ECOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES APPLIQUÉES
*UMR CNRS 7641*

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 41 50. Fax: 01 69 33 30 11
http://www.cmap.polytechnique.fr/

# On the band gap structure of Hill's equation

Grégoire Allaire, Rafael Orive

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# On the band gap structure of Hill's equation

G. Allaire*and R. Orive†

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**Abstract**

We revisit the old problem of finding the stability and instability intervals of a second-order elliptic equation on the real line with periodic coefficients (Hill's equation). It is well known that the stability intervals correspond to the spectrum of the Bloch family of operators defined on a single period. Here we propose a characterization of the instability intervals. We introduce a new family of non self-adjoint operators, formally equivalent to the Bloch ones but with an imaginary Bloch parameter, that we call exponential. We prove that they admit a countable infinite number of eigenvalues which, when they are real, completely characterize the intervals of instability of Hill's equation.

## 1 Introduction

We consider the following Hill's equation

$$-\{a(x)u'(x)\}' + \Sigma(x)u(x) = \lambda\sigma(x)u(x) \quad x \in \mathbb{R}, \tag{1.1}$$

with real, piecewise continuous, periodic coefficients, and where $\lambda \in \mathbb{C}$ is a parameter (or an eigenvalue). We also assume that $a(x)$ and $\sigma(x)$ do not vanish. One of the main concern of Floquet theory is to classify the solutions of (1.1) according to the value of $\lambda$. Indeed, restricting $\lambda$ to $\mathbb{R}$, it is proved [8], [15] that there exist intervals of $\mathbb{R}$ such that, if $\lambda$ belongs to them, then any solution of (1.1) is bounded (and said to be stable), while, if $\lambda$ belongs to a complementary family of intervals, then any solution of (1.1) is unbounded (and said to be unstable). We refer to Section 2 for a precise statement.

There is a well-known connection between Floquet theory and the so-called Bloch spectral problems which, for any Bloch parameter $\theta \in (-1/2, +1/2]$, is concerned with the eigenvalues and eigenfunctions of

$$\left\{ \begin{array}{l} -(\nabla + i2\pi\theta)a(x)(\nabla + i2\pi\theta)\phi(x;\theta) + \Sigma(x)\phi(x;\theta) = \lambda(\theta)\sigma(x)\phi(x;\theta) \quad \text{in} \quad (0,1], \\ \qquad \phi(x+1;\theta) = \phi(x;\theta) \quad \forall x. \end{array} \right. \tag{1.2}$$

Indeed, the (closure of the) stability intervals of (1.1) are exactly the range of all eigenvalues of (1.2) when the Bloch parameter runs in $(-1/2, +1/2]$.

Our main contribution in this paper is to characterize in a similar way the instability intervals of (1.1). To do so, we introduce new spectral problems, similar to (1.2), that we call

*Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau Cedex, Paris (France). E-mail address: gregoire.allaire@polytechnique.fr

†Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid (Spain). E-mail address: rafael.orive@uam.es

exponential spectral problems. Actually there are two such families parameterized again by $\theta \in (-1/2, +1/2]$

$$\begin{cases} -(\nabla + 2\pi\theta)a(x)(\nabla + 2\pi\theta)\phi(x) + \Sigma(x)\phi(x;\theta) = \mu(\theta)\sigma(x)\phi(x;\theta) \quad \text{in} \quad (0,1], \\ \qquad \phi(x+1;\theta) = \phi(x;\theta) \quad \forall x, \end{cases} \tag{1.3}$$

and

$$\begin{cases} -(\nabla + 2\pi\theta)a(x)(\nabla + 2\pi\theta)\phi(x) + \Sigma(x)\phi(x;\theta) = \nu(\theta)\sigma(x)\phi(x;\theta) \quad \text{in} \quad (0,1], \\ \qquad \phi(x+1;\theta) = -\phi(x;\theta) \quad \forall x. \end{cases} \tag{1.4}$$

We prove that these non self-adjoint, compact problems on $L^2(0,1)$ admit a countable infinite number of eigenvalues (that may be complex), and that the (closure of the) instability intervals of (1.1) is precisely the range of their real eigenvalues. Problems (1.3) and (1.4) are coined "exponentials" because upon multiplication by $e^{2\pi\theta x}$ their solutions are (unbounded) solutions of (1.1).

There are many well-known motivations for studying (1.1) (see e.g. [8], [15]) but let us explain our motivation for introducing the exponential problems (1.3) and (1.4). A common feature of periodic media, perturbed by some defaults, is to exhibit localized solutions, i.e. solutions which decay exponentially away from the defaults. For example, this property is used in photonic crystals to create wave guides. A photonic crystal is a periodic media in which polarized electromagnetic waves propagate according to an equation of the type (1.1). Actually only those waves having a frequency compatible with the stability intervals of (1.1) do propagate, the other ones being quickly attenuated. However, if there is a perturbation of the periodic microstructure localized along a line or a plane, then it is possible that waves, having a forbidden frequency for the purely periodic crystal, propagate along the perturbation and decay exponentially away from it. In one space dimension, such localized solutions can be built by combining solutions of (1.3) and (1.4) in the periodic bulk. This idea is not new. It was used in control theory for exhibiting counter-examples to the uniform controllability of periodic media [2], or in diffusion problems to decide in which cases the fundamental mode is localized or not [1].

Another possible occurrence of (1.3) and (1.4) is in the study of boundary layers in periodic homogenization (see e.g. [3], [14]). A classical result states that the boundary layers stabilize exponentially fast away from the boundary. Once again the solutions of (1.3) and (1.4) could be used to build such boundary layers. Problem (1.3) has also some connections with homogenization theory for non self-adjoint equations (see section 5.8 in chapter 4 of [4], [6]). Nevertheless, our goal here is not to study applications but rather to give a detailed description of all solutions of the exponential spectral problems (1.3) and (1.4). The contents of the paper is as follows. Section 2 is a brief overview of classical results on Hill's equation. Similarly, Section 3 recalls known results in Bloch wave analysis. Our main results are given in Section 4.

## 2 Some results on Hill's equation

In this section we recall some classical results (see [8] or [15]). Let us first consider the following Hill's equation without parameter

$$-\{a(x)u'(x)\}' + q(x)u(x) = 0, \tag{2.1}$$

where we assume that $a(x)$ is real, uniformly coercive, i.e. $a(x) \geq a_0 > 0$, $q(x)$ is complex-valued, and both are piecewise continuous and have the same normalized period 1, i.e.,

$a(x+1) = a(x)$, $q(x+1) = q(x)$ for any $x \in \mathbb{R}$. The study of this equation is known as *Floquet theory* [9]. A first useful lemma in this work is (see [8] pp. 1):

**Lemma 2.1** *There exist at least one non-zero constant $\rho$ and one non-trivial solution $u$ of (2.1) which satisfy*

$$u(x+1) = \rho u(x), \qquad \forall x \in \mathbb{R}. \tag{2.2}$$

We are interested in Hill's equation when $q(x)$ involves a parameter $\lambda \in \mathbb{C}$ in the form

$$q(x) = \lambda\sigma(x) - \Sigma(x)$$

where $\sigma(x)$ and $\Sigma(x)$ are real-valued and piecewise continuous with period 1 and there is a constant $\sigma_0 > 0$ such that $\sigma(x) \geq \sigma_0$. Now, equation (2.1) is

$$-\{a(x)u'(x)\}' + \Sigma(x)u(x) = \lambda\sigma(x)u(x). \tag{2.3}$$

Considered as an ordinary differential equation, (2.3) has a basis of two linearly independent solutions that depend on the parameter $\lambda$. It is convenient to introduce the so-called normalized solutions $y_1(x,\lambda)$, $y_2(x,\lambda)$ of (2.3) which satisfy the following initial conditions

$$\begin{aligned} y_1(0,\lambda) &= 1, \qquad & a(0)y_1'(0,\lambda) &= 0, \\ y_2(0,\lambda) &= 0, \qquad & a(0)y_2'(0,\lambda) &= 1. \end{aligned} \tag{2.4}$$

The discriminant of (2.3) is defined as

$$D(\lambda) = y_1(1,\lambda) + a(1)y_2'(1,\lambda). \tag{2.5}$$

Recall that $a$ may be discontinuous, so $y'$ may also be discontinuous and only $ay'$ is continuous. Lemma 2.1 implies the following classical result on Hill's equation (see [8] pp. 5–9 or [15]):

**Lemma 2.2** *For $\lambda \in \mathbb{C}$, there exist two complex constants $\rho_1$ and $\rho_2$, called the characteristic multipliers of (2.3), such that $\rho_1\rho_2 = 1$, $D(\lambda) = \rho_1 + \rho_2$, and $\rho_1, \rho_2$ are continuous functions of $\lambda$. Moreover, there exist two non-trivial solutions $u_1$ and $u_2$ of (2.3) such that, if $\rho_1 \neq \rho_2$,*

$$u_1(x+1) = \rho_1 u_1(x) \quad \textit{and} \quad u_2(x+1) = \rho_2 u_2(x), \tag{2.6}$$

*and, if $\rho_1 \neq \rho_2$, either (2.6) is satisfied or*

$$u_1(x+1) = \rho_1 u_1(x) \quad \textit{and} \quad u_2(x+1) - \rho_1 u_2(x) = \rho_1 u_1(x). \tag{2.7}$$

When we restrict ourselves to $\lambda \in \mathbb{R}$, it is possible to classify the solutions of (2.3): a solution is said to be stable if it is uniformly bounded, and unstable if it is unbounded. According to [8], [15], there exist a countably infinite sequence $\{\alpha_n\}_{n\geq 1}$ of real roots of $D(\lambda) = 2$ and a countably infinite sequence $\{\beta_n\}_{n\geq 1}$ of real roots of $D(\lambda) = -2$ such that

$$-\infty < \alpha_1 < \beta_1 \leq \beta_2 < \alpha_2 \leq \alpha_3 < \beta_3 \leq \beta_4 < \cdots \to +\infty,$$

and the collection of disjoint open intervals

$$]\alpha_1, \beta_1[,\ ]\beta_2, \alpha_2[,\ ]\alpha_3, \beta_3[,\ ]\beta_4, \alpha_4[, \cdots$$

are called the stability intervals of (2.3). Their union is called the region of stability

$$S = ]\alpha_1, \beta_1[\cup]\beta_2, \alpha_2[\cup]\alpha_3, \beta_3[\cup]\beta_4, \alpha_4[\cup \cdots,$$

and

$$U = ]-\infty, \alpha_1[\cup]\beta_1, \beta_2[\cup]\alpha_2, \alpha_3[\cup]\beta_3, \beta_4[\cup \cdots$$

is called the unstable region. Accordingly, $|D(\lambda)| < 2$ for $\lambda \in S$, and $|D(\lambda)| > 2$ for $\lambda \in U$. As a consequence of Lemma 2.2 one can identify the stable and unstable solutions of (2.3), which is the central result of Floquet's theory [8].

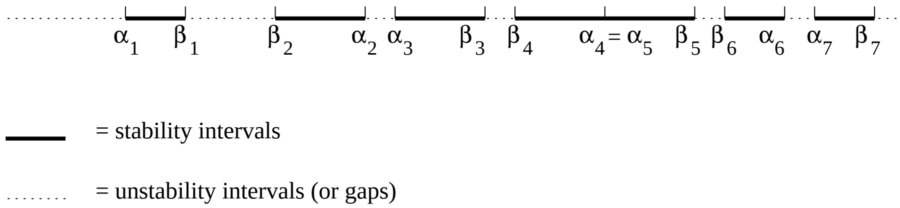


Figure 1: Intervals of stability or instability on the real line for Hill's equation (2.3). In this sketch we assume that there is no gap between the 4th and 5th Bloch bands.

**Theorem 2.3**

*(i) For $\lambda \in S$ there exists a unique $\theta \in ]0,1[$ such that (2.3) has two linear independent solutions of the type $p_1(x)e^{i2\pi\theta x}$ and $p_2(x)e^{-i2\pi\theta x}$ with $p_1$, $p_2$ 1-periodic, and $e^{i2\pi\theta}$, $e^{-i2\pi\theta}$ are the characteristic multipliers defined in Lemma 2.2.*

*(ii) For $\lambda \in U$ there exists a unique $\theta > 0$ such that (2.3) has two linear independent solutions of the type $p_1(x)e^{2\pi\theta x}$ and $p_2(x)e^{-2\pi\theta x}$ with either $p_1$, $p_2$ 1-periodic, or $p_1$, $p_2$ semi-periodic, and $e^{2\pi\theta}$, $e^{-2\pi\theta}$ (or $e^{2\pi\theta+i\pi}$, $e^{-2\pi\theta-i\pi}$, respectively) are the characteristic multipliers defined in Lemma 2.2.*

*In both cases $\theta$ depends continuously on $\lambda$.*

Recall that a function $p(x)$ is said to be semi-periodic if $p(x+1) = -p(x)$ for any $x \in \mathbb{R}$. The next result covers the case of the end points of the intervals in $S$ and $U$.

**Theorem 2.4**

*(i) If $\alpha_{2i} < \alpha_{2i+1}$, then, for $\lambda = \alpha_{2i}$ and for $\lambda = \alpha_{2i+1}$, (2.3) has two linearly independent solutions of the type $p_1(x)$ and $xp_1(x) + p_2(x)$ with $p_1$, $p_2$ 1-periodic. If $\alpha_{2i} = \alpha_{2i+1}$, then, for $\lambda = \alpha_{2i} = \alpha_{2i+1}$, (2.3) has two linearly independent 1-periodic solutions.*

*(ii) If $\beta_{2i-1} < \beta_{2i}$, then for $\lambda = \beta_{2i-1}$ and for $\lambda = \beta_{2i}$, (2.3) has two linearly independent solutions of the type $p_1(x)$ and $xp_1(x) + p_2(x)$ with $p_1$, $p_2$ semi-periodic. If $\beta_{2i-1} = \beta_{2i}$, then, for $\lambda = \beta_{2i-1} = \beta_{2i}$, (2.3) has two linearly independent semi-periodic solutions.*

**Remark 2.5** *If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then any solution of (2.3) is unstable (i.e. unbounded).* ▪

## 3 Bloch spectrum

In this section we recall other classical results of a somewhat different theory on the eigenvalue structure of (2.3), known as the Bloch decomposition theory [5]. If we consider (2.3) as a spectral problem posed in $L^2(\mathbb{R})$, it is a natural question to determine its spectrum. To this end, we consider the following Bloch spectral problem parameterized by $\theta \in \mathbb{R}$: find $\lambda = \lambda(\theta) \in \mathbb{R}$ and $\psi = \psi(x;\theta)$ (not identically zero) such that

$$\begin{cases} -\{a(x)\psi'(x;\theta)\}' + \Sigma(x)\psi(x;\theta) = \lambda(\theta)\sigma(x)\psi(x;\theta) \quad \text{in} \quad \mathbb{R}, \\ \psi(\cdot;\theta) \text{ is } (\theta,1)\text{-periodic, i.e, } \psi(x+m;\theta) = e^{i2\pi m\theta}\psi(x;\theta) \quad \forall m \in \mathbb{Z},\ x \in \mathbb{R}. \end{cases} \tag{3.1}$$

It is clear from (3.1) that the $(\theta,1)$ periodicity condition is unaltered if we replace $\theta$ by $(\theta+q)$ with $q \in \mathbb{Z}$ and $\theta$ can therefore be confined to the *dual cell* $\theta \in Y' = ]-1/2, 1/2]$. It is well known (see [4] and [7]) that for each $\theta \in ]-1/2, 1/2]$, the above spectral problem admits a discrete sequence of eigenvalues with the following properties

$$\begin{cases} 0 \leq \lambda_1(\theta) \leq \cdots \leq \lambda_m(\theta) \leq \cdots \to \infty, \\ \forall m \geq 1, \ \lambda_m(\theta) \text{ is a Lipschitz function of } \theta \in Y'. \end{cases}$$

We can write $\psi(x;\theta) = e^{ix2\pi\theta}\phi(x;\theta)$, $\phi$ being 1-periodic in the variable x. Thus, (3.1) is equivalent to:

$$\begin{cases} -(\nabla + i2\pi\theta)a(x)(\nabla + i2\pi\theta)\phi(x;\theta) + \Sigma(x)\phi(x;\theta) = \lambda(\theta)\sigma(x)\phi(x;\theta) \quad \text{in} \quad \mathbb{R}, \\ \qquad \phi(\cdot;\theta) \text{ is 1-periodic} \end{cases} \tag{3.2}$$

By taking the complex conjugate of (3.2) it is clear that $\lambda_m(\theta) = \lambda_m(-\theta)$ for any $m \geq 1$. Furthermore, by writing the min-max variational principle, $\lambda_m(\theta)$ is a Lipschitz function of $\theta$. We define the Bloch spectrum as

$$\sigma_B = \{\lambda_m(\theta) \mid \quad \theta \in Y' \quad m \geq 1\} = \bigcup_{m\geq 1} [\min_{\theta \in Y'} \lambda_m(\theta), \max_{\theta \in Y'} \lambda_m(\theta)].$$

The following result relates the Bloch spectrum with the stability intervals of (2.3).

**Proposition 3.1** *For any $i \geq 1$, $\alpha_i = \lambda_i(0)$, $\beta_i = \lambda_i(1/2)$, and*

$$\sigma_B = [\alpha_1, \beta_1] \cup [\beta_2, \alpha_2] \cup [\alpha_3, \beta_3] \cup [\beta_4, \alpha_4] \cup \cdots$$

*In particular, $\lambda_{2i-1}(0)$ (respectively $\lambda_{2i}(0)$) is the minimum of $\lambda_{2i-1}(\theta)$ (respectively maximum of $\lambda_{2i}(\theta)$), while $\lambda_{2i-1}(1/2)$ (respectively $\lambda_{2i}(1/2)$) is the maximum of $\lambda_{2i-1}(\theta)$ (respectively minimum of $\lambda_{2i}(\theta)$).*

**Proof**. For $\theta = 0$, the pair $(\lambda_m(0), \phi_m(x;0))$ satisfies (2.3) with $\lambda = \lambda_m(0)$ and $u(x) = \phi_m(x;0)$. Then, thanks to $(i)$ Theorem 2.4, we get that

$$\alpha_m = \lambda_m(0) \qquad \forall m \geq 1.$$

For $\theta = 1/2$, the pair $(\lambda_m(1/2), e^{ix\pi}\phi_m(x;1/2))$ satisfies (2.3) with $\lambda = \lambda_m(1/2)$ and $u(x) = e^{ix\pi}\phi_m(x;1/2)$ where $e^{ix\pi}\phi_m(x;1/2)$ is semi-periodic. Then, thanks to $(ii)$ Theorem 2.4, we get that

$$\beta_m = \lambda_m(1/2) \qquad \forall m \geq 1.$$

Now, let $\lambda \in ]\alpha_{2k+1}, \beta_{2k+1}[$ $k = 0, 1, \ldots$ Thanks to $(i)$ Theorem 2.3, there exists an unique $\theta \in ]0,1[$ such that (2.3) has two linear independent solutions of the type $p_1(x)e^{i2\pi\theta x}$ and $p_2(x)e^{-i2\pi\theta x}$ with $p_1$, $p_2$ 1-periodic. Then, $p_1$ and $p_2$ satisfy (3.2) with $\lambda(\theta) = \lambda$ and $\lambda(-\theta) = \lambda$, respectively. Thus, since $\lambda_{2k+1}(0) = \alpha_{2k+1}$, $\lambda_{2k+1}(1/2) = \beta_{2k+1}$ and $\lambda_{2k+1}(\theta)$ is a Lipschitz function of $\theta \in Y'$, we get

$$[\alpha_{2k+1}, \beta_{2k+1}] = \{\lambda_{2k+1}(\theta) \mid \quad \theta \in Y'\} \qquad k = 0, 1 \ldots$$

Analogously, we have

$$[\beta_{2k}, \alpha_{2k}] = \{\lambda_{2k+1}(\theta) \mid \quad \theta \in Y'\} \qquad k = 1, 2, \ldots,$$

and we conclude the proof. ▪

**Remark 3.2** *It is also known [12], [16], [17] and [18] that, upon a suitable relabeling, the eigenvalues $\lambda_m(\theta)$ are analytic (holomorphic) functions of $\theta$. If one insists in using the usual labeling by increasing order, then each $\lambda_m(\theta)$ is analytic except possibly at $\theta = 0$ or $\theta = 1/2$ when the eigenvalue is of multiplicity two.* ▪

## 4 Exponential spectrum

In this section we define the exponential spectrum of (1.1). Let us consider the following two spectral problems parameterized by $\theta \in \mathbb{R}$: find $\mu = \mu(\theta) \in \mathbb{C}$ and $\phi = \phi(x;\theta)$ (not identically zero) such that

$$\left\{ \begin{array}{l} -(\nabla + 2\pi\theta)a(x)(\nabla + 2\pi\theta)\phi(x) + \Sigma(x)\phi(x;\theta) = \mu(\theta)\sigma(x)\phi(x;\theta) \quad \text{in} \quad \mathbb{R}, \\ \qquad \phi(\cdot;\theta) \text{ is 1-periodic,} \end{array} \right. \tag{4.1}$$

and find $\nu = \nu(\theta) \in \mathbb{C}$ and $\phi = \phi(x;\theta)$ (not identically zero) such that

$$\left\{ \begin{array}{l} -(\nabla + 2\pi\theta)a(x)(\nabla + 2\pi\theta)\phi(x) + \Sigma(x)\phi(x;\theta) = \nu(\theta)\sigma(x)\phi(x;\theta) \quad \text{in} \quad \mathbb{R}, \\ \qquad \phi(\cdot;\theta) \text{ is semi-periodic.} \end{array} \right. \tag{4.2}$$

These problems are compact in $L^2(0,1)$ but not self-adjoint: for a given $\theta$ the adjoint of (4.1) or (4.2) is simply the same problem with the opposite parameter $-\theta$. Nevertheless, we shall prove that they admit a countable infinite number of eigenvalues (that may be complex). The range of their real eigenvalues is called the exponential spectrum of (1.1).

The existence of the first eigenvalue of problem (4.1) has already been addressed in [4], [6]. In particular, Lemma 4.1 in [6] (based on the Krein-Rutman theorem) implies:

**Lemma 4.1** *For any $\theta \in \mathbb{R}$, there exists a minimal first eigenvalue $\mu_1$ for (4.1) which is real, simple and such that*

$$\begin{array}{l} \theta \longrightarrow \mu_1(\theta) \qquad \textit{is analytic, concave and even, } \mu_1(\theta) = \mu_1(-\theta), \\ \lim\limits_{|\theta| \to +\infty} \mu_1(\theta) = -\infty, \\ \max\limits_{\theta \in \mathbb{R}} \mu_1(\theta) = \mu_1(0) = \lambda_1(0) = \alpha_1. \end{array}$$

This lemma shows that the first unstable interval of (2.3) is $\{\mu_1(\theta)\,|\theta \in \mathbb{R}^*\} = ]-\infty, \alpha_1)$ which is thus part of the exponential spectrum of (1.1). To continue with the description of the exponential spectrum structure, we use the following spectral continuity result (see e.g. [10] p. 14).

**Theorem 4.2** *Let $\{A_n\}_{n\geq 1}$ be a sequence of compact operators in a Hilbert space $H$ converging uniformly to a compact limit $A$. Let $\gamma$ be a smooth contour enclosing $j$ eigenvalues of $A$ (counting their multiplicity) and such that any $\lambda \in \gamma$ does not belong to the spectrum of $A$. Then, there exists $n_0$ such that, for any $n \geq n_0$, $\gamma$ contains exactly $j$ eigenvalues of $A_n$.*

We shall apply Theorem 4.2 to the Hilbert spaces $H = L^2(\mathbb{T})$ and $H = L^2(\mathbb{T}_s)$ which denotes the subspaces of $L^2_{loc}(\mathbb{R})$ made of 1-periodic or semi-periodic functions, respectively. Theorem 4.2 allows us to characterize the other unstable intervals of $U$. There are two different type of instability intervals: $[\beta_{2k-1}, \beta_{2k}]$ and $[\alpha_{2k}, \alpha_{2k+1}]$ for $k \geq 1$.

**Proposition 4.3** *Assume that $\beta_{2k-1} < \beta_{2k}$. Then, there exist $\theta_{2k-1,2k} > 0$ such that, for $\theta \in ]-\theta_{2k-1,2k}, \theta_{2k-1,2k}[$, there exist $\nu_{2k-1}(\theta)$ and $\nu_{2k}(\theta)$, real and simple eigenvalues of (4.2)* *which satisfy*

$$
\begin{aligned}
&\theta \longrightarrow \nu_{2k-1}(\theta), \nu_{2k}(\theta) \qquad \textit{are analytic and even},\\
&\nu_{2k-1}(\theta), \nu_{2k}(\theta) \qquad \textit{are strictly monotone on } [0, \theta_{2k-1,2k}[,\\
&\min_{\theta\in[0,\theta_{2k-1,2k}[} \nu_{2k-1}(\theta) = \nu_{2k-1}(0) = \lambda_{2k-1}\left(\frac{1}{2}\right) = \beta_{2k-1},\\
&\max_{\theta\in[0,\theta_{2k-1,2k}[} \nu_{2k}(\theta) = \nu_{2k}(0) = \lambda_{2k}\left(\frac{1}{2}\right) = \beta_{2k},\\
&\lim_{|\theta|\to\theta_{2k-1,2k}} \nu_{2k-1}(\theta) = \lim_{|\theta|\to\theta_{2k-1,2k}} \nu_{2k}(\theta) = \nu_{2k-1,2k},\\
&\{\nu_{2k-1}(\theta)\,|\theta \in [0, \theta_{2k-1,2k}]\} = [\beta_{2k-1}, \nu_{2k-1,2k}],\\
&\{\nu_{2k}(\theta)\,|\theta \in [0, \theta_{2k-1,2k}]\} = [\nu_{2k-1,2k}, \beta_{2k}].
\end{aligned}
$$

The assumption $\beta_{2k-1} < \beta_{2k}$ means that there is a gap in the spectrum. When $\beta_{2k-1} = \beta_{2k}$, i.e. there is no gap here, we extend the notations of Proposition 4.3 by taking $\theta_{2k-1,2k} = 0$.

**Proposition 4.4** *Assume that $\alpha_{2k} < \alpha_{2k+1}$. Then, there exist $\theta_{2k,2k+1} > 0$ such that, for $\theta \in ]-\theta_{2k,2k+1}, \theta_{2k,2k+1}[$, there exist $\mu_{2k}(\theta)$ and $\mu_{2k+1}(\theta)$, real and simple eigenvalues of (4.1) which satisfy*

$$
\begin{aligned}
&\theta \longrightarrow \mu_{2k}(\theta), \mu_{2k+1}(\theta) \qquad \textit{are analytic and even},\\
&\mu_{2k}(\theta), \mu_{2k+1}(\theta) \qquad \textit{are strictly monotone on } [0, \theta_{2k,2k+1}[,\\
&\min_{\theta\in[0,\theta_{2k,2k+1}[} \mu_{2k}(\theta) = \mu_{2k}(0) = \lambda_{2k}(0) = \alpha_{2k},\\
&\max_{\theta\in[0,\theta_{2k,2k+1}[} \mu_{2k+1}(\theta) = \mu_{2k+1}(0) = \lambda_{2k+1}(0) = \alpha_{2k+1},\\
&\lim_{|\theta|\to\theta_{2k,2k+1}} \mu_{2k}(\theta) = \lim_{|\theta|\to\theta_{2k,2k+1}} \mu_{2k+1}(\theta) = \mu_{2k,2k+1},\\
&\{\mu_{2k}(\theta)\,|\theta \in [0, \theta_{2k,2k+1}]\} = [\alpha_{2k}, \mu_{2k,2k+1}],\\
&\{\mu_{2k+1}(\theta)\,|\theta \in [0, \theta_{2k,2k+1}]\} = [\mu_{2k,2k+1}, \alpha_{2k+1}].
\end{aligned}
$$

If $\alpha_{2k} = \alpha_{2k+1}$ (no gap), we extend the notations of Proposition 4.4 by taking $\theta_{2k,2k+1} = 0$.

**Proofs**. We prove Proposition 4.4. The proof of Proposition 4.3 is similar.

First, we study the case of the eigenvalue problem (4.1) with $\theta = 0$. Thanks to $(i)$ Theorem 2.4, (4.1) with $\theta = 0$ has only two simple real eigenvalues in $[\alpha_{2k}, \alpha_{2k+1}]$ that we denote by $\mu_{2k}(0)(= \alpha_{2k})$ and $\mu_{2k+1}(0)(= \alpha_{2k+1})$, respectively.

Applying Theorem 4.2 in a vicinity of $\theta = 0$, problem (4.1) admits a unique simple eigenvalue $\mu_{2k}(\theta)$ close to $\mu_{2k}(0) = \alpha_{2k}$. Similarly (4.1) admits a unique simple eigenvalue $\mu_{2k+1}(\theta)$ close to $\mu_{2k+1}(0) = \alpha_{2k+1}$. If these eigenvalues were not real, since (4.1) has real coefficients, they would go by complex conjugate pairs, $\overline{\mu_{2k}(\theta)} \neq \mu_{2k}(\theta)$ or $\overline{\mu_{2k+1}(\theta)} \neq \mu_{2k+1}(\theta)$, which is a contradiction because Theorem 4.2 implies that there is one and only eigenvalue of (4.1) close to $\mu_{2k}(0)$ or $\mu_{2k+1}(0)$. Therefore, $\mu_{2k}(\theta)$ and $\mu_{2k+1}(\theta)$ are real and simple for $\theta$ near 0. Thanks to the perturbation theory of eigenvalues in a finite-dimensional space (see [12] pp. 62–63 and [17] pp. 29–39), simple eigenvalues are analytic (holomorphic) functions of $\theta$.

By $(ii)$ of Theorem 2.3, it is immediate that

$$\mu_{2k}(\theta) = \mu_{2k}(-\theta) \quad \text{and} \quad \mu_{2k+1}(\theta) = \mu_{2k+1}(-\theta).$$

Furthermore, $\theta \to \mu_{2k}(\theta), \mu_{2k+1}(\theta)$ are strictly monotone for $\theta > 0$ or $\theta < 0$, since otherwise we would have obtained two different values $\theta$ and $\widetilde{\theta}$ (having the same sign) yielding the same eigenvalue $\lambda \in ]\alpha_{2k}, \alpha_{2k+1}[$, which is impossible by $(ii)$ of Theorem 2.3.

Now, we can continue this perturbation argument for larger and larger values of $\theta$ until it breaks down, namely until the eigenvalues $\mu_{2k}(\theta), \mu_{2k+1}(\theta)$ remain simple and real. The only possibility for a change in their multiplicity is when they are equal. More precisely, let us prove that

$$\{\mu_{2k}(\theta) \mid \quad \text{s.t. } \theta > 0\} \cup \{\mu_{2k+1}(\theta) \mid \quad \text{s.t. } \theta > 0\} = ]\alpha_{2k}, \alpha_{2k+1}[. \tag{4.3}$$

We define

$$\theta_{2k} = \max\{\theta \mid \text{ s.t. there exists } \mu_{2k}(\theta) \in \mathbb{R}\}, \tag{4.4}$$

which satisfies $\theta_{2k} > 0$, and analogously we define $\theta_{2k+1} > 0$.

We can not have $\mu_{2k}(\theta_{2k}) < \mu_{2k+1}(\theta_{2k+1})$ because, for any $\lambda \in (\mu_{2k}(\theta_{2k}), \mu_{2k+1}(\theta_{2k+1}))$, part $(ii)$ of Theorem 2.3 implies the existence of $\theta > 0$ such that either $\lambda = \mu_{2k}(\theta)$ or $\lambda = \mu_{2k+1}(\theta)$, which is a contradiction with the definition of $\theta_{2k}$ or $\theta_{2k+1}$. Thus, $\mu_{2k}(\theta_{2k}) = \mu_{2k+1}(\theta_{2k+1})$, and by uniqueness of the exponent $\theta > 0$ in part $(ii)$ of Theorem 2.3 we also have $\theta_{2k} = \theta_{2k+1} < +\infty$ and we denote their common value by $\theta_{2k,2k+1} = \theta_{2k} = \theta_{2k+1}$. This proves (4.3). ∎

**Remark 4.5** *As a consequence of the results of Section 3 on the Bloch spectrum, of Lemma 4.1 and Propositions 4.3, 4.4, we obtain a complete characterization of (the closure of) the instability region (or equivalently the complement of the Bloch spectrum)*

$$\mathbb{R} = \sigma_B \cup \sigma_e, \quad \sigma_B \cap \sigma_e = \bigcup_{i=1}^{+\infty} \{\alpha_i\} \cup \{\beta_i\},$$

*where $\sigma_e$ is the exponential spectrum defined by*

$$\sigma_e = \{\mu_1(\theta) | \theta \in \mathbb{R}\} \bigcup_{k \geq 1} \{\mu_{2k}(\theta), \mu_{2k+1}(\theta) | \theta \in [0, \theta_{2k,2k+1}]\} \cup \{\nu_{2k-1}(\theta), \nu_{2k}(\theta) | \theta \in [0, \theta_{2k-1,2k}]\}.$$

Now, we study the eigenvalue problems (4.1) and (4.2) for $\theta > \theta_{2k-1,2k}$ and $\theta > \theta_{2k,2k+1}$, respectively.

**Proposition 4.6** *For $k \geq 1$, let $\theta_{2k-1,2k}$ and $\theta_{2k,2k+1}$ be defined in Propositions 4.3 and 4.4, respectively. Then,*

*(i) for $|\theta| > \theta_{2k,2k+1}$ there exist two complex (non-real), simple eigenvalues $\mu_{2k}(\theta)$ and $\mu_{2k+1}(\theta)$ of (4.1),*

*(ii) for $|\theta| > \theta_{2k-1,2k}$ there exist two complex (non-real), simple eigenvalues $\nu_{2k-1}(\theta)$ and $\nu_{2k}(\theta)$ of (4.2).*

*Moreover, $\theta \longrightarrow \mu_k(\theta), \nu_k(\theta)$ are analytic, even functions which satisfy*

$$\mu_{2k}(\theta) = \overline{\mu_{2k+1}(\theta)} \quad \text{and} \quad \nu_{2k-1}(\theta) = \overline{\nu_{2k}(\theta)},$$
$$\lim_{|\theta| \to \infty} |\mu_k(\theta)| = \infty \quad \text{and} \quad \lim_{|\theta| \to \infty} |\nu_k(\theta)| = \infty,$$

*and none of these branches of eigenvalues intersects another one.*

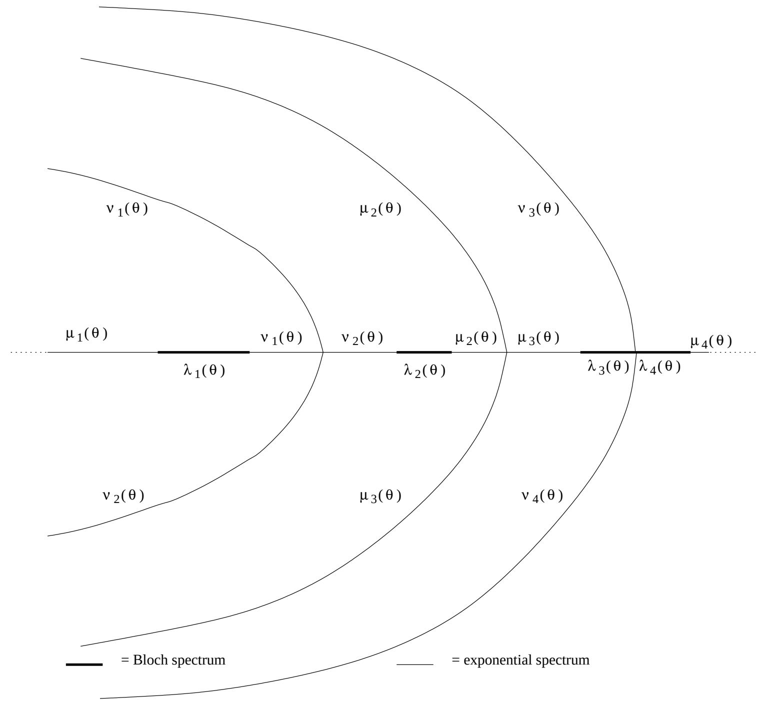


Figure 2: Spectrum of (4.1) and (4.2) in the complex plane, as the $\theta$ varies in $\mathbb{R}$. In this sketch we assume that there is no gap between the 3rd and 4th Bloch bands.

**Proof**. We prove $(i)$ (the proof of $(ii)$ is similar). First, we observe by Proposition 4.4 that, for $\theta_{2k,2k+1}$, (4.1) admits a double eigenvalue

$$\mu_{2k}(\theta_{2k,2k+1}) = \mu_{2k+1}(\theta_{2k,2k+1}).$$

Applying Theorem 4.2, in the vicinity of $\theta_{2k,2k+1}$ (4.1) admits two eigenvalues $\mu_{2k}(\theta)$ and $\mu_{2k+1}(\theta)$, which are continuous with respect to $\theta$. By definition of $\theta_{2k,2k+1}$, these two eigenvalues can not be real. Now, if $\mu_{2k}(\theta) \in \mathbb{C}$ is eigenvalue of (4.1), then $\overline{\mu_{2k}(\theta)}$ is also eigenvalue of (4.1). Therefore, $\mu_{2k+1}(\theta) = \overline{\mu_{2k}(\theta)}$ and both are simple non-real eigenvalues. Furthermore, by Lemma 2.2, $\mu_{2k}(\theta) = \mu_{2k}(-\theta)$.

We can reiterate the application of the perturbation argument of Theorem 4.2 and define two branches of eigenvalues $\mu_{2k}$ and $\mu_{2k+1}$ for any $\theta > \theta_{2k,2k+1}$.

Now, we check the behavior of $\mu_{2k}(\theta)$ as $\theta \to +\infty$ (the same argument applies to $\mu_{2k+1}(\theta)$). Assume that there exists a sequence $\theta_n \to \infty$ such that $\{|\mu_{2k}(\theta_n)|\}$ is bounded. Therefore, the sequence of normalized solutions of (2.3) $\{y_1(x, \mu_{2k}(\theta_n)), y_1(x, \mu_{2k}(\theta_n))\}$ and its derivative are bounded in $[0, 1]$. However, by definition of the discriminant in Lemma 2.2, we have that

$$D(\mu_{2k}(\theta)) = y_1(1, \mu_{2k}(\theta_n)) + a(1)y_2'(1, \mu_{2k}(\theta_n)) = e^{2\pi\theta} + e^{-2\pi\theta}.$$

Thus, $D(\mu_{2k}(\theta_n)) \to \infty$ as $\theta_n \to \infty$ and we get a contradiction.

Finally, we check that $\mu_{2k}(\theta)$ remains a simple eigenvalue for $\theta > \theta_{2k,2k+1}$. First, we remark that a branch of eigenvalues $\mu_j(\theta)$ can never intersects a branch of $\nu_l(\theta)$ for, if it were true, we would obtain both periodic and semi-periodic solutions of (2.3), which is impossible by virtue of Theorems 2.3 and 2.4. Second, $\mu_{2k}(\theta)$ and $\mu_{2k+1}(\theta)$ obviously do not intersect since $\mu_{2k+1}(\theta) = \overline{\mu_{2k}(\theta)}$. Then, two branches $\mu_j(\theta)$ and $\mu_l(\theta)$ (with $(j, l) \neq (2k, 2k+1)$) can not intersect because in between there is another branch $\nu_m(\theta)$ which can not cross them either. Therefore, since the branch $\mu_{2k}(\theta)$ starts at $\theta_{2k,2k+1}$ as simple and never meets another eigenvalue, it remains simple for all $\theta > \theta_{2k,2k+1}$. Finally, since they are simple, the eigenvalues $\mu_k(\theta)$ and $\nu_k(\theta)$ are analytic (holomorphic) functions of $\theta$. ▪

**Remark 4.7** *Globally, for $\theta \in \mathbb{R}$, thanks to the perturbation theory of eigenvalues in a finite-dimensional space (see [12] pp. 62–63 and [17] pp. 29–39), the eigenvalues $\mu_k(\theta)$ and $\nu_k(\theta)$ are analytic functions of $\theta$ with only algebraic singularities. In particular, each $\mu_{2k}(\theta)$ and $\nu_{2k}(\theta)$ are holomorphic except where the eigenvalue is of multiplicity two, i.e., in $|\theta| = \theta_{2k,2k+1}$ and $|\theta| = \theta_{2k-1,2k}$, respectively, which are the only possible singularities.* ▪

**Remark 4.8** *In higher space dimensions, some of our results (but not all of them) still hold true. Of course, in such a case the periodicity condition in (4.1) and the semi-periodicity condition in (4.2) have to be understood for each component of the space variable. In particular, Lemma 4.1 about the existence of the first eigenvalue is still valid [6]. To prove the existence of higher order eigenvalues of (4.1) and (4.2) we used a perturbation argument that still works in higher dimensions under additional assumptions. More precisely, if we assume that the bottom of the odd Bloch bands (respectively the top of the even bands) are uniquely attained at the Bloch parameter $\theta = 0$, that the bottom of the even bands (respectively the top of the odd bands) are uniquely attained at the Bloch parameter $\theta = (1/2, ..., 1/2)$, and further that the corresponding eigenvalues are simple, then our perturbation argument still allows to prove a local version of Propositions 4.3 and 4.4, i.e., the existence of simple real eigenvalues for $\theta$ close to 0.* ▪

# 5 Example: The model of Krönig-Penney

In this section we illustrate our previous analysis by computing the spectrum of the so-called Krönig-Penney model (see [13]). This is an example of Hill's equation for which there exists an infinite number of gaps or instability intervals (see [11] and [16] p. 381). We consider (1.1) with the following coefficients (extended by 1-periodicity)

$$a(x) = \begin{cases} 1, & 0 \le x < \delta, \\ a^2, & \delta \le x < 1, \end{cases} \qquad \Sigma(x) = \begin{cases} \Sigma_1, & 0 \le x < \delta, \\ \Sigma_2, & \delta \le x < 1, \end{cases} \qquad \sigma(x) = \begin{cases} \sigma_1, & 0 \le x < \delta, \\ \sigma_2, & \delta \le x < 1. \end{cases}$$

Without loss of generality we assume that $\Sigma_1/\sigma_1 < \Sigma_2/\sigma_2$. Clearly any solution $u$ of the Krönig-Penney equation (1.1) is continuous and smooth in each interval $(n, n+\delta)$ or $(n+\delta, n+1)$, for $n \in \mathbb{Z}$, and its derivative satisfies the following jump conditions:

$$a^2 u'(n+\delta^+) = u'(n+\delta^-), \qquad u'(n^+) = a^2 u'(n^-), \qquad \forall n \in \mathbb{Z}.$$

First, we consider the case $\lambda \in \mathbb{R}$ in (1.1). The spectrum of Krönig-Penney's model can easily be computed by solving the Cauchy problem for (1.1) with the two initial data (2.4). Introducing the $2 \times 2$ matrix

$$T = \begin{pmatrix} y_1(1,\lambda) & a^2 y_1'(1^-,\lambda) \\ y_2(1,\lambda) & a^2 y_2'(1^-,\lambda) \end{pmatrix}, \tag{5.1}$$

the trace of which is the discriminant $D(\lambda)$ defined by (2.5), $\lambda$ belongs to the Bloch spectrum (defined in Proposition 3.1) if and only if $T$ has one eigenvalue $\exp(i2\pi\theta)$ with $\theta \in (-1/2, +1/2]$. On the other hand, $\lambda$ belongs to the exponential spectrum (defined in Remark 4.5) if $T$ has one eigenvalue $\exp(2\pi\theta)$ or $-\exp(2\pi\theta)$ with $\theta \in \mathbb{R}$. In other words, if $-2 \le D(\lambda) \le 2$, then there exist a unique integer $m \ge 1$ and a unique Bloch parameter

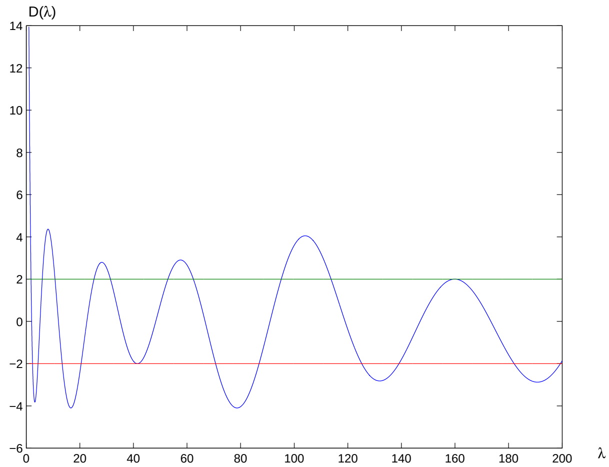


Figure 3: Discriminant $D(\lambda)$, defined by (2.5), when $\sigma_1 = \sigma_2 = 1$, $\Sigma_1 = \sqrt{2}$, $\Sigma_2 = \sqrt{5}$, $a = 0.25$, $\delta = 0.5$.

$\theta \in [0, +1/2]$ such that $\lambda = \lambda_m(\theta) = \lambda_m(-\theta)$, satisfying $2\cos(2\pi\theta) = D(\lambda)$. If $D(\lambda) < -2$, then there exist a unique integer $m \geq 1$ and a unique exponential parameter $\theta > 0$ such that $\lambda = \mu_m(\theta) = \mu_m(-\theta)$, satisfying $2\cosh(2\pi\theta) = D(\lambda)$. Finally, if $D(\lambda) > 2$, then there exist a unique integer $m \geq 1$ and a unique exponential parameter $\theta > 0$ such that $\lambda = \nu_m(\theta) = \nu_m(-\theta)$, satisfying $2\cosh(2\pi\theta) = -D(\lambda)$. The discriminate $D(\lambda)$ is plotted on Figure 3 for one instance of the coefficients. There are five regimes in the explicit formula for $D(\lambda)$ when $\lambda$ runs in $\mathbb{R}$.

**1.** Assume $\lambda \in (-\infty, \Sigma_1/\sigma_1)$. We define two real numbers

$$w_1 = \sqrt{\Sigma_1 - \lambda\sigma_1}, \qquad w_2 = \sqrt{\Sigma_2 - \lambda\sigma_2}. \tag{5.2}$$

In such a case the discriminant, defined by (2.5), is precisely

$$D(\lambda) = 2\cosh(w_1\delta)\cosh(w_2(1-\delta)/a) + 2\cosh(\log(aw_2/w_1))\sinh(w_1\delta)\sinh(w_2(1-\delta)/a),$$

which satisfies $D(\lambda) > 2$. Thus, $(-\infty, \Sigma_1/\sigma_1)$ is included in an unstable interval and in the exponential spectrum.

**2.** Assume $\lambda = \Sigma_1/\sigma_1$. In this case, the discriminant is

$$D(\Sigma_1/\sigma_1) = 2\cosh(w_2(1-\delta)/a) + aw_2\delta\sinh(w_2(1-\delta)/a) > 2,$$

with $w_2 = \sqrt{\Sigma_2 - (\Sigma_1\sigma_2/\sigma_1)}$. Thus, $\lambda = \Sigma_1/\sigma_1$ is unstable.

**3.** Assume $\lambda \in (\Sigma_1/\sigma_1, \Sigma_2/\sigma_2)$. Now, the discriminant is

$$D(\lambda) = 2\cos(w_1\delta)\cosh(w_2(1-\delta)/a) + 2\sinh(\log(aw_2/w_1))\sin(w_1\delta)\sinh(w_2(1-\delta)/a),$$

with

$$w_1 = \sqrt{\lambda\sigma_1 - \Sigma_1}, \qquad w_2 = \sqrt{\Sigma_2 - \lambda\sigma_2}.$$

Remark that the minimal eigenvalue $\alpha_1 = \lambda_1(0)$ of the Bloch spectrum belongs to this region, i.e., $\alpha_1 \in (\Sigma_1/\sigma_1, \Sigma_2/\sigma_2)$. Therefore, there is at least one stability interval that intersects $(\Sigma_1/\sigma_1, \Sigma_2/\sigma_2)$. The precise number of Bloch intervals in this region depends on the parameters of the model.

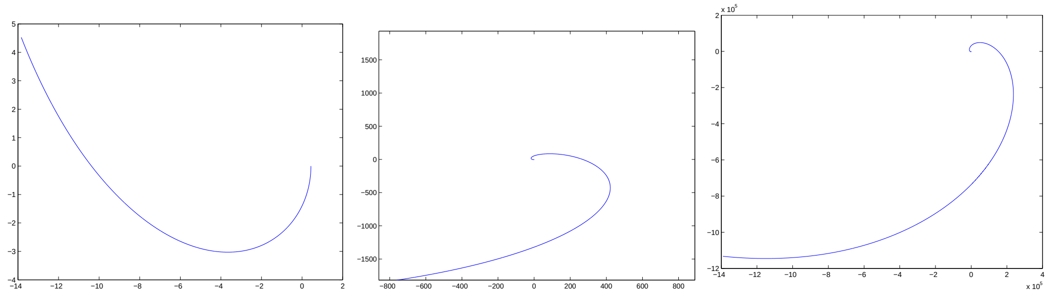

Figure 4: Three views in the complex plane of the function $\lambda_{imag} \to D(\lambda)/2$ when $\lambda_{real} = \Sigma_2/\sigma_2$, $a = \delta = 1/2$, $\sigma_1 = \sigma_2 = 1$ and $\Sigma_2 = 2$. We zoom the picture from right to left: $\lambda_{imag}$ runs into $(0, 200)$, $(0, 50)$, $(0, 10)$, and the unit scale of the graphic is $200,000$, $200$, $2$ respectively.

**4.** Assume $\lambda = \Sigma_2/\sigma_2$. The discriminant is

$$D(\Sigma_2/\sigma_2) = 2\cos(w_1\delta) - (1-\delta)w_1 a^{-2}\sin(w_1\delta),$$

with $w_1 = \sqrt{\Sigma_2\sigma_1/\sigma_2 - \Sigma_1}$.
**5.** Assume $\lambda \in (\Sigma_2/\sigma_2, \infty)$. The discriminant is

$$D(\lambda) = 2\cos(w_1\delta)\cos(w_2(1-\delta)/a) - 2\cosh(\log(aw_2/w_1))\sin(w_1\delta)\sin(w_2(1-\delta)/a),$$

with

$$w_1 = \sqrt{\lambda\sigma_1 - \Sigma_1}, \qquad w_2 = \sqrt{\lambda\sigma_2 - \Sigma_2}, \tag{5.3}$$

and there exists an infinite number of gaps in this region.

Second, we study the case $\lambda \in \mathbb{C} \setminus \mathbb{R}$ in (1.1). This allows us to compute the branches of eigenvalues in the complex plane for the exponential problems (4.1) and (4.2). The matrix $T$, defined by (5.1) is now complex valued, and $\lambda = \mu_m(\theta) = \mu_m(-\theta)$ if $T$ has a positive eigenvalue $\exp(2\pi\theta)$, while $\lambda = \nu_m(\theta) = \nu_m(-\theta)$ if $T$ has a negative eigenvalue $-\exp(2\pi\theta)$, with $\theta \in \mathbb{R}$.

Applying Lemma 2.2, there exist $\vartheta \in \mathbb{C}$ and two complex solutions $u_1$ and $u_2$ of (1.1) such that

$$u_1(x+1) = e^{2\pi\vartheta}u_1(x), \qquad u_2(x+1) = e^{-2\pi\vartheta}u_2(x),$$

and $e^{2\pi\vartheta}$ is an eigenvalue of $T$ which satisfies

$$\cosh(2\pi\vartheta) = D(\lambda)/2 = \cos(w_1\delta)\cos(w_2(1-\delta)/a) - \cosh(\log(aw_2/w_1))\sin(w_1\delta)\sin(w_2(1-\delta)/a), \tag{5.4}$$

where $w_1, w_2 \in \mathbb{C}$ are defined by (5.3). We are interested in finding those values of $\lambda \in \mathbb{C} \setminus \mathbb{R}$ such that $\cosh(2\pi\vartheta) \in \mathbb{R}$. If $\cosh(2\pi\vartheta) > 1$, then $\theta = \vartheta \in \mathbb{R}$ and $\lambda = \mu_k(\theta)$ for some $k \geq 2$. Otherwise, if $\cosh(2\pi\vartheta) < -1$, then $\theta = \vartheta - i/2 \in \mathbb{R}$ and $\lambda = \nu_k(\theta)$ for some $k \geq 1$.

Introducing $\lambda = \lambda_{real} + i\lambda_{imag}$, we draw on Figure 4 the function

$$\lambda_{imag} \longrightarrow D(\lambda)/2 \tag{5.5}$$

which looks like a spiral when $\lambda_{real}$ is fixed and $\lambda_{imag}$ runs into $\mathbb{R}^+$. If $\lambda_{real} \leq \nu_1(\theta_{1,2})$ (the largest first real eigenvalue of (4.2)), the intersection of this spiraling function with the real axis correspond to the following values of $\lambda$: $\nu_1(\theta_{\lambda_{real},1})$, $\mu_2(\theta_{\lambda_{real},2})$, $\nu_3(\theta_{\lambda_{real},3})$, $\mu_4(\theta_{\lambda_{real},4})$,

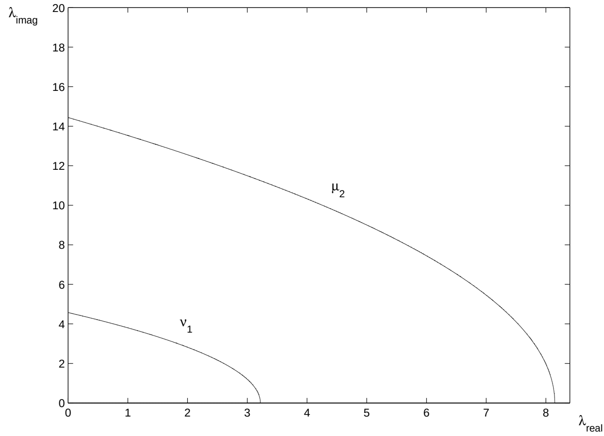


Figure 5: Branches of eigenvalues $\nu_1(\theta)$ and $\mu_2(\theta)$ in the complex plane when $\sigma_1 = \sigma_2 = 1$, $\Sigma_1 = \sqrt{2}$, $\Sigma_2 = \sqrt{5}$, $a = 0.25$, $\delta = 0.5$ (they should be symmetrized with respect to the real axis.

and so on, which all belong to the exponential spectrum. If $\nu_1(\theta_{1,2}) < \lambda_{real} \leq \mu_2(\theta_{2,3})$, the intersections with the real axis are obtained when $\lambda$ is equal to the exponential eigenvalues $\mu_2(\theta_{\lambda_{real},2})$, $\nu_3(\theta_{\lambda_{real},3})$, $\mu_4(\theta_{\lambda_{real},4})$, and so on. Increasing further $\lambda_{real}$ decreases step by step the number of obtained exponential eigenvalues. A similar result is obtained when $\lambda_{imag}$ runs into $\mathbb{R}^-$. By varying $\lambda_{real}$ and plotting the values of $\lambda_{imag}$ for which $D(\lambda)/2$ is real, we obtain the behavior of two first non-real branches of exponential eigenvalues $\nu_1(\theta)$ and $\mu_2(\theta)$ in Figure 5.


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