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Second-harmonic generation in the undepleted-pump approximation

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Abstract

We study electromagnetic diffraction by a dielectric object surrounded by a nonlinear thin layer. The geometry of the problem is two dimensional and the incident wave is TM polarized. We derive a first order expantion of the fundamental field, then we derive the leading term of the second harmonic field. Our approach is based on layer potential techniques through integral representation formulas of the fields.

1 Introduction

In this paper, we study the electromagnetic theory of diffraction from nonlinear thin layers in the undepleted-pump approximation. Since its birth in the early 1960s, rapid and continuous andvances have been made in the field of nonlinear optics. One of the many important applications of nonlinear optical phenomena is a method for obtaining coherent radiation at a wavelength shorter than that of available lasers, through the process of second-harmonic generation (SHG).

All optical media are nonlinear. However, the nonlinearity is generally so weak that it is impossible to be observed without the use of high intensity laser beams. Mathematical modeling of nonlinear optics in thin layers is more difficult than that of linear optics studied in the literature by many authors [10, 11, 12, 3, 4]. In the nonlinear case, the electromagnetic wave propagation is now governed by the system of nonlinear Maxwell's equations, *i.e.*, nonlinear PDEs need to be studied. Also, the amplitude of the incident wave, which has no role in the linear case, plays an important role in the nonlinear case. Further, since nonlinear material properties are usually characterized by tensors, vectorial models become inevitable in the general situation.

The main aim of this paper is to rigorously derive the effect of thin layers of nonlinear material in the undepleted-pump approximation. In this approximation, the nonlinear Maxwell equations reduce to two coupled Helmholtz equations for the fundamental and the second-harmonic fields. We derive asymptotic formulas for two-dimensional fundamental and second-harmonic fields associated with thin layers of nonlinear materials. Our approach is based on layer potential techniques through integral representation formulas of the fields, avoiding the use (and the adaptation to our context) of the highly-nontrivial

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regularity results of Li and Vogelius [24]. See for a similar approach Beretta and Francini [10].

Our interest in such asymptotic formulas owes to the fact that they provide extremely powerful tools to solve optimization problems [9, 25, 26]. In [31, 6, 5, 3], asymptotic expansions of this kind for electromagnetic inclusions (of linear material) of small diameter have been already derived. However, they are by nature completely different from those derived in this paper. The degeneracy of the curves associated with the thin layer complicates a mathematically rigorous derivation, based on layer potentials, of the leading-order perturbations in the fundamental and second-harmonic fields.

This paper is organized in the following way. In Section 2 we formulate the problem and state our main results. Section 3 is devoted to the proof of existence and uniqueness of the fundamental and second-harmonic fields that are solution of two coupled Helmholtz equations. In Section 4 we review some well-known properties of the layer potentials and prove some useful identities. In Sections 5 and 6 we give an integral representation of the fundamental field and prove a regularity result necessary to show existence of the second-harmonic field. In Section 7 we provide a rigorous derivation of the leading-order perturbation term in its asymptotic expansion due to the nonlinear thin layer. Our aim in Sections 8 and 9 is to provide a rigorous derivation of leading-order term in the asymptotic expansion of the second-harmonic field.

2 Problem formulation

We start from the following Maxwell's equations, which are the general laws governing electromagnetic fields interacting with (nonmagnetic) matter

$$\nabla \times \mathbf{E} = -\frac{i\omega}{c}\mathbf{H}, \tag{2.1}$$

$$\nabla \times \mathbf{H} = \frac{i\omega}{c}(\mathbf{E} + 4\pi\mathbf{P}), \tag{2.2}$$

where c is the speed of light, ω is the angular frequency, $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, respectively, and $\mathbf{P}$ is the polarization.

These two Maxwell equations combine into

$$\nabla \times \nabla \times \mathbf{E} - \frac{\omega^2}{c^2}(\mathbf{E} + 4\pi\mathbf{P}) = 0.$$

It is obvious that we need the information on the relationship between $\mathbf{P}$ and $\mathbf{E}$ to proceed further. This is where the optical nonlinearities are introduced. In general, the nonlinear responses are orders of magnitude smaller than the linear response and the displacement vector of a medium can be expanded according to the power of the applied electric field $\mathbf{E}$. The case of most general interest, which is the subject of the investigations described later in the paper, is the second-harmonic generation (SHG). In this case we have

$$4\pi\mathbf{P} = (\epsilon - 1)\mathbf{E} + \chi^{(2)}(x,\omega) : \mathbf{E}\mathbf{E},$$

where ϵ is the dielectric coefficient, and $\chi^{(2)}$ is the second-order nonlinear susceptibility tensor of third rank, *i.e.*, $\chi^{(2)} : \mathbf{E}\mathbf{E}$ is a vector whose jth component is $\sum_{k,l=1}^{3} \chi^{(2)}_{jkl} : \mathbf{E}_k\mathbf{E}_l$.

For simplicity, we assume that the nonlinear polarization term $\mathbf{P}$ contains only the sum-frequency generation of the second-harmonic from the fundamental frequency and ignore all other $\chi^{(2)}$ phenomena such as difference-frequency generation, optical rectification, or cascaded nonlinear effects, as is consistent with the undepleted-pump approximation. Thus the polarization $\mathbf{P}$ at the fundamental frequency $\omega_1 = \omega$ and the second-harmonic frequency $\omega_2 = 2\omega$ may be written as

$$4\pi \mathbf{P}(x, \omega_1) = (\epsilon(x, \omega_1) - 1)\mathbf{E}(x, \omega_1),$$

and

$$4\pi \mathbf{P}(x, \omega_2) = (\epsilon(x, \omega_2) - 1)\mathbf{E}(x, \omega_2) + \chi^{(2)}(x, \omega_2) : \mathbf{E}(x, \omega_1)\mathbf{E}(x, \omega_1).$$

Assume that the depletion of energy from the pump waves (at the fundamental frequency ω_1) may be neglected. Then, using the above expression of the undepleted-pump nonlinear polarization, we can decompose the Maxwell equations (2.1)-(2.2) into two sets of coupled partial differential equations at the fundamental and second-harmonic frequencies.

Suppose all fields to be invariant in the x_3 direction. In the linear case, in transverse electric (TE) polarization the electric field is transversal to the (x_1, x_2)-plane, and in transverse magnetic (TM) polarization the magnetic field is transversal to the (x_1, x_2)-plane. In the nonlinear case, however, the polarization is determined by group symmetry properties of $\chi^{(2)} = (\chi^{(2)}_{njl})^3_{n,j,l=1}$. In this work, we assume that the electromagnetic fields are TM polarized at the fundamental frequency ω_1 and TE polarized at the second-harmonic frequency ω_2. This polarization assumption is known to support a large class of nonlinear optical materials, for example, crystals with cubic symmetry structures. See [30].

Therefore,

$$\mathbf{H}(x, \omega_1) = u(x_1, x_2, \omega_1)\vec{x}_3,$$
$$\mathbf{E}(x, \omega_2) = v(x_1, x_2, \omega_2)\vec{x}_3.$$

Define for the sake of simplicity

$$\epsilon_j = \epsilon(x_1, x_2, \omega_j), \quad j = 1, 2,$$

$$\kappa_j = \frac{\omega_j}{c}\sqrt{\epsilon_j}, \quad \Im \kappa_j \geq 0, \quad j = 1, 2.$$

At the fundamental frequency ω_1, the system (2.1)-(2.2) can be simplified to

$$\nabla \cdot \left(\frac{1}{\kappa_1^2}\nabla u\right) + u = 0.$$

We deduce the expression of the electric field at the fundamental frequency ω_1

$$\begin{aligned}\mathbf{E}(x, \omega_1) &= \frac{c}{i\omega_1\epsilon_1}\nabla \times \mathbf{H}(x, \omega_1) \\ &= \frac{c}{i\omega_1\epsilon_1}(\partial_{x_2}u, -\partial_{x_1}u, 0).\end{aligned}$$

Hence the second-harmonic field satisfies

$$\begin{aligned}\left(\Delta+\kappa_2^2\right) v &= -\frac{4\pi\omega_2^2}{c^2} \sum_{j,l=1,2,3} \chi^{(2)}_{3jl}(x,\omega_2)(\mathbf{E}(x,\omega_1))_j(\mathbf{E}(x,\omega_1))_l \\ &= \sum_{j,l=1,2} \chi_{jl}\partial_{x_j}u\partial_{x_l}u,\end{aligned}$$

where $\chi_{jl}=(-1)^{j+l}(16\pi/\epsilon_1^2)\chi^{(2)}_{3jl}(x,\omega_2)$.

Then (u,v) satisfies in the nonlinear material the following two coupled Helmholtz equations

$$\nabla\cdot\left(\frac{1}{\kappa_1^2}\nabla u\right)+u = 0, \tag{2.3}$$

$$\Delta v+\kappa_2^2 v = \sum_{j,l=1,2} \chi_{jl}\partial_{x_j}u\partial_{x_l}u. \tag{2.4}$$

Let us now specify the geometry of the problem. Let Ω be a bounded $\mathcal{C}^2$-domain in $\mathbb{R}^2$. Let $\tau(x)$ and $\nu(x)$ denote a unit tangential and a unit (exterior to Ω) normal field to $\partial\Omega$. For a function f defined on $\mathbb{R}^2\setminus\partial\Omega$, we denote $[f(x)]_{\partial\Omega}= f|_+ - f|_-$ where $f|_+=\lim_{\delta\to 0^+} f(x+\delta\nu(x))$ and $f|_-=\lim_{\delta\to 0^+} f(x-\delta\nu(x))$, if the limits exist.

We consider a layer of nonlinear material of the form

$$\mathcal{O}_\delta=\Big\{x+\eta\nu(x) : x\in\partial\Omega, \eta\in(0,\delta)\Big\},$$

where the thickness, $\delta>0$, is a small parameter. Let $\Omega^i_\delta=\overline{\Omega}\cup\mathcal{O}_\delta$, $\Omega^e_\delta=\mathbb{R}^2\setminus\overline{\Omega^i_\delta}$.

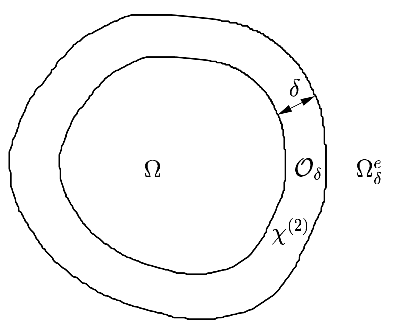

Figure 1: The dielectric medium.

Throughout this paper we suppose that the susceptibility tensor is of the form

$$\chi_{jl}(x+\eta\nu)=\tilde{\chi}_{jl}\left(x,\frac{\eta}{\delta}\right), \qquad x\in\partial\Omega\ ,\ 0<\eta<\delta\ ,$$

where $\tilde{\chi}_{jl}\in L^\infty(\overline{\mathcal{O}}_\delta)$ are independent of δ, and define

$$k(x)=\begin{cases} k_1 & \text{for } x\in\Omega, \\ k_2 & \text{for } x\in\mathcal{O}_\delta, \\ k_0 & \text{for } x\in\Omega^e_\delta,\end{cases}$$

where k_1, k_2, k_0 are positive constants. We also introduce the function $k'(x)$ defined analogously with positive constants k'_1, k'_2, k'_0. We assume in all what follows that $k_2 \neq k_1$, $k_2 \neq k_0$, $k'_2 \neq k'_1$, and $k'_2 \neq k'_0$.

By $\partial\Omega_\eta$, for $\eta \in (0, \delta)$, we denote

$$\partial\Omega_\eta = \Big\{ x + \eta\nu(x) : x \in \partial\Omega \Big\},$$

with the convention $\partial\Omega_0 = \partial\Omega$. We denote by $\rho(x)$ the curvature at the point $x \in \partial\Omega$. If ds denotes the surface measure on $\partial\Omega$ then the corresponding surface measure on $\partial\Omega_\eta$ is related to ds at the point $x \in \partial\Omega$ through the relation $ds_\eta(x + \eta\nu(x)) = (1 + \eta\rho(x))\, ds(x)$.

Consider an incident plane wave given by $u_I(x) = U_I e^{ik_I \cdot x}$ where $k_I \in \mathbb{R}^2$ is the wave-vector with $|k_I| = k_0$ and $U_I \in \mathbb{R}$ is a positive constant. Then (u, v) is solution of the following problem

$$\left\{\begin{array}{lcc} \nabla \cdot \dfrac{1}{k^2} \nabla u + u & = & 0, \\ \Delta v + k'^2 v & = & \displaystyle\sum_{j,l=1,2} \chi_{jl} \partial_{x_j} u \partial_{x_l} u \, \mathrm{I}_{\mathcal{O}_\delta}, \\ \displaystyle\lim_{|x| \to +\infty} \sqrt{|x|} \left(\frac{\partial (u - u_I)}{\partial |x|} - i k_0 (u - u_I) \right) & = & 0, \\ \displaystyle\lim_{|x| \to +\infty} \sqrt{|x|} \left(\frac{\partial v}{\partial |x|} - i k'_0 v \right) & = & 0, \end{array}\right. \tag{2.5}$$

where $\mathrm{I}_{\mathcal{O}_\delta}$ is the characteristic function of $\mathcal{O}_\delta$.

The equations (2.5) may alternatively be formulated as follows

$$\left\{\begin{array}{l} \Delta u + k_1^2 u = 0 \qquad \text{in } \Omega, \\ \Delta u + k_2^2 u = 0 \qquad \text{in } \mathcal{O}_\delta, \\ \Delta u + k_0^2 u = 0 \qquad \text{in } \Omega_\delta^e, \\ [u]_{\partial\Omega} = [u]_{\partial\Omega_\delta} = 0, \\ \left[\dfrac{1}{k^2} \dfrac{\partial u}{\partial \nu} \right]_{\partial\Omega} = \left[\dfrac{1}{k^2} \dfrac{\partial u}{\partial \nu} \right]_{\partial\Omega_\delta} = 0, \\ \displaystyle\lim_{|x| \to \infty} \sqrt{|x|} \left| \frac{\partial (u - u_I)}{\partial |x|} - i k_0 (u - u_I) \right| = 0, \qquad \text{uniformly in } \frac{x}{|x|}, \end{array}\right. \tag{2.6}$$

and

$$\begin{cases} \Delta v + k_1^{'2} v = 0 & \text{in } \Omega, \\ \Delta v + k_2^{'2} v = \displaystyle\sum_{j,l=1,2} \chi_{jl} \partial_{x_j} u \partial_{x_l} u & \text{in } \mathcal{O}_\delta, \\ \Delta v + k_0^{'2} v = 0 & \text{in } \Omega_\delta^e, \\ [v]_{\partial\Omega} = [v]_{\partial\Omega_\delta} = 0, \\ \left[\dfrac{\partial v}{\partial \nu}\right]_{\partial\Omega} = \left[\dfrac{\partial v}{\partial \nu}\right]_{\partial\Omega_\delta} = 0, \\ \lim_{|x|\to\infty} \sqrt{|x|} \left| \dfrac{\partial v}{\partial |x|} - i k_0' v \right| = 0, & \text{uniformly in } \dfrac{x}{|x|}. \end{cases} \tag{2.7}$$

In the remainder of this paper U shall always refer to the solution of

$$\begin{cases} \Delta U + k_1^2 U = 0 & \text{in } \Omega, \\ \Delta U + k_0^2 U = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ [U]_{\partial\Omega} = 0, \\ \left[\dfrac{1}{\tilde{k}^2} \dfrac{\partial U}{\partial \nu}\right]_{\partial\Omega} = 0, \\ \lim_{|x|\to\infty} \sqrt{|x|} \left| \dfrac{\partial (U - u_I)}{\partial |x|} - i k_0 (U - u_I) \right| = 0, & \text{uniformly in } \dfrac{x}{|x|}, \end{cases} \tag{2.8}$$

where the function $\tilde{k}(y)$ is given by

$$\tilde{k}(y) = \begin{cases} k_1 & \text{for } y \in \Omega \,, \\ k_0 & \text{for } y \in \mathbb{R}^2 \setminus \overline{\Omega} \,. \end{cases}$$

Before giving a precise formulation of the main results of this paper we need to introduce some additional notation. By G, we denote the fundamental solution of the following transmission problem

$$\begin{cases} \Delta_y G(x,y) + k_0^2 G(x,y) = \delta_x(y) & \text{for } y \in \mathbb{R}^2 \setminus \overline{\Omega} \,, \\ \Delta_y G(x,y) + k_1^2 G(x,y) = \delta_x(y) & \text{for } y \in \Omega \,, \\ \left[\tilde{k}^2 G(x,\cdot)\right] = 0 \quad \text{on } \partial\Omega \,, \\ \left[\dfrac{\partial G(x,\cdot)}{\partial \nu(y)}\right] = 0 \quad \text{on } \partial\Omega \,, \\ \lim_{|y|\to\infty} \sqrt{|y|} \left| \dfrac{\partial G(x,y)}{\partial |y|} - i k_0 G(x,y) \right| = 0, & \text{uniformly in } \dfrac{y}{|y|}. \end{cases}$$

We will also need the function G' that is the solution to

$$\begin{cases} \Delta_y G'(x,y) + (k_0')^2 G'(x,y) = \delta_x(y) \quad \text{for } y \in \mathbb{R}^2 \setminus \overline{\Omega}\,, \\ \Delta_y G'(x,y) + (k_1')^2 G'(x,y) = \delta_x(y) \quad \text{for } y \in \Omega\,, \\ [G'(x,\cdot)] = 0 \quad \text{on } \partial\Omega\,, \\ \left[\dfrac{\partial G'(x,\cdot)}{\partial \nu(y)}\right] = 0 \quad \text{on } \partial\Omega\,, \\ \lim\limits_{|y|\to\infty} \sqrt{|y|}\left|\dfrac{\partial G'(x,y)}{\partial |y|} - ik_0' G'(x,y)\right| = 0, \qquad \text{uniformly in } \dfrac{y}{|y|}, \end{cases}$$

where the function $\tilde{k}'(y)$ is defined analogously to $\tilde{k}(y)$.

Define the symmetric matrix $\mathcal{A}(x), x \in \partial\Omega$, by

$$\begin{aligned} &\mathcal{A} \text{ has eigenvectors } \tau(x) \text{ and } \nu(x), \\ &\text{the eigenvalue corresponding to } \tau(x) \text{ is } (\frac{k_0}{k_2})^2 - 1, \\ &\text{the eigenvalue corresponding to } \nu(x) \text{ is } 1 - (\frac{k_2}{k_0})^2. \end{aligned} \tag{2.9}$$

It is clear that $\mathcal{A}$ is positive definite if $k_0 > k_2$, and negative definite if $k_0 < k_2$. We also need the matrix $\mathcal{A}'(x), x \in \partial\Omega$, defined by

$$\begin{aligned} &\mathcal{A}' \text{ has eigenvectors } \tau(x) \text{ and } \nu(x), \\ &\text{the eigenvalue corresponding to } \tau(x) \text{ is } 1, \\ &\text{the eigenvalue corresponding to } \nu(x) \text{ is } (\frac{k_2}{k_0})^2. \end{aligned} \tag{2.10}$$

The main achievement of this paper consists in the following asymptotic formulas concerning the perturbation, $u - U$, and the second-harmonic field v, enhanced by the thin layer of nonlinear material $\mathcal{O}_\delta$ in the undepleted-pump approximation.

Theorem 2.1 *Let u and v be the solutions to (2.6) and (2.7), respectively, and let $\mathcal{A}$ and $\mathcal{A}'$ be the matrices defined by (2.9) and (2.10), respectively. Then, for $x \in \mathbb{R}^2 \setminus \overline{\Omega}$ bounded away from $\partial\Omega$, the following pointwise expansions hold:*

$$u(x) = U(x) + \delta \int_{\partial\Omega} \nabla_y G(x,y)\Big|_+ \cdot \mathcal{A}\nabla U(y)\Big|_+ \, ds(y) + o(\delta)\,, \tag{2.11}$$

and

$$\begin{aligned} v(x) = \delta \sum_{j,l=1,2} \int_{\partial\Omega} & G'(x,y)\Big|_+ \\ & \left(\textstyle\int_0^1 \tilde{\chi}_{jl}(y,\theta)\, d\theta\right) \left(\mathcal{A}'\nabla U(y)\Big|_+\right)_j \left(\mathcal{A}'\nabla U(y)\Big|_+\right)_l ds(y) \\ & + o(\delta)\,, \end{aligned} \tag{2.12}$$

where the remainder terms $o(\delta)$ are independent of x.

It is worth noticing that from the nature of our derivations it follows that we cannot expect the remainder terms in (2.11) and (2.12) to be uniform in $\mathbb{R}^2 \setminus \partial\Omega$. Rather, these terms are uniform at fixed distance away from $\partial\Omega$, but with the estimates (2.11) and (2.12) degenerate as x approaches $\partial\Omega$. Indeed, the transmission problem for U and the first order correction

$$u_1 = \int_{\partial\Omega} \nabla_y G(x,y)\Big|_+ \cdot \mathcal{A}\nabla U(y)\Big|_+ ds(y)$$

are not posed on the same domain – the transmission problem for U is posed on the whole $\mathbb{R}^2$, but the one for u_1 is naturally posed on $\mathbb{R}^2 \setminus \partial\Omega$. This significantly complicates our mathematically rigorous derivation of the expansions (2.11) and (2.12) and makes our analysis nontrivial.

3 Well-posedness

In this section, we will prove existence and uniqueness of the fundamental field u. Even though these results are classical we give their proof for the reader's convenience. The proof of existence and uniqueness of the second-harmonic field v is exactly the same as for the fundamental field u since, as will be shown later in Corollary 6.1, $\sum_{j,l=1,2} \chi_{jl} \partial_{x_j} u \partial_{x_l} u$ belongs to $L^2(\mathcal{O}_\delta)$.

We start by formulating the problem (2.6) in a bounded domain. Consider the disc B_R centered at the origin with radius R large enough to have $\overline{\Omega} \subset B_R$ and denote by S_R its boundary. The scattered field $u - u_I$ satisfy in $\mathbb{R}^2 \setminus \overline{B}_R$ the Helmholtz equation

$$\Delta(u-u_I) + k_0^2(u-u_I) = 0 ,$$

together with the (outgoing) radiation condition

$$\lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial(u-u_I)}{\partial r} - ik_0(u-u_I)\right) = 0 .$$

Taking the Fourier series $(u^n - u_I^n)(r)$ with respect to the angular variable θ, where (r,θ) are the polar coordinates, we get

$$(u^n - u_I^n)''(r) + \frac{1}{r}(u^n - u_I^n)'(r) + (k_0^2 - \frac{n^2}{r^2})(u^n - u_I^n)(r) = 0 \quad \text{for } r > R.$$

Therefore

$$(u^n - u_I^n)(r) = A_n H_n^{(1)}(k_0 r) + B_n H_n^{(2)}(k_0 r) ,$$

where A_n and B_n are constants, and $H_n^{(1)}$ and $H_n^{(2)}$ denote the Hankel functions of the first and the second kind, respectively. However, only $H_n^{(1)}(k_0 r)$ satisfies the above radiation condition. Thus,

$$(u-u_I)(r,\theta) = \sum_{n\in\mathbb{Z}} A_n H_n^{(1)}(k_0 r) e^{in\theta} \quad \text{for } r > R \;\text{ and }\; \theta \in [0, 2\pi) .$$

Using this Fourier expansion we can express the trace of $(u-u_I)$ and $\dfrac{\partial(u-u_I)}{\partial\nu}$ on S_R as follows

$$(u-u_I)(R,\theta) = \sum_{n\in\mathbb{Z}} A_n H_n^{(1)}(k_0 R) e^{in\theta} ,$$

$$\frac{\partial(u-u_I)}{\partial\nu}(R,\theta)=\sum_{n\in\mathbb{Z}}A_n k_0 H_n^{(1)\prime}(k_0R)e^{in\theta}\ ,$$

from which we readily get that

$$\left(\frac{\partial(u-u_I)}{\partial\nu}\right)^n(R)=k_0\frac{H_n^{(1)\prime}(k_0R)}{H_n^{(1)}(k_0R)}(u-u_I)^n(R)\ .$$

Let C_R be the mapping defined by

$$\begin{array}{llll} C_R: & H^{1/2}(S_R) & \to & H^{-1/2}(S_R) \\ & f=\sum_{n\in\mathbb{Z}}f^ne^{in\theta} & \mapsto & C_R(f)=\displaystyle\sum_{n\in\mathbb{Z}}k_0\frac{H_n^{(1)\prime}(k_0R)}{H_n^{(1)}(k_0R)}f^ne^{in\theta}\ . \end{array}$$

The proof of uniqueness of a solution to (2.6) relies on the following classical properties of the so-called Dirichlet-to-Neumann map C_R.

Lemma 3.1 *The mapping C_R defines a bounded operator from $H^{1/2}(S_R)$ into $H^{-1/2}(S_R)$. Furthermore, we have*

$$\Im\int_{S_R}C_R(u)\overline{u}>0\quad\forall u\in H^{1/2}(S_R)\ ,u\neq0\ ,\tag{3.13}$$

$$\Re\int_{S_R}C_R(u)\overline{u}\leq0\quad\forall u\in H^{1/2}(S_R)\ .\tag{3.14}$$

Now we can formulate (2.6) in the bounded domain B_R using the Dirichlet-to-Neumann map C_R. Introduce the following transmission problem

$$\begin{cases} \Delta u+k_1^2u=0 & \text{in } \Omega, \\ \Delta u+k_2^2u=0 & \text{in } \mathcal{O}_\delta, \\ \Delta u+k_0^2u=0 & \text{in } \Omega_\delta^e\cap B_R, \\ [u]_{\partial\Omega}=[u]_{\partial\Omega_\delta}=0, & \\ \left[\dfrac{1}{k^2}\dfrac{\partial u}{\partial\nu}\right]_{\partial\Omega}=\left[\dfrac{1}{k^2}\dfrac{\partial u}{\partial\nu}\right]_{\partial\Omega_\delta}=0\ , & \\ \dfrac{\partial u}{\partial\nu}=C_R(u)+g & \text{on } S_R, \end{cases}\tag{3.15}$$

where $g:=\dfrac{\partial u_I}{\partial\nu}-C_R(u_I)$ on S_R.

Lemma 3.2 *To each solution u to the problem (2.6) corresponds one and only one solution u^i to the problem (3.15) that is its restriction to B_R.*

Proof. Let u be a solution to (2.6). Since $(u-u_I)$ satisfies the Helmholtz equation $\Delta(u-u_I)+k_0^2(u-u_I)=0$ in $\mathbb{R}^2\setminus\overline{B}_R$ and the radiation condition, it immediately follows that

$$\frac{\partial(u-u_I)}{\partial\nu}=C_R(u-u_I)\quad\text{on } S_R,$$

which is equivalent to

$$\frac{\partial u}{\partial \nu} = C_R(u) + g \quad \text{on } S_R.$$

The restriction of u to B_R is then a solution to (2.6).

Conversely, let u^i be a solution to (3.15). Let $f = u^i|_{S_R}$. It is well known from the potential theory that the following exterior problem

$$\begin{cases} \Delta u^e + k_0^2 u^e = 0 & \text{in } \mathbb{R}^2 \setminus \overline{B}_R, \\ u^e = f - u_I & \text{on } S_R\ , \\ \lim_{|x|\to\infty} \sqrt{|x|} \left| \frac{\partial u^e}{\partial |x|} - ik_0 u^e \right| = 0 & \text{uniformly in } \frac{x}{|x|}, \end{cases} \tag{3.16}$$

has a unique solution u^e. Define u by

$$u = \begin{cases} u^i & \text{in } B_R\ , \\ u^e + u_I & \text{in } \mathbb{R}^2 \setminus \overline{B}_R\ , \end{cases}$$

It is easy to check that u satisfies (2.6). □

We are now ready to prove the well-posedness of the problem (3.15). We introduce the bilinear form $a(u,w)$ on $H^1(B_R) \times H^1(B_R)$ by

$$a(u,w) = \int_{B_R} \frac{1}{k^2} \nabla u \cdot \nabla \overline{w} - \int_{B_R} u\overline{w} - \frac{1}{k_0^2} \int_{S_R} C_R(u)\overline{w}\ . \tag{3.17}$$

We can immediately see that a function $u \in H^1(B_R)$ is a weak solution to (3.15) if and only if it is a solution to the variational problem

$$a(u,w) = \frac{1}{k_0^2} \int_{S_R} g\overline{w} \quad \forall w \in H^1(B_R). \tag{3.18}$$

The following existence and uniqueness result holds.

Proposition 3.1 *There exists a unique weak solution to the problem (3.15) in $H^1(B_R)$.*

Proof. Since $k^2(x)$ is bounded away from 0 and ∞, there exists a constant $C > 0$ such that

$$\Re a(u,u) \geq C \int_{B_R} |\nabla u|^2 - \int_{B_R} |u|^2\ . \tag{3.19}$$

It is also obvious that the bilinear form a is bounded. Since the embedding of $H^1(B_R)$ into $L^2(B_R)$ is compact, the Fredholm alternative holds and the existence will follow from the uniqueness.

In order to prove the uniqueness, suppose that there exists $u \in H^1(B_R)$ satisfying

$$a(u,w) = 0 \quad \forall w \in H^1(B_R)\ .$$

Therefore

$$\Im a(u,u) = 0 = \Im \int_{S_R} C_R(u)\overline{u}\ ,$$

and thus, using (3.13), we deduce that u belongs to $H^1_0(B_R)$ and satisfies

$$\int_{B_R} \frac{1}{k^2}\nabla u \cdot \nabla \overline{w} - \int_{B_R} u\overline{w} = 0 \ , \forall w \in H^1(B_R) \ . \tag{3.20}$$

This means that u is a weak solution to

$$\begin{cases} \nabla \cdot \dfrac{1}{k^2}\nabla u + u = 0 & \text{in } B_R \ , \\ u = 0 \quad \text{on } S_R \ , \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } S_R \ . \end{cases}$$

Finally, since k^2 is piecewise constant in B_R, the unique continuation theorem for the Helmholtz equation applies to ensure the uniqueness of a solution. The proof of the proposition is complete. □

4 Preliminary results

Let us first review some well-known properties of the layer potentials for the Helmholtz equation and prove some useful identities.

Let $k > 0$ be a given constant and let Γ_k and Γ_0 be the fundamental (outgoing) solutions of $\Delta + k^2$ and Δ, respectively, which are defined by

$$\Gamma_k(x) = -\frac{i}{4} H^1_0(k|x|), \quad x \in \mathbb{R}^2 ,$$

$$\Gamma_0(x) = \frac{1}{2\pi} \log(|x|), \quad x \in \mathbb{R}^2 ,$$

for $x \neq 0$.

Let $\eta \geq 0$ be small enough. We define the slightly modified single layer potential S^k_η and double layer potential D^k_η for a density $\varphi \in L^2(\partial\Omega)$ by the following

$$S^k_\eta \varphi(x) = \int_{\partial\Omega} \Gamma_k(x - y - \eta\nu(y))(1 + \eta\rho(y))\varphi(y)\, ds(y) \ , \quad x \in \mathbb{R}^2 ,$$

$$D^k_\eta \varphi(x) = \int_{\partial\Omega} \frac{\partial \Gamma_k(x - y - \eta\nu(y))}{\partial \nu(y)}(1 + \eta\rho(y))\varphi(y)\, ds(y) \ , \quad x \in \mathbb{R}^2 \setminus \partial\Omega_\eta,$$

We also define the operators K^k_η, its L^2-adjoint $(K^k_\eta)^*$, and M^k_η by

$$K^k_\eta \varphi(x) = \int_{\partial\Omega} \frac{\partial \Gamma_k(x - y + \eta(\nu(x) - \nu(y)))}{\partial \nu(y)}(1 + \eta\rho(y))\varphi(y)\, ds(y) \ , \quad x \in \partial\Omega \ ,$$

$$(K^k_\eta)^* \varphi(x) = \int_{\partial\Omega} \frac{\partial \Gamma_k(x - y + \eta(\nu(x) - \nu(y)))}{\partial \nu(x)}(1+\eta\rho(y))\varphi(y)\, ds(y) \ , \quad x \in \partial\Omega \ ,$$

$$M^k_\eta \varphi(x) = \int_{\partial\Omega} \left(\frac{\partial^2}{\partial \nu(x)^2} + \frac{\partial^2}{\partial \nu(x)\partial \nu(y)} \right) \Gamma_k(x - y + \eta(\nu(x) - \nu(y))) \\ (1 + \eta\rho(y))\varphi(y)\, ds(y) \ ,$$

for $x \in \partial\Omega$.

Finally, we introduce the following notations for $\eta, \delta \geq 0$ small enough and $x \in \partial\Omega$:

$$S^k_{\eta,\delta}\varphi(x) = S^k_\eta\varphi(x + \delta\nu(x)) , \quad x \in \mathbb{R}^2 ,$$

and

$$D^k_{\eta,\delta}\varphi(x) = D^k_\eta\varphi(x + \delta\nu(x)) , \quad x \in \mathbb{R}^2 \setminus \partial\Omega_{\eta-\delta} .$$

The functions $S^k_\eta\varphi$ and $D^k_\eta\varphi$ are in fact the single and double layer potentials of the density $\tilde{\varphi}(y + \eta\nu(y)) = \varphi(y)$ on the curve $\partial\Omega_\eta$.

We recall the following classical result.

Lemma 4.1 *For any $k > 0$, the function $\Gamma_k - \Gamma_0$ is continuous.*

Proof. From

$$\Delta(\Gamma_k - \Gamma_0) + k^2(\Gamma_k - \Gamma_0) = -k^2\Gamma_0 ,$$

and since Γ_0 is in $L^2_{\text{loc}}(\mathbb{R}^2)$, we deduce by applying classical results on elliptic regularity [17] and the Sobolev embedding theorem [2] that $\Gamma_k - \Gamma_0$ is a continuous function. □

From the properties of S^0_η and D^0_η, see [13], we can obtain that

$$\frac{\partial(S^k_\eta\varphi)_\pm}{\partial\nu}(x) = \left(\pm\frac{1}{2}I + (K^k_\eta)^*\right)\varphi(x) , \quad \text{a.e. } x \in \partial\Omega_\eta, \tag{4.21}$$

$$(D^k_\eta\varphi)_\pm(x) = \left(\mp\frac{1}{2}I + K^k_\eta\right)\varphi(x) , \quad \text{a.e. } x \in \partial\Omega_\eta, \tag{4.22}$$

for $\varphi \in L^2(\partial\Omega)$, where

$$\frac{\partial(u)_\pm}{\partial\nu}(x) := \lim_{h\to+0} \nu(x)\cdot\nabla u(x \pm h\nu(x)) ,$$

and

$$u_\pm(x) := \lim_{h\to+0} u(x \pm h\nu(x)) .$$

From the standard potential theory, we also have the following results.

Lemma 4.2 *For $\eta \geq 0$ small enough, the following operators*

$$S^k_{\eta,\eta} : L^2(\partial\Omega) \to H^1(\partial\Omega) ,$$

$$K^k_\eta, (K^k_\eta)^* : L^2(\partial\Omega) \to H^1(\partial\Omega) ,$$

$$M^k_\eta, (\partial_\nu S^k_{\eta,\eta})_\pm, (D^k_{\eta,\eta})_\pm : L^2(\partial\Omega) \to L^2(\partial\Omega) ,$$

are bounded.

We can prove that the following expansions hold. See Appendix A.2 for the proof.

Lemma 4.3 *For any $\eta > 0$ small enough and for any $\varphi \in L^2(\partial\Omega)$, we have*

$$S^k_{\delta,\delta}\varphi(x) = S^k_{0,0}\varphi(x) + \delta\left(K^k_0\varphi(x) + (K^k_0)^*\varphi(x) + S^k_{0,0}(\rho\varphi)(x)\right) + O(\delta^2) ,$$

$$(K^k_\delta)^*\varphi(x) = (K^k_0)^*\varphi(x) + \delta\left(M^k_0\varphi(x) + (K^k_0)^*(\rho\varphi)(x)\right) + O(\delta^2) ,$$

where $O(\delta^2)$ is in $H^1(\partial\Omega)$ in the first equation and in $L^2(\partial\Omega)$ in the second one.

The following lemma is of importance to us. We refer the reader to Appendix A.3 for its proof.

Lemma 4.4 *There exists $\varepsilon_\delta > 0$ satisfying $\lim_{\delta\to 0} \varepsilon_\delta = 0$ such that for any $\varphi \in L^2(\partial\Omega)$ and $s = 0, 1$, the following estimates hold:*

$$||S^k_{0,\delta}\varphi - S^k_{0,0}\varphi||_{H^{s+1}(\partial\Omega)} \le \varepsilon_\delta ||\varphi||_{H^s(\partial\Omega)} \ ,$$

$$||S^k_{\delta,\delta}\varphi - S^k_{\delta,0}\varphi||_{H^{s+1}(\partial\Omega)} \le \varepsilon_\delta ||\varphi||_{H^s(\partial\Omega)} \ ,$$

$$\left\| \frac{\partial (S^k_{0,\delta}\varphi)}{\partial \nu} - \frac{\partial (S^k_{0,0}\varphi)_+}{\partial \nu} \right\|_{H^s(\partial\Omega)} \le \varepsilon_\delta ||\varphi||_{H^s(\partial\Omega)} \ ,$$

$$\left\| \frac{\partial (S^k_{\delta,\delta}\varphi)_-}{\partial \nu} - \frac{\partial (S^k_{\delta,0}\varphi)}{\partial \nu} \right\|_{H^s(\partial\Omega)} \le \varepsilon_\delta ||\varphi||_{H^s(\partial\Omega)} \ .$$

The estimates for $s = 1$ hold under the assumption that $\partial\Omega$ is of class $\mathcal{C}^3$.

The following result on the spectral radius of K^k_0 is also of use to us.

Lemma 4.5 *For any λ satisfying $|\lambda| > \frac{1}{2}$ and any $k > 0$, the operator $\lambda I + (K^k_\eta)^*$ defined from $L^2(\partial\Omega)$ into $L^2(\partial\Omega)$ is invertible.*

Proof. It has been proved by Kellog in [22] that when $|\lambda| > \frac{1}{2}$, the operator $\lambda I + (K^0_\eta)^*$ is invertible on $L^2(\partial\Omega)$. Lemma 4.1 shows that $(K^k_\eta)^* - (K^0_\eta)^*$ is compact on $L^2(\partial\Omega)$. Therefore the Fredholm alternative holds. It remains then to prove the injectivity of $\lambda I + (K^k_\eta)^*$. Let us suppose that we have $\varphi \in L^2(\partial\Omega)$ satisfying

$$\left(\lambda I + (K^k_\eta)^*\right)\varphi = 0 \ .$$

Define u on $\mathbb{R}^2$ by $u(x) = S^k_\eta \varphi(x)$. It is clear that u satisfies the Helmholtz equation in $\mathbb{R}^2 \setminus \partial\Omega_\eta$ together with the radiation condition as $|x| \to +\infty$. Moreover, it can be easily seen that

$$u_+ = u_- \quad \text{on } \partial\Omega_\eta \ , \tag{4.23}$$

$$\frac{1}{-\frac{1}{2} - \lambda}\partial_\nu(u)_- = \frac{1}{\frac{1}{2} - \lambda}\partial_\nu(u)_+ = \varphi \quad \text{on } \partial\Omega_\eta \ . \tag{4.24}$$

Consequently

$$\begin{aligned} \Im \int_{\partial\Omega_\eta} \partial_\nu(u)_+ \overline{u} &= \frac{\frac{1}{2} - \lambda}{-\frac{1}{2} - \lambda} \Im \int_{\partial\Omega_\eta} \partial_\nu(u)_- \overline{u} \\ &= \frac{\frac{1}{2} - \lambda}{-\frac{1}{2} - \lambda} \Im \int_{\partial\Omega_\eta} (\Delta u \overline{u} + |\nabla u|^2) = 0 \ . \end{aligned}$$

Applying Lemma A.1.2, we obtain that $u \equiv 0$ in Ω^e_η. From the expression of $\partial_\nu(u)_+$ we finally conclude that $\varphi \equiv 0$ which ends the proof. $\square$

5 Representation formula for the fundamental field

In this section, we state and prove a representation formula of the solution of (2.6) which will be the main tool for deriving the asymptotic expansions of the fundamental and second-harmonic fields. A similar representation formula for the transmission problem for the harmonic equation was found in [19, 20]. See also [3].

By X and Y let us denote

$$X := L^2(\partial\Omega)^2\ , \quad Y := H^1(\partial\Omega) \times L^2(\partial\Omega).$$

The following theorem is of particular importance to us for establishing our representation formula.

Theorem 5.1 *Suppose k_0^2, k_2^2 are not Dirichlet eigenvalues for $-\Delta$ on Ω. There exists $\delta_0 > 0$ such that, for $0 < \delta < \delta_0$, for each $(f_1, f_2, g_1, g_2) \in Y^2$, there exists a unique solution $\Phi = (\varphi_1, \varphi_2, \psi_2, \varphi_0) \in X^2$ to the system of integral equations*

$$\begin{cases} S^{k_1}_{0,0}\varphi_1 - S^{k_2}_{0,0}\varphi_2 - S^{k_2}_{\delta,0}\psi_2 = f_1, \\ \dfrac{1}{k_1^2}\dfrac{\partial(S^{k_1}_{0,0}\varphi_1)_-}{\partial\nu} - \dfrac{1}{k_2^2}\dfrac{\partial(S^{k_2}_{0,0}\varphi_2)_+}{\partial\nu} - \dfrac{1}{k_2^2}\dfrac{\partial(S^{k_2}_{\delta,0}\psi_2)}{\partial\nu} = f_2, \\ S^{k_2}_{0,\delta}\varphi_2 + S^{k_2}_{\delta,\delta}\psi_2 - S^{k_0}_{\delta,\delta}\varphi_0 = g_1, \\ \dfrac{1}{k_2^2}\dfrac{\partial(S^{k_2}_{0,\delta}\varphi_2)}{\partial\nu} + \dfrac{1}{k_2^2}\dfrac{\partial(S^{k_2}_{\delta,\delta}\psi_2)_-}{\partial\nu} - \dfrac{1}{k_0^2}\dfrac{\partial(S^{k_0}_{\delta,\delta}\varphi_0)_+}{\partial\nu} = g_2 \end{cases} \tag{5.25}$$

on $\partial\Omega$.

To prove this theorem, we need some preliminary results. First, define the operator T from X^2 into Y^2 by $T(\Phi) = (f_1, f_2, g_1, g_2)$ where (f_1, f_2, g_1, g_2) is given as in (5.25), and let the operator T_0 from X^2 into Y^2 be given by $T_0(\Phi) = (f_1, f_2, g_1, g_2)$ where

$$\begin{cases} S^{k_2}_{0,0}\varphi_1 - S^{k_2}_{0,0}\varphi_2 = f_1, \\ \dfrac{1}{k_1^2}\dfrac{\partial(S^{k_2}_{0,0}\varphi_1)_-}{\partial\nu} - \dfrac{1}{k_2^2}\dfrac{\partial(S^{k_2}_{0,0}\varphi_2)_+}{\partial\nu} = f_2, \\ S^{k_2}_{\delta,\delta}\psi_2 - S^{k_2}_{\delta,\delta}\varphi_0 = g_1, \\ \dfrac{1}{k_2^2}\dfrac{\partial(S^{k_2}_{\delta,\delta}\psi_2)_-}{\partial\nu} - \dfrac{1}{k_0^2}\dfrac{\partial(S^{k_2}_{\delta,\delta}\varphi_0)_+}{\partial\nu} = g_2 \end{cases}$$

on $\partial\Omega$.

Then the following lemma holds.

Lemma 5.1 *The operator $T_0 : X^2 \to Y^2$ is invertible.*

Proof. Let us solve the equation $T_0(\Phi) = (f_1, f_2, g_1, g_2)$. Since the two first equations are decoupled from the two last ones, we start by solving the system

$$\left\{\begin{array}{l} S^{k_2}_{0,0}\varphi_1 - S^{k_2}_{0,0}\varphi_2 = f_1 \\ \dfrac{1}{k_1^2}\dfrac{\partial(S^{k_2}_{0,0}\varphi_1)_-}{\partial\nu} - \dfrac{1}{k_2^2}\dfrac{\partial(S^{k_2}_{0,0}\varphi_2)_+}{\partial\nu} = f_2 \end{array}\right. \quad \text{on } \partial\Omega.$$

Since k_2^2 is not a Dirichlet eigenvalue of $-\Delta$ in Ω, the operator $S^{k_2}_{0,0} : L^2(\partial\Omega) \to H^1(\partial\Omega)$ is invertible and so we have

$$\varphi_1 = \varphi_2 + \left(S^{k_2}_{0,0}\right)^{-1} f_1 .$$

Substituting this into the second equation, we get

$$\left(\frac{1}{k_1^2} - \frac{1}{k_2^2}\right)\left(\lambda + \left(K^{k_2}_{0,0}\right)^*\right)\varphi_2 = f_2 - \frac{1}{k_1^2}\left(-\frac{1}{2} + \left(K^{k_2}_{0,0}\right)^*\right)\left(S^{k_2}_{0,0}\right)^{-1} f_1 ,$$

where λ is given by

$$\lambda = -\frac{1}{2}\frac{\frac{1}{k_1^2} + \frac{1}{k_2^2}}{\frac{1}{k_1^2} - \frac{1}{k_2^2}}.$$

Since $|\lambda| > \dfrac{1}{2}$ for any positive constants $k_1 \neq k_2$, applying Lemma 4.5 yields that $\lambda I + \left(K^{k_2}_{0,0}\right)^* : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is invertible. We can then express φ_2 in terms of (f_1, f_2), and the expression of φ_1 follows immediately.

On the other hand, it is well-known that for δ small enough, k_2^2 is not a Dirichlet eigenvalue of $-\Delta$ in $\overline{\Omega} \cup \mathcal{O}_\delta$. See, for example, [21]. We can then express analogously (ψ_2, φ_0) in terms of (g_1, g_2). □

Lemma 5.2 *The operator $T - T_0 : X^2 \to Y^2$ is compact.*

Proof. Let $\Phi = (\varphi_1, \varphi_2, \psi_2, \varphi_0) \in X^2$, then $(T - T_0)\Phi$ is given by

$$(T - T_0)\Phi = \begin{pmatrix} \left(S^{k_1}_{0,0} - S^{k_2}_{0,0}\right)\varphi_1 - S^{k_2}_{\delta,0}\psi_2 \\ \dfrac{1}{k_1^2}\dfrac{\partial\left((S^{k_1}_{0,0} - S^{k_2}_{0,0})\varphi_1\right)_-}{\partial\nu} - \dfrac{1}{k_2^2}\dfrac{\partial S^{k_2}_{\delta,0}\psi_2}{\partial\nu} \\ S^{k_2}_{0,\delta}\varphi_2 - \left(S^{k_0}_{\delta,\delta} - S^{k_2}_{\delta,\delta}\right)\varphi_0 \\ \dfrac{1}{k_2^2}\dfrac{\partial S^{k_2}_{0,\delta}\varphi_2}{\partial\nu} - \dfrac{1}{k_0^2}\dfrac{\partial\left((S^{k_0}_{\delta,\delta} - S^{k_2}_{\delta,\delta})\varphi_0\right)_+}{\partial\nu} \end{pmatrix}.$$

Since $\Gamma_k - \Gamma_{k_1}$ is smooth, we can easily see that $S^k_{\eta,\eta} - S^{k_2}_{\eta,\eta} : L^2(\partial\Omega) \to H^1(\partial\Omega)$ is a compact operator, and so is $\dfrac{\partial(S^k_{\eta,\eta})_\pm}{\partial\nu} - \dfrac{\partial(S^{k_2}_{\eta,\eta})_\pm}{\partial\nu} : L^2(\partial\Omega) \to L^2(\partial\Omega)$. It is also clear, since the layer potential of a curve is smooth away from its curve, that $S^k_{0,\delta}, S^k_{\delta,0} : L^2(\partial\Omega) \to H^1(\partial\Omega)$ and $\dfrac{\partial S^k_{\delta,0}}{\partial\nu}, \dfrac{\partial S^k_{0,\delta}}{\partial\nu} : L^2(\partial\Omega) \to L^2(\partial\Omega)$ are compact operators which ends the proof. □

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. Since T_0 is invertible and $T-T_0$ is compact, the Fredholm alternative holds and existence follows from uniqueness.

Let $\Phi = (\varphi_1, \varphi_2, \psi_2, \varphi_0) \in X^2$ satisfy $T\Phi = 0$. Consider the function u defined as follows

$$u(x) = \begin{cases} S_0^{k_1}\varphi_1(x) & x \in \Omega \, , \\ S_0^{k_2}\varphi_2(x) + S_\delta^{k_2}\psi_2(x) & x \in \mathcal{O}_\delta \, , \\ S_\delta^{k_0}\varphi_0(x) & x \in \Omega_\delta^e . \end{cases}$$

This function satisfies the equations in (2.6) with the incident field $u_I \equiv 0$. Moreover,

$$\begin{aligned} \int_{\partial\Omega_\delta} \frac{\partial u}{\partial \nu}\bigg|_+ \overline{u}\, ds &= \frac{k_0^2}{k_2^2}\int_{\partial\Omega_\delta} \frac{\partial u}{\partial \nu}\bigg|_- \overline{u}\, ds \\ &= \frac{k_0^2}{k_2^2}\int_{\mathcal{O}_\delta} \left(|\nabla u|^2 - k_2^2|u|^2\right)\, dy + \frac{k_0^2}{k_2^2}\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\bigg|_+ \overline{u}\, ds \\ &= \frac{k_0^2}{k_2^2}\int_{\mathcal{O}_\delta} \left(|\nabla u|^2 - k_2^2|u|^2\right)\, dy + \frac{k_0^2}{k_1^2}\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\bigg|_- \overline{u}\, ds \\ &= \frac{k_0^2}{k_2^2}\int_{\mathcal{O}_\delta} \left(|\nabla u|^2 - k_2^2|u|^2\right)\, dy + \frac{k_0^2}{k_1^2}\int_{\Omega} \left(|\nabla u|^2 - k_1^2|u|^2\right)\, dy. \end{aligned}$$

Thus

$$\Im \int_{\partial\Omega_\delta} \frac{\partial u}{\partial \nu}\bigg|_+ \overline{u}\, ds = 0.$$

Since u satisfies the radiation condition, using Lemma A.1.2 we deduce that $u \equiv 0$ in Ω_δ^e. Then, u satisfies the Helmholtz equation in $\mathcal{O}_\delta$ with $u = \frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega_\delta$. By the unique continuation theorem, we conclude that $u \equiv 0$ in $\mathcal{O}_\delta$ and in the same way we get $u \equiv 0$ in Ω.

Now let us define $\tilde{u}$ by

$$\tilde{u}(x) = S_\delta^{k_0}\varphi_0(x) \quad \text{ for } x \in \mathbb{R}^2 .$$

Then $\tilde{u}$ is a solution to $\Delta\tilde{u} + k_0^2\tilde{u} = 0$ in Ω_δ^i with the Dirichlet boundary condition. Since k_0^2 is not a Dirichlet eigenvalue for $-\Delta$ on Ω, there exists $\delta_0 > 0$ such that, for $0 \le \delta \le \delta_0$, k_0^2 is not a Dirichlet eigenvalue for $-\Delta$ on Ω_δ^i, and for such δ, we have necessarily $\tilde{u} = 0$ in Ω_δ^i. From the jump of the normal derivative, we obtain

$$\varphi_0 = \frac{\partial \tilde{u}}{\partial \nu}\bigg|_+ - \frac{\partial \tilde{u}}{\partial \nu}\bigg|_- = 0 \quad \text{on } \partial\Omega_\delta .$$

Consider now the function $\tilde{v}$ defined by

$$\tilde{v}(x) := S_0^{k_2}\varphi_2(x) + S_\delta^{k_2}\psi_2(x) \quad \text{ for } x \in \mathbb{R}^2 ,$$

which satisfies the Helmholtz equation on $\Omega \cup \mathcal{O}_\delta \cup \Omega_\delta^e$ together with the radiation condition. Since $\tilde{v}(x) = 0$ on $\partial\Omega_\delta$, it follows by Lemma A.1.2 that $\tilde{v} = 0$ in Ω_δ^e.

We also notice that $\tilde{v} = 0$ on $\partial\Omega$. Since k_2^2 is not a Dirichlet eigenvalue for $-\Delta$ on Ω, $\tilde{v} = 0$ in Ω, and so $\tilde{v} \equiv 0$ in $\mathbb{R}^2$. Then, we get

$$\varphi_2 = \frac{\partial(\tilde{v})_+}{\partial\nu} - \frac{\partial(\tilde{v})_-}{\partial\nu} = 0 \quad \text{on } \partial\Omega.$$

$$\psi_2 = \frac{\partial(\tilde{v})_+}{\partial\nu} - \frac{\partial(\tilde{v})_-}{\partial\nu} = 0 \quad \text{on } \partial\Omega_\delta.$$

Define $\hat{v}(x) = S_0^{k_1}\varphi_1(x)$ in $\mathbb{R}^2$. It is already proved that $\hat{v}(x) = 0$ and

$$\frac{\partial(S_0^{k_1}\varphi_1)_-}{\partial\nu}(x) = 0$$

on $\partial\Omega$. We deduce by Lemma A.1.2 that $\dfrac{\partial(S_0^{k_1}\varphi_1)_+}{\partial\nu}(x) = 0$ on $\partial\Omega$. It then follows that

$$\varphi_0 = \frac{\partial(S_0^{k_1}\varphi_1)_+}{\partial\nu} - \frac{\partial(S_0^{k_1}\varphi_1)_-}{\partial\nu} = 0 \quad \text{on } \partial\Omega.$$

This ends the proof of the theorem. □

At this point we have all the necessary ingredients to state and prove the following representation formula.

Theorem 5.2 *Suppose k_0^2, k_2^2 are not Dirichlet eigenvalues for $-\Delta$ on Ω. There exists $\delta_0 > 0$ such that for $0 < \delta < \delta_0$, if u is the solution of the problem (2.6) and $\Phi = (\varphi_1, \varphi_2, \psi_2, \varphi_0) \in X^2$ is the unique solution of*

$$\begin{cases} S_{0,0}^{k_1}\varphi_1 - S_{0,0}^{k_2}\varphi_2 - S_{\delta,0}^{k_2}\psi_2 = 0, \\ \dfrac{1}{k_1^2}\dfrac{\partial(S_{0,0}^{k_1}\varphi_1)_-}{\partial\nu} - \dfrac{1}{k_2^2}\dfrac{\partial(S_{0,0}^{k_2}\varphi_2)_+}{\partial\nu} - \dfrac{1}{k_2^2}\dfrac{\partial(S_{\delta,0}^{k_2}\psi_2)}{\partial\nu} = 0, \\ S_{0,\delta}^{k_2}\varphi_2 + S_{\delta,\delta}^{k_2}\psi_2 - S_{\delta,\delta}^{k_0}\varphi_0 = u_I(x+\delta\nu(x)), \\ \dfrac{1}{k_2^2}\dfrac{\partial(S_{0,\delta}^{k_2}\varphi_2)}{\partial\nu} + \dfrac{1}{k_2^2}\dfrac{\partial(S_{\delta,\delta}^{k_2}\psi_2)_-}{\partial\nu} - \dfrac{1}{k_0^2}\dfrac{\partial(S_{\delta,\delta}^{k_0}\varphi_0)_+}{\partial\nu} = \dfrac{1}{k_0^2}\dfrac{\partial u_I}{\partial\nu}(x+\delta\nu(x)), \end{cases} \tag{5.26}$$

where $x \in \partial\Omega$, then u can be represented as

$$u(x) = \begin{cases} S_0^{k_1}\varphi_1(x) & \text{for } x \in \Omega\ , \\ S_0^{k_2}\varphi_2(x) + S_\delta^{k_2}\psi_2(x) & \text{for } x \in \mathcal{O}_\delta \\ u_I + S_\delta^{k_0}\varphi_0(x)\ , & \text{for } x \in \Omega_\delta^e\ . \end{cases} \tag{5.27}$$

Proof. In fact, the function defined as in (5.27) clearly satisfies the Helmholtz equations, the transmission conditions and the radiation condition in (2.6). □

6 Regularity result

In order to rigorously derive the asymptotic expansions of the fundamental and the second-harmonic fields u and v, we will need to prove a more refined regularity result for the solution Φ in X^2 of the system of integral equations (5.26).

Lemma 6.1 *Let Φ be the solution in X^2 of the system of integral equations (5.26), then $\Phi \in \left(\mathcal{C}^1(\partial\Omega)\right)^4$.*

Proof. From (5.26), we have

$$T_0\Phi = \begin{pmatrix} S^{k_2}_{\delta,0}\psi_2 + (S^{k_2}_{0,0} - S^{k_1}_{0,0})\varphi_1 \\ \dfrac{1}{k_2^2}\dfrac{\partial S^{k_2}_{\delta,0}\psi_2}{\partial\nu} + \dfrac{1}{k_2^2}\left((K_0^{k_2})^* - (K_0^{k_1})^*\right)\varphi_1 \\ -S^{k_2}_{0,\delta}\varphi_2 + \left(S^{k_0}_{\delta,\delta} - S^{k_2}_{\delta,\delta}\right)\varphi_0 + u_I(x+\delta\nu(x)) \\ -\dfrac{1}{k_2^2}\dfrac{\partial S^{k_2}_{0,\delta}\varphi_2}{\partial\nu} + \dfrac{1}{k_0^2}\left((K_\delta^{k_0})^* - (K_\delta^{k_2})^*\right)\varphi_0 + \dfrac{1}{k_0^2}\dfrac{\partial u_I}{\partial\nu}(x+\delta\nu(x)) \end{pmatrix}.$$

Since $\partial\Omega$ is $\mathcal{C}^2$ we can deduce from the right-hand side of the previous identity that $T_0\Phi \in \left(\mathcal{C}^2(\partial\Omega)\times\mathcal{C}^1(\partial\Omega)\right)^2$. Another immediate consequence of the regularity of $\partial\Omega$ is that the following operators

$$\begin{aligned} S^k_{0,0} &: \mathcal{C}^1(\partial\Omega) \to \mathcal{C}^2(\partial\Omega) , \\ \lambda I + (K^k_{0,0})^* &: \mathcal{C}^1(\partial\Omega) \to \mathcal{C}^1(\partial\Omega) , \end{aligned}$$

with $|\lambda| > \frac{1}{2}$ and $k > 0$, are invertible with bounded inverse. The proof can be found, for example, in [22].

From the expression of $(T_0)^{-1}$ in the proof of Lemma 5.1 we can then deduce that $\Phi \in \left(\mathcal{C}^1(\partial\Omega)\right)^4$. □

As a direct consequence of the previous lemma and the following integral representation for u in $\mathcal{O}_\delta$:

$$u(x) = S_0^{k_2}\varphi_2(x) + S_\delta^{k_2}\psi_2(x) ,$$

the following regularity result holds.

Corollary 6.1 *Let u be the solution to the problem (2.6). Then $\nabla u \in L^\infty(\overline{\mathcal{O}}_\delta)$.*

This result is important to us for establishing the well-posedness of problem (2.7).

Corollary 6.2 *Let Φ be the solution in X^2 of the system (5.26), then $\Phi \in \left(H^1(\partial\Omega)\right)^4$.*

With this higher regularity for Φ, we can define higher-order derivatives of the single layer potential. The following lemma holds.

Lemma 6.2 *Let $\varphi \in H^1(\partial\Omega)$. Then the first and second order normal derivatives of the single layer potential exist and are continuous from $H^1(\partial\Omega)$ into $H^1(\partial\Omega)$ and $L^2(\partial\Omega)$, respectively. In particular, we have*

$$\begin{aligned}
S^k_{0,\delta}\varphi &= S^k_{0,0}\varphi + \delta\frac{\partial(S^k_{0,0}\varphi)_+}{\partial\nu} + o(\delta)\ , \\
\frac{\partial S^k_{0,\delta}\varphi}{\partial\nu} &= \frac{\partial(S^k_{0,0}\varphi)_+}{\partial\nu} + \delta\frac{\partial^2(S^k_{0,0}\varphi)_+}{\partial\nu^2} + o(\delta)\ , \\
S^k_{\delta,0}\varphi &= S^k_{\delta,\delta}\varphi + \delta\frac{\partial(S^k_{0,0}\varphi)_-}{\partial\nu} + o(\delta)\ , \\
\frac{\partial S^k_{\delta,0}\varphi}{\partial\nu} &= \frac{\partial(S^k_{\delta,\delta}\varphi)_-}{\partial\nu} + \delta\frac{\partial^2(S^k_{0,0}\varphi)_-}{\partial\nu^2} + o(\delta)\ ,
\end{aligned}$$

where $o(\delta)$ is in $H^1(\partial\Omega)$ in the first and third equations and in $L^2(\partial\Omega)$ in the remaining ones.

7 Asymptotic expansion of the fundamental field

Given sufficient regularity of $\partial\Omega$, we rigorously establish in this section the asymptotic formula (2.11) for the fundamental field u. We first introduce some notations. Define the operators Q_δ, R_δ and W_δ from X into Y by

$$\begin{aligned}
Q_\delta(\varphi,\psi) &:= \left(S^{k_1}_{0,0}\varphi - S^{k_0}_{\delta,\delta}\psi,\ \frac{1}{k_1^2}\frac{\partial(S^{k_1}_{0,0}\varphi)_-}{\partial\nu} - \frac{1}{k_0^2}\frac{\partial(S^{k_0}_{\delta,\delta}\psi)_+}{\partial\nu}\right)\ , \\
R_\delta(\varphi,\psi) &:= \Bigg((S^{k_2}_{0,\delta} - S^{k_2}_{0,0})\varphi + (S^{k_2}_{\delta,\delta} - S^{k_2}_{\delta,0})\psi, \\
&\qquad \frac{1}{k_2^2}\frac{\partial}{\partial\nu}\Big((S^{k_2}_{0,\delta}\varphi) - (S^{k_2}_{0,0}\varphi)_+ + (S^{k_2}_{\delta,\delta}\psi)_- - (S^{k_2}_{\delta,0}\psi)\Big)\Bigg)\ , \\
W_\delta(\varphi,\psi) &:= \left(S^{k_2}_{0,0}\varphi + S^{k_2}_{\delta,0}\psi,\ \frac{1}{k_2^2}\frac{\partial(S^{k_2}_{0,0}\varphi)_+}{\partial\nu} + \frac{1}{k_2^2}\frac{\partial(S^{k_2}_{\delta,0}\psi)}{\partial\nu}\right)\ , \\
W_0(\varphi,\psi) &:= \left(S^{k_2}_{0,0}\varphi + S^{k_2}_{0,0}\psi,\ \frac{1}{k_2^2}\frac{\partial(S^{k_2}_{0,0}\varphi)_+}{\partial\nu} + \frac{1}{k_2^2}\frac{\partial(S^{k_2}_{0,0}\psi)_-}{\partial\nu}\right)\ ,
\end{aligned}$$

and the function U^δ_I on $\partial\Omega$ by

$$U^\delta_I(x) := \left(u_I(x+\delta\nu(x)),\ \frac{1}{k_0^2}\frac{\partial u_I}{\partial\nu}(x+\delta\nu(x))\right)\ .$$

The following lemma holds.

Lemma 7.1 *Let $\Phi = (\varphi_1, \varphi_2, \psi_2, \varphi_0) \in X^2$ be the unique solution of (5.26), then (φ_1, φ_0) and (φ_2, ψ_2) are solutions of the following equations*

$$\begin{aligned} Q_\delta(\varphi_1, \varphi_0) &= U_I^\delta - R_\delta(\varphi_2, \psi_2)\ , \\ W_\delta(\varphi_2, \psi_2) &= \left((S_{0,0}^{k_1}\varphi_1), \frac{1}{k_1^2}\frac{\partial(S_{0,0}^{k_1}\varphi_1)_-}{\partial\nu} \right)\ . \end{aligned} \tag{7.28}$$

In order to expand Φ with respect to the thickness δ, we need to prove a stability result when δ goes to 0, provided $\partial\Omega$ is sufficiently regular.

Proposition 7.1 *The operators Q_δ and W_δ converge uniformly to Q_0 and W_0 respectively. Moreover, assuming $\partial\Omega$ of class $\mathcal{C}^3$, Q_0 and W_0 are invertible from X into Y and from $(H^1(\partial\Omega))^2$ into $H^2(\partial\Omega) \times H^1(\partial\Omega)$.*

Proof. The uniform convergence is a consequence of Lemmas 4.3 and 4.4. Let us prove the invertibility of Q_0. We write Q_0 as

$$\begin{aligned} Q_0(\varphi, \psi) &= \left(S_{0,0}^{k_1}\varphi - S_{0,0}^{k_1}\psi, \frac{1}{k_1^2}\frac{\partial(S_{0,0}^{k_1}\varphi)_-}{\partial\nu} - \frac{1}{k_0^2}\frac{\partial(S_{0,0}^{k_1}\psi)_+}{\partial\nu} \right) \\ &\quad + \Bigg(-(S_{0,0}^{k_0} - S_{0,0}^{k_1})\psi, \\ &\quad -\frac{1}{k_0^2}\frac{\partial((S_{0,0}^{k_0} - S_{0,0}^{k_1})\psi)_+}{\partial\nu} \Bigg)\ , \end{aligned}$$

where the first operator is invertible and the second one is compact. The Fredholm alternative holds. It remains then to prove the injectivity of Q_0. Let $(\varphi, \psi) \in X$ be satisfying $Q_0(\varphi, \psi) = 0$. We define u in $\mathbb{R}^2$ by

$$u(x) = \begin{cases} S_0^{k_1}\varphi(x) & \text{for } x \in \Omega, \\ S_0^{k_0}\psi(x) & \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega}. \end{cases}$$

In a similar way as for Theorem 5.1, we prove that

$$\Im \int_{\partial\Omega} \frac{\partial(u)_+}{\partial\nu}\overline{u}\, ds = 0\ ,$$

from which we get that, since u satisfies the outgoing radiation condition, $u \equiv 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$ and so, by the unique continuation theorem, we obtain that $u \equiv 0$ in $\mathbb{R}^2$. Since k_0^2 is not a Dirichlet eigenvalue for $-\Delta$ on Ω, it follows that $S_0^{k_0}\psi \equiv 0$ in $\mathbb{R}^2$ and from the jump of its normal derivative on $\partial\Omega$, we can deduce that $\psi \equiv 0$. We prove in a similar way that $S_0^{k_1}\varphi \equiv 0$ in $\mathbb{R}^2$ and then from the jump of its normal derivative, we get $\varphi \equiv 0$. The invertibility of Q_0 is then proved.

To prove that W_0 is invertible, let us suppose that we have $(\varphi, \psi) \in X^2$ and $(f, g) \in Y$ satisfying

$$\begin{cases} S_{0,0}^{k_2}\varphi + S_{0,0}^{k_2}\psi & = f\ , \\ \dfrac{1}{k_2^2}\dfrac{\partial(S_{0,0}^{k_2}\varphi)_+}{\partial\nu} + \dfrac{1}{k_2^2}\dfrac{\partial(S_{0,0}^{k_2}\psi)_-}{\partial\nu} & = g\ . \end{cases}$$

Then we deduce from the first equation, since k_2^2 is not a Dirichlet eigenvalue of $-\Delta$ on Ω, that

$$\varphi + \psi = (S_{0,0}^{k_2})^{-1} f \ .$$

Inserting this into the second equation together with the expression of the jump of the normal derivative of the single layer potential, we obtain

$$\varphi = k_2^2 g - (-\frac{1}{2} + (K_{0,0}^{k_2})^*)(S_{0,0}^{k_2})^{-1} f),$$

which gives us the expression of ψ and proves the invertibility of W_0.

The invertibility of Q_0 and W_0 from $(H^1(\partial\Omega))^2$ into $H^2(\partial\Omega) \times H^1(\partial\Omega)$ can be proved in the same way. □

Proposition 7.2 *Suppose that $\partial\Omega$ is of class $\mathcal{C}^3$. Let $\Phi = (\varphi_1, \varphi_2, \psi_2, \varphi_0) \in X^2$ be the unique solution of (5.26), then there exists a constant $C > 0$ such that*

$$\|\Phi\|_{H^1(\partial\Omega)^4} \leq C \ .$$

Proof. Since W_δ converges uniformly to W_0 and since W_0 is invertible from $(H^1(\partial\Omega))^2$ into $H^2(\partial\Omega) \times H^1(\partial\Omega)$, then in view of (7.28) it can be seen that there exist two constants $C, C_1 > 0$ such that

$$\begin{aligned}
\|(\varphi_2, \psi_2)\|_{(H^1(\partial\Omega))^2} &\leq C_1 \left\| \left(S_{0,0}^{k_1}\varphi_1, \frac{\partial (S_{0,0}^{k_1}\varphi_1)_-}{\partial \nu} \right) \right\|_{H^2(\partial\Omega)\times H^1(\partial\Omega)} \\
&\leq C \|\varphi_1\|_{H^1(\partial\Omega)} \ .
\end{aligned}$$

Combining the facts that Q_δ converges to Q_0 uniformly together with the fact that Q_0 is invertible, we show that there exist constants $C', C_1' > 0$ and ε_δ small such that for δ small enough, we have from Lemma 4.3 that

$$\begin{aligned}
\|(\varphi_1, \varphi_0)\|_{(H^1(\partial\Omega))^2} &\leq C_1' \left\| U_I^\delta + R_\delta(\varphi_2, \psi_2) \right\|_{H^2(\partial\Omega)\times H^1(\partial\Omega)} \\
&\leq C_1 + \varepsilon_\delta \|\varphi_1\|_{H^1(\partial\Omega)} \ .
\end{aligned}$$

Here $\varepsilon_\delta \to 0$ as $\delta \to 0$. It then follows that φ_1 and φ_0 are bounded in $H^1(\partial\Omega)$ which also implies that φ_2 and ψ_2 are bounded in $H^1(\partial\Omega)$. □

Proposition 7.3 *Let $\Phi^\delta = (\varphi_1^\delta, \varphi_2^\delta, \psi_2^\delta, \varphi_0^\delta) \in X^2$ be the unique solution of (5.26), then $(\varphi_1^\delta, \varphi_0^\delta)$ and $(\varphi_2^\delta, \psi_2^\delta)$ converge to $(\varphi_1^0, \varphi_0^0)$ and (φ_2^0, ψ_2^0) respectively in $(H^1(\partial\Omega))^2$ where $(\varphi_1^\delta, \varphi_0^\delta, \varphi_2^\delta, \psi_2^\delta)$ are the unique solutions to the decoupled systems of integral equations*

$$Q_0(\varphi_1^0, \varphi_0^0) = U_I^0 \ ,$$

and

$$W_0(\varphi_2^0, \psi_2^0) = \left((S_{0,0}^{k_1}\varphi_1^0), \frac{1}{k_1^2} \frac{\partial (S_{0,0}^{k_1}\varphi_1^0)_-}{\partial \nu} \right) \ . \tag{7.29}$$

Proof. Recalling that φ_2^δ and ψ_2^δ are bounded in $H^1(\partial\Omega)$, we have

$$U_I^\delta - R_\delta(\varphi_2, \psi_2) \to U_I^0$$

uniformly in $H^2(\partial\Omega) \times H^1(\partial\Omega)$. Since Q_δ converges uniformly to Q_0, $(\varphi_1^\delta, \varphi_0^\delta)$ converges to $(\varphi_1^0, \varphi_0^0)$ in $(H^1(\partial\Omega))^2$. It follows that, since W_δ converges uniformly to W_0, $(\varphi_2^\delta, \psi_2^\delta)$ converges to (φ_2^0, ψ_2^0) in $(H^1(\partial\Omega))^2$ which ends the proof of the proposition. $\square$

It is worth noticing that the limit $(\varphi_1^0, \varphi_0^0)$ represents the solution of the problem without the thin coating. In fact, if we define U by

$$U(x) := \begin{cases} S_0^{k_1}\varphi_1^0(x) & \text{for } x \in \Omega\,, \\ S_0^{k_0}\varphi_0^0(x) + u_I(x) & \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega}\,, \end{cases} \tag{7.30}$$

then U is the unique solution to the problem (2.8).

The following proposition is a direct consequence of Lemmas 4.3 and 6.2.

Proposition 7.4 *The following expansions hold.*

$$\begin{aligned} Q_\delta(\varphi, \psi) &= Q_0(\varphi, \psi) - \delta Q_1(\psi) + O(\delta^2)\,, \quad \forall\, \varphi, \psi \in H^1(\partial\Omega)\,, && (7.31)\\ R_\delta(\varphi, \psi) &= \delta R_1(\varphi, \psi) + o(\delta)\,, \quad \forall\, \varphi, \psi \in H^1(\partial\Omega)\,, && (7.32) \end{aligned}$$

where the remainder terms $O(\delta^2)$ and $o(\delta)$ are in $H^1(\partial\Omega) \times L^2(\partial\Omega)$, and

$$\begin{aligned} R_1(\varphi, \psi) \;:=\; & \Bigg(\frac{\partial (S_{0,0}^{k_2}\varphi)_+}{\partial \nu} + \frac{\partial (S_{0,0}^{k_2}\psi)_-}{\partial \nu}, && (7.33)\\ & -\frac{1}{k_2^2}\left(\rho \frac{\partial (S_{0,0}^{k_2}\varphi)_+}{\partial \nu} + \rho \frac{\partial (S_{0,0}^{k_2}\psi)_-}{\partial \nu} + \frac{\partial^2 S_{0,0}^{k_2}(\varphi+\psi)}{\partial \tau^2} + k_2^2 S_{0,0}^{k_2}(\varphi+\psi) \right) \Bigg)\,, \\ Q_1(\varphi) \;:=\; & \Bigg(\frac{\partial (S_{0,0}^{k_0}\varphi)_+}{\partial \nu} + S_{0,0}^{k_0}(\rho\varphi) + (D_{0,0}^{k_0}\varphi)_+, && (7.34)\\ & \frac{1}{k_0^2}\left(-\rho \frac{\partial (S_{0,0}^{k_0}\varphi)_+}{\partial \nu} - \frac{\partial^2 (S_{0,0}^{k_0}\varphi)}{\partial \tau^2} - k_0^2 (S_{0,0}^{k_0}\varphi) + \frac{\partial (S_{0,0}^{k_0}\rho\varphi)_+}{\partial \nu} + \frac{\partial (D_{0,0}^{k_0}\varphi)}{\partial \nu} \right) \Bigg)\,. \end{aligned}$$

Proof. Since, for $\varphi \in H^1(\partial\Omega)$, $S_0^k\varphi$ satisfies the Helmholtz equation in $\mathbb{R}^2 \setminus \partial\Omega$ then

$$\frac{\partial^2 (S_{0,0}^k\varphi)_\pm}{\partial \nu^2} = -\rho \frac{\partial (S_{0,0}^k\varphi)_\pm}{\partial \nu} - \frac{\partial^2 (S_{0,0}^k\varphi)}{\partial \tau^2} - k^2 S_{0,0}^k\varphi \quad \text{on } \partial\Omega,$$

and equation (7.32) follows immediately from Lemma 6.2. Here we have expressed the Laplacian in the local coordinates

$$\Delta = \frac{\partial^2}{\partial \nu^2} + \rho \frac{\partial}{\partial \nu} + \frac{\partial^2}{\partial \tau^2} \quad \text{on } \partial\Omega. \tag{7.35}$$

Applying Lemma 4.3, we obtain (7.31) for $(\varphi, \psi) \in (L^2(\partial\Omega))^2$ with

$$Q_1(\psi) = \left(K_0^k\psi + (K_0^k)^*\psi + S_{0,0}^k(\rho\psi), \frac{1}{k_0^2}(M_0^k\psi + (K_0^k)^*(\rho\psi)) \right)\,.$$

It remains then to prove that for $\psi \in H^1(\partial\Omega)$ this expression is identical to the one defined in (7.34).

In view of identities (4.21) and (4.22), it is easy to see that

$$K_0^k\psi + (K_0^k)^*\psi = \frac{\partial(S_{0,0}^k\psi)_+}{\partial\nu} + (D_{0,0}^k\psi)_+ \ .$$

On the other hand, by using the local coordinates (7.35) it follows that

$$\begin{aligned} M_0^k\psi + (K_0^k)^*(\rho\psi) &= -\rho\frac{\partial(S_{0,0}^{k_0}\varphi)_+}{\partial\nu} - \frac{\partial^2(S_{0,0}^{k_0}\varphi)}{\partial\tau^2} - k_0^2(S_{0,0}^{k_0}\varphi) \\ &\quad + \frac{\partial(S_{0,0}^{k_0}\rho\varphi)_+}{\partial\nu} + \frac{\partial(D_{0,0}^{k_0}\varphi)}{\partial\nu} \ , \end{aligned}$$

as desired. The proof is complete. □

Proposition 7.5 *Let* $(\varphi_1^{1,\delta}, \varphi_0^{1,\delta}) \in X$ *be defined as*

$$(\varphi_1^{1,\delta}, \varphi_0^{1,\delta}) := \left(\frac{\varphi_1^\delta - \varphi_1^0}{\delta}, \frac{\varphi_0^\delta - \varphi_0^0}{\delta}\right) \ .$$

Then, for $\delta \to 0$, $(\varphi_1^{1,\delta}, \varphi_0^{1,\delta})$ *converges in* X *to the pair* $(\varphi_1^{1,0}, \varphi_0^{1,0})$ *which satisfies*

$$Q_0(\varphi_1^{1,0}, \varphi_0^{1,0}) = U_I'(x) + Q_1(\varphi_0^0) - Z(\varphi_1^0) \ , \tag{7.36}$$

where

$$\begin{aligned} U_I'(x) &:= \left(\frac{\partial u_I}{\partial\nu}(x), -\frac{1}{k_0^2}\rho\frac{\partial u_I}{\partial\nu}(x) - \frac{1}{k_0^2}\frac{\partial^2 u_I}{\partial\tau^2}(x) - u_I(x)\right) \ , \quad x \in \partial\Omega \ , \\ Z(\varphi) &:= \left(\frac{k_2^2}{k_1^2}\frac{\partial(S_{0,0}^{k_1}\varphi)_-}{\partial\nu}, -\frac{1}{k_1^2}\rho\frac{\partial(S_{0,0}^{k_1}\varphi)_-}{\partial\nu} - \frac{1}{k_2^2}\frac{\partial^2 S_{0,0}^{k_1}(\varphi)}{\partial\tau^2} - S_{0,0}^{k_1}(\varphi)\right) \ . \end{aligned}$$

Proof. Since $(\varphi_2^\delta, \psi_2^\delta)$ converges to (φ_2^0, ψ_2^0) and $(\varphi_1^\delta, \varphi_0^\delta)$ converges to $(\varphi_1^0, \varphi_0^0)$ in $H^1(\partial\Omega)$, the following equation holds.

$$Q_0(\varphi_1^{1,\delta}, \varphi_0^{1,\delta}) = U_I'(x) + Q_1(\varphi_0^0) - R_1(\varphi_2^0, \psi_2^0) + o(1) \ ,$$

where $o(1)$ is in Y. We can then state that $(\varphi_1^{1,\delta}, \varphi_0^{1,\delta})$ converges to $(\varphi_1^{1,0}, \varphi_0^{1,0})$ that is a solution of

$$Q_0(\varphi_1^{1,0}, \varphi_0^{1,0}) = U_I'(x) + Q_1(\varphi_0^0) - R_1(\varphi_2^0, \psi_2^0) \ .$$

From equation (7.29), we see that

$$\begin{aligned} S_{0,0}^{k_2}(\varphi_2^0 + \psi_2^0) &= S_{0,0}^{k_1}\varphi_1^0 \ , \\ \frac{\partial(S_{0,0}^{k_2}\varphi_2^0)_+}{\partial\nu} + \frac{\partial(S_{0,0}^{k_2}\psi_2^0)_-}{\partial\nu} &= \frac{k_2^2}{k_1^2}\frac{\partial(S_{0,0}^{k_1}\varphi_1^0)_-}{\partial\nu} \ , \end{aligned}$$

which gives

$$R_1(\varphi_2^0, \psi_2^0) = Z(\varphi_1^0) \ .$$

The proposition is then proved. $\square$

Finally, the following proposition provides the expansion of $(\varphi_1^\delta, \varphi_0^\delta)$ as δ goes to zero.

Proposition 7.6 *The following expansions hold.*

$$\begin{aligned} \varphi_1^\delta &= \varphi_1^0 + \delta \varphi_1^{1,0} + o(\delta) , \\ \varphi_0^\delta &= \varphi_0^0 + \delta \varphi_0^{1,0} + o(\delta) . \end{aligned}$$

Proof of formula (2.11) in Theorem 2.1. From the representation formula and the expansion of (φ_1, φ_0) we can write

$$u(x) = U(x) + \delta u_1(x) + o(\delta) ,$$

where

$$u_1(x) = \begin{cases} S_0^{k_1}\varphi_1^{1,0}(x) & \text{for } x \in \Omega, \\ S_0^{k_0}\varphi_0^{1,0}(x) + D_0^{k_0}\varphi_0^0(x) + S_0^{k_0}(\rho\varphi_0^0)(x) & \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega}. \end{cases}$$

Consider now the unique solution to the following problem

$$\begin{cases} \Delta w_1 + k_1^2 w_1 = 0 & \text{for } x \in \Omega, \\ \Delta w_1 + k_0^2 w_1 = 0 & \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega}, \\ [w_1]_{\partial\Omega} = (D_{0,0}^{k_0}\varphi_0^0)_+ + S_{0,0}^{k_0}(\rho\varphi_0^0) & \text{on } \partial\Omega, \\ \left[\dfrac{1}{k^2}\dfrac{\partial w_1}{\partial \nu}\right]_{\partial\Omega} = \dfrac{1}{k_0^2}\dfrac{\partial (S_{0,0}^{k_0}(\rho\varphi_0^0))_+}{\partial \nu} + \dfrac{1}{k_0^2}\dfrac{\partial (D_{0,0}^{k_0}\varphi_0^0)}{\partial \nu} & \text{on } \partial\Omega, \\ w_1 \text{ satisfies the (outgoing) radiation condition.} \end{cases} \tag{7.37}$$

The function w_1 can be expressed using the Green's function G as follows

$$\begin{aligned} w_1(x) \;=\; & -\int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x,y)\left((D_{0,0}^{k_0}\varphi_0^0)_+ + S_{0,0}^{k_0}(\rho\varphi_0^0)\right) ds(y) \\ & + \int_{\partial\Omega} k_0^2 (G(x,y))_+ \left(\frac{1}{k_0^2}\frac{\partial (S_{0,0}^{k_0}(\rho\varphi_0^0))_+}{\partial \nu} + \frac{1}{k_0^2}\frac{\partial (D_{0,0}^{k_0}\varphi_0^0)}{\partial \nu}\right) ds(y). \end{aligned}$$

On the other hand, we can see that

$$w_1(x) = \begin{cases} 0 & \text{for } x \in \Omega , \\ D_0^{k_0}\varphi_0^0(x) + S_0^{k_0}(\rho\varphi_0^0)(x) & \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega} , \end{cases}$$

since this last function satisfies all the conditions in (7.37). We also introduce the function w_2 defined on $\mathbb{R}^2$ by

$$w_2(x) = \begin{cases} S_0^{k_1}\varphi_1^{1,0}(x) & \text{for } x \in \Omega , \\ S_0^{k_0}\varphi_0^{1,0}(x) & \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega} . \end{cases}$$

Since $(\varphi_1^{1,0}, \varphi_0^{1,0})$ satisfies the system (7.36), we can express w_2 using G as follows

$$
\begin{aligned}
w_2(x) \;=\; & \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x,y) \\
& \left(\frac{\partial u_I}{\partial \nu} + \frac{\partial (S_{0,0}^{k_0}\varphi_0^0)_+}{\partial \nu} + S_{0,0}^{k_0}(\rho\varphi_0^0) + (D_{0,0}^{k_0}\varphi_0^0)_+ - \frac{k_2^2}{k_1^2}\frac{\partial (S_{0,0}^{k_1}\varphi_1^0)_-}{\partial \nu} \right) ds(y) \\
& + \int_{\partial\Omega} k_0^2 (G(x,y))_+ \left(\frac{1}{k_0^2}\rho \frac{\partial u_I}{\partial \nu}(x) + \frac{1}{k_0^2}\frac{\partial^2 u_I}{\partial \tau^2}(x) + u_I + \frac{1}{k_0^2}\rho \frac{\partial (S_{0,0}^{k_0}\varphi_0^0)_+}{\partial \nu} \right. \\
& + \frac{1}{k_0^2}\frac{\partial^2 (S_{0,0}^{k_0}\varphi_0^0)}{\partial \tau^2} + (S_{0,0}^{k_0}\varphi_0^0) - \frac{1}{k_0^2}\frac{\partial (S_{0,0}^{k_0}(\rho\varphi_0^0))_+}{\partial \nu} - \frac{1}{k_0^2}\frac{\partial (D_{0,0}^{k_0}\varphi_0^0)}{\partial \nu} \\
& \left. - \frac{1}{k_1^2}\rho \frac{\partial (S_{0,0}^{k_1}\varphi_1^0)_-}{\partial \nu} - \frac{1}{k_2^2}\frac{\partial^2 S_{0,0}^{k_1}(\varphi_1^0)}{\partial \tau^2} - S_{0,0}^{k_1}(\varphi_1^0) \right) ds(y).
\end{aligned}
$$

Since

$$
u_1(x) = w_1(x) + w_2(x) \; ,
$$

we conclude that

$$
\begin{aligned}
u_1(x) \;=\; & (k_0^2 - k_2^2) \int_{\partial\Omega} \left[\frac{\partial G}{\partial \nu}(x,y) \frac{1}{k_0^2} \left(\frac{\partial U}{\partial \nu}(y) \right)_+ \right. \\
& \left. - \left(\frac{1}{k_0^2} - \frac{1}{k_2^2} \right) \int_{\partial\Omega} k_0^2 \left(\frac{\partial G}{\partial \tau}(x,y) \right)_+ \frac{\partial U}{\partial \tau}(y) \right] ds(y) \; ,
\end{aligned}
$$

which ends the proof of the first asymptotic expansion in Theorem 2.1. □

8 Representation formula for the second-harmonic field

In this section, we derive a representation formula for the solution of (2.7). The formula is essentially the same as for the fundamental field. However, we give its proof in order to make sure that the assumptions $k_2 \neq k_0$ and $k_2 \neq k_1$ do not play any role in the proof of Theorem 5.2.

The following holds.

Theorem 8.1 *Suppose $k_0'^2$, $k_2'^2$ are not Dirichlet eigenvalues for $-\Delta$ on Ω. Then, there exists $\delta_0 > 0$ such that, for $0 < \delta < \delta_0$, for each $(f_1, f_2, g_1, g_2) \in Y^2$, there exists a unique solution $\tilde{\Phi} = (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\psi}_2, \tilde{\varphi}_0) \in X^2$ to the system of integral equations*

$$
\left\{
\begin{aligned}
& S_{0,0}^{k_1'}\tilde{\varphi}_1 - S_{0,0}^{k_2'}\tilde{\varphi}_2 - S_{\delta,0}^{k_2'}\tilde{\psi}_2 = f_1, \\
& \frac{\partial (S_{0,0}^{k_1'}\tilde{\varphi}_1)_-}{\partial \nu} - \frac{\partial (S_{0,0}^{k_2'}\tilde{\varphi}_2)_+}{\partial \nu} - \frac{\partial (S_{\delta,0}^{k_2'}\tilde{\psi}_2)}{\partial \nu} = f_2, \\
& S_{0,\delta}^{k_2'}\tilde{\varphi}_2 + S_{\delta,\delta}^{k_2'}\tilde{\psi}_2 - S_{\delta,\delta}^{k_0'}\tilde{\varphi}_0 = g_1, \\
& \frac{\partial (S_{0,\delta}^{k_2'}\tilde{\varphi}_2)}{\partial \nu} + \frac{\partial (S_{\delta,\delta}^{k_2'}\tilde{\psi}_2)_-}{\partial \nu} - \frac{\partial (S_{\delta,\delta}^{k_0'}\tilde{\varphi}_0)_+}{\partial \nu} = g_2.
\end{aligned}
\right. \tag{8.38}
$$

The proof of this theorem is basically the same as the one of Theorem 5.1. First, we define the operator T' from X^2 into Y^2 by $T'(\tilde{\Phi}) = (f_1, f_2, g_1, g_2)$ where (f_1, f_2, g_1, g_2) is given as in (8.38), and the operator T_0' from X^2 into Y^2 by

$$T_0'(\tilde{\Phi}) = \begin{pmatrix} S_{0,0}^{k_2'}\tilde{\varphi}_1 - S_{0,0}^{k_2'}\tilde{\varphi}_2 \\ \dfrac{\partial(S_{0,0}^{k_2'}\tilde{\varphi}_1)_-}{\partial\nu} - \dfrac{\partial(S_{0,0}^{k_2'}\tilde{\varphi}_2)_+}{\partial\nu} \\ S_{\delta,\delta}^{k_2'}\tilde{\psi}_2 - S_{\delta,\delta}^{k_2'}\tilde{\varphi}_0 \\ \dfrac{\partial(S_{\delta,\delta}^{k_2'}\tilde{\psi}_2)_-}{\partial\nu} - \dfrac{\partial(S_{\delta,\delta}^{k_2'}\tilde{\varphi}_0)_+}{\partial\nu} \end{pmatrix}.$$

Then the following lemma holds.

Lemma 8.1 *The operator $T_0' : X^2 \to Y^2$ is invertible.*

Proof. Let us solve the equation $T_0'(\tilde{\Phi}) = (f_1, f_2, g_1, g_2)$. Since the two first equations are decoupled from the two last ones, we start by solving the following system of integral equations

$$\begin{cases} S_{0,0}^{k_2'}\tilde{\varphi}_1 - S_{0,0}^{k_2'}\tilde{\varphi}_2 = f_1, \\ \dfrac{\partial(S_{0,0}^{k_2'}\tilde{\varphi}_1)_-}{\partial\nu} - \dfrac{\partial(S_{0,0}^{k_2'}\tilde{\varphi}_2)_+}{\partial\nu} = f_2. \end{cases}$$

Since $k_2'^2$ is not a Dirichlet eigenvalue of $-\Delta$ in Ω, the operator $S_{0,0}^{k_2'} : L^2(\partial\Omega) \to H^1(\partial\Omega)$ is invertible and we have

$$\tilde{\varphi}_1 = \tilde{\varphi}_2 + \left(S_{0,0}^{k_2'}\right)^{-1} f_1 .$$

Substituting this into the second equation, we readily get

$$\tilde{\varphi}_2 = -f_2 + \left(-\frac{1}{2} + \left(K_0^{k_2'}\right)^*\right)\left(S_{0,0}^{k_2'}\right)^{-1} f_1 .$$

The expression of $\tilde{\varphi}_1$ follows immediately. Analogously, we can easily express $(\tilde{\psi}_2, \tilde{\varphi}_0)$ in terms of (g_1, g_2). □

Lemma 8.2 *The operator $T' - T_0' : X^2 \to Y^2$ is compact.*

Proof. The proof is exactly the same as the one of Lemma 5.2. □

Proof of Theorem 8.1. Since T_0' is invertible and $T' - T_0'$ is compact, the Fredholm alternative holds and existence will follow from uniqueness.

Let $\tilde{\Phi} = (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\psi}_2, \tilde{\varphi}_0) \in X^2$ satisfy $T'\tilde{\Phi} = 0$. Consider the function v defined by

$$v(x) = \begin{cases} S_0^{k_1'}\tilde{\varphi}_1(x) & \text{for } x \in \Omega, \\ S_0^{k_2'}\tilde{\varphi}_2(x) + S_\delta^{k_2'}\tilde{\psi}_2(x) & \text{for } x \in \mathcal{O}_\delta, \\ S_\delta^{k_0'}\tilde{\varphi}_0 & \text{for } x \in \Omega_\delta^e. \end{cases} \tag{8.39}$$

This function satisfies the equations in (2.7) where the source term

$$\sum_{j,l=1,2} \chi_{jl} \partial_{x_j} u \partial_{x_l} u \equiv 0.$$

Moreover, we can easily prove in a similar way as for the fundamental field u that

$$\Im \int_{\partial\Omega_\delta} \frac{\partial v}{\partial \nu} v \, ds = 0 ,$$

from which we obtain, by using Lemma A.1.2, that $v \equiv 0$ in Ω_δ^e, since v satisfies the outgoing radiation condition. Thus, v satisfies the Helmholtz equation in $\mathcal{O}_\delta$ with $v = \partial v / \partial \nu = 0$ on $\partial\Omega_\delta$. By the unique continuation theorem, we deduce that $v \equiv 0$ in $\mathcal{O}_\delta$ and in the same way, we get $v \equiv 0$ in Ω.

Then, as for Theorem 5.1, there exists $\delta_0 > 0$ such that, for $0 \leq \delta \leq \delta_0$, $k_0^{'2}$ is not a Dirichlet eigenvalue for $-\Delta$ on Ω_δ^i, and, for such δ, we have necessarily $S_0^{k_0'} \tilde{\varphi}_0 \equiv 0$ in Ω_δ^i. From the jump of the normal derivative of $S_0^{k_0'} \tilde{\varphi}_0$ on $\partial\Omega_\delta$, we immediately deduce that $\tilde{\varphi}_0 = 0$.

Then we can easily find that $S_0^{k_2'} \tilde{\varphi}_2(x) + S_\delta^{k_2'} \tilde{\psi}_2 \equiv 0$ in Ω_δ^e. The jump of the normal derivative of this function on $\partial\Omega_\delta$ gives $\tilde{\psi}_2 = 0$. Since $k_2^{'2}$ is not a Dirichlet eigenvalue for $-\Delta$ on Ω, we arrive at $S_0^{k_2'} \tilde{\varphi}_2(x) + S_\delta^{k_2'} \tilde{\psi}_2 \equiv 0$ in Ω. From the jump of its normal derivative on $\partial\Omega$, we arrive at $\tilde{\varphi}_2 = 0$.

Finally, since $S_0^{k_1'} \tilde{\varphi}_1$ has a null trace on $\partial\Omega$, we obtain from Lemma A.1.2 that $S_0^{k_1'} \tilde{\varphi}_1 \equiv 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$ and from the jump of its normal derivative on $\partial\Omega$, we deduce that $\tilde{\varphi}_1 = 0$. The uniqueness of $\tilde{\Phi}$ is then proved which ends the proof of the theorem. □

Theorem 8.2 *Suppose $(k_0')^2, (k_2')^2$ are not Dirichlet eigenvalues for $-\Delta$ on Ω. Let V be the unique solution of*

$$\Delta V + k_2^{'2} V = \sum_{j,l=1,2} \chi_{jl} \partial_{x_j} u \partial_{x_l} u \, \mathrm{I}_{\mathcal{O}_\delta} \quad in \ \mathbb{R}^2 ,$$

with the outgoing radiation condition, and let $V_0 = V|_{\partial\Omega}$, $V_\delta = V|_{\partial\Omega_\delta}$, $V_0' = \frac{\partial V}{\partial \nu}|_{\partial\Omega}$ and $V_\delta' = \frac{\partial V}{\partial \nu}|_{\partial\Omega_\delta}$.

Then, there exists $\delta_0 > 0$ such that for $0 < \delta < \delta_0$, if v is the solution of the problem (2.7) and $\tilde{\Phi} = (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\psi}_2, \tilde{\varphi}_0) \in X^2$ is the unique solution of

$$\left\{ \begin{aligned} & S_{0,0}^{k_1'} \tilde{\varphi}_1 - S_{0,0}^{k_2'} \tilde{\varphi}_2 - S_{\delta,0}^{k_2'} \tilde{\psi}_2 = V_0, \\ & \frac{\partial (S_{0,0}^{k_1'} \tilde{\varphi}_1)_-}{\partial \nu} - \frac{\partial (S_{0,0}^{k_2'} \tilde{\varphi}_2)_+}{\partial \nu} - \frac{\partial (S_{\delta,0}^{k_2'} \tilde{\psi}_2)}{\partial \nu} = V_0', \\ & S_{0,\delta}^{k_2'} \tilde{\varphi}_2 + S_{\delta,\delta}^{k_2'} \tilde{\psi}_2 - S_{\delta,\delta}^{k_0'} \tilde{\varphi}_0 = -V_\delta, \\ & \frac{\partial (S_{0,\delta}^{k_2'} \tilde{\varphi}_2)}{\partial \nu} + \frac{\partial (S_{\delta,\delta}^{k_2'} \tilde{\psi}_2)_-}{\partial \nu} - \frac{\partial (S_{\delta,\delta}^{k_0'} \tilde{\varphi}_0)_+}{\partial \nu} = -V_\delta', \end{aligned} \right. \tag{8.40}$$

then v can be represented as

$$v(x) = \begin{cases} S_0^{k_1'}\tilde{\varphi}_1(x) & \textit{for } x \in \Omega\,, \\ V(x) + S_0^{k_2'}\tilde{\varphi}_2(x) + S_\delta^{k_2'}\tilde{\psi}_2(x) & \textit{for } x \in \mathcal{O}_\delta\,, \\ S_\delta^{k_0'}\tilde{\varphi}_0(x) & \textit{for } x \in \Omega_\delta^e\,. \end{cases} \tag{8.41}$$

Proof. Recalling Corollary 6.1, we can express explicitly V by setting

$$V(x) = \int_{\mathcal{O}_\delta} \Gamma_{k_2'}(x-y) \sum_{j,l=1,2} \chi_{jl}\partial_{x_j}u(y)\partial_{x_l}u(y)\,dy\,. \tag{8.42}$$

Then it is clear that the function defined as in (8.41) satisfies the Helmholtz equations, the transmission conditions and the radiation condition in (2.6). $\square$

9 Asymptotic expansion of the second-harmonic field

We proceed as for the fundamental field u. We first define the operators $\tilde{Q}_\delta$, $\tilde{R}_\delta$ and $\tilde{W}_\delta$ from X into Y by

$$\begin{aligned}
\tilde{Q}_\delta(\tilde{\varphi},\tilde{\psi}) &:= \left(S_{0,0}^{k_1'}\tilde{\varphi} - S_{\delta,\delta}^{k_0'}\tilde{\psi}, \frac{\partial(S_{0,0}^{k_1'}\tilde{\varphi})_-}{\partial\nu} - \frac{\partial(S_{\delta,\delta}^{k_0'}\tilde{\psi})_+}{\partial\nu} \right), \\
\tilde{R}_\delta(\tilde{\varphi},\tilde{\psi}) &:= \Bigg((S_{0,\delta}^{k_2'} - S_{0,0}^{k_2'})\tilde{\varphi} + (S_{\delta,\delta}^{k_2'} - S_{\delta,0}^{k_2'})\tilde{\psi}, \\
&\qquad \frac{\partial}{\partial\nu}\Big((S_{0,\delta}^{k_2'}\tilde{\varphi}) - (S_{0,0}^{k_2'}\tilde{\varphi})_+ + (S_{\delta,\delta}^{k_2'}\tilde{\psi})_- - (S_{\delta,0}^{k_2'}\tilde{\psi}) \Big) \Bigg), \\
\tilde{W}_\delta(\tilde{\varphi},\tilde{\psi}) &:= \left(S_{0,0}^{k_2'}\tilde{\varphi} + S_{\delta,0}^{k_2'}\tilde{\psi}, \frac{\partial(S_{0,0}^{k_2'}\tilde{\varphi})_+}{\partial\nu} + \frac{\partial(S_{\delta,0}^{k_2'}\tilde{\psi})}{\partial\nu} \right), \\
\tilde{W}_0(\tilde{\varphi},\tilde{\psi}) &:= \left(S_{0,0}^{k_2'}\tilde{\varphi} + S_{0,0}^{k_2'}\tilde{\psi}, \frac{\partial(S_{0,0}^{k_2'}\tilde{\varphi})_+}{\partial\nu} + \frac{\partial(S_{0,0}^{k_2'}\tilde{\psi})_-}{\partial\nu} \right),
\end{aligned}$$

and the function $\tilde{V}^\delta$ on $\partial\Omega$ by

$$\tilde{V}_\delta := \Big(V_\delta - V_0, V_\delta' - V_0' \Big)\,.$$

The following proposition holds.

Proposition 9.1 *Let $\tilde{\Phi} = (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\psi}_2, \tilde{\varphi}_0) \in X^2$ be the unique solution of (8.40), then $(\tilde{\varphi}_1, \tilde{\varphi}_0)$ and $(\tilde{\varphi}_2, \tilde{\psi}_2)$ are solutions of the following equations*

$$\tilde{Q}_\delta(\tilde{\varphi}_1, \tilde{\varphi}_0) = -\tilde{V}_\delta - \tilde{R}_\delta(\tilde{\varphi}_2, \tilde{\psi}_2) , \tag{9.43}$$

$$\tilde{W}_\delta(\tilde{\varphi}_2, \tilde{\psi}_2) = \left((S_{0,0}^{k_1'} \tilde{\varphi}_1), \frac{\partial (S_{0,0}^{k_1'} \tilde{\varphi}_1)_-}{\partial \nu} \right) - (V_0, V_0') . \tag{9.44}$$

In order to expand $\tilde{\Phi}$, we need to prove its stability when δ goes to 0.

Proposition 9.2 *Let u be the solution to problem (2.6). Then, for δ small enough, there exists a constant $C > 0$ independent of δ and $0 < \eta < \delta$ such that*

$$\begin{aligned} \| u(x + \eta \nu(x)) - U(x) \|_{\mathcal{C}^1(\partial\Omega)} &\leq C\delta , \\ \left\| \frac{\partial u(x + \eta \nu(x))}{\partial \nu} - \frac{k_2^2}{k_0^2} \frac{\partial (U)_+}{\partial \nu}(x) \right\|_{\mathcal{C}^0(\partial\Omega)} &\leq C\delta . \end{aligned}$$

Proof. It is easy to prove that $(\varphi_1^0, \varphi_0^0)$ and (φ_2^0, ψ_2^0) belong to $(\mathcal{C}^1(\partial\Omega))^2$. The inequalities in Lemma 4.4 are also true when replacing the norms in $L^2(\partial\Omega)$ and $H^1(\partial\Omega)$ by $\mathcal{C}^0(\partial\Omega)$ and $\mathcal{C}^1(\partial\Omega)$, respectively. See, for example, [13]. Analogously to Proposition 7.5 we then deduce that

$$\begin{aligned} \left\| \varphi_2 - \varphi_2^0 \right\|_{\mathcal{C}^0(\partial\Omega)} &\leq C\delta , \\ \left\| \psi_2 - \psi_2^0 \right\|_{\mathcal{C}^0(\partial\Omega)} &\leq C\delta , \end{aligned}$$

for some constant $C > 0$. It then follows that

$$\begin{aligned} u(x + \eta \nu(x)) &= S_{0,\eta}^{k_2} \varphi_2 + S_{\delta,\eta}^{k_2} \psi_2 \\ &= S_{0,0}^{k_2} (\varphi_2^0 + \psi_2^0) + O(\delta) \\ &= U(x) + O(\delta) , \end{aligned}$$

where $O(\delta)$ is in $\mathcal{C}^1(\partial\Omega)$. We also have

$$\begin{aligned} \frac{\partial u(x + \eta \nu(x))}{\partial \nu} &= \frac{\partial S_{0,\eta}^{k_2} \varphi_2}{\partial \nu} + \frac{\partial S_{\delta,\eta}^{k_2} \psi_2}{\partial \nu} \\ &= \frac{\partial (S_{0,0}^{k_2} \varphi_2^0)_+}{\partial \nu} + \frac{\partial (S_{0,0}^{k_2} \psi_2^0)_-}{\partial \nu} + O(\delta) \\ &= \frac{k_2^2}{k_0^2} \frac{\partial (U)_+}{\partial \nu}(x) + O(\delta) , \end{aligned}$$

where $O(\delta)$ is in $\mathcal{C}^0(\partial\Omega)$. The proposition is then proved. □

Now we give an expansion for the source term defined for $x + \eta\nu \in \mathcal{O}_\delta$ ($x \in \partial\Omega$ and $0 < \eta < \delta$) by

$$\Pi(x) = \sum_{j,l=1,2} \chi_{jl} \partial_{x_j} u(x) \partial_{x_l} u(x).$$

First, we recall the following assumption on the susceptibility tensor

$$\chi_{jl}(x+\eta\nu) = \tilde{\chi}_{jl}\left(x, \frac{\eta}{\delta}\right), \qquad x \in \partial\Omega\ ,\ 0 < \eta < \delta\ ,$$

where $\tilde{\chi}_{jl}$ are independent of δ. We define then Π_0 on $\partial\Omega$ by

$$\Pi_0(x) = \sum_{j,l=1,2} \left(\int_0^1 \tilde{\chi}_{jl}(x,\theta)\, d\theta \right) w_j^0(x) w_l^0(x)\ ,$$

where w^0 is given by

$$w^0(x) = \frac{\partial U}{\partial \tau}(x)\tau(x) + \frac{k_2^2}{k_0^2}\frac{\partial (U)_+}{\partial \nu}(x)\nu(x)\ .$$

The next proposition is a direct consequence of the expansion of the fundamental field u.

Proposition 9.3 *The following expansion holds.*

$$\partial_{x_j} u(x+\eta\nu(x))\partial_{x_l} u(x+\eta\nu(x)) = w_j^0(x)w_l^0(x) + O(\delta)\ ,$$

where $x \in \partial\Omega$, $0 < \eta < \delta$, and $O(\delta)$ is in $\mathcal{C}^0(\partial\Omega)$.

Now, we give expansions of the function V defined by (8.42).

Proposition 9.4 *There exists $\varepsilon_\delta \to 0$ as $\delta \to 0$ such that*

$$\begin{aligned}
\|V_0 - \delta S_{0,0}^{k_2'}\Pi_0\|_{H^1(\partial\Omega)} &\leq \varepsilon_\delta \delta\ ,\\
\|V_0' - \delta \frac{\partial (S_{0,0}^{k_2'}\Pi_0)_-}{\partial\nu}\|_{L^2(\partial\Omega)} &\leq \varepsilon_\delta \delta\ ,\\
\|V_\delta - V_0\|_{H^1(\partial\Omega)} &\leq \varepsilon_\delta \delta\ ,\\
\|V_\delta' - V_0' - \delta\Pi_0\|_{L^2(\partial\Omega)} &\leq \varepsilon_\delta \delta\ .
\end{aligned}$$

Proof. Recall that

$$V(x) = \int_0^\delta S_\eta^{k_2'}\Pi(x)\, d\eta \quad \text{for } x \in \mathcal{O}_\delta\ ,$$

to obtain that for $x \in \partial\Omega$,

$$\begin{aligned}
V_0(x) &= \int_0^\delta S_{\eta,0}^{k_2'}\Pi(x+\eta\nu(x))\, d\eta\\
&= \delta S_{0,0}^{k_2'}\Pi_0(x)\\
&+ \int_0^\delta (S_{\eta,0}^{k_2'} - S_{0,0}^{k_2'})\Pi(x+\eta\nu(x))\, d\eta + S_{0,0}^{k_2'}\left(\int_0^\delta \left(\Pi(x+\eta\nu(x)) - \Pi_0(x)\right) d\eta\right).
\end{aligned}$$

But Π is uniformly bounded in $L^2(\mathcal{O}_\delta)$. Therefore, as an application of Lemma 4.4, the argument of the integral in the second term can be bounded in $H^1(\partial\Omega)$

by ε_δ and the second term is bounded by $\delta\varepsilon_\delta$. Concerning the third term, we notice that

$$\int_0^\delta \big(\Pi(x+\eta\nu(x)) - \Pi_0(x)\big)\, d\eta$$

$$= \sum_{j,l} \int_0^\delta \chi_{jl}(x+\eta\nu(x))\partial_{x_j} u(x+\eta\nu(x))\partial_{x_l} u(x+\eta\nu(x))\, d\eta$$

$$- \sum_{j,l} \delta \left(\int_0^1 \tilde{\chi}_{jl}(x,\theta)\, d\theta \right) w_j^0(x) w_l^0(x)\, d\eta$$

$$= \sum_{j,l} \int_0^\delta \chi_{jl}(x+\eta\nu(x))$$

$$\left(\partial_{x_j} u(x+\eta\nu(x))\partial_{x_l} u(x+\eta\nu(x)) - w_j^0(x) w_l^0(x) \right) d\eta.$$

Therefore, it follows from Proposition 9.3 that

$$\left\| \int_0^\delta \big(\Pi(x+\eta\nu(x)) - \Pi_0(x)\big)\, d\eta \right\|_{L^2(\partial\Omega)} \le C\delta^2 ,$$

for some constant C. Hence, the third term in the previous expression of V_0 is bounded in $H^1(\partial\Omega)$ by $C\delta^2$. The first inequality is then proved. The second inequality can be proved in exactly the same way.

Now we turn to the last two inequalities

$$V_\delta(x) - V_0(x) = \int_0^\delta (S_{\eta,\delta}^{k_2'} - S_{\eta,0}^{k_2'})\Pi(x+\eta\nu(x))\, d\eta$$

$$= \int_0^\delta (S_{\eta,\delta}^{k_2'} - S_{\eta,\eta}^{k_2'})\Pi(x+\eta\nu(x))\, d\eta + \int_0^\delta (S_{\eta,\eta}^{k_2'} - S_{\eta,0}^{k_2'})\Pi(x+\eta\nu(x))\, d\eta.$$

Since $\Pi(x+\eta\nu(x))$ is bounded in $L^2(\partial\Omega)$, it follows that the arguments of each integral is bounded in $H^1(\partial\Omega)$ by ε_δ and $V_\delta - V_0$ is bounded by $\delta\varepsilon_\delta$. Thus

$$V_\delta'(x) - V_0'(x) = \int_0^\delta \left(\frac{\partial S_{\eta,\delta}^{k_2'}}{\partial\nu} - \frac{\partial S_{\eta,0}^{k_2'}}{\partial\nu} \right) \Pi(x+\eta\nu(x))\, d\eta$$

$$= \int_0^\delta \left(\frac{\partial S_{\eta,\delta}^{k_2'}}{\partial\nu} - \frac{\partial (S_{\eta,\eta}^{k_2'})_+}{\partial\nu} \right) \Pi(x+\eta\nu(x))\, d\eta$$

$$+ \int_0^\delta \left(\frac{\partial (S_{\eta,\eta}^{k_2'})_-}{\partial\nu} - \frac{\partial S_{\eta,0}^{k_2'}}{\partial\nu} \right) \Pi(x+\eta\nu(x))\, d\eta$$

$$+ \int_0^\delta \Big(\Pi(x+\eta\nu(x)) - \Pi_0(x) \Big)\, d\eta + \delta\Pi_0(x).$$

Again, since $\Pi(x+\eta\nu(x))$ is bounded in $L^2(\partial\Omega)$, the argument of the first and second integrals are bounded in $L^2(\partial\Omega)$ by ε_δ while we have already proved that the last integral is bounded in $L^2(\partial\Omega)$ by $C\delta^2$. This proves the last inequality in the proposition. $\square$

Next, we state and prove the following convergence result for $\tilde{\Phi}$.

Proposition 9.5 *Let $\tilde{\Phi}^\delta = (\tilde{\varphi}_1^\delta, \tilde{\varphi}_2^\delta, \tilde{\psi}_2^\delta, \tilde{\varphi}_0^\delta) \in X^2$ be the unique solution of (8.40). Then $(\frac{\tilde{\varphi}_1^\delta}{\delta}, \frac{\tilde{\varphi}_0^\delta}{\delta})$ and $(\frac{\tilde{\varphi}_2^\delta}{\delta}, \frac{\tilde{\psi}_2^\delta}{\delta})$ converge in X to $(\tilde{\varphi}_1^1, \tilde{\varphi}_0^1)$ and $(\tilde{\varphi}_2^1, \tilde{\psi}_2^1)$, respectively, where*

$$
\begin{aligned}
&\tilde{Q}_0(\tilde{\varphi}_1^1, \tilde{\varphi}_0^1) = -(0, \Pi_0) , \\
&\tilde{W}_0(\tilde{\varphi}_2^1, \tilde{\psi}_2^1) = \left(S_{0,0}^{k_1'} \tilde{\varphi}_1^1, \frac{\partial (S_{0,0}^{k_1'} \tilde{\varphi}_1^1)_-}{\partial \nu} \right) - \left(S_{0,0}^{k_2'} \Pi_0, \frac{\partial (S_{0,0}^{k_2'} \Pi_0)_-}{\partial \nu} \right) .
\end{aligned}
\tag{9.45}
$$

Proof. The proof is very similar to the one of Proposition 7.3. From Propositions 9.4 and (7.28) it follows that

$$
\left\| \tilde{Q}_\delta \left(\frac{\tilde{\varphi}_1^\delta}{\delta}, \frac{\tilde{\varphi}_0^\delta}{\delta} \right) \right\|_Y \le C_1 + \varepsilon_\delta \left(\left\| \frac{\tilde{\varphi}_2^\delta}{\delta} \right\|_{L^2(\partial\Omega)} + \left\| \frac{\tilde{\psi}_2^\delta}{\delta} \right\|_{L^2(\partial\Omega)} \right) .
$$

On the other hand, (9.44) yields

$$
\left\| \tilde{W}_\delta \left(\frac{\tilde{\varphi}_2^\delta}{\delta}, \frac{\tilde{\psi}_0^\delta}{\delta} \right) \right\|_Y \le C_1' + C_2' \left\| \frac{\tilde{\varphi}_1^\delta}{\delta} \right\|_{L^2(\partial\Omega)} ,
$$

where C_1, C_1' and C_2' are some positive constants independent of δ and ε_δ. It is then easy to see that $\frac{\tilde{\varphi}_1^\delta}{\delta}$, $\frac{\tilde{\varphi}_2^\delta}{\delta}$, $\frac{\tilde{\psi}_0^\delta}{\delta}$, and $\frac{\tilde{\varphi}_0^\delta}{\delta}$ are bounded in $L^2(\partial\Omega)$. Therefore

$$
\lim_{\delta \to 0} \left\| \tilde{R}_\delta \left(\frac{\tilde{\varphi}_2^\delta}{\delta}, \frac{\tilde{\psi}_0^\delta}{\delta} \right) \right\|_Y = 0,
$$

and (9.45) is straightforward. $\square$

Now we can give an expansion of the second-harmonic field away from the thin layer of nonlinear material $\mathcal{O}_\delta$.

Theorem 9.1 *Let v be the solution to problem (2.7). Then, the following expansion holds uniformly in $H^1_{\text{loc}}(\mathbb{R}^2 \setminus \partial\Omega)$:*

$$
v(x) = \delta v_0(x) + o(\delta) , \tag{9.46}
$$

where v_0 is given by

$$
v_0(x) = \begin{cases} S_0^{k_1'} \tilde{\varphi}_1^1(x) & \text{for } x \in \Omega , \\ S_0^{k_0'} \tilde{\varphi}_0^1(x) & \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega} . \end{cases} \tag{9.47}
$$

Proof. From the representation formula (8.41), we have

$$
v(x) = \begin{cases} S_0^{k_1'} \tilde{\varphi}_1(x) & \text{for } x \in \Omega , \\ S_\delta^{k_0'} \tilde{\varphi}_0(x) & \text{for } x \in \Omega_\delta^e . \end{cases}
$$

Recalling (9.45) we immediately obtain (9.46). $\square$

The proof of the asymptotic expansion (2.12) is now immediate.

10 Appendices

A.1 Uniqueness results

We recall the following important result from the theory of the Helmholtz equation known as Rellich's lemma.

Lemma A.1.1 *Let $R_0 > 0$, $B_R = \{x : |x| < R\}$, and $S_R = \{x : |x| = R\}$. Let u satisfy the Helmholtz equation $\Delta u + k_0^2 = 0$ for $|x| > R_0$. Assume, furthermore that*

$$\lim_{R \to +\infty} \int_{S_R} |u(x)|^2 \, ds(x) = 0 \, .$$

Then, $u \equiv 0$ for $|x| > R_0$.

Let $W^{1,2}_{\text{loc}}(\mathbb{R}^2 \setminus \overline{\Omega})$ denote the space of functions $f \in L^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{\Omega})$ such that

$$hf \in W^{1,2}(\mathbb{R}^2 \setminus \overline{\Omega}), \forall \, h \in \mathcal{C}_0^\infty(\mathbb{R}^2 \setminus \overline{\Omega}) \, .$$

The following uniqueness result is a consequence of the previous lemma.

Lemma A.1.2 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$. Let $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2 \setminus \overline{\Omega})$ satisfy*

$$\begin{cases} \Delta u + k_0^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega} \, , \\ \left| \dfrac{\partial u}{\partial |x|} - i k_0 u \right| = O\left(|x|^{-3/2} \right) \quad \text{as } |x| \to +\infty \quad \text{uniformly in } \dfrac{x}{|x|} \, , \\ \Im \displaystyle\int_{\partial \Omega} \overline{u} \frac{\partial u}{\partial \nu} \, ds = 0 \, . \end{cases}$$

Then, $u \equiv 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$.

A.2 Proof of Lemma 4.3

In view of Lemma 4.1 it suffices to prove Lemma 4.3 for $k = 0$. We have

$$2 S^0_{\delta,\delta} \varphi(x) = \int_{\partial \Omega} \log |x - y + \delta(\nu(x) - \nu(y))|^2 (1 + \delta \rho(y)) \varphi(y) \, ds(y)$$

$$= \int_{\partial \Omega} \log(|x - y|^2 + \delta^2 |\nu(x) - \nu(y)|^2 + 2\delta < \nu(x) - \nu(y), x - y >)$$
$$(1 + \delta \rho(y)) \varphi(y) \, ds(y)$$

$$= 2 S^0_{0,0} \varphi(x) + 2\delta S^0_{0,0} (\rho \varphi)(x)$$
$$+ \int_{\partial \Omega} \log \left(1 + 2\delta \frac{< \nu(x) - \nu(y), x - y >}{|x - y|^2} + \delta^2 \frac{|\nu(x) - \nu(y)|^2}{|x - y|^2} \right)$$
$$(1 + \delta \rho(y)) \varphi(y) \, ds(y) \, .$$

Since $\partial\Omega$ is of class $\mathcal{C}^2$, $\dfrac{< \nu(x)-\nu(y), x-y >}{|x-y|^2}$ and $\dfrac{|\nu(x)-\nu(y)|^2}{|x-y|^2}$ are bounded. Therefore,

$$\begin{aligned} S^0_{\delta,\delta}\varphi(x) &= S^0_{0,0}\varphi(x) + \delta S^0_{0,0}(\rho\varphi)(x) + \delta \int_{\partial\Omega} \frac{< \nu(x)-\nu(y), x-y >}{|x-y|^2}\varphi(y)\, ds(y) \\ &\quad +O(\delta^2) \\ &= S^0_{0,0}\varphi(x) + \delta S^0_{0,0}(\rho\varphi)(x) + \delta (K^0_0)^*\varphi(x) + \delta K^0_0\varphi(x) + O(\delta^2)\ . \end{aligned}$$

On the other hand,

$$\begin{aligned} (K^0_\delta)^*\varphi(x) &= \int_{\partial\Omega} \frac{< \nu(x), x-y+\delta(\nu(x)-\nu(y)) >}{|x-y+\delta(\nu(x)-\nu(y))|^2}(1+\delta\rho)\varphi(y)\ ds(y) \\ &= \int_{\partial\Omega} \frac{< \nu(x), x-y > +\delta < \nu(x), \nu(x)-\nu(y) >}{|x-y|^2} \\ &\quad \left(1-2\delta\frac{< \nu(x)-\nu(y), x-y >}{|x-y|^2}\right)(1+\delta\rho)\varphi(y)\ ds(y) + O(\delta^2), \end{aligned}$$

or equivalently,

$$\begin{aligned} (K^0_\delta)^*\varphi(x) &= (K^0_0)^*\varphi(x) + \delta (K^0_0)^*(\rho\varphi)(x) - 2\delta \int_{\partial\Omega} \frac{< \nu(x), x-y >^2}{|x-y|^4}\varphi(y)\ ds(y) \\ &\quad +2\delta \int_{\partial\Omega} \frac{< \nu(x), x-y >< \nu(y), x-y >}{|x-y|^4}\varphi(y)\ ds(y) \\ &\quad +\delta \int_{\partial\Omega} \frac{< \nu(x), \nu(x)-\nu(y) >}{|x-y|^2}\varphi(y)\ ds(y) + O(\delta^2) \\ &= (K^0_0)^*\varphi(x) + \delta (K^0_0)^*(\rho\varphi)(x) + \delta M^0_0\varphi(x) + O(\delta^2)\ . \end{aligned}$$

Indeed, for $\delta > 0$ small enough and $k = 0$, we have

$$
\begin{aligned}
&2\Big(\frac{\partial S^0_{0,\delta}\varphi}{\partial \nu} + D^0_{0,\delta}\varphi + S^0_{0,\delta}(\rho\varphi)\Big)(x) - 2\Big(\frac{\partial (S^0_{0,0}\varphi)_+}{\partial \nu} + (D^0_{0,0}\varphi)_+ \\
&+ S^0_{0,0}(\rho\varphi)\Big)(x) = \int_{\partial\Omega} \frac{< \nu(x), x-y+\delta\nu(x) >}{|x-y|^2 + \delta^2 + 2\delta < \nu(x), x-y >} \varphi(y)\ ds(y) \\
&- \int_{\partial\Omega} \frac{< \nu(y), x-y+\delta\nu(x) >}{|x-y|^2 + \delta^2 + 2\delta < \nu(x), x-y >} \varphi(y)\ ds(y) \\
&- \int_{\partial\Omega} \frac{< \nu(x)-\nu(y), x-y >}{|x-y|^2} \varphi(y)\ ds(y) + S^0_{0,\delta}(\rho\varphi)(x) - S^0_{0,0}(\rho\varphi)(x) \\
&= \int_{\partial\Omega} \frac{< \nu(x)-\nu(y), x-y >}{|x-y|^2+\delta^2} \Big(1 - 2\delta \frac{< \nu(x), x-y >}{|x-y|^2+\delta^2}\Big) \varphi(y)\ ds(y) \\
&+ \int_{\partial\Omega} \frac{\delta < \nu(x)-\nu(y), \nu(x) >}{|x-y|^2+\delta^2} \Big(1 - 2\delta \frac{< \nu(x), x-y >}{|x-y|^2+\delta^2}\Big) \varphi(y)\ ds(y) \\
&- \int_{\partial\Omega} \frac{< \nu(x)-\nu(y), x-y >}{|x-y|^2} \varphi(y)\ ds(y) + \delta \frac{\rho\varphi}{2}(x) + \delta (K^0_0)^*(\rho\varphi)(x) + o(\delta) \\
&= -\delta \int_{\partial\Omega} \frac{\delta}{|x-y|^2+\delta^2} \frac{< \nu(x)-\nu(y), x-y >}{|x-y|^2} \varphi(y)\ ds(y) + \delta \frac{\rho\varphi}{2}(x) \\
&- 2\delta \int_{\partial\Omega} \frac{< \nu(x), x-y >^2}{\big(|x-y|^2+\delta^2\big)^2} \varphi(y)\ ds(y) + 2\delta \int_{\partial\Omega} \frac{< \nu(x), x-y >< \nu(y), x-y >}{\big(|x-y|^2+\delta^2\big)^2} \\
&\varphi(y)\ ds(y) + \delta \int_{\partial\Omega} \frac{< \nu(x)-\nu(y), \nu(x) >}{|x-y|^2+\delta^2} \varphi(y)\ ds(y) \\
&- 2\delta^2 \int_{\partial\Omega} \frac{< \nu(x)-\nu(y), \nu(x) >< \nu(x), x-y >}{\big(|x-y|^2+\delta^2\big)^2} \varphi(y)\ ds(y) \\
&+ \delta (K^0_0)^*(\rho\varphi)(x) + o(\delta) = \delta M^0_0 \varphi(x) + \delta (K^0_0)^*(\rho\varphi)(x) + o(\delta)\ .
\end{aligned}
$$

Here we have used the fact that

$$
\frac{< \nu(x)-\nu(y), x-y >}{|x-y|^2} \rightarrow \rho(x) \quad \text{when } y \rightarrow x\ .
$$

A.3 Proof of Lemma 4.4

We start by proving the inequalities for $s = 0$. Since the normal derivative of $\dfrac{\partial (S^k_{0,0}\varphi)_+}{\partial \nu}$ exists and is bounded from $L^2(\partial\Omega)$ into $L^2(\partial\Omega)$, we have

$$
\|S^k_{0,\delta}\varphi - S^k_{0,0}\varphi\|_{L^2(\partial\Omega)} \leq C\delta \|\varphi\|_{L^2(\partial\Omega)}\ ,
$$

for some $C > 0$. It remains then to show that $\left\| \dfrac{\partial S^k_{0,\delta}\varphi}{\partial \tau} - \dfrac{\partial S^k_{0,0}\varphi}{\partial \tau} \right\|_{L^2(\partial\Omega)}$ goes uniformly to 0 as $\delta \rightarrow 0$. Once again, in view of Lemma 4.1 it suffices to prove this result for $k=0$. Denoting by $\oint$ the Cauchy principal value, we compute

$$\frac{\partial S^k_{0,\delta}\varphi}{\partial \tau}(x) - \frac{\partial S^k_{0,0}\varphi}{\partial \tau}(x) = \int_{\partial\Omega} \frac{< \tau(x), x-y >}{|x-y|^2 + \delta^2 + 2\delta < \nu(x), x-y >}$$

$$(1-\delta\rho(x))\varphi(y)\ ds(y) - \oint_{\partial\Omega} \frac{< \tau(x), x-y >}{|x-y|^2}\varphi(y)\ ds(y)$$

$$= \oint_{\partial\Omega} \frac{< \tau(x), x-y >}{|x-y|^2+\delta^2}\left(-2\delta\frac{< \nu(x), x-y >}{|x-y|^2+\delta^2}(1-\delta\rho(x)) - \frac{\delta^2}{|x-y|^2} + O(\delta^2)\right)$$

$$\varphi(y)\ ds(y)$$

$$= -2\delta\oint_{\partial\Omega} \frac{< \tau(x), x-y >}{|x-y|^2}\frac{|x-y|^2}{|x-y|^2+\delta^2}\frac{< \nu(x), x-y >}{|x-y|^2+\delta^2}(1-\delta\rho(x))\varphi(y)\ ds(y)$$

$$- \oint_{\partial\Omega} \frac{< \tau(x), x-y >}{|x-y|^2}\frac{\delta^2}{|x-y|^2+\delta^2}\varphi(y)\ ds(y) + \delta^2\oint_{\partial\Omega}\frac{< \tau(x), x-y >}{|x-y|^2}O(1)$$

$$\varphi(y)\ ds(y).$$

The two first integrals can be bounded by $\delta\|\varphi\|_{L^2(\partial\Omega)}$ while the last one is bounded by $\varepsilon_\delta\|\varphi(x)\|_{L^2(\partial\Omega)}$ for $\delta \le \delta_0$. This ends the proof of the first inequality.

Let us now prove the third inequality. We write

$$\frac{\partial S^k_{0,\delta}\varphi}{\partial \nu} - \frac{\partial (S^k_{0,0}\varphi)_+}{\partial \nu} = \int_{\partial\Omega}\frac{< \nu(x), x-y+\delta\nu(x) >}{|x-y|^2+\delta^2+2\delta < \nu(x), x-y >}\varphi(y)\ ds(y)$$

$$- \int_{\partial\Omega}\frac{< \nu(x), x-y >}{|x-y|^2}\varphi(y)\ ds(y) - \frac{1}{2}\varphi(x)$$

$$= \int_{\partial\Omega}\frac{< \nu(x), x-y > +\delta}{|x-y|^2+\delta^2}\left(1 - 2\delta\frac{< \nu(x), x-y >}{|x-y|^2+\delta^2} + O(\delta^2)\right)\varphi(y)\ ds(y)$$

$$- \int_{\partial\Omega}\frac{< \nu(x), x-y >}{|x-y|^2}\varphi(y)\ ds(y) - \frac{1}{2}\varphi(x)$$

$$= -\delta\int_{\partial\Omega}\frac{\delta}{|x-y|^2+\delta^2}\frac{< \nu(x), x-y >}{|x-y|^2}\varphi(y)\ ds(y)$$

$$+\delta\int_{\partial\Omega}\frac{< \nu(x), x-y > +\delta}{|x-y|^2+\delta^2}\left(-2\frac{< \nu(x), x-y >}{|x-y|^2+\delta^2} + O(\delta)\right)\varphi(y)\ ds(y)$$

$$+ \int_{\partial\Omega}\frac{\delta}{|x-y|^2+\delta^2}\varphi(y)\ ds(y) - \frac{1}{2}\varphi(x)\ .$$

Since the Poisson kernel [17]

$$\int_{\partial\Omega}\frac{\delta}{|x-y|^2+\delta^2}\varphi(y)\ ds(y)$$

converges uniformly in $L^2(\partial\Omega)$ as $\delta \to 0$, the first integral is bounded by $C\delta\varphi(x)$, the second integral is bounded by $C\delta(\|\varphi\|_{L^2(\partial\Omega)} + \varphi(x))$, and the last one is uniformly bounded by $\varepsilon_\delta\varphi(x)$. Here C is a positive constant independent of x and $\varepsilon_\delta \to 0$ as $\delta \to 0$.

The second and the last inequalities can be proved in a very similar way. For the reader's convenience we give here the proof for the second one, that is,

$$S^k_{\delta,\delta}\varphi - S^k_{\delta,0}\varphi = \big(S^k_{\delta,\delta} - S^k_{0,0}\big)\varphi + \big(S^k_{0,\delta} - S^k_{\delta,0}\big)\varphi - \big(S^k_{0,\delta} - S^k_{0,0}\big)\varphi\ .$$

Using Lemma 4.3, the first and the third terms can be bounded uniformly in $H^1(\partial\Omega)$ by $\varepsilon_\delta \|\varphi\|_{L^2(\partial\Omega)}$. Then, in view of Lemma 4.1, we only need to investigate the second term for $k = 0$. We compute

$$
\begin{aligned}
&2\big(S^0_{0,\delta} - S^0_{\delta,0}\big)\varphi(x) = \int_{\partial\Omega} \log\big(|x-y|^2 - 2\delta < \nu(y), x-y > + \delta^2\big) \\
&(1+\delta\rho)\varphi(y)\ ds(y) - \int_{\partial\Omega} \log\big(|x-y|^2 + 2\delta < \nu(x), x-y > + \delta^2\big)\varphi(y)\ ds(y) \\
&= \delta \int_{\partial\Omega} \log\big(|x-y|^2 - 2\delta < \nu(y), x-y > + \delta^2\big)\rho\varphi(y)\ ds(y) \\
&+ \int_{\partial\Omega} \log\left(1 + 2\delta \frac{< \nu(x), x-y >}{|x-y|^2 + \delta^2}\right)\varphi(y)\ ds(y) \\
&- \int_{\partial\Omega} \log\left(1 - 2\delta \frac{< \nu(y), x-y >}{|x-y|^2 + \delta^2}\right)\varphi(y)\ ds(y) \\
&= \delta \int_{\partial\Omega} \log\big(|x-y|^2 + \delta^2\big)\rho\varphi(y)\ ds(y) \\
&+ 2\delta \int_{\partial\Omega} \frac{< \nu(x)+\nu(y), x-y >}{|x-y|^2+\delta^2}\varphi(y)\ ds(y) + O(\delta^2) \\
&= \delta \int_{\partial\Omega} \log\big(|x-y|^2 + \delta^2\big)\rho\varphi(y)\ ds(y) - 2\delta K^0_0\varphi(x) + 2\delta (K^0_0)^*\varphi(x) \\
&- 2\delta^2 \int_{\partial\Omega} \frac{\delta}{|x-y|^2+\delta^2} \frac{< \nu(x)+\nu(y), x-y >}{|x-y|^2}\varphi(y)\ ds(y) + O(\delta^2)\ .
\end{aligned}
$$

The first integral converges uniformly to $S^0_{0,0}(\rho\varphi)$ in $H^1(\partial\Omega)$ and the last integral converges uniformly to $\frac{\rho\varphi}{2}$ in $H^1(\partial\Omega)$. The proof is then complete.

For $s = 1$, we need to suppose that $\partial\Omega$ is $\mathcal{C}^3$ or equivalently that ρ is $\mathcal{C}^1$. For the first inequality, we need then to proof that

$$
\left\| \frac{\partial^2 S^k_{0,\delta}\varphi}{\partial\tau^2} - \frac{\partial^2 S^k_{0,0}\varphi}{\partial\tau^2} \right\|_{L^2(\partial\Omega)} \leq \varepsilon_\delta \left\| \frac{\partial\varphi}{\partial\tau} \right\|_{L^2(\partial\Omega)} .
$$

Let us then study the term $\frac{\partial^2 S^k_{0,\delta}\varphi}{\partial\tau^2}$. We rewrite this term as

$$
\begin{aligned}
&2\frac{\partial^2 S^0_{0,\delta}\varphi}{\partial\tau^2}(x) = \\
&\frac{\partial}{\partial\tau(x)} \int_{\partial\Omega} \left(\frac{\partial}{\partial\tau(x)} + \frac{\partial}{\partial\tau(y)}\right) \log(|x-y|^2 + \delta^2 + 2\delta < \nu(x), x-y >)\varphi(y)\ ds(y) \\
&+ \frac{\partial}{\partial\tau(x)} \int_{\partial\Omega} \log(|x-y|^2 + \delta^2 + 2\delta < \nu(x), x-y >) \frac{\partial\varphi(y)}{\partial\tau(y)}\ ds(y)\ .
\end{aligned}
$$

The second term in the above identity converges uniformly to $\frac{\partial S^0_{0,0}(\partial_\tau\varphi)}{\partial\tau}$ for $\partial_\tau\varphi$ bounded in $L^2(\partial\Omega)$. It remains then to find the limit of the first term that we denote by $2I_\delta$. Direct computations give

$$\begin{aligned}
I_\delta &= \int_{\partial\Omega}\left(\frac{\partial}{\partial\tau(x)}+\frac{\partial}{\partial\tau(y)}\right)(1-\delta\rho(x)) \\
&\qquad\qquad \frac{<\tau(x),x-y>}{|x-y|^2+\delta^2+2\delta<\nu(x),x-y>}\varphi(y)\ ds(y) \\
&= -\delta\partial_{\tau(x)}\rho(x)\int_{\partial\Omega}\frac{<\tau(x),x-y>}{|x-y|^2+\delta^2+2\delta<\nu(x),x-y>}\varphi(y)\ ds(y) \\
&\quad +(1-\delta\rho(x))\int_{\partial\Omega}\left(\frac{\partial}{\partial\tau(x)}+\frac{\partial}{\partial\tau(y)}\right) \\
&\qquad\qquad \frac{<\tau(x),x-y>}{|x-y|^2+\delta^2+2\delta<\nu(x),x-y>}\varphi(y)\ ds(y).
\end{aligned}$$

In a similar way as done for $s=0$, we show that the first term in the expression of I_δ is uniformly bounded in $L^2(\partial\Omega)$, for φ in the unit ball of $L^2(\partial\Omega)$, by

$$\delta\partial_{\tau(x)}\rho(x)\oint_{\partial\Omega}\frac{<\tau(x),x-y>}{|x-y|^2}\varphi(y)\ ds(y)\ .$$

We look now into the integral in the second term of I_δ which we denote by J_δ. We have

$$\begin{aligned}
J_\delta &= \int_{\partial\Omega}\frac{\rho(x)<\nu(x),x-y>}{|x-y|^2+\delta^2+2\delta<\nu(x),x-y>}\varphi(y)\ ds(y) \\
&+\int_{\partial\Omega}\frac{<\tau(x),\tau(x)-\tau(y)>}{|x-y|^2+\delta^2+2\delta<\nu(x),x-y>}\varphi(y)\ ds(y) \\
&-2\int_{\partial\Omega}\frac{<\tau(x),x-y><\tau(x)-\tau(y),x-y>}{(|x-y|^2+\delta^2+2\delta<\nu(x),x-y>)^2}\varphi(y)\ ds(y) \\
&+2\delta\int_{\partial\Omega}\frac{<\tau(x),x-y>(\rho(x)<\tau(x),x-y>+<\tau(x),\nu(x)-\nu(y)>)}{(|x-y|^2+\delta^2+2\delta<\nu(x),x-y>)^2} \\
&\qquad\qquad \varphi(y)\ ds(y).
\end{aligned}$$

The last term is uniformly bounded in $L^2(\partial\Omega)$, for φ in the unit ball of $L^2(\partial\Omega)$, by

$$\delta\oint_{\partial\Omega}\frac{<\tau(x),x-y>}{|x-y|^2}\frac{\rho(x)<\tau(x),x-y>+<\tau(x),\nu(x)-\nu(y)>}{|x-y|^2}\varphi(y)\ ds(y)\ .$$

We prove then in a similar way as done for $s=0$ that J_δ converges uniformly in $L^2(\partial\Omega)$, for φ in the unit ball of $L^2(\partial\Omega)$ to J_0, given by

$$\begin{aligned}
J_0 &= \int_{\partial\Omega}\frac{\rho(x)<\nu(x),x-y>}{|x-y|^2}\varphi(y)\ ds(y)+\oint_{\partial\Omega}\frac{<\tau(x),\tau(x)-\tau(y)>}{|x-y|^2} \\
&\quad \varphi(y)\ ds(y)-2\oint_{\partial\Omega}\frac{<\tau(x),x-y>}{|x-y|^2}\frac{<\tau(x)-\tau(y),x-y>}{|x-y|^2}\varphi(y)\ ds(y).
\end{aligned}$$

Therefore, it follows that $\dfrac{\partial^2 S^0_{0,\delta}\varphi}{\partial\tau^2}$ converges uniformly to $\dfrac{\partial^2 S^0_{0,0}\varphi}{\partial\tau^2}$ in $L^2(\partial\Omega)$, for φ in the unit ball of $L^2(\partial\Omega)$, since it can be easily checked that

$$\begin{aligned}
\frac{\partial^2 S^0_{0,0}\varphi}{\partial\tau^2}(x) &= \frac{\partial S^0_{0,0}(\partial_\tau\varphi)}{\partial\tau}(x) + \int_{\partial\Omega} \frac{\rho(x) < \nu(x), x-y >}{|x-y|^2}\varphi(y)\ ds(y) \\
&+ \oint_{\partial\Omega} \frac{< \tau(x), \tau(x)-\tau(y) >}{|x-y|^2}\varphi(y)\ ds(y) \\
&-2\oint_{\partial\Omega} \frac{< \tau(x), x-y >}{|x-y|^2}\frac{< \tau(x)-\tau(y), x-y >}{|x-y|^2}\varphi(y)\ ds(y).
\end{aligned}$$

The proofs for the other inequalities essentially follow the same arguments.

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