsilon_{\mathrm{p}}(y)-\frac{\beta^2}{\omega^2})(\varepsilon(y)-\frac{\beta^2}{\omega^2})}\nabla G_{\frac{\beta}{\omega}}(\omega^2;x,y)\cdot\nabla u(y)\,dy\ .\end{aligned}$$

Denoting by $\epsilon_{\rm p}^i$, ε^i and $(\delta\varepsilon)^i$ the values of $\epsilon_{\rm p}$, ε and $(\delta\varepsilon)$ in D_i, we get

$$\begin{aligned} u(x) &= -\sum_{i\in I}\int_{D_i}\frac{(\delta\varepsilon)^i}{(\epsilon_{\rm p}^i-\frac{\beta^2}{\omega^2})}G_{\frac{\beta}{\omega}}(\omega^2;x,y)\nabla\cdot\frac{1}{\varepsilon^i-\frac{\beta^2}{\omega^2}}\nabla u(y)\,dy \\ &\quad+\sum_{i\in I}\int_{\partial D_i}G_{\frac{\beta}{\omega}}(\omega^2;x,y)\frac{(\delta\varepsilon)^i}{(\epsilon_{\rm p}^i-\frac{\beta^2}{\omega^2})(\varepsilon^i-\frac{\beta^2}{\omega^2})}\partial_\nu u(y)\,dl_y \\ &= \omega^2\int_\Omega\frac{(\delta\varepsilon)(y)}{(\epsilon_{\rm p}(y)-\frac{\beta^2}{\omega^2})}G_{\frac{\beta}{\omega}}(\omega^2;x,y)u(y)\,dy \\ &\quad+\int_\Pi G_{\frac{\beta}{\omega}}(\omega^2;x,y)\left[\frac{(\delta\varepsilon)}{(\epsilon_{\rm p}-\frac{\beta^2}{\omega^2})}\right]\frac{1}{(\varepsilon(y)-\frac{\beta^2}{\omega^2})}\partial_\nu u(y)\,dl_y\ , \end{aligned}$$

which ends the proof. □

6.2 The TM polarization

Now we consider the TM polarization for which the results are mainly the same. Suppose that we have a guided mode. Then u is an eigenfunction of $B_{\frac{\beta}{\omega}}$ for the eigenvalue ω^2 where B_α is the operator defined for $\alpha^2 < \varepsilon_-$ as

$$\begin{aligned} B_\alpha : L^2(\mathbb{R}^2) &\to L^2(\mathbb{R}^2) \\ u &\mapsto B_\alpha u = -\frac{1}{\varepsilon}\nabla\cdot\frac{\varepsilon}{\varepsilon(x)-\alpha^2}\nabla u\ . \end{aligned} \tag{6.24}$$

The counterpart of Proposition 6.1 is the following.

Proposition 6.3 *For any $\alpha\in]0,\varepsilon_-[$, the operators $B_\alpha^{\rm p}$ and B_α have the same essential spectrum.*

We consider only guided modes for which $\omega^2\notin\Sigma_{\frac{\beta}{\omega}}$. The following proposition holds.

Proposition 6.4 *The guided modes $(\omega^2,\beta^2,u)\in\mathbb{R}^+\times\mathbb{R}^+\times L^2(\mathbb{R}^2)$ satisfying $\omega^2\notin\Sigma_{\frac{\beta}{\omega}}$ are exactly the functions u satisfying*

$$\left[\frac{\varepsilon}{\varepsilon-\frac{\beta^2}{\omega^2}}\partial_\nu u\right]=0\ ,$$

and are solutions of the following integral equation:

$$\begin{aligned} u(x) &= \omega^2\int_\Omega\frac{\epsilon_{\rm p}(\delta\varepsilon)}{(\epsilon_{\rm p}-\frac{\beta^2}{\omega^2})}G_{\frac{\beta}{\omega}}(\omega^2;x,y)u(y)\,dy \\ &\quad+\frac{\beta^2}{\omega^2}\int_\Pi G_{\frac{\beta}{\omega}}(\omega^2;x,y)\left[\frac{(\delta\varepsilon)}{\varepsilon(\epsilon_{\rm p}-\frac{\beta^2}{\omega^2})}\right]\frac{\varepsilon}{(\varepsilon-\frac{\beta^2}{\omega^2})}\partial_\nu u(y)\,dl_y\ , \end{aligned} \tag{6.25}$$

where $\partial_\nu u$ is the normal derivative of u on Π and $[f]$ represents the jump of f across Π in the ν direction.

Proof. Suppose that u satisfies the conditions above. Then it is clear that u satisfies (3.10) in Ω^c. Now let us consider u in a domain D_i. Since $\epsilon_{\rm p}$, ε, and $(\delta\varepsilon)$ are constant in D_i, we have for any $x \in D_i$ and any $y \in \mathbb{R}^2$:

$$\Big(\frac{\varepsilon(x)}{\varepsilon(x) - \frac{\beta^2}{\omega^2}}\Delta_x + \omega^2\varepsilon(x)\Big)G_{\frac{\beta}{\omega}}(\omega^2; x, y) = \frac{(\epsilon_{\rm p}(x) - \frac{\beta^2}{\omega^2})\varepsilon(x)}{(\varepsilon(x) - \frac{\beta^2}{\omega^2})\epsilon_{\rm p}(x)}\delta(x-y) \qquad (6.26)$$
$$-\omega^2 \frac{\varepsilon(x)(\delta\varepsilon)(x)}{(\varepsilon(x) - \frac{\beta^2}{\omega^2})} G_{\frac{\beta}{\omega}}(\omega^2; x, y)\ ,$$

from which we deduce in a similar way as done in the TE case that u satisfies equation (3.10) in $\Omega \setminus \Pi$. Recalling the jump relation it satisfies, we deduce that u satisfies (3.10) in $\mathbb{R}^2$.

Conversely, suppose that u solves equation (3.10). Then u satisfies the jump relation

$$\Big[\frac{\varepsilon}{(\varepsilon - \frac{\beta}{\omega})}\partial_\nu u\Big] = 0$$

on Π.

Moreover, we have:

$$\begin{aligned}
u(x) &= \int_{\mathbb{R}^2}\Big(\nabla\cdot \frac{\epsilon_{\rm p}(y)}{\epsilon_{\rm p}(y) - \frac{\beta^2}{\omega^2}}\nabla + \omega^2\epsilon_{\rm p}(y)\Big)G_{\frac{\beta}{\omega}}(\omega^2; x, y)u(y)\, dy \\
&= -\int_{\mathbb{R}^2}\frac{\epsilon_{\rm p}(y)}{\epsilon_{\rm p}(y) - \frac{\beta^2}{\omega^2}}\nabla G_{\frac{\beta}{\omega}}(\omega^2; x, y)\cdot\nabla u(y)\, dy \\
&\quad +\omega^2\int_{\mathbb{R}^2} G_{\frac{\beta}{\omega}}(\omega^2; x, y)u(y)\epsilon_{\rm p}(y)\, dy \\
&= \frac{\beta^2}{\omega^2}\int_\Omega \frac{(\delta\varepsilon)(y)}{(\epsilon_{\rm p}(y) - \frac{\beta^2}{\omega^2})(\varepsilon(y) - \frac{\beta^2}{\omega^2})}\nabla G_{\frac{\beta}{\omega}}(\omega^2; x, y)\cdot\nabla u(y)\, dy \\
&\quad +\omega^2\int_\Omega G_{\frac{\beta}{\omega}}(\omega^2; x, y)u(y)(\delta\varepsilon)(y)\, dy \\
&\quad +\int_{\mathbb{R}^2} G_{\frac{\beta}{\omega}}(\omega^2; x, y)\nabla\cdot\frac{\epsilon_{\rm p}(y)}{\varepsilon(y) - \frac{\beta^2}{\omega^2}}\nabla u(y)\, dy \\
&\quad +\omega^2\int_{\mathbb{R}^2} G_{\frac{\beta}{\omega}}(\omega^2; x, y)u(y)\epsilon_{\rm p}(y)\, dy \\
&= -\frac{\beta^2}{\omega^2}\sum_{i\in I}\int_{D_i}\frac{(\delta\varepsilon)^i}{\varepsilon^i(\epsilon_{\rm p}^i - \frac{\beta^2}{\omega^2})}G_{\frac{\beta}{\omega}}(\omega^2; x, y)\nabla\cdot\frac{\varepsilon^i}{(\varepsilon^i - \frac{\beta^2}{\omega^2})}\nabla u(y)\, dy \\
&\quad +\frac{\beta^2}{\omega^2}\sum_{i\in I}\int_{\partial D_i}\frac{(\delta\varepsilon)^i}{(\epsilon_{\rm p}^i - \frac{\beta^2}{\omega^2})(\varepsilon^i - \frac{\beta^2}{\omega^2})}G_{\frac{\beta}{\omega}}(\omega^2; x, y)\partial_\nu u(y)\, dy \\
&\quad +\omega^2\int_\Omega G_{\frac{\beta}{\omega}}(\omega^2; x, y)u(y)(\delta\varepsilon)(y)\, dy\ ,
\end{aligned}$$

and therefore

$$
\begin{aligned}
u(x) &= \beta^2 \sum_{i\in I} \int_{D_i} \frac{(\delta\varepsilon)^i}{(\epsilon_{\rm p}^i - \frac{\beta^2}{\omega^2})} G_{\frac{\beta}{\omega}}(\omega^2; x, y) u(y)\, dy \\
&+ \frac{\beta^2}{\omega^2} \sum_{i\in I} \int_{\partial D_i} \frac{(\delta\varepsilon)^i}{(\epsilon_{\rm p}^i - \frac{\beta^2}{\omega^2})(\varepsilon^i - \frac{\beta^2}{\omega^2})} G_{\frac{\beta}{\omega}}(\omega^2; x, y) \partial_\nu u(y)\, dy \\
&+ \omega^2 \int_\Omega G_{\frac{\beta}{\omega}}(\omega^2; x, y) u(y) (\delta\varepsilon)(y)\, dy \\
&= \omega^2 \int_\Omega \frac{\epsilon_{\rm p}(y)}{(\epsilon_{\rm p}(y) - \frac{\beta^2}{\omega^2})} G_{\frac{\beta}{\omega}}(\omega^2; x, y) u(y) (\delta\varepsilon)(y)\, dy \\
&+ \frac{\beta^2}{\omega^2} \int_\Pi G_{\frac{\beta}{\omega}}(\omega^2; x, y) \left[\frac{(\delta\varepsilon)}{(\epsilon_{\rm p} - \frac{\beta^2}{\omega^2})\varepsilon} \right] \frac{\varepsilon(y)}{(\varepsilon(y) - \frac{\beta^2}{\omega^2})} \partial_\nu u(y)\, dy \ .
\end{aligned}
$$

The proposition is then proved. □

7 Preliminary results

We introduce in this section new integral operators that will be useful for finding guided modes. We start by orienting the curves $\overline{D_i} \cap \overline{D_j}$ and $\overline{D_i} \cap \Omega^c$ and define a normal vector ν on each one.

Definition 7.1 *We define the operator $\mathcal{A}_{\omega,\beta}$ for $\omega^2 \notin \Sigma_{\frac{\beta}{\omega}}$ by*

$$
\begin{aligned}
\mathcal{A}_{\omega,\beta} : L^2(\Omega) \times L^2(\Pi) &\rightarrow L^2(\Omega) \times L^2(\Pi) \\
(u, \varphi) &\mapsto \mathcal{A}_{\omega,\beta}(u, \varphi) = (v, \psi) \ ,
\end{aligned}
$$

such that

$$
\begin{aligned}
v(x) &= \omega^2 \int_\Omega \frac{(\delta\varepsilon)}{(\epsilon_{\rm p} - \frac{\beta^2}{\omega^2})} G_{\frac{\beta}{\omega}}(\omega^2; x, y) u(y)\, dy \\
&+ \int_\Pi G_{\frac{\beta}{\omega}}(\omega^2; x, y) \left[\frac{(\delta\varepsilon)}{(\epsilon_{\rm p} - \frac{\beta^2}{\omega^2})} \right] \varphi(y)\, dl_y \ , \quad x \in \Omega \ ,
\end{aligned} \tag{7.27}
$$

and

$$
\psi(x) = \frac{1}{\varepsilon(x)^\mu - \frac{\beta^2}{\omega^2}} \partial_\nu^\mu v(x) \ , \quad x \in \Pi \ , \tag{7.28}
$$

for some $\mu \in \{+, -\}$, where for $x \in \Pi$ and f defined on a neighborhood of Π, $f^\pm(x) = \lim_{\tau \to 0^\pm} f(x + \tau\nu_x)$.

In fact, the parameter μ has just to take a fixed value + or − on each component $\overline{D_i} \cap \overline{D_j}$ and $\overline{D_i} \cap \Omega^c$, i, j in I. The following proposition holds.

Proposition 7.1 *The operator $\mathcal{A}_{\omega,\beta}$ is compact.*

Proof. Let $(v, \psi) = \mathcal{A}_{\omega,\beta}(u, \varphi)$. It is obvious that in each subdomain D_i, v solves a Helmholtz equation with an L^2 right hand side. It follows that $v \in$

$\prod_{i\in I} H^2(D_i)$ and $\psi \in \prod_{i,j\in I} H^{1/2}(\overline{D_i}\cap\overline{D_j})\times\prod_{i\in I} H^{1/2}(\overline{D_i}\cap\Omega^c)$. The compactness of $\mathcal{A}_{\omega,\beta}$ is then a consequence of the compact embedding of $\prod_{i\in I} H^2(D_i)$ in $L^2(\Omega)$ and of $\prod_{i,j\in I} H^{1/2}(\overline{D_i}\cap\overline{D_j})\times\prod_{i\in I} H^{1/2}(\overline{D_i}\cap\Omega^c)$ in $L^2(\Pi)$. □

Now we define the analogous operator that will be useful in the TM polarization.

Definition 7.2 *We define the operator $\mathcal{B}_{\omega,\beta}$ for $\omega^2 \notin \Sigma_{\frac{\beta}{\omega}}$ by*

$$\begin{aligned} \mathcal{B}_{\omega,\beta} : L^2(\Omega)\times L^2(\Pi) &\rightarrow L^2(\Omega)\times L^2(\Pi) \\ (u,\varphi) &\mapsto \mathcal{B}_{\omega,\beta}(u,\varphi) = (v,\psi)\ , \end{aligned}$$

such that

$$\begin{aligned} v(x) &= \omega^2 \int_\Omega \frac{\epsilon_{\mathrm{p}}(\delta\varepsilon)}{(\epsilon_{\mathrm{p}}-\frac{\beta^2}{\omega^2})} G_{\frac{\beta}{\omega}}(\omega^2;x,y)u(y)\,dy \\ &\quad + \frac{\beta^2}{\omega^2}\int_\Pi G_{\frac{\beta}{\omega}}(\omega^2;x,y)\left[\frac{(\delta\varepsilon)}{\varepsilon(\epsilon_{\mathrm{p}}-\frac{\beta^2}{\omega^2})}\right]\varphi(y)\,dl_y\ , \quad x\in\Omega\ , \end{aligned} \tag{7.29}$$

and

$$\psi(x) = \frac{\varepsilon(x)^\mu}{\varepsilon(x)^\mu - \frac{\beta^2}{\omega^2}}\partial_\nu^\mu v(x)\ , \quad x\in\Pi\ , \tag{7.30}$$

for some $\mu \in \{+,-\}$.

The following proposition holds.

Proposition 7.2 *The operator $\mathcal{B}_{\omega,\beta}$ is compact.*

The proof is exactly the same as for $\mathcal{A}_{\omega,\beta}$.

8 Guided modes in the photonic fiber

Now we are going to give the main result of this paper. Actually, we characterize the guided modes in the photonic fiber as a spectral problem on a compact operator.

Theorem 8.1 *The guided modes (ω^2,β^2,u) in the TE-polarization satisfying $\omega^2\notin\Sigma_{\frac{\beta^2}{\omega^2}}$ are exactly the solutions to the following spectral problem:*

$$\mathcal{A}_{\omega,\beta}(u,\varphi) = (u,\varphi) \tag{8.31}$$

for some $\varphi\in L^2(\mathbb{R}^2)$.

Proof. Suppose that (ω^2,β^2,u) is a guided mode and that $\omega^2\notin\Sigma_{\frac{\beta^2}{\omega^2}}$. Then from Proposition 6.2, we have

$$\mathcal{A}_{\omega,\beta}(u, \frac{1}{\varepsilon(x)^\mu-\frac{\beta^2}{\omega^2}}\partial_\nu^\mu u) = (u, \frac{1}{\varepsilon(x)^\mu-\frac{\beta^2}{\omega^2}}\partial_\nu^\mu u)\ . \tag{8.32}$$

Conversely, suppose that (u,φ) is an eigenfunction of $\mathcal{A}_{\omega,\beta}$ for the eigenvalue 1. Then, recalling Proposition 6.2, we need only to prove that

$$\left[\frac{1}{\varepsilon-\frac{\beta^2}{\omega^2}}\partial_\nu u\right]=0\ .$$

to establish that (ω^2,β^2,u) is a guided mode.

From the equation satisfied by $G_{\frac{\beta}{\omega}}(\omega^2;x,y)$ we deduce that for any $x\in\Pi$ and any $y\in\Omega\setminus\Pi$ we have

$$\left[\frac{1}{\epsilon_{\rm p}-\frac{\beta^2}{\omega^2}}\partial_\nu G_{\frac{\beta}{\omega}}(\omega^2;x,y)\right]=0\ . \tag{8.33}$$

It follows that

$$\left[\frac{1}{\varepsilon-\frac{\beta^2}{\omega^2}}\partial_\nu G_{\frac{\beta}{\omega}}(\omega^2;x,y)\right]=\left[\frac{\epsilon_{\rm p}-\frac{\beta^2}{\omega^2}}{\varepsilon-\frac{\beta^2}{\omega^2}}\right]\frac{1}{\epsilon_{\rm p}^\mu-\frac{\beta^2}{\omega^2}}\partial_\nu^\mu G_{\frac{\beta}{\omega}}(\omega^2;x,y)\ . \tag{8.34}$$

Let us consider

$$\frac{1}{\varepsilon^\mu-\frac{\beta^2}{\omega^2}}\partial_\nu^\mu\int_\Pi G_{\frac{\beta}{\omega}}(\omega^2;x,y)\left[\frac{(\delta\varepsilon)}{(\epsilon_{\rm p}-\frac{\beta^2}{\omega^2})}\right]\varphi(y)\,dl_y\ .$$

From Lemma 5.2 we deduce that

$$\begin{aligned}&\frac{1}{\varepsilon^\mu-\frac{\beta^2}{\omega^2}}\partial_\nu^\mu\int_\Pi G_{\frac{\beta}{\omega}}(\omega^2;x,y)\left[\frac{(\delta\varepsilon)}{(\epsilon_{\rm p}-\frac{\beta^2}{\omega^2})}\right]\varphi(y)\,dl_y\\
=&\int_\Pi\frac{1}{\varepsilon^\mu-\frac{\beta^2}{\omega^2}}\partial_\nu^\mu\left(G_{\frac{\beta}{\omega}}(\omega^2;x,y)-\frac{\epsilon_{\rm p}^\mu-\frac{\beta^2}{\omega^2}}{2\pi}\log|x-y|\right)\left[\frac{(\delta\varepsilon)}{(\epsilon_{\rm p}-\frac{\beta^2}{\omega^2})}\right]\varphi(y)\,dl_y\\
&+\frac{1}{\varepsilon^\mu-\frac{\beta^2}{\omega^2}}\partial_\nu^\mu\int_\Pi\frac{\epsilon_{\rm p}^\mu-\frac{\beta^2}{\omega^2}}{2\pi}\log|x-y|\left[\frac{(\delta\varepsilon)}{(\epsilon_{\rm p}-\frac{\beta^2}{\omega^2})}\right]\varphi(y)\,dl_y\ .\end{aligned}$$

The following identity is a classical result in the potential theory:

$$\partial_\nu^\mu\int_\Pi\frac{1}{2\pi}\log|x-y|\varphi(y)\,dl_y=\mu\frac{1}{2}\varphi(x)+\int_\Pi\frac{1}{2\pi}\partial_\nu^\mu\log|x-y|\varphi(y)dl_y\ . \tag{8.35}$$

Therefore

$$\begin{aligned}&\frac{1}{\varepsilon^\mu-\frac{\beta^2}{\omega^2}}\partial_\nu^\mu\int_\Pi G_{\frac{\beta}{\omega}}(\omega^2;x,y)\left[\frac{(\delta\varepsilon)}{(\epsilon_{\rm p}-\frac{\beta^2}{\omega^2})}\right]\varphi(y)\,dl_y\\
&\qquad=\int_\Pi\frac{1}{\varepsilon^\mu-\frac{\beta^2}{\omega^2}}\partial_\nu^\mu G_{\frac{\beta}{\omega}}(\omega^2;x,y)\left[\frac{(\delta\varepsilon)}{(\epsilon_{\rm p}-\frac{\beta^2}{\omega^2})}\right]\varphi(y)\,dl_y\\
&\qquad\quad+\mu\frac{1}{2}\frac{\epsilon_{\rm p}^\mu-\frac{\beta^2}{\omega^2}}{\varepsilon^\mu-\frac{\beta^2}{\omega^2}}\left[\frac{(\delta\varepsilon)}{\epsilon_{\rm p}-\frac{\beta^2}{\omega^2}}\right]\varphi(x)\ .\end{aligned}$$

We then deduce the expression of the jump we are looking for:

$$\begin{aligned}\left[\frac{1}{\varepsilon-\frac{\beta^2}{\omega^2}}\partial_\nu u\right]&=\left[\frac{\epsilon_{\rm p}-\frac{\beta^2}{\omega^2}}{\varepsilon-\frac{\beta^2}{\omega^2}}\right]\frac{1}{\epsilon_{\rm p}^\mu-\frac{\beta^2}{\omega^2}}\partial_\nu^\mu u\\
&\quad+\left(\left[\mu\frac{1}{2}\frac{\epsilon_{\rm p}-\frac{\beta^2}{\omega^2}}{\varepsilon-\frac{\beta^2}{\omega^2}}\right]-\mu\frac{1}{2}\left[\frac{\epsilon_{\rm p}-\frac{\beta^2}{\omega^2}}{\varepsilon-\frac{\beta^2}{\omega^2}}\right]\right)\left[\frac{(\delta\varepsilon)}{\epsilon_{\rm p}-\frac{\beta^2}{\omega^2}}\right]\varphi\ .\end{aligned}$$

After making the necessary simplifications, we get

$$\left[\frac{1}{\varepsilon - \frac{\beta^2}{\omega^2}}\partial_\nu u\right] = \left[\frac{\epsilon_{\rm p} - \frac{\beta^2}{\omega^2}}{\varepsilon - \frac{\beta^2}{\omega^2}}\right]\frac{\varepsilon^\mu - \frac{\beta^2}{\omega^2}}{\epsilon_{\rm p}^\mu - \frac{\beta^2}{\omega^2}}\left(\frac{1}{\varepsilon^\mu - \frac{\beta^2}{\omega^2}}\partial_\nu^\mu u - \varphi\right) . \tag{8.36}$$

The second identity in $\mathcal{A}_{\omega,\beta}(u,\varphi) = (u,\varphi)$ gives the desired result. □

Here is the analogous result concerning the TM polarization.

Theorem 8.2 *The guided modes (ω^2, β^2, u) in the TM-polarization satisfying $\omega^2 \notin \Sigma_{\frac{\beta^2}{\omega^2}}$ are exactly the solutions to the following spectral problem.*

$$\mathcal{B}_{\omega,\beta}(u,\varphi) = (u,\varphi) , \tag{8.37}$$

for some $\varphi \in L^2(\mathbb{R}^2)$.

Proof. If (ω^2, β^2, u) is a guided mode and $\omega^2 \notin \Sigma_{\frac{\beta^2}{\omega^2}}$, then from Proposition 6.4, we have clearly

$$\mathcal{B}_{\omega,\beta}(u, \frac{1}{\varepsilon(x)^\mu - \frac{\beta^2}{\omega^2}}\partial_\nu^\mu u) = (u, \frac{1}{\varepsilon(x)^\mu - \frac{\beta^2}{\omega^2}}\partial_\nu^\mu u) . \tag{8.38}$$

Conversely, suppose that (u,φ) is an eigenfunction of $\mathcal{B}_{\omega,\beta}$ for the eigenvalue 1. Recalling Proposition 6.4, we just have to prove that

$$\left[\frac{\varepsilon}{\varepsilon - \frac{\beta^2}{\omega^2}}\partial_\nu u\right] = 0 .$$

From the equation satisfied by $G_{\frac{\beta}{\omega}}(\omega^2; x, y)$ we deduce that for any $x \in \Pi$ and any $y \in \Omega \setminus \Pi$ we have

$$\left[\frac{\epsilon_{\rm p}}{\epsilon_{\rm p} - \frac{\beta^2}{\omega^2}}\partial_\nu G_{\frac{\beta}{\omega}}(\omega^2; x, y)\right] = 0 . \tag{8.39}$$

As a consequence, we have

$$\left[\frac{\varepsilon}{\varepsilon - \frac{\beta^2}{\omega^2}}\partial_\nu G_{\frac{\beta}{\omega}}(\omega^2; x, y)\right] = \left[\frac{\varepsilon(\epsilon_{\rm p} - \frac{\beta^2}{\omega^2})}{\epsilon_{\rm p}(\varepsilon - \frac{\beta^2}{\omega^2})}\right]\frac{\epsilon_{\rm p}}{\epsilon_{\rm p}^\mu - \frac{\beta^2}{\omega^2}}\partial_\nu^\mu G_{\frac{\beta}{\omega}}(\omega^2; x, y) . \tag{8.40}$$

Using again the classical potential theory result mentioned in the previous proof, we get

$$\begin{aligned}
&\frac{\varepsilon^\mu}{\varepsilon^\mu - \frac{\beta^2}{\omega^2}}\partial_\nu^\mu \int_\Pi G_{\frac{\beta}{\omega}}(\omega^2; x, y)\left[\frac{(\delta\varepsilon)}{\varepsilon(\epsilon_{\rm p} - \frac{\beta^2}{\omega^2})}\right]\varphi(y)\, dl_y \\
&\qquad = \int_\Pi \frac{\varepsilon^\mu}{\varepsilon^\mu - \frac{\beta^2}{\omega^2}}\partial_\nu^\mu G_{\frac{\beta}{\omega}}(\omega^2; x, y)\left[\frac{(\delta\varepsilon)}{\varepsilon(\epsilon_{\rm p} - \frac{\beta^2}{\omega^2})}\right]\varphi(y)\, dl_y \\
&\qquad\quad + \mu\frac{1}{2}\frac{\varepsilon^\mu(\epsilon_{\rm p}^\mu - \frac{\beta^2}{\omega^2})}{\epsilon_{\rm p}^\mu(\varepsilon^\mu - \frac{\beta^2}{\omega^2})}\left[\frac{(\delta\varepsilon)}{\varepsilon(\epsilon_{\rm p} - \frac{\beta^2}{\omega^2})}\right]\varphi(x) .
\end{aligned}$$

We deduce the expression of the jump we are looking for:

$$\begin{aligned}\left[\frac{\varepsilon}{\varepsilon-\frac{\beta^2}{\omega^2}}\partial_\nu u\right] &= \left[\frac{\varepsilon(\epsilon_{\rm p}-\frac{\beta^2}{\omega^2})}{\epsilon_{\rm p}(\varepsilon-\frac{\beta^2}{\omega^2})}\right]\frac{\epsilon_{\rm p}^\mu}{\epsilon_{\rm p}^\mu-\frac{\beta^2}{\omega^2}}\partial_\nu^\mu u \\ &\quad+\left(\left[\mu\frac{1}{2}\frac{\varepsilon(\epsilon_{\rm p}-\frac{\beta^2}{\omega^2})}{\epsilon_{\rm p}(\varepsilon-\frac{\beta^2}{\omega^2})}\right]-\mu\frac{1}{2}\left[\frac{\varepsilon(\epsilon_{\rm p}-\frac{\beta^2}{\omega^2})}{\epsilon_{\rm p}(\varepsilon-\frac{\beta^2}{\omega^2})}\right]\right)\left[\frac{(\delta\varepsilon)}{\epsilon_{\rm p}-\frac{\beta^2}{\omega^2}}\right]\varphi\,,\end{aligned}$$

and therefore we get

$$\left[\frac{\varepsilon}{\varepsilon-\frac{\beta^2}{\omega^2}}\partial_\nu u\right]=\left[\frac{\varepsilon(\epsilon_{\rm p}-\frac{\beta^2}{\omega^2})}{\epsilon_{\rm p}(\varepsilon-\frac{\beta^2}{\omega^2})}\right]\frac{\epsilon_{\rm p}^\mu(\varepsilon^\mu-\frac{\beta^2}{\omega^2})}{\varepsilon^\mu(\epsilon_{\rm p}^\mu-\frac{\beta^2}{\omega^2})}\left(\frac{\varepsilon^\mu}{\varepsilon^\mu-\frac{\beta^2}{\omega^2}}\partial_\nu^\mu u-\varphi\right)\,. \tag{8.41}$$

The second identity in $\mathcal{B}_{\omega,\beta}(u,\varphi)=(u,\varphi)$ gives the desired result. □

In both cases, because of the exponential decay of the corresponding Green's function, it is clear that the guided modes are exponentially confined.

9 Gaps opening in Σ_α: TE polarization

In this section we are interested in the existence of gaps in the spectrum of the operator $A_\alpha^{\rm p}$ and especially in the asymptotic behaviour under some limit conditions on α.

Our approach is inspired by the work of Hempel and Lienau in [23] where almost all the results of this section can be found with weaker conditions on the smoothness of what will be denoted the domain Ω. We give here all the proofs adapting them to our problem for the sake of clarity.

The structure of the 2D-photonic crystal considered here is simple, but the results could be generalized to many other structures.

9.1 Medium description

For $\mathbf{n}=(n_1,n_2)\in\mathbb{Z}^2$ we define $Q_{\mathbf{n}}=(n_1,n_1+1)\times(n_2,n_2+1)$. Let $\mathcal{O}_0$ be a connected open domain with smooth boundary such that $\mathcal{O}_0\subset\subset Q_0$. We define

$$\mathcal{O}_{\mathbf{n}}=\mathcal{O}_0+\mathbf{n}\,,$$
$$\mathcal{O}=\cup_{\mathbf{n}\in\mathbb{Z}^2}\mathcal{O}_{\mathbf{n}}\,,$$
$$\mathcal{O}_{\mathbf{n}}^c=Q_{\mathbf{n}}\setminus\overline{\mathcal{O}_{\mathbf{n}}}\,,$$

and

$$\mathcal{O}^c=\mathbb{R}^2\setminus\overline{\mathcal{O}}\,.$$

Finally, ∂D denotes the boundary of the domain D.

We consider the photonic crystal which dielectric permittivity is given by $\epsilon_{\rm p}(x)$ that satisfies

$$\epsilon_{\rm p}(x)=\begin{cases}1 & x\in\mathcal{O}^c\,,\\ \epsilon+1 & x\in\mathcal{O}\,,\end{cases} \tag{9.42}$$

where ϵ is a positive constant.

This dielectric function represents a photonic fiber made of rods of dielectric $1+\epsilon>1$ with section $\mathcal{O}_0$ placed periodically in air or more generally in a homogeneous dielectric medium with permittivity strictly lower than that of the rods (after scaling, we come back to the problem with $\epsilon_{\rm p}(x)$).

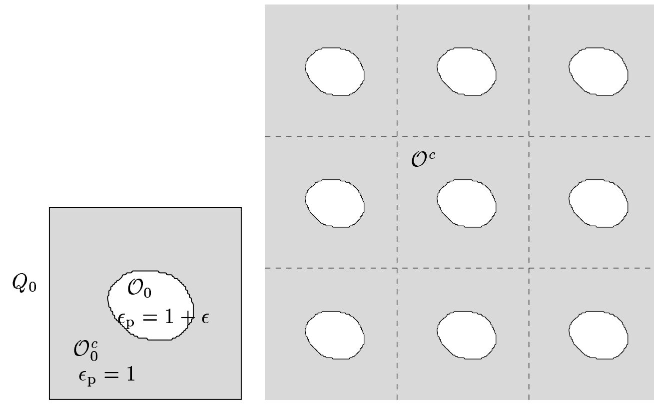

Figure 1: Section of the photonic fiber cladding.

9.2 The spectral problem

We are interested in the spectrum Σ_α of the operator $A^{\rm p}_\alpha$ and more precisely in the existence of gaps in Σ_α. Our idea is the following: suppose that α^2 goes to 1, then the coefficient of $A^{\rm p}_\alpha$ that takes the value $1-\alpha^2 \ll 1$ in $\mathcal{O}^c$ and $\epsilon + 1 - \alpha^2 \approx \epsilon > 0$ will have a very high contrast. It can then be expected to find gaps in the corresponding spectrum. Of course, this is very far from the final proof, since we do not even know where these gaps will appear. It can for example appear around values that diverge which will be useless since we look for gaps around finite values of ω^2.

Our original spectral problem is then

$$\nabla \cdot \frac{1}{\epsilon_{\rm p}(x) - \alpha^2} \nabla u + \omega^2 u = 0 \, . \tag{9.43}$$

Since $\alpha^2 \to 1^-$, we introduce the small positive parameter $\eta = 1 - \alpha^2$. We also define the operator $\tilde{A}_\eta$ as

$$\tilde{A}_\eta = -\nabla \cdot (1 + \frac{\epsilon}{\eta} \chi_{\mathcal{O}^c}) \nabla u \, , \tag{9.44}$$

and the new spectral parameter $\lambda = (\epsilon + \eta)\omega^2$. It can be easily seen that $\tilde{A}_\eta = (\epsilon + \eta) A^{\rm p}_\alpha$.

Our new spectral problem consists now in finding gaps in the spectrum $\tilde{\Sigma}_\eta$ of $\tilde{A}_\eta$ when η goes to 0^+.

9.3 Asymptotic behaviour of the spectrum

We introduce the quadratic form $\mathfrak{a}_\eta[u]$, also denoted $\mathfrak{a}_\eta[u,u]$, in the Hilbert space $L^2(\mathbb{R}^2)$ defined by

$$\mathfrak{a}_\eta[u] = \int_{\mathbb{R}^2} (1 + \frac{\epsilon}{\eta} \chi_{\mathcal{O}^c}) |\nabla u|^2 \, dx \, , \tag{9.45}$$

for $u \in D(\mathfrak{a}_\eta) = H^1(\mathbb{R}^2)$, the usual Sobolev space with the norm $\|u\|_1 = \|u\|_{L^2(\mathbb{R}^2)} + \|\nabla u\|_{L^2(\mathbb{R}^2)}$. It is obvious that this quadratic form is positive, densely defined and closed. It then defines a unique self-adjoint operator in $L^2(\mathbb{R}^2)$ that is $\tilde{A}_\eta$ since

$$(\tilde{A}_\eta u, v) = \mathfrak{a}_\eta[u,v] , \quad u \in D(\tilde{A}_\eta) , \quad v \in D(\mathfrak{a}_\eta) . \tag{9.46}$$

The operator $\tilde{A}_\eta$ is then uniquely determined by its quadratic form. This allows us to study the limit of the quadratic form $\mathfrak{a}_\eta$ in order to determine the limiting spectrum of $\tilde{A}_\eta$.

It is clear that the quadratic form $\mathfrak{a}_\eta$, whose domain is independent of η, increases monotonically when $\eta \to 0^+$. The monotone convergence theorem for an increasing sequence of quadratic forms [36] yields a closed quadratic form $\mathfrak{a}_0$ defined by

$$D(\mathfrak{a}_0) = \{u \in H^1(\mathbb{R}^2) \, ; \sup_{\eta>0} \mathfrak{a}_\eta[u] < \infty\} , \tag{9.47}$$

and

$$\mathfrak{a}_0[u] = \lim_{\eta\to 0^+} \mathfrak{a}_\eta[u] = \sup_{\eta>0} \mathfrak{a}_\eta[u] , \quad u \in D(\mathfrak{a}_0) . \tag{9.48}$$

Furthermore, this quadratic form defines a unique self-adjoint operator $\tilde{A}_0$ which satisfies

$$\tilde{A}_\eta \to \tilde{A}_0 \quad \text{in the strong resolvent sense,} \quad \eta \to 0^+ . \tag{9.49}$$

This operator acts in a (possibly smaller) Hilbert space given by the closure of $D(\mathfrak{a}_0)$ in $L^2(\mathbb{R}^2)$, and we think of the resolvent of $\tilde{A}_0$ as the zero operator on the orthogonal complement of $D(\mathfrak{a}_0)$ in $L^2(\mathbb{R}^2)$.

We recall that $\tilde{A}_\eta$ converges to $\tilde{A}_0$ in the strong resolvent sense if and only if:

$$(\tilde{A}_\eta + I)^{-1} f \to (\tilde{A}_0 + I)^{-1} f , \quad \forall f \in L^2(\mathbb{R}^2) . \tag{9.50}$$

Now let us prove that $\tilde{A}_0$ is the Dirichlet Laplacian on $\mathcal{O}$.

Lemma 9.1 *Suppose $u \in H^1(\mathbb{R}^2)$ is such that $\mathfrak{a}_\eta[u] < C$ for all $\eta > 0$ and some positive constant C. Then $u = 0$ a.e. in $\mathcal{O}^c$.*

Proof. Suppose that $u \in H^1(\mathbb{R}^2)$ is such that for any $\eta > 0$ and for some positive constant C we have $\mathfrak{a}_\eta[u] < C$. It follows that

$$\frac{\epsilon}{\eta} \int_{\mathcal{O}^c} |\nabla u|^2 \, dx < C , \quad \forall \eta > 0 . \tag{9.51}$$

This implies that $\nabla u = 0$ a.e. in $\mathcal{O}^c$ which is connected and so u is constant in $\mathcal{O}^c$. Since $u \in L^2(\mathbb{R}^2)$, it follows that $u = 0$ a.e. in $\mathcal{O}^c$. □

As a consequence, we have the following corollary.

Corollary 9.1 *The domain of $\mathfrak{a}_0$ is the space*

$$\tilde{H}_0^1(\mathcal{O}) = \{u \in H^1(\mathbb{R}^2) \, ; \, u(x) = 0 \ \text{a.e. in} \ \mathcal{O}^c\} , \tag{9.52}$$

which coincides with the classical space $H_0^1(\mathcal{O})$ defined as the closure of $C_c^\infty(\mathcal{O})$ in the $\|\cdot\|_1$-norm provided $\mathcal{O}$ is regular, which we suppose (note that an exterior cone condition is sufficient).

This determines the self-adjoint operator $\tilde{A}_0$.

Corollary 9.2 *The limiting operator $\tilde{A}_0$ is the Dirichlet Laplacian in the domain $\mathcal{O}$ denoted $-\Delta_{\mathcal{O}}$ with the domain given by the closure of $H^2(\mathcal{O})$ in $H_0^1(\mathcal{O})$.*

The following proposition holds.

Proposition 9.1 *The operator $\tilde{A}_\eta$ converges to $-\Delta_{\mathcal{O}} = \bigoplus_{\mathbf{n}\in\mathbb{Z}^2}(-\Delta_{\mathcal{O}_\mathbf{n}})$ in the strong resolvent sense.*

It is clear that the operator $-\Delta_{\mathcal{O}_0}$ has compact resolvent and then its spectrum consists in a sequence of discrete eigenvalues of finite multiplicity. We denote these (repeated) eigenvalues, ordered by min – max, as δ_k, $k \in \mathbb{N}^*$, or

$$0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_k \leq \delta_{k+1} \leq \cdots , \quad k \in \mathbb{N}^* , \tag{9.53}$$

where $\delta_k \to +\infty$ as $k \to +\infty$. The spectrum of $-\Delta_{\mathcal{O}}$ is then the set $\{\delta_k \ , \ k \in \mathbb{N}^*\}$, each point in the spectrum being an eigenvalue of infinite multiplicity.

Determining the strong resolvent limit is however not sufficient to determine the limit of the spectrum of $\tilde{A}_\eta$. Actually we need a norm resolvent convergence to determine the uniform limit of any compactly supported part of the spectrum of $\tilde{A}_\eta$.

Let us now turn to the Floquet theory and look into the operator $\tilde{A}_\eta$ as the "direct integral" of the operators $\tilde{A}_\eta^\gamma$:

$$\tilde{A}_\eta = \int_{\gamma\in(-\pi,\pi]^2}^{\oplus} \tilde{A}_\eta^\gamma \, d\gamma, \tag{9.54}$$

where $\tilde{A}_\eta^\gamma$ denotes the operator $\nabla \cdot (1 + \frac{\epsilon}{\eta})\nabla$ acting on $L^2_\gamma(Q_0)$, the subspace of $L^2(Q_0)$ with γ-periodic boundary condition. We denote by $\mathfrak{a}_\eta^\gamma$ its associated quadratic form which domain is the space of γ-periodic functions in $H^1(Q_0)$.

It is obvious that each $\tilde{A}_\eta^\gamma$ has compact resolvent. Let $(\lambda_{\eta,k}^\gamma)_{k\in\mathbb{N}^*}$ be its (finite multiplicity) eigenvalues ordered by the min – max, *i.e.*, $\lambda_{\eta,k}^\gamma \leq \lambda_{\eta,k+1}^\gamma$. We recall that the (continuous) spectrum of $\tilde{A}_\eta$ consists in the union of the intervals corresponding to the range of each $\gamma \mapsto \lambda_{\eta,k}^\gamma$ when γ varies in $(-\pi,\pi]^2$, *i.e.*,

$$\tilde{\Sigma}_\eta = \cup_{k\in\mathbb{N}^*}\{\lambda_{\eta,k}^\gamma \mid \gamma \in (-\pi,\pi]^2\} . \tag{9.55}$$

Let us now introduce the Dirichlet and Neumann operators on Q_0, denoted by $\tilde{A}_\eta^{(D)}$ and $\tilde{A}_\eta^{(N)}$, respectively, acting like $-\nabla \cdot (1 + \frac{\epsilon}{\eta})\nabla$ on $L^2(Q_0)$ and their respective associated quadratic forms $\mathfrak{a}_\eta^{(D)}$ and $\mathfrak{a}_\eta^{(N)}$, with domains $H_0^1(Q_0)$ and $H^1(Q_0)$, respectively.

As for $\tilde{A}_\eta^\gamma$, the operators $\tilde{A}_\eta^{(D)}$ and $\tilde{A}_\eta^{(N)}$ have compact resolvent and we denote by $\lambda_{\eta,k}^{(D)}$ and $\lambda_{\eta,k}^{(N)}$ their respective ordered eigenvalues. From the min – max principle, we deduce that

$$\lambda_{\eta,k}^{(N)} \leq \lambda_{\eta,k}^\gamma \leq \lambda_{\eta,k}^{(D)} , \quad k \in \mathbb{N}^* , \ \gamma \in (-\pi,\pi]^2 , \ \eta > 0 . \tag{9.56}$$

It follows that

$$\tilde{\Sigma}_\eta \subset \bigcup_{k\in\mathbb{N}^*} [\lambda_{\eta,k}^{(N)}, \lambda_{\eta,k}^{(D)}] . \tag{9.57}$$

Again, we apply the monotone convergence theorem for quadratic forms to the forms $\mathfrak{a}_\eta^\gamma$, $\mathfrak{a}_\eta^{(N)}$, and $\mathfrak{a}_\eta^{(D)}$ and we obtain the limiting quadratic forms $\mathfrak{a}_0^\gamma$, $\mathfrak{a}_0^{(N)}$, and $\mathfrak{a}_0^{(D)}$, respectively. The self-adjoint operators associated to these quadratic forms are $\tilde{A}_0^\gamma$, $\tilde{A}_0^{(N)}$, and $\tilde{A}_0^{(D)}$, respectively.

The operator $\tilde{A}_0^{(D)}$ is a self-adjoint operator on $H_0^1(\mathcal{O}_0)$ and $\tilde{A}_0^{(N)}$ acts on the subspace $H_0^1(\mathcal{O}_0) \oplus \mathbf{1}_{Q_0}$ of the functions $u = \tilde{u} + c \in H^1(Q_0)$ where $\tilde{u} \in H_0^1(\mathcal{O}_0)$ and $c \in \mathbb{R}$ is a constant.

The operators $\tilde{A}_0^\gamma$, $\tilde{A}_0^{(N)}$, and $\tilde{A}_0^{(D)}$ are the strong resolvent limits of $\tilde{A}_\eta^\gamma$, $\tilde{A}_\eta^{(N)}$, and $\tilde{A}_\eta^{(D)}$, respectively. We recall that these operators have compact resolvent and then purely discrete spectrum. By a result of Kato [25] (cf. [Thm. VIII-3.5]), compactness implies the convergence in the norm resolvent sense. Then we have

$$\tilde{A}_\eta^\gamma \to \tilde{A}_0^\gamma \ , \quad \text{in norm resolvent sense,} \quad \eta \to 0 \ , \tag{9.58}$$

for each $\gamma \in (-\pi, \pi]^2$, and

$$\tilde{A}_\eta^{(D)} \to \tilde{A}_0^{(D)} \ , \quad \tilde{A}_\eta^{(N)} \to \tilde{A}_0^{(N)} \ , \quad \text{in norm resolvent sense,} \quad \eta \to 0 \ . \tag{9.59}$$

Let us denote by $(\nu_k)_{k\in\mathbb{N}}$ the eigenvalues of the operator $\tilde{A}_0^{(N)}$ ordered by the min − max principle.

$$0 = \nu_1 < \nu_2 \le \nu_3 \le \cdots \le \nu_k \le \nu_k + 1 \le \cdots \ , \qquad k \in \mathbb{N}^* \ . \tag{9.60}$$

From equation (9.59) we deduce that

$$\lambda_{\eta,k}^{(N)} \to \nu_k \ , \quad \eta \to 0 \ , \ k \in \mathbb{N}^* \ . \tag{9.61}$$

We recall that the domain of the form limit $\mathfrak{a}_0^{(N)}$ is the subspace of the functions in $H^1(\mathbb{R}^2)$ that are constant in $Q_0 \setminus \overline{\mathcal{O}_0}$ and then the operator $\tilde{A}_0^{(N)}$ is not the Neumann operator on $\mathcal{O}_0$.

The following proposition gives the operator limit of $\tilde{A}_\eta^\gamma$.

Proposition 9.2 *(i) The limit of the Dirichlet operator $\tilde{A}_\eta^{(D)}$ is the Dirichlet operator $\tilde{A}_0^{(D)} = -\Delta_{\mathcal{O}_0}$. Moreover,*

$$\lambda_{\eta,k}^{(D)} \to \delta_k \ , \quad \eta \to 0 \ , \ k \in \mathbb{N}^* \ . \tag{9.62}$$

(ii) For any $\gamma \in (-\pi, \pi]^2 \setminus \{(0,0)\}$, we have $\tilde{A}_0^\gamma = \tilde{A}_0^{(D)} = -\Delta_{\mathcal{O}_0}$ and

$$\lambda_{\eta,k}^\gamma \to \delta_k \ , \quad \eta \to 0 \ , \ k \in \mathbb{N}^* \ , \ \gamma \neq (0,0) \ . \tag{9.63}$$

(iii) For $\gamma_0 = (0,0)$, we have $\tilde{A}_0^{\gamma_0} = \tilde{A}_0^{(N)}$ and

$$\lambda_{\eta,k}^{\gamma_0} \to \nu_k \ , \quad \eta \to 0 \ , \ k \in \mathbb{N}^* \ . \tag{9.64}$$

Proof. From (9.58) and (9.59) it follows that we only need to identify the limiting operators that is equivalent to identifying the corresponding quadratic forms.

It is obvious that, since all the quadratic forms considered are defined by the same expression

$$\int_{Q_0} (1+\frac{\epsilon}{\eta})|\nabla u|^2 \, dx \; ,$$

we only need to determine the form domains of the limiting quadratic forms.

Concerning $\tilde{A}_0^{(D)}$, it is clear that it is defined for the subspace of $H_0^1(Q_0)$ of the functions with null gradient in $Q_0 \setminus \overline{\mathcal{O}_0}$, which corresponds to $H_0^1(\mathcal{O}_0)$. We deduce then that $\tilde{A}_0^{(D)} = -\Delta_{\mathcal{O}_0}$.

Now let us consider $\mathfrak{a}_\eta^\gamma$ for $\gamma \neq \gamma_0$. The form domain of $\mathfrak{a}_0^\gamma$ is the subspace of γ-periodic functions in $H^1(Q_0)$ that are constant in $Q_0 \setminus \overline{\mathcal{O}_0}$. Since $\gamma \neq (0,0)$, this constant is necessarily 0 and the form domain of $\mathfrak{a}_0^\gamma$ is then $H_0^1(Q_0)$. It follows that for $\gamma \neq \gamma_0$, $\tilde{A}_0^\gamma = \tilde{A}_0^{(D)} = -\Delta_{\mathcal{O}_0}$.

Finally the form domain of $\mathfrak{a}_0^{\gamma_0}$ is the subspace of periodic functions in $H^1(Q_0)$ that are constant in $Q_0 \setminus \overline{\mathcal{O}_0}$ or simply the subspace of functions in $H^1(Q_0)$ that are constant in $Q_0 \setminus \overline{\mathcal{O}_0}$, that is exactly the form domain of $\tilde{A}_0^{(N)}$.
□

Now we can state the following result on the convergence of the spectrum of A_η.

Theorem 9.1 *The spectrum of A_α^{p} converges to $\cup_{k\in\mathbb{N}^*}[\epsilon^{-1}\nu_k, \epsilon^{-1}\delta_k]$ as $\alpha \to 1$, in the sense that if $[\lambda_{\eta,k}^-, \lambda_{\eta,k}^+]$ is the k^{th} band of the spectrum of A_α^{p}, then*

$$\lambda_{\eta,k}^- \to \epsilon^{-1}\nu_k \; , \quad \lambda_{\eta,k}^+ \to \epsilon^{-1}\delta_k \; , \quad \eta \to 0 \; . \tag{9.65}$$

The convergence of the spectrum of A_α^{p} is uniform on any compact of $\mathbb{R}^+$, i.e., *for any compact I of $\mathbb{R}^+$ and any $C > 0$, there exists $0 < \alpha_0 < 1$ such that if $\alpha_0 < \alpha < 1$,*

$$dist_{\mathrm{H}}(\Sigma_\alpha \cap I, \bigcup_{k\in\mathbb{N}^*} [\epsilon^{-1}\nu_k, \epsilon^{-1}\delta_k] \cap I) < C \; . \tag{9.66}$$

Here $\mathrm{dist}_{\mathrm{H}}(E,F)$ denotes the Hausdorff distance between the subsets E and F.

Now we can see clearly the emergence of gaps in Σ_α as α goes to 1.

Corollary 9.3 *Suppose that for some $k \in \mathbb{N}^*$, $\delta_k < \nu_{k+1}$, then for any compact $I \subset\subset (\epsilon^{-1}\delta_k, \epsilon^{-1}\nu_{k+1})$, there exists $\alpha_0 > 0$ such that for any $\alpha_0 < \alpha < 1$,*

$$\Sigma_\alpha \cap I = \emptyset \; . \tag{9.67}$$

9.4 Existence of gaps in the limiting spectrum

The existence of gaps in Σ_α for α close enough to 1 is an obvious consequence of the existence of $k \in \mathbb{N}^*$ such that $\delta_k < \nu_{k+1}$. In a first step, let us prove that the eigenvalues ν_k and δ_k are enlaced.

Proposition 9.3 *The eigenvalues ν_k and δ_k of the operators respective $\tilde{A}_0^{(N)}$ and $\tilde{A}_0^{(D)}$ enlace,* i.e.,

$$0 = \nu_1 < \delta_1 \leq \nu_2 \leq \delta_2 \leq \cdots \leq \nu_k \leq \delta_k \leq \nu_{k+1} \leq \delta_{k+1} \leq \cdots \; . \tag{9.68}$$

Proof. Recalling the inclusion of the form domains $D(\mathfrak{a}_0^{(D)}) \subset D(\mathfrak{a}_0^{(N)})$, we deduce immediately that for any $k \in \mathbb{N}^*$,

$$\nu_k \leq \delta_k \ . \tag{9.69}$$

Let us denote $u_k^{(N)}$ the eigenvector of $\tilde{A}_0^{(N)}$ related to the eigenvalue ν_k. We recall that $\nu_1 = 0$ and $u_1^{(N)} = \mathbf{1}_{Q_0}$. Let $D_k^{(N)} = \mathrm{Vect}(u_l^{(N)})_{1\leq l \leq k}$. From the definition of $D(\mathfrak{a}_0^{(N)})$ there exist constants $c_l \in \mathbb{R}$ for $l \geq 2$ such that $u_l^{(N)} = c_l u_1^{(N)} + \tilde{u}_l^{(D)}$, where $\tilde{u}_l^{(D)} \in H_0^1(\mathcal{O}_0)$. It is also obvious that the dimension of $\tilde{D}_k^{(D)} = \mathrm{Vect}(\tilde{u}_l^{(D)})_{2\leq l \leq k+1}$ is k and that $\tilde{D}_k^{(D)} \subset D_{k+1}^{(N)}$

Finally, since

$$\mathfrak{a}_0^{(D)}[\tilde{u}] = \mathfrak{a}_0^{(N)}[\tilde{u}] \leq \nu_{k+1} \|\tilde{u}\|^2_{L^2(\mathcal{O}_0)} \ , \qquad \forall \tilde{u} \in \tilde{D}_k^{(D)} \ , \tag{9.70}$$

we deduce that

$$\delta_k \leq \nu_{k+1} \ , \tag{9.71}$$

which ends the proof. □

Now we give a condition for the existence of gaps in $\tilde{\Sigma}_\eta$ when η is sufficiently small.

Proposition 9.4 *Let $(\delta_k)_{k\geq 0}$ be the eigenvalues of $\tilde{A}_0^{(D)}$ ordered by the* min − max *principle where formally $\delta_0 = -\infty$. Suppose that for some $k, m \geq 0$,*

$$\delta_{k-1} < \delta_k = \cdots = \delta_{k+m} < \delta_{k+m+1} \ . \tag{9.72}$$

(i) If there exists an eigenvector $u_0 \in H_0^1(\mathcal{O}_0)$ corresponding to the eigenvalue δ_k and satisfying

$$\int_{\mathcal{O}_0} u_0 \, dx \neq 0 \ , \tag{9.73}$$

then

$$\nu_k < \delta_k \ , \qquad \delta_{k+m} < \nu_{k+m+1} \ . \tag{9.74}$$

(ii) If all functions $u \in \ker(\tilde{A}_0^{(D)} - \delta_k)$ have zero mean value, then

$$\nu_k = \delta_k \ , \qquad or \qquad \delta_{k+m} = \nu_{k+m+1} \ . \tag{9.75}$$

The proof of this proposition can be found in [23, Proposition 3.4.].

10 Gaps opening in Σ_α: TM polarization

In the TM polarization we have exactly the same results replacing the Hilbert space $L^2(\mathbb{R}^2, dx)$ by the weighted Hilbert space $L^2(\mathbb{R}^2, \epsilon_{\mathrm{p}}(x)\, dx)$.

Theorem 10.1 *The spectrum of B_α^{p} converges to $\cup_{k\in\mathbb{N}}[\epsilon^{-1}\nu_k, \epsilon^{-1}\delta_k]$ as $\alpha \to 1$, in the sense that if $[\lambda_{\eta,k}^-, \lambda_{\eta,k}^+]$ is the k^{th} band of the spectrum of A_α^{p}, then*

$$\lambda_{\eta,k}^- \to \epsilon^{-1}\nu_k \ , \quad \lambda_{\eta,k}^+ \to \epsilon^{-1}\delta_k \ , \quad \eta \to 0 \ . \tag{10.76}$$

The convergence of the spectrum of B_α^{p} is uniform on any compact of $\mathbb{R}^+$, i.e., for any compact I of $\mathbb{R}^+$, there exists a positive constant C independent of η such that

$$dist_{\mathrm{H}}(\Sigma_\alpha \cap I, \bigcup_{k\in\mathbb{N}} [\epsilon^{-1}\nu_k, \epsilon^{-1}\delta_k] \cap I) < C \ . \tag{10.77}$$

The eigenvalues (δ_k) and (ν_k) are exactly the same as those defined in the previous section. This is because $\epsilon_{\rm p}(x) = \epsilon + 1$ in $\mathcal{O}$ and so the modification of the Hilbert space does not change the limiting operators.

11 Numerical experiments

In this section, we consider only the TE polarization. The periodic structures considered here are conformal to those in the previous section. The dielectric permittivity takes the value 5 in the domain $\mathcal{O}_0$ and 1 otherwise. We give numerical results for different shapes of the domain $\mathcal{O}_0$. The numerical tool used here is the *MIT Photonic-Bands* (MPB) package [24].

We compute the continuous spectrum of $A^{\rm p}_\alpha$ for different values of $\alpha \in [0,1]$. The 16 first bands are represented. The results are shown with the corresponding periodic medium in the figures below. The dark regions correspond dielectric permittivity 5.

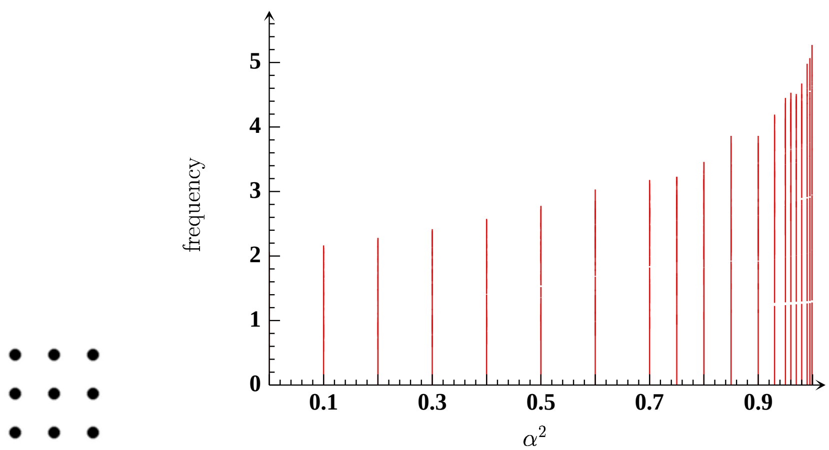

Figure 2: Spectrum of the structure with discs of radius 0.15.

All the structures shown here have no gaps for planar propagation, i.e. $\alpha = 0$. Figure 9 shows the bands of the structure with discs of radius 0.3 in the planar propagation. These bands are computed on the boundary of the irreductible Brillouin zone.

We notice clearly the appearance of one or more gaps in the spectra of each structure when α^2 approaches 1 ($\alpha^2 \geq 0.9$). The bottom of the first gap goes to the first eigenvalue of the Dirichlet-Laplacian in the domain $\mathcal{O}_0$ when $\alpha^2 \to 1$. Actually, if f_0 is the limit of the bottom of the first gap and d_0^2 is the first eigenvalue of the Dirichlet-Laplacian in $\mathcal{O}_0$, then

$$2\pi f_0 = \frac{1}{\sqrt{\epsilon}} d_0 \ , \tag{11.78}$$

where ϵ is defined in the previous section and is equal to 4 in our case.

When the size of $\mathcal{O}_0$ increases, the midgap defect decreases and the width of the gap which corresponds to the interval (δ_1, ν_2) (defined in the previous section) increases.

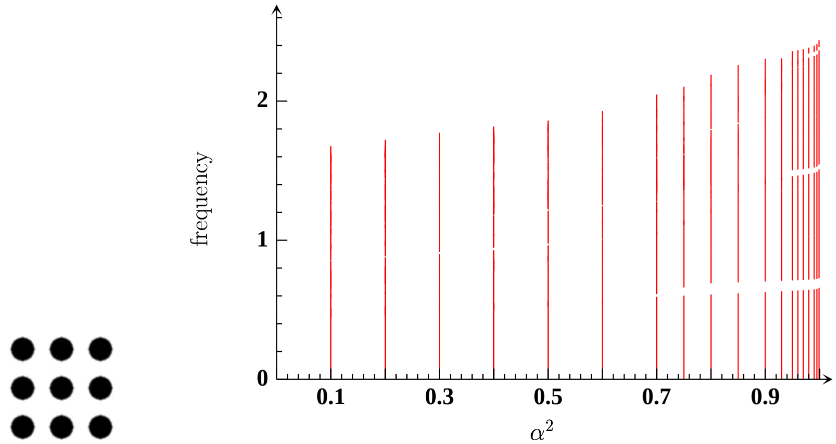

Figure 3: Spectrum of the structure with discs of radius 0.3.

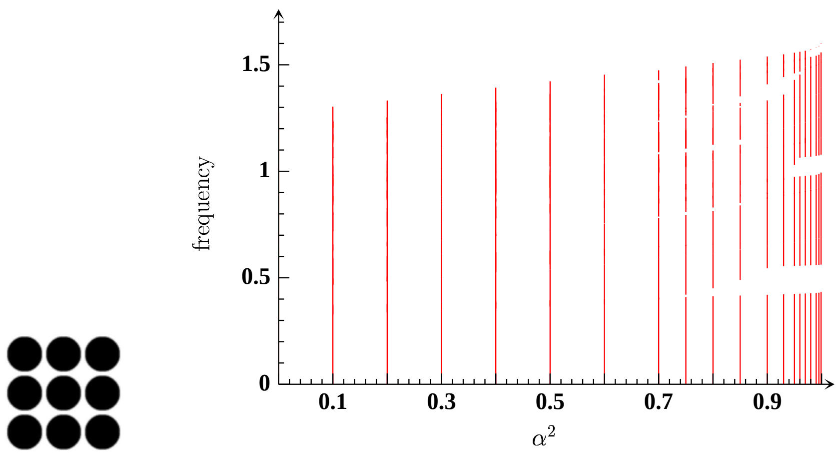

Figure 4: Spectrum of the structure with discs of radius 0.45.

When the gap is wider, the exponential decay of the electromagnetic energy in the periodic structure is higher which allows the use of very few periods in the cladding of the photonic fiber.

Next, we introduce a defect to the structure shown in Figure 3 and we compute the spectrum for different values of α^2. Two defects are investigated, the first one called "negative" consists in removing one rod from the structure, the second one called "positive" consists in increasing the radius of one rod in the structure. The way we call the defect comes from the sign of $(\delta\varepsilon)$. The method used for determining the spectrum is the "supercell" method. The size of the supercell is 5 or 7.

The parameter α^2 takes values in $[0.75, 0.90]$ for the negative defect and in $[0.80, 0.93]$ in the positive defect. When α^2 is too close to 1, the contrast between the coefficients is too high (more than 50) which, added to the complexity of the supercell, makes it impossible to get convergence to reliable results. For such limits we need dedicated preconditioners. We find one defect state for each

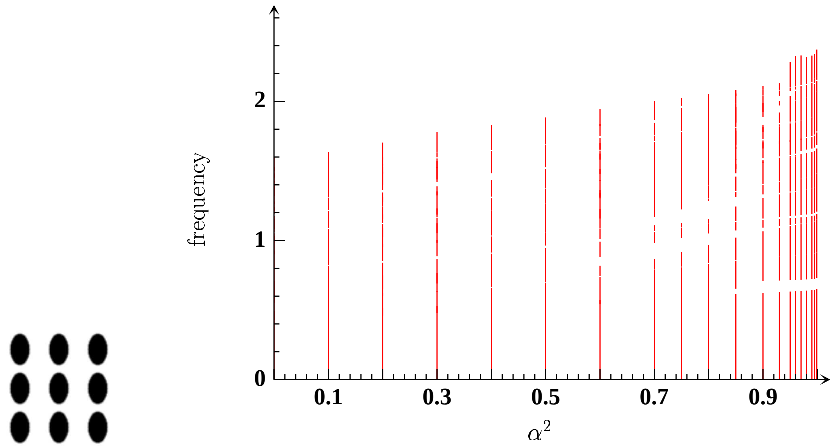

Figure 5: Spectrum of the structure with ellipses of axes 0.50 and 0.80.

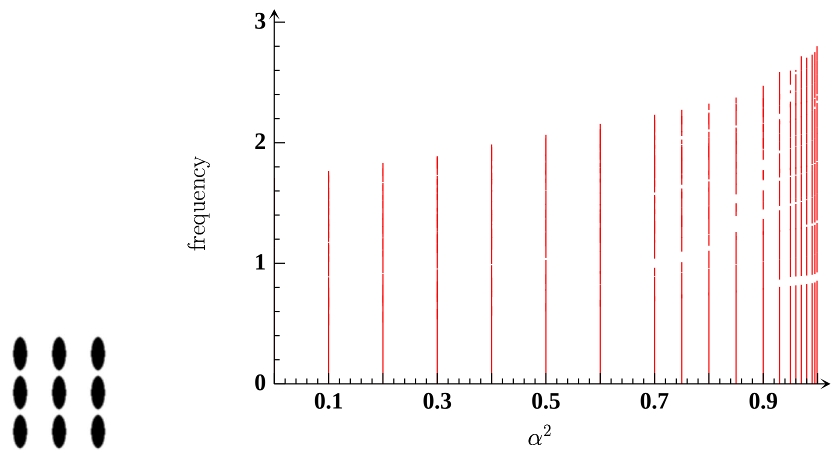

Figure 6: Spectrum of the structure with ellipses of axes 0.85 and 0.35.

case. The corresponding spectra are shown in Figures 10 and 11.

We notice that the defect frequency goes from the top to the bottom of the gap when $\alpha^2 \to 1$ in the positive defect and from the bottom to the top of the gap in the negative defect. When it is too close to the edge, the decay of the electromagnetic energy away from the defect is very weak.

Figures 12-19 represent the energy distribution of the defect modes in the supercell. The horizontal graduations represent the limits of unit cells.

Finally, we give in Tables 1-2 the percentage of the electromagnetic energy located in the defect region and the four closest dielectric rods.

12 Conclusion

We gave a rigorous proof for the origin of polarized guided modes in a photonic fiber. For a parameter α and a defect frequency ω_{d} , the corresponding guided

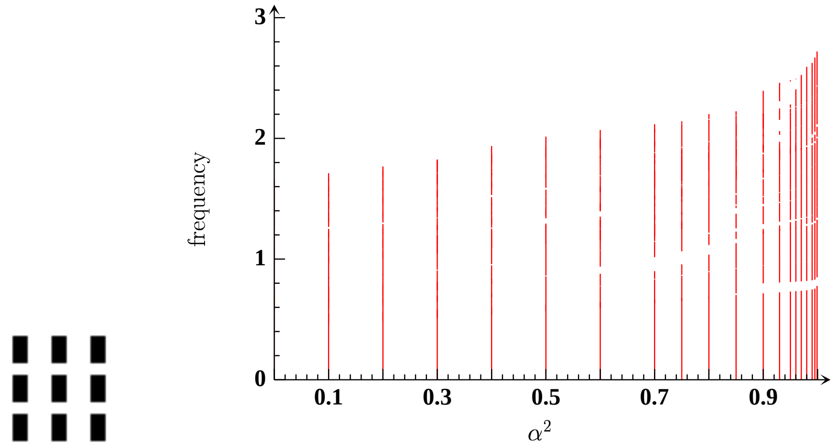

Figure 7: Spectrum of the structure with ellipses of axes 0.70 and 0.40.

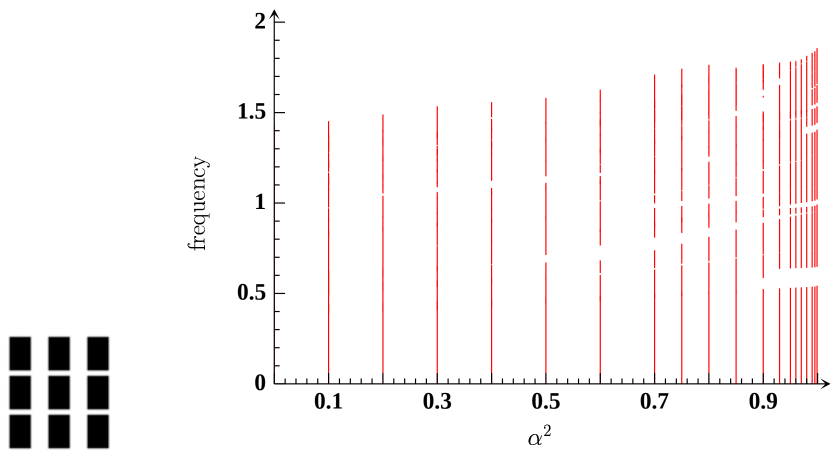

Figure 8: Spectrum of the structure with ellipses of axes 0.85 and 0.55.

mode will have the propagation constant $\beta = \alpha\omega_{\mathrm{d}}$.

It is also important to notice that we can get gaps and guide electromagnetic waves without any need to high dielectric contrast nor thin structures which is hard to achieve. The dielectric perturbation in the core of the fiber can be either positive or negative while the case of the classical fiber we can guide waves only with positive defects.

The integral formulation of guided modes could be used to achieve numerical tools for determining the defect frequencies in the fiber. This represents an alternative to the supercell method that could have some advantages. Actually, the supercell method does not distinguish defect eigenvalues from regular eigenvalues and computes all. But the degeneracy of the regular eigenvalues grows as the square of the supercell size. This fact added to the growth of the computational domain makes the method slow. In the integral formulation, however, we have to compute an approximation of the Green's function once for every α value and then with this function we can determine the defect modes

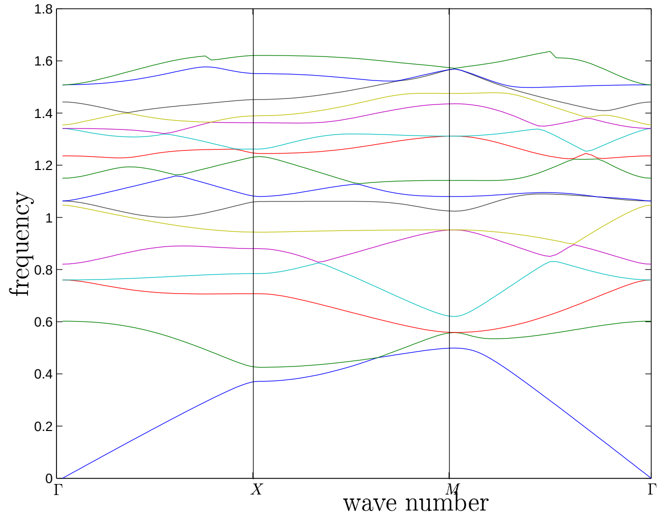

Figure 9: Band spectrum of the structure with discs of radius 0.3 in the planar propagation.

α^2	defect frequency	% of energy around the defect
0.80	0.650	72.9
0.85	0.659	70.6
0.90	0.660	70.9
0.93	0.662	74.8

Table 1: Energy of positive defect modes located around the defect area.

for different defects.

α^2	defect frequency	% of energy around the defect
0.75	0.617	87.7
0.80	0.639	92.9
0.85	0.656	95.1
0.90	0.685	64.0

Table 2: Energy of negative defect modes located around the defect area.

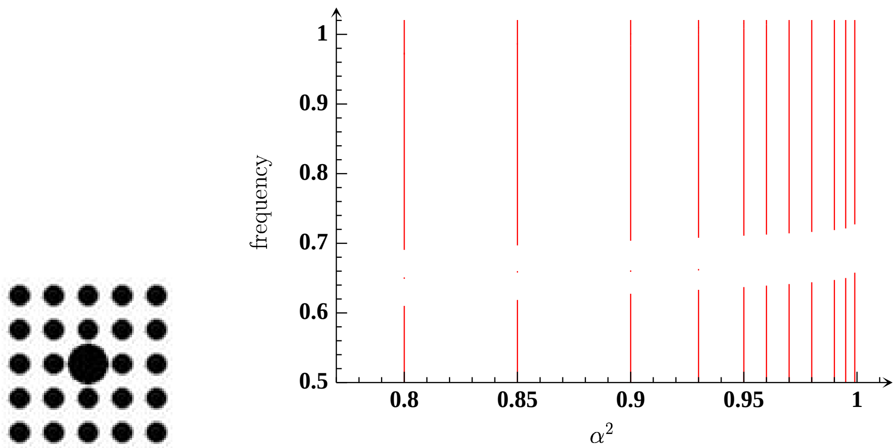

Figure 10: Spectrum of the structure with positive defect ($R_{\text{def}} = 0.6$).

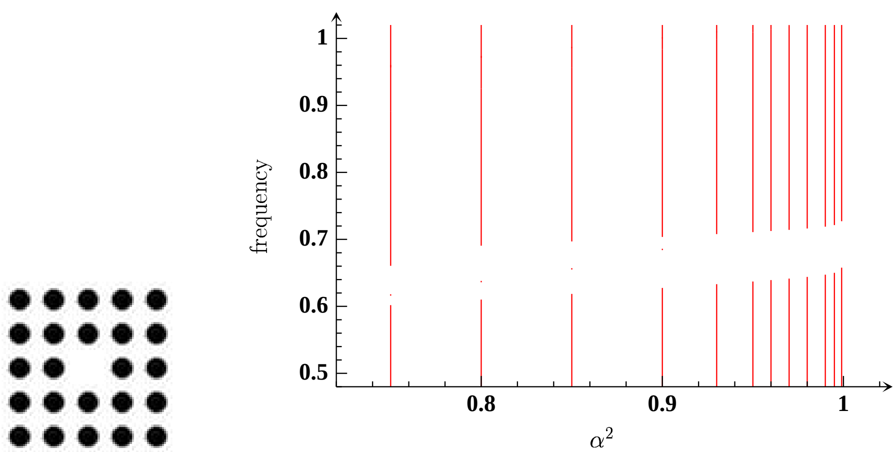

Figure 11: Spectrum of the structure with negative defect.

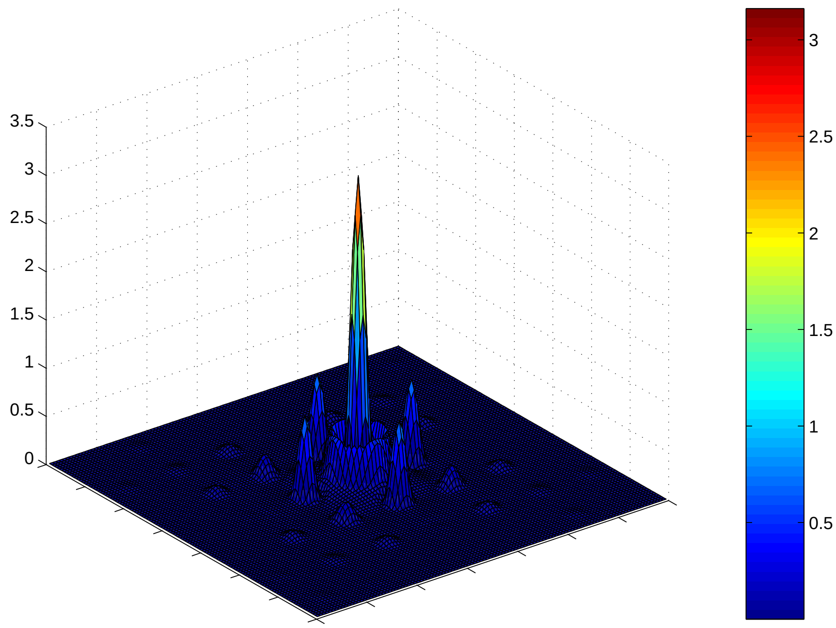

Figure 12: Energy density of the positive defect mode ($\alpha^2 = 0,80$).

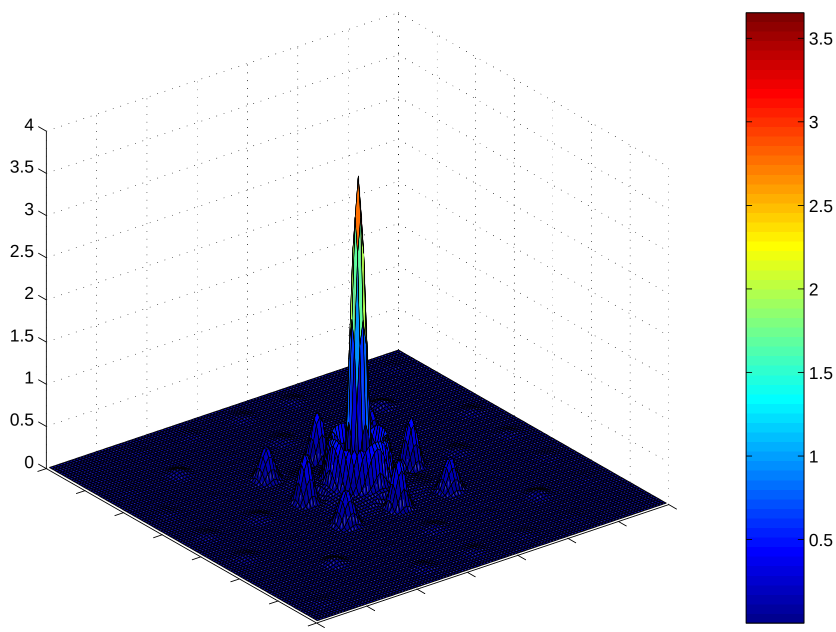

Figure 13: Energy density of the positive defect mode ($\alpha^2 = 0,85$).

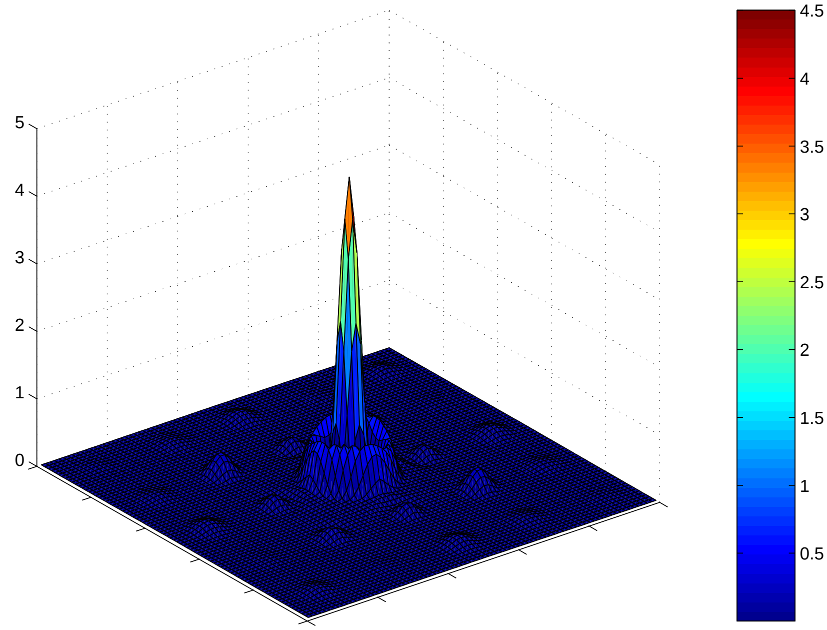

Figure 14: Energy density of the positive defect mode ($\alpha^2 = 0,90$).

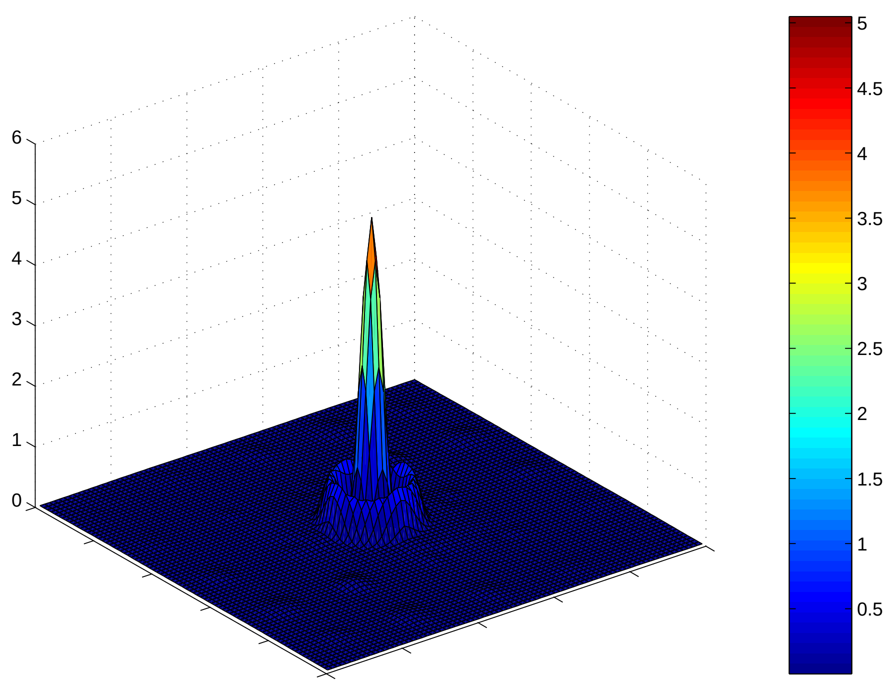

Figure 15: Energy density of the positive defect mode ($\alpha^2 = 0,93$).

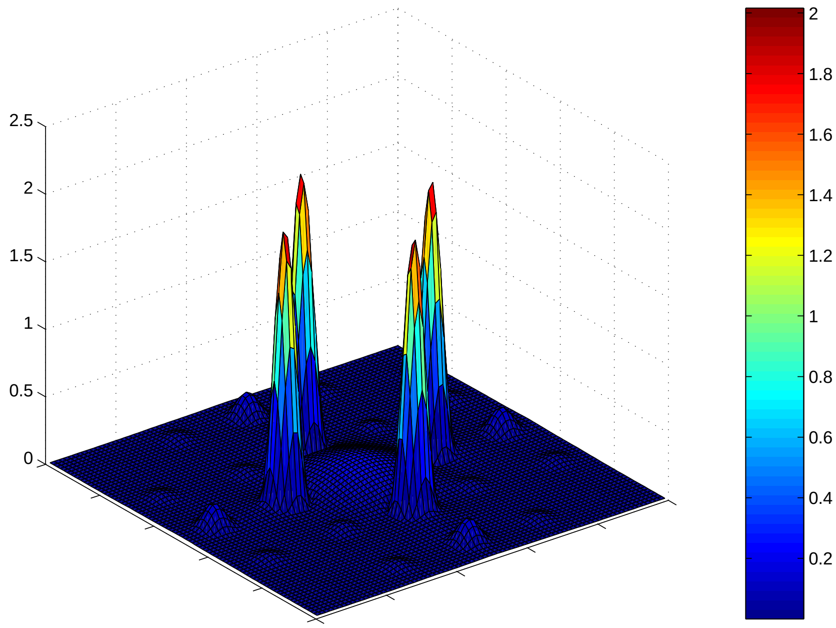

Figure 16: Energy density of the negative defect mode ($\alpha^2 = 0,75$).

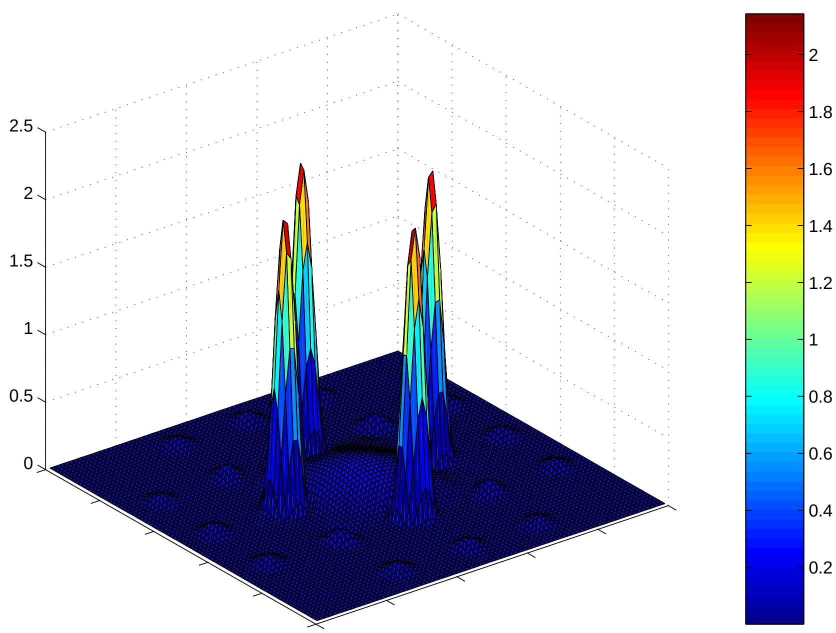

Figure 17: Energy density of the negative defect mode ($\alpha^2 = 0,80$).

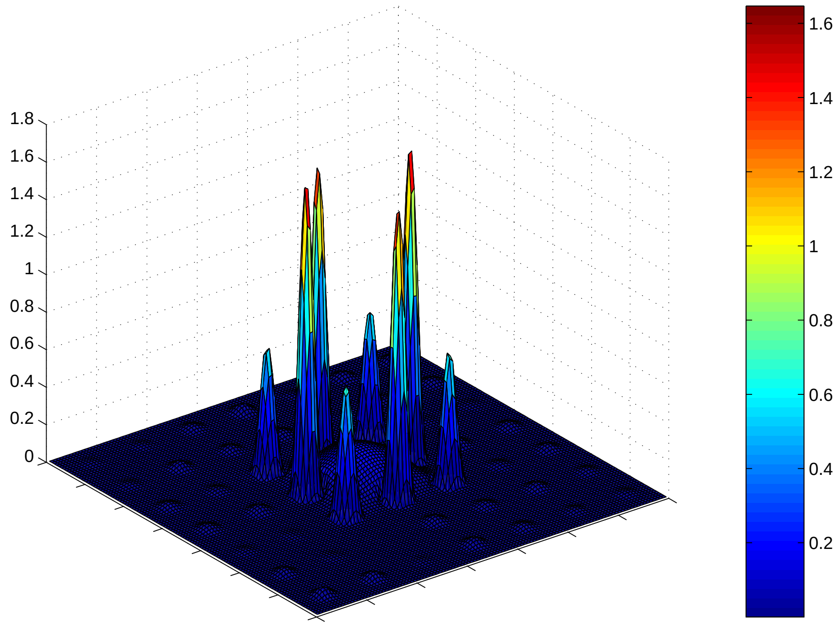

Figure 18: Energy density of the negative defect mode ($\alpha^2 = 0,85$).

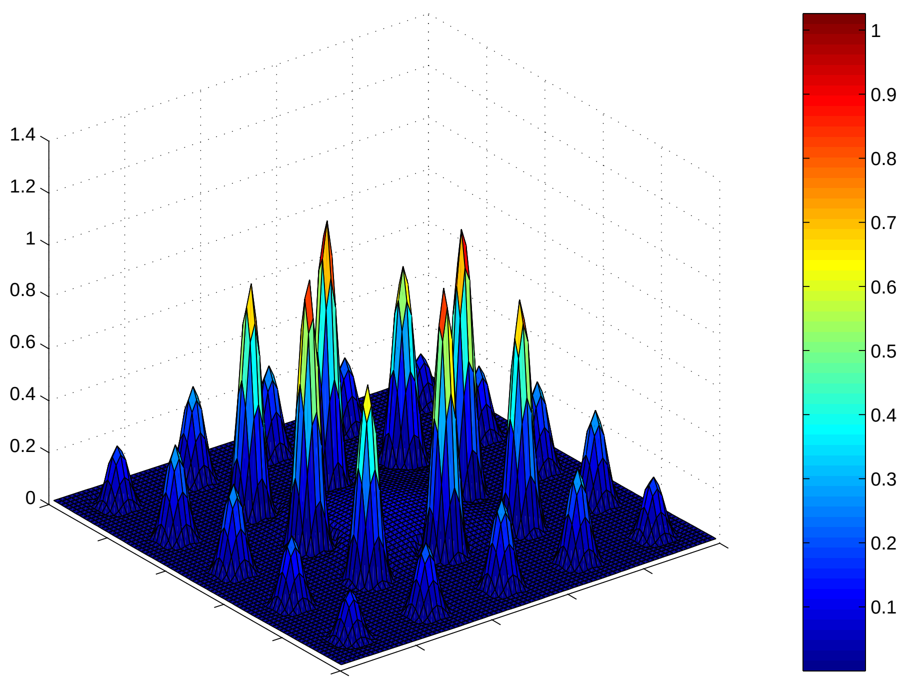

Figure 19: Energy density of the negative defect mode ($\alpha^2 = 0,90$).

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