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A free Girsanov property for free Brownian motions

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Abstract

A "free Girsanov" property is proved for free Brownian motions. It is reminiscent of the classical Girsanov theorem in probability theory.

In the free probability context, we prove that if $(\sigma_s)_{s \in \mathbb{R}^+}$ is a free Brownian motion in (M, τ), if x is a process free from the σ_s, if $\tilde{\sigma_s} = \sigma_s + \int_0^s x(u)du$, then there is a trace $\tilde{\tau}$ such that$(\tilde{\sigma_s})_{s \in \mathbb{R}^+}$ is a free Brownian motion for $\tilde{\tau}$ and the two traces are "asymptotically equivalent". This means that τ resp. $\tilde{\tau}$ are asymptotic limits of states Ψ_n resp.$\tilde{\Psi}_n$ and that for each n $\tilde{\Psi}_n$ is obtained from Ψ_n by a change of probability given by an exponential density.

keywords

Free probability theory, Free products of C^* algebras, Free Brownian motion, Girsanov theorem.

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1 Introduction

The context of the present work is that of free probability theory. D.Voiculescu has introduced and studied the theory of free probability, giving a meaning to free random variables, free product of states and free Brownian motions (see the book by Voiculescu, Dykema and Nica [8], for a survey).

In classical probability theory the Girsanov theorem is a very important theorem for stochastic calculus (see for exemple [3]).

In view of stochastic calculus for free Brownian motions, Biane and

Speicher (see [1]) have proved an Ito formula for free stochastic integrals. The purpose of this paper is to obtain for free Brownian motions a property which is reminiscent of the classical Girsanov property.

The usual Girsanov theorem says that if one translates a Brownian motion by an adapted stochastic process ($\tilde{W}_s = W_s + \int\limits_0^s \theta(u)du$) one can find a change of probability given by an exponential density such that $(\tilde{W}_s)_{s\in I\!\!R^+}$ is a Brownian motion for this new probability.

In the context of free probability we want to prove a result which is in the same vein.

Let $(\sigma_s)_{s\in I\!\!R^+}$ be a free Brownian motion in (M,τ). Let x be a measurable process with values in N a commutative sub-algebra of M free from the $(\sigma_s)_{s\in I\!\!R^+}$. Assume that $x(u) = x(u)^*$ for all u. Let $\tilde{\sigma}_s = \sigma_s + \int\limits_0^s x(u)du$. We want to prove the existence of a new trace $\tilde{\tau}$ closely related to the trace τ such that $(\tilde{\sigma}_s)_{s\in I\!\!R^+}$ is a free Brownian motion for the new trace $\tilde{\tau}$ and such that the joint distribution of $(\tilde{\sigma}_s, x(u))_{s,u\in I\!\!R^+}$ for $\tilde{\tau}$ is the same as the joint distribution of $(\sigma_s, x(u))_{s,u\in I\!\!R^+}$ for τ

Unfortunately as the von Neumann algebra generated by a free Brownian motion is a factor, there is only one normalized trace on it. Thus it is impossible to find a new trace on the von Neumann algebra generated by N and the σ_s satisfying the required properties.

Nevertheless notice that a free Brownian motion $(\sigma_s)_{s\in I\!\!R^+}$ in (M,τ) is just defined by the joint distribution of the σ_s for τ. And Voiculescu has proved that a free Brownian motion is an asymptotic limit of matrices of random processes. Using this point of view, we prove the following result:

There is a new trace $\tilde{\tau}$ on $N * \mathbb{C}[(\sigma_s)_{s\in I\!\!R^+}]$ such that the joint distribution of $((\tilde{\sigma}_s)_{s\in I\!\!R^+}, x(u)_{u\in I\!\!R^+})$ for this new trace $\tilde{\tau}$ is the same as the joint distribution of $(\sigma_{s\in I\!\!R^+}, x(u)_{u\in I\!\!R^+})$ for the trace τ (in particular $(\tilde{\sigma}_s)_{s\in I\!\!R^+}$ is a free Brownian motion for the new trace) and the two traces are asymptotically equivalent

This has the following meaning:

-There is a family $(\tilde{Z}_n(s))_{n\in I\!\!N^*}$ of matrices of random processes $\tilde{Z}_n(s) \in \mathcal{M}_n(L^\infty[0,1] * L)$ and a family $(D_n(u))_{n\in I\!\!N^*}$ of diagonal matrices of real processes

such that

$$(\mathbb{C}[\sigma_s, x(u)]_{s,u\in\mathbb{R}^+}, \tilde{\tau}) = \lim_{n\to\infty}(\mathbb{C}[\tilde{Z}_n(s), D_n(u)]_{s,u\in\mathbb{R}^+}, \tilde{\Psi}_n)$$

and

$$(\mathbb{C}[\sigma_s, x(u)]_{s,u\in\mathbb{R}^+}, \tau) = \lim_{n\to\infty}(\mathbb{C}[\tilde{Z}_n(s), D_n(u)]_{s,u\in\mathbb{R}^+}, \Psi_n)$$

where $\tilde{\Psi}_n$ and Ψ_n are two traces on $\mathcal{M}_n(L^\infty[0,1] * L)$. $\tilde{\Psi}_n$ is obtained from Ψ_n by a change of probability given by an exponential density h_n

$$\Psi_n = \frac{1}{n} Tr_n(\phi * \phi_0)$$

and

$$\tilde{\Psi}_n = \frac{1}{n} Tr_n(\phi * \phi_0(h_n.))$$

(and for all $p \in \mathbb{N}$, $\sup_{n\in\mathbb{N}} \phi_0(h_n^p) < \infty$) i.e. the limit joint distribution of $(\tilde{Z}_n(s), D_n(u))$ for $\tilde{\Psi}_n$ is the joint distribution of $(\sigma_s, x(u))$ for $\tilde{\tau}$ and the limit joint distribution of $(\tilde{Z}_n(s), D_n(u))$ for Ψ_n is the joint distribution of $(\sigma_s, x(u))$ for τ.

In order to prove this result we make use, as already mentioned, of the asymptotic model of matrices of random processes, and we modelize the process $(x(u))$ by diagonal matrices. For each $n \in \mathbb{N}$ we can apply the classical Girsanov theorem and this gives rise to a change of probability given by an exponential density d_n. Unfortunately, these densities d_n explode as n tends to infinity and so we have to renormalize the asymptotic model of matrices of random processes in order to get densities h_n which do not explode.

The paper is organised as follows:

After a few recalls in section 2, we construct in section 3, a new asymptotic model of random matrices with values in a free product algebra, in order to make the renormalization. This is a technical part making use of computation of free cumulants and non crossing partitions introduced by Speicher [5].

In section 4 making use of this new asymptotic model, we prove our main result: A free Girsanov property for free Brownian motions.

2 Some recalls

Free probability theory

We recall some definitions and results in free probability theory which can be found in the references [6],[7], [8].

Def: A $*$free probability space (A, ϕ) is an involutive unital algebra A over $\mathbb{C}$ with a state $\phi : A \to \mathbb{C}$ i.e. a linear functional such that $\phi(1) = 1$ and $\phi(x^*) = \overline{\phi(x)}$. Elements of A are called random variables.

Def: A family $(f_i)_{i\in I}$ of random variables of A is free if the family $(A_i)_{i\in I}$ of $*$algebras generated by 1 and f_i is free: i.e. if $\phi(a_1 a_2 ... a_n) = 0$ whenever $a_j \in A_{i(j)}$ with $i(j) \neq i(j+1)$ $(1 \leq j \leq n-1)$ and $\phi(a_j) = 0$ $(1 \leq j \leq n)$.

Def: A random variable σ in (A, ϕ) is semicircular centered of variance r^2 if the distribution of σ is

$$\phi(\sigma^\alpha) = \frac{2}{\pi r^2} \int_{-r}^{r} t^\alpha \sqrt{r^2 - t^2} dt$$

Def: A free Brownian motion in (A, ϕ) is a family $(\sigma_s)_{s\in \mathbb{R}^+}$ of random variables such that

- $\sigma_0 = 0$
- if $0 \leq s' \leq s \leq t$, $\sigma_t - \sigma_s$ is semicircular centered of variance $t - s$ and is free from $\sigma_{s'}$.

One has also the following very important connection between free semicircular random variables and Gaussian random matrices:

Consider a probability space $(\Sigma, d\sigma)$. $L^\infty(\Sigma, d\sigma)$ is a unital algebra with the state ϕ_0 defined by $\phi_0(f) = E_0(f) = \int_\Sigma f d\sigma$. Let

$$L = \bigcap_{p \geq 1} L^p(\Sigma).$$

We denote by ϕ_n the state defined on $\mathcal{M}_n(L)$ by

$$\phi_n \left(\sum_{1\leq i,j\leq n} b_{ij} e(i,j,n) \right) = \frac{1}{n} \sum_{1\leq i\leq n} (\phi_0)(b_{ii}) = \frac{1}{n} Tr_n((\phi_0(b_{i,j})_{1\leq i,j\leq n}))$$

(where $(e(i,j,n))_{1\leq i,j\leq n}$ is the canonical basis and $b_{i,j} \in L$)

Voiculescu has then proved the following theorem(theorem 2.2 of [7])

let $Y(s,n) = \sum_{1\leq i,j\leq n} a(i,j,s,n)e(i,j,n)$ with $a(i,j,s,n) \in L$. Assume that

$$a(i,j,s,n) = \overline{a(j,i,s,n)}$$

and that $Re(a(i,j,s,n)), 1 \leq i \leq j \leq n, s \in I\!N, Im(a(i,j,s,n)), 1 \leq i < j \leq n, s \in I\!N$ are independent Gaussian random variables such that

$$E_0(a(i,j,s,n)) = 0$$

$$E_0(Re(a(i,j,s,n))^2) = \frac{1}{2n}\ for\ 1 \leq i < j \leq n$$

$$E_0(Im(a(i,j,s,n))^2) = \frac{1}{2n}\ for\ 1 \leq i < j \leq n$$

and

$$E_0((a(i,i,s,n))^2) = \frac{1}{n}\ for\ 1 \leq i \leq n$$

Consider the trace ϕ_n defined above. Let $D(j,n)$ be elements in Δ_n, the set of constant diagonal matrices, such that $\sup_{n\in I\!N} \|D(j,n)\| < \infty$, for each j; and such that for all j, $(D(j,n))$ has a limit distribution as $n \to \infty$. Then the family of subsets of ramdom variables $\{Y(s,n), s \in I\!N\}\{D(j,n), j \in I\!N\}$ is asymptotically free, and the limit distributions of the $Y(s,n)$ are semicircle laws as $n \to \infty$.

It follows that a model for the free Brownian motion is the following one:

$$(I\!\!C[(\sigma_s)_{s\in I\!R^+}], \tau) = \lim_{n\to\infty} I\!\!C[(B_{n,s})_{s\in I\!R^+}], \frac{1}{n}Tr_n(\phi_0)]$$

where $B_{n,s} = (\frac{1}{\sqrt{n}} W_{n,i,j,s})_{1\leq i,j\leq n}$; the $(W_{n,i,j,s})_{1\leq i\leq j\leq n}$ being independent Brownian motions.

Classical Girsanov theorem

For this we refer to Karatzas and Shreve [3].

Let $(\Omega, (\mathcal{F}_s)_{0\leq s}, P)$ be a filtered probability space. Let $(W_s)_{0\leq s}$ be a Brownian motion adapted to $(\mathcal{F}_s)$. let $(\theta_u)_{0\leq u}$ be an adapted process such that

$$E(\exp \int_0^\infty \theta_u^2 du) < \infty$$

Then $\tilde{W}_s = W_s - \int_0^s \theta_u du$ is a Brownian motion for the probability Q equivalent to the probability P defined by $Q(A) = \int_A Z(s)dP$ for all A in $\mathcal{F}_s$, where $Z(s) = \exp(\int_0^s \theta_u dW_u - \frac{1}{2}\int_0^s (\theta_u)^2 du)$.

3 A new asymptotic model for the free Brownian motion

In this section we construct for the free Brownian motion an asymptotic model of random matrices with coefficients in a free product algebra. The motivation for the construction of this new model is to use a free product algebra in order to make a renormalization.

Consider a probability space $(\Sigma, d\sigma)$. $L^\infty(\Sigma, d\sigma)$ is a unital algebra with the state ϕ_0 defined by $\phi_0(f) = E_0(f) = \int f d\sigma$. Let

$$L = \bigcap_{p \geq 1} L^p(\Sigma)$$

Let μ be the Lebesgue measure on $[0,1]$. Consider the state ϕ defined on $L^\infty([0,1], \mu)$ by

$$\phi(f) = \int f d\mu$$

Now we consider the free product state $\phi * \phi_0$ on $L^\infty([0,1], \mu) * L^\infty(\Sigma, d\sigma)$. We can extend $\phi * \phi_0$ to $L^\infty([0,1]) * L$. We then get a state still noted $\phi * \phi_0$ such that $L^\infty([0,1])$ is free from L for this state. We denote Ψ_n the state defined on $\mathcal{M}_n(L^\infty([0,1]) * L)$ by

$$\Psi_n\left(\sum_{1 \leq i,j \leq n} b_{ij} e(i,j,n)\right) = \frac{1}{n} \sum_{1 \leq i \leq n} (\phi * \phi_0)(b_{ii}) = \frac{1}{n} Tr_n((\phi * \phi_0)(b_{i,j})_{1 \leq i,j \leq n})$$

We keep the same notations as in section 2.

We now prove the existence of a new family of matrices of random processes with coefficients in $L^\infty([0,1]) * L$ which are asymptotically free and whose limit distributions are semi-circular laws. More precisely:

Proposition 3.1 *For all $s \in I\!N$, and $n \in I\!N$, let*

$$\tilde{Y}(s,n) = \sum_{1 \leq i,j \leq n} \tilde{a}(i,j,s,n) e(i,j,n)$$

with $\tilde{a}(i,j,s,n) \in L^\infty([0,1]) * L$. Assume that

$$\tilde{a}(i,j,s,n) = \sum_{k=1}^{n^2} q_{k,n}\sqrt{n}a(i,j,s,n)\ q_{k,n}$$

where the $q_{k,n}$ are orthogonal projectors in $L^\infty[0,1]$, $\sum_{k=1}^{n^2} q_{k,n} = 1$ such that

$$\phi(q_{k,n}) = \frac{1}{n^2}$$

and the $(a(i,j,s,n))_{s\in I\!N, 1\le i\le j\le n, n\in I\!N}$ are independent real normal Gaussian variables, i.e.,in particular

$$E_0(a(i,j,s,n)) = 0$$

$$E_0((a(i,j,s,n)^2) = 1$$

And

$$a(i,j,s,n) = a(j,i,s,n)$$

Consider the trace Ψ_n defined above. Let $D_n(j)$ be elements in Δ_n, the set of diagonal matrices, such that $\sup_{n\in I\!N} \|D_n(j)\| < \infty$, for each j; and such that for all j, $(D_n(j))$ has a limit distribution as $n \to \infty$.

Then the family of subsets $\{\tilde{Y}(s,n), s \in I\!N\}$ and $\{D_n(j), j \in I\!N\}$ are asymptotically free, and the limit distribution of the $\tilde{Y}(s,n)$ are semicircular laws.

This proposition is comparable with the theorem 2.2 of Voiculescu [7] recalled in section 2. The important property of this new asymptotic model is that it is renormalized: we have replaced the Gaussian random variables of variance $\frac{1}{n}$ of the theorem of Voiculescu by Gaussian random variables $(\sqrt{n}a(i,j,s,n))$ of variance n.

Although the proof follows the same lines of reasonning, the proof of Voiculescu must be significantly amended because the entries of these new matrices are in a free product algebra. We have to use the free calculus developped by Speicher [5].

We start with the following results.

Lemma 3.2 *Let $i \in \{1,..,j\}$; let $(y_1, y_2, ..., y_j)$ be random variables in L. Assume there is one $i \in \{1,..,j\}$ such that $y_i = az_i$, where a is independent of all others y_k for $k \neq i$ and of z_i, and such that $E_0(a) = 0$.*
Then $(\phi * \phi_0)(qy_1q...qy_jq) = 0$ for each q projector in $L^\infty([0,1])$.

Proof:

The proof is done by recurrence on j using the freeness.

For $j=1$:

$(\phi*\phi_0)(qy_1)=\phi(q)E_0(y_1)=\phi(q)E_0(a)E_0(z_1)=0.$

Assume now that the result is true for j and prove it for $j+1$. From the freeness of $L^\infty([0,1])$ and L for $\phi*\phi_0$, we get that

$$(\phi*\phi_0)((q-\phi(q))(y_1-E_0(y_1))(q-\phi(q))(y_2-E_0(y_2))...(q-\phi(q))(y_{j+1}-E_0(y_{j+1})))=0$$

If we develop the preceeding expression there is the term $(\phi*\phi_0)(qy_1qy_2...qy_{j+1})$, and in all the other terms there is at least one $\phi(q)$ or one $E_0(y_k)$. So that all these terms can be written either $\alpha(\phi*\phi_0)(qt_1qt_2...qt_k)$ with $k\leq j$ and $\alpha\in\mathbb{C}$; and the $(t_l)_{l\leq k}$ satisfie the same hypothesis as the y_l or $E_0(y_i)\alpha$ with $\alpha\in\mathbb{C}$ By recurrence each of these terms is equal to 0. So $(\phi*\phi_0)(qy_1qy_2...qy_{j+1})=0$. i.e. $(\phi*\phi_0)(qy_1qy_2...qy_{j+1}q)=0$ as $\phi*\phi_0$ is a trace and $q^2=q$.

Q.E.D.

Corollary 3.3

$$\Psi_n(\tilde{Y}(s_1,n)D_n(t_1)...\tilde{Y}(s_m,n)D_n(t_m))=0 \ \ if \ \ s_i\neq s_1 \ \ \forall i\neq 1.$$

Proof:

$\Psi_n(\tilde{Y}(s_1,n)D_n(t_1)...\tilde{Y}(s_m,n)D_n(t_m))$ is a sum of terms

$$(\phi*\phi_0)(q\sqrt{n}a(i_1,j_1,s_1,n)d_1q\sqrt{n}a(i_2,j_2,s_2,n)d_2q...(q\sqrt{n}a(i_m,j_m,s_m,n)d_m).$$

$a(i_1,j_1,s_1,n)$ is independent of all others $a(i_k,j_k,s_k,n)$ for $k\neq 1$. And $E_0(a(i_1,j_1,s_1,n)=0$. So the result follows immediately from lemma 3.2

We prove now the following technical lemma making use of the computation of free cumulants introduced by Speicher [5].

Lemma 3.4 *Let $y_1,...,y_j\in L$. Let $a\in L$, with $E_0(a)=0$. Assume that a is independent of $y_1,...,y_j$. Let q be a projector in $L^\infty[0,1]$. Denote $Y=y_1qy_2q...qy_j$.*

1. $k_\phi(q,q)=\phi(q)-\phi(q)^2$
2. $k_{(\phi*\phi_0)}(q,Y)=(\phi*\phi_0)(qY)-\phi(q)(\phi*\phi_0)(Y)$

3. $k_{(\phi*\phi_0)}(q,q,Y) = (1-2\phi(q))[(\phi*\phi_0)(qY) - \phi(q)(\phi*\phi_0)(Y)]$

4. $(\phi*\phi_0)(a^2Y) = E_0(a^2)(\phi*\phi_0)(Y)$

5. $k_{(\phi*\phi_0)}(a,a,Y) = 0$

Proof of the lemma:

1. $\phi(q) = \phi(q^2) = \phi(q)^2 + k_\phi(q,q)$. Q.E.D.

2. $(\phi*\phi_0)(qY) = k_{(\phi*\phi_0)}(q,Y) + \phi(q)(\phi*\phi_0)(Y)$. Q.E.D.

3.
$$(\phi*\phi_0)(qY) = (\phi*\phi_0)(q^2Y)$$
$$= k_\phi(q,q)(\phi*\phi_0)(Y) + 2\phi(q)k_{\phi*\phi_0}(q,Y) + k_{\phi*\phi_0}(q,q,Y) + \phi(q)^2(\phi*\phi_0)(Y).$$
Using 1. and 2. we get 3.

4. From [5] as $L^\infty[0,1]$ and L are free for $\Phi*\phi_0$, we know that the cumulants mixing elements of $L^\infty[0,1]$ and L are 0; and furthermore, as $E_0(a)=0$, for a non crossing partition giving a non zero contribution, a cannot be alone. So
$$(\phi*\phi_0)(a^2Y) = (\phi*\phi_0)(a^2y_jy_1qy_2q...y_{j-1}q)$$
$$= \sum_\pi k_\pi(q,q...q)k_\pi(q,...q)k_\pi(a^2y_jy_1,y_{i_1},..)k_\pi(y_{i_l},...)...$$
But a is independent of all the y_i so
$$k_\pi(a^2y_jy_1,y_{i_1},..) = E_0(a^2)k_\pi(y_jy_1,y_{i_1},..)$$
Q.E.D.

5. $(\phi*\phi_0)(a^2Y) = k_{(\phi*\phi_0)}(a,a)(\phi*\phi_0)(Y) + k_{(\phi*\phi_0)}(a,a,Y)$.
$E_0(a^2) = (\phi*\phi_0)(a^2) = k_{(\phi*\phi_0)}(a,a)$ as $E_0(a)=0$.
It follows then from 4. that $k_{(\phi*\phi_0)}(a,a,Y)=0$. Q.E.D.

Lemma 3.5 *Let a q and $Y \in L^\infty[0,1]*L$ as in lemma 3.4. Then*
$$(\phi*\phi_0)(qaqaqY) = \phi(q)E_0(a^2)(\phi*\phi_0)(qY)$$

Proof:

$$Y = y_1 q y_2 q ... q y_j$$

$$(\phi * \phi_0)(qaqaqY) = \sum_{\pi \in NC(6)} k_\pi(q, a, q, a, q, Y)$$

Using the same arguments as in the proof of lemma 3.4, and also the equality $k_{\phi * \phi_0}(a, a, Y) = 0$, we get

$$(\phi*\phi_0)(qaqaqY) = k_{\phi_0}(a, a)\phi(q)^3(\phi*\phi_0)(Y) + k_{\phi_0}(a, a)\phi(q)k_\phi(q, q)(\phi*\phi_0)(Y) +$$

$$k_{\phi_0}(a, a)\phi(q)k_{(\phi*\phi_0)}(q, q, Y) + 2k_{\phi_0}(a, a)\phi(q)^2 k_{(\phi*\phi_0)}(q, Y)$$

And now the result follows easily from the lemma 3.4, and the equality $k_{\phi_0}(a, a) = E_0(a^2)$.

Lemma 3.6 *Let $a(i, j, s, n)_{(1 \leq i \leq j \leq n)}$ be independent normal Gaussian variables in (L, ϕ_0). Let (B, ϕ) a $*$ free probability space.*

Let $q \in B$ be a projector such that $\phi(q) = \frac{1}{n^2}$. Let $d(t, j, n)$ be elements in B commuting with q and uniformly bounded. Then

$$\phi * \phi_0(qa(i_1, i_2, s_1, n)d(t_1, i_2, n)qa(i_2, i_3, s_2, n)d(t_2, i_3, n)...$$

$$qa(i_m, i_1, s_m, n)d(t_m, i_1, n)) =$$

$$\sum_\pi k_\pi[\ a(i_1, i_2, s_1, n),\ a(i_2, i_3, s_2, n), ...\ a(i_m, i_1, s_m, n)]O((\frac{1}{n^2})^{|\pi_B|})$$

where $|\pi_B|$ denotes the number of blocks of the restriction π_B of π to B;

Denote E_n the set of $(i_1, i_2, ...i_m) \in \{1, ..., n\}^m$ such that

$$(\phi * \phi_0)(qa(i_1, i_2, s_1, n)d(t_1, i_2, n)qa(i_2, i_3, s_2, n)d(t_2, i_3, n)q...$$

$$qa(i_m, i_1, s_m, n)d(t_m, i_1, n)) \neq 0;$$

$Card(E_n) = O(n^{\frac{m}{2}+1})$.

Proof:

We use another time the free cumulants to compute

$$(\Phi * \phi_0)(qa(i_1, i_2, s_1, n)d(t_1, i_2, n)qa(i_2, i_3, s_2, n)d(t_2, i_3, n)...$$
$$qa(i_m, i_1, s_1, n)d(t_m, i_1, n))$$

Using the hypothesis on the independence of the $a(i, j, s, n)$ and the lemma 3.2, it follows exactly as in the proof of theorem 2.2 of [7] that $Card(E_n) = O(n^{\frac{m}{2}+1})$.

Now, from the theorem 8.2 of [5] as the $a(i, j, s, n)$ are free from B for $\phi * \phi_0$, and $d(t, j, n)$ and q are in B, we can write:

$$(\phi * \phi_0)(qa(i_1, i_2, s_1, n)d(t_1, i_2, n)qa(i_2, i_3, s_2, n)d(t_2, i_3, n)...$$
$$qa(i_m, i_1, s_m, n)d(t_m, i_1, n)) =$$
$$\sum_{\pi} k_\pi[\ a(i_1, i_2, s_1, n), ..., a(i_m, i_1, s_m, n)]\phi_{\pi_B}[d(t_m, i_1, n)q, ..., d(t_{m-1}, i_m, n)q]$$

But $\phi_{\pi_B}[d(t_m, i_1, n)q, ..., d(t_{m-1}, i_m, n)q] = O((\frac{1}{n^2})^{|\pi_B|})$ where $|\pi_B|$ is the number of blocks of π_B (as the $d(t, j, n)$ are uniformly bounded).

Q.E.D.

Lemma 3.7 *Let k fixed. If π is a non crossing partition giving a non zero contribution in lemma 3.6, the number of different blocks of π_B (i.e. $|\pi_B|$) is greater or equal to $[\frac{m}{2}] + 1$.*

Proof:

We do it by recurrence on m.

- if $m = 1$, we always obtain 0.
- if $m = 2$, if the term associated to the partition π is non zero, the number of components containing the q is 2 (because $E_0(a(i, j, s, n)) = 0$).
- let $m \geq 2$. Assume that the result is true for m and prove it for $m + 1$. Let r be the minimal distance between two q which are in a same block of π. Two successive q can never be in the same component of π,(because $E_0(a(i, j, s, n)) = 0$). So $2 \leq r$. So there is l such that the

$l^{th}q$ and the $(l+r)^{th}q$ are in the same block and all the q between are alone in one block of π. As the cumulants must be non crossing, π can be decomposed in a partition π' on

$$(a(i_l, i_{l+1}, s_l, n), a(i_{l+1}, i_{l+2}, s_{l+1}, n), ..., a(i_{l+r-1}, i_{l+r}, s_{l+r-1}, n))$$

and a non crossing partition π'' on

$$(d(t_m, i_1, n)q, a(i_1, i_2, s_1, n), d(t_1, i_2, n)q, ...,$$

$$a(i_{l-1}, i_l, s_{l-1}, n), (a(i_{l+r}, i_{l+r+1}, s_{l+r}, n), d(t_{l+r}, i_{l+r+1}, n)q, .., a(i_m, i_1, s_m, n))$$

and blocks reduced to q.

It follows that the blocks of π_B are either reduced to one element q or are blocks of the restriction of the partition π'' to B. By recurrence, we know that the number of components of π'' containing the q is greater or equal to $[\frac{m-r}{2}]+1$. So the number of components of π_B is greater or equal to $[\frac{m-r}{2}]+1+r-1$; and as $2 \leq r$, $[\frac{m}{2}]+1 \leq [\frac{m-r}{2}]+1+r-1$.

Q.E.D.

Before proving the proposition 3.1, we give two other lemmas.

Lemma 3.8 *There is a constant C_m (independent of n) such that for all π, for all (i_k, j_k, s_k),*

$$|k_\pi[a(i_1, j_1, s_1, n), ...a(i_m, j_m, s_m, n)]| \leq C_m$$

Proof:

As two $a(i,j,s,n)_{i \leq j \leq n}$ are either equal or independent and $E_0(a(i,j,s,n)) = 0$, and $E_0((a(i,j,s,n)^2) = 1$, it follows that

$$|\phi_0(a(i'_1, j'_1, s'_1, n)...a(i'_l, j'_l, s'_l, n))| \leq 1$$

We now prove by recurrence on l that there is a constant C_l such that for each block of length l,

$$|k_l[a(i'_1, j'_1, s'_1, n), ..., a(i'_l, j'_l, s'_l, n)]| \leq C_l$$

- $l=1$
$$k_1[a(i'_1,j'_1,s'_1,n)] = \phi_0(na(i'_1,j'_1,s'_1,n)) = 0$$

- $l=2$
$$k_2[a(i'_1,j'_1,s'_1,n),a(i'_2,j'_2,s'_2,n)]$$
$$= \phi_0(a(i'_1,j'_1,s'_1,n)a(i'_2,j'_2,s'_2,n)) - \phi_0(a(i'_1,j'_1,s'_1,n))\phi_0(a(i'_2,j'_2,s'_2,n))$$
So $|k_2[a(i'_1,j'_1,s'_1,n),na(i'_2,j'_2,s'_2,n)]| \leq 1$

- Assume that the result is true for l and prove it for $l+1$
$$k_{l+1}[a(i'_1,j'_1,s'_1,n),...,a(i'_{l+1},j'_{l+1},s'_{l+1},n)]$$
$$= \phi_0(a(i'_1,j'_1,s'_1,n)...a(i'_{l+1},j'_{l+1},s'_{l+1},n))$$
$$- \sum_{\pi\in NC(l+1),\pi\neq 1_{l+1}} k_\pi[a(i'_1,j'_1,s'_1,n),...,a(i'_{l+1},j'_{l+1},s'_{l+1},n)]$$
For each π in $NC(l+1)$ such that $\pi\neq 1_{l+1}$,
$$k_\pi[a(i'_1,j'_1,s'_1,n),...,a(i'_{l+1},j'_{l+1},s'_{l+1},n)]$$
$$= \Pi_{i=1}^r k_{|\nu_i|}[a(i_{\alpha(1)},j_{\alpha(1)},s_{\alpha(1)},n)...a(i_{\alpha(1)},j_{\alpha(1)},s_{\alpha(1)},n)]$$
By recurrence $|k_{|\nu_i|}[...]| \leq C_{|\nu_i|}$ as $|\nu_i|\leq l$. And this proves the existence of the constant C_{l+1}

Lemma 3.9 *Let* $s_1,s_2,...,s_m\in N$. *Let* $t_1,t_2,...,t_m\in N$. *Then*
$$sup_{n\in N}|\Psi_n(\tilde{Y}(s_1,n)D_n(t_1)...\tilde{Y}(s_m,n)D_n(t_m))| = K(m) < \infty$$

Proof:
The $(q_{k,n})_{k\leq n^2}$ are orthogonal projectors in $(L^\infty[0,1],\phi)$ such that $\phi(q_{k,n}) = \frac{1}{n^2}$ We apply lemma 3.6
$$|\Psi_n(\tilde{Y}(s_1,n)D(t_1,n)...\tilde{Y}(s_m,n)D(t_m,n))| \leq$$
$$\frac{1}{n}n^2 \sum_{(i_1,...,i_m)\in E_n}\sum_\pi |k_\pi[\sqrt{n}a(i_1,i_2,s_1,n),...,\sqrt{n}a(i_m,i_1,s_m,n))]|K(\frac{1}{n^2})^{|\pi_B|})$$
From lemma 3.7, $|\pi_B|$ is greater or equal to $[\frac{m}{2}]+1$.

Now from lemma 3.8, we get

$$|\Psi_n(\tilde{Y}(s_1,n)D_n(t_1)...\tilde{Y}(s_m,n)D_n(t_m))| \leq n\ Card(E_n)Card(\{\pi\})C_m n^{\frac{m}{2}}(\frac{1}{n^2})^{[\frac{m}{2}]+1}$$

But from lemma 3.6, $Card(E_n) = O(n^{\frac{m}{2}+1})$. So

$$|\Psi_n(\tilde{Y}_n(s_1)D_n(t_1)...\tilde{Y}_n(s_m)D_n(t_m))| = O(n^{m-2[\frac{m}{2}]})$$

- if m is odd, for each π, at least one block ν of π_L is of length $|\nu|$ odd. It follows then from an obvious recurrence on $|\nu|$ odd, using the fact that the $a(i,j,s,n)$ are either independent or equal, that

$$k_{|\nu|}[(\sqrt{n}a(i_1,j_1,s_1,n),..,\sqrt{n}a(i_{|\nu|},j_{|\nu|},s_{|\nu|},n)] = 0$$

  so

$$k_\pi[\sqrt{n}a(i_1,i_2,s_1,n),\sqrt{n}a(i_2,i_3,s_2,n),...,\sqrt{n}a(i_m,i_1,s_m,n))] = 0.$$

  Hence

$$\Psi_n(\tilde{Y}(s_1,n)D_n(t_1)...\tilde{Y}(s_m,n)D_n(t_m)) = 0$$

- if m is even, $m - 2[\frac{m}{2}] = 0$. So

$$\sup_{n\in I\!N}|\Psi_n(\tilde{Y}(s_1,n)D(t_1,n)...\tilde{Y}(s_m,n)D(t_m,n))| = K(m) < \infty$$

Q.E.D

Now we are able to prove the proposition 3.1. We follow the proof of Theorem 2.2 of [7].

Proof of proposition 3.1:

Step I

It is to prove that

$$sup_{n\in I\!N}|\Psi_n(\tilde{Y}(s_1,n)D(t_1,n)...\tilde{Y}(s_m,n)D(t_m,n))| < \infty$$

This is exactly lemma 3.9

Step II

1. first we prove that

$$\Psi_n(\tilde{Y}(s_{\alpha(1)},n)D(t_1,n)\tilde{Y}(s_{\alpha(2)},n)D(t_2,n)...\tilde{Y}(s_{\alpha(m)},n)D(t_m,n)) = 0$$

   if $\alpha(1) \neq \alpha(j)$ for all $j \neq 1$

   This is corollary 3.3.

2. Let $\alpha(1) = \alpha(2)$ and $Card(\alpha^{-1}(p) \leq 2$ for all p in $I\!N$. We want to prove that

$$\Psi_n(\tilde{Y}(s_{\alpha(1)},n)D(t_1,n)\tilde{Y}(s_{\alpha(2)},n)D(t_2,n)...\tilde{Y}(s_{\alpha(m)},n)D(t_m,n)) =$$

$$\Psi_n(D(t_1,n)\Psi_n(D(t_2,n)\tilde{Y}(s_{\alpha(3)},n)D(t_3,n)...\tilde{Y}(s_{\alpha(m)},n)D(t_m,n))$$

   Indeed

$$\Psi_n(\tilde{Y}(s_{\alpha(1)},n)D(t_1,n)\tilde{Y}(s_{\alpha(2)},n)D(t_2,n)...\tilde{Y}(s_{\alpha(m)},n)D(t_m,n)) =$$

$$\frac{1}{n}\sum_{k=1}^{n^2}\sum_{i_1,i_2,...i_m}(\phi*\phi_0)(q_{k,n}a(i_1,i_2,s_{\alpha(1)},n)d(t_1,i_2)q_{k,n}a(i_2,i_3,s_{\alpha(1)},n)d(t_2,i_3)...$$

$$q_{k,n}a(i_m,i_1,s_{\alpha(m)},n)d(t_m,i_1))n^{\frac{m}{2}}$$

   From lemma 3.2 and lemma 3.5, this is equal to

$$\frac{1}{n}\sum_{k=1}^{n^2}\phi(q_{k,n})\sum_{i_1,i_2,i_4,...i_m} d(t_1,i_2)(E_0(a(i_1,i_2,s_{\alpha(1)},n)^2)(\phi*\phi_0)(d(t_2,i_1)q_{k,n}a(i_1,i_4,s_{\alpha(3)},n)$$

$$d(t_3,i_4))...q_{k,n}a(i_m,i_1,s_{\alpha(m)},n)d(t_m,i_1))n^{\frac{m}{2}} =$$

$$\frac{1}{n^3}\sum_{k=1}^{n^2}\sum_{i_2} d(t_1,i_2)\sum_{i_1,i_4,...i_m}(\phi*\phi_0)(d(t_2,i_1)q_{k,n}a(i_1,i_4,s_{\alpha(3),n}d(t_3,i_4))$$

$$...q_{k,n}a(i_m,i_1,s_{\alpha(m)},n)d(t_m,i_1))n^{\frac{m}{2}} =$$

$$\frac{1}{n}\sum_{i_2} d(t_1,i_2)\frac{1}{n}\sum_{k=1}^{n^2}\sum_{i_1,i_4,...i_m}(\phi*\phi_0)(d(t_2,i_1)q_{k,n}a(i_1,i_4,s_{\alpha(3)},n)d(t_3,i_4))$$

$$...q_{k,n}a(i_m,i_1,s_{\alpha(m)},n)d(t_m,i_1))n^{\frac{m}{2}-1} =$$

$$\Psi_n(D(t_1,n))\Psi_n(D(t_2,n)\tilde{Y}(s_{\alpha(3)},n)D(t_3,n)...\tilde{Y}(s_{\alpha(m)},n)D(t_m,n))$$

   Q.E.D.

3. One proves then that

$$\lim_{n\to\infty}\Psi_n(\tilde{Y}(s_1,n)D(t_1,n)...\tilde{Y}(s_m,n)D(t_m,n))=0$$

if $s_k\neq s_{k+1}(1\leq k\leq m-1)$ and $s_m\neq s_1$
As in the proof of Theorem 2.2 of [7] if

$$\Psi_n(\tilde{Y}(s_1,n)D(t_1,n)...\tilde{Y}(s_m,n)D(t_m,n))\neq 0$$

there is an automorphism γ of order 2 of $1,...,m$ without fixed point such that for $p\neq q$, $s_p=s_q$ if and only if $p=\gamma(q)$ And then as in lemma 3.6

$$\Psi_n(\tilde{Y}(s_1,n)D(t_1,n)...\tilde{Y}(s_m,n)D(t_m,n))=$$

$$\frac{1}{n}\sum_{k=1}^{n^2}\sum_{(i_1,i_2,...i_m)\in E_n(\gamma)}\sum_{\pi}k_\pi[\ a(i_1,i_2,s_1,n),\ a(i_2,i_3,s_2,n),$$

$$...\,a(i_m,i_1,s_m,n)]O((\frac{1}{n^2})^{|\pi_B|})n^{\frac{m}{2}}$$

As in [7] $E_n(\gamma)$ denote the set of $(i_1,i_2,...,i_m)\in(1,2,...,n)^m$ such that $i_k=i_{\gamma(k)+1}$, $i_{k+1}=i_{\gamma(k)}$ where $\gamma(k)$ and $\gamma(k)+1$ are considered modulo m. From [7] $Card(E_n(\gamma))\leq n^{\frac{m}{2}}$. So from lemma 3.6 lemma 3.7 and lemma 3.8, we get

$$\Psi_n(\tilde{Y}(s_1,n)D(t_1,n)...\tilde{Y}(s_m,n)D(t_m,n))=O(n^{(m-2[\frac{m}{2}]-1)}$$

- So if m is even,

  $$\Psi_n(\tilde{Y}(s_1,n)D(t_1,n)...\tilde{Y}(s_m,n)D(t_m,n))=O(\frac{1}{n})$$

  and we get the result.
- On the other hand, if m is odd, as in the proof of lemma 3.9,

  $$\Psi_n(\tilde{Y}(s_1,n)D(t_1,n)...\tilde{Y}(s_m,n)D(t_m,n))=0$$

This ends step II.

Step III.

Exactly as in step III of the proof of theorem 2.2 of [7], we apply the theorem 2.1. of [7], to prove that the family of random variables $\tilde{Y}(s,n)$ and $D(j,n)$ are asymptotically free, with the $\tilde{Y}(s,n)$ having limit distributions given by semicircular laws of variance 1.

This ends the proof of proposition 3.1.

This gives now the following renormalized model for the free Brownian motion:

Theorem 3.10 *For all $s \in I\!R^+$, and $n \in I\!N^*$, let*

$$\tilde{Z}_n(s) = \sum_{1 \le i,j \le n} \tilde{W}_{(i,j,n)}(s) e(i,j,n)$$

with $\tilde{W}_{(i,j,n)}(s) \in L^\infty([0,1]) * L$. Assume that

$$\tilde{W}_{(i,j,n)}(s) = \sum_{k=1}^{n^2} q_{k,n} \sqrt{n} W_{(i,j,n)}(s)\, q_{k,n}$$

where the $q_{k,n}$ are orthogonal projectors in $L^\infty[0,1]$, $\sum_{k=1}^{n^2} q_{k,n} = 1$ such that

$$\phi(q_{k,n}) = \frac{1}{n^2}$$

and the $(W_{(i,j,n)}(s)_{s \in R^+})_{1 \le i \le j \le n, n \in I\!N^*}$ are independent Brownian motions, i.e.,in particular

$$if\, s_0 = 0 < s_1 < s_2 ... < s_k \;\; (W_{(i,j,n)}(s_{l+1}) - W_{(i,j,n)}(s_l))_{0 \le l \le k-1}$$

are independent Gaussian random variables centered of variance $s_{l+1} - s_l$ and

$$W(i,j,n)(s) = W(j,i,n)(s)$$

Consider the trace Ψ_n defined at the begining of the section. Let $D_n(j)$ be elements in Δ_n,

the set of diagonal matrices, such that $\sup_{n \in N} \|D_n(j)\| < \infty$, for each j; and such that for all j, $(D_n(j))$ has a limit distribution as $n \to \infty$.

Then the family of subsets $\{\tilde{Z}_n(s)\}$ and $\{D_n(j), j \in N\}$ are asymptotically free, and the limit distribution of the $\tilde{Z}_n(s)$ is the distribution of the free Brownian motion.

Proof: For $i \in \{0, ..., k-1\}$ and $n \in I\!N$, let $0 = s_0 < s_1 < s_2 < ... < s_k$. Let $\tilde{Y}(i,n) = \frac{1}{\sqrt{s_{i+1}-s_i}}(\tilde{Z}_n(s_{i+1}) - \tilde{Z}_n(s_i))$

We apply the proposition 3.1 to $\tilde{Y}(i,n)$ and we get the result.

4 A free Girsanov property

Hypothesis: Let $(\sigma_s)_{s \in I\!R^+}$ be a free Brownian motion in (M, τ). Let N be a commutative C^*sub-algebra of M free from the $(\sigma_s)_{s \in I\!R^+}$. Let x be a measurable process with values in N. Assume that $x(u) = x(u)^*$ for all u and that $\int_0^\infty ||x(u)||^2 du < \infty$. Let $\tilde{\sigma}_s = \sigma_s + \int_0^s x(u)du)$.

We want first associate to the system $(\sigma_s, x(u))_{s,u \in R^+}$ an asymptotic system $(\tilde{Z}_n(s), D_n(u))$ in the set of random matrices with coefficients in a free product algebra $\mathcal{M}_n(L^\infty[0,1] * L)$ and then define on $\mathcal{M}_n(L^\infty[0,1] * L)$ two traces Ψ_n and $\tilde{\Psi}_n$ such that their asymptotic limits are respectively the traces τ and $\tilde{\tau}$ where τ is the given trace and $\tilde{\tau}$ is a new trace such that $(\tilde{\sigma}_s)_{s \in I\!R^+}$ is a free Brownian motion for $\tilde{\tau}$.

An asymptotic system in $\mathcal{M}_n(L^\infty[0,1] * L)$

$((\sigma_s)_{s \in I\!R^+})$ is a free Brownian motion and the $x(u) = x(u)^*$ belong to a commutative subalgebra of M free from the σ_s. In view of theorem 3.10 , we will associate to the process σ_s the process of random matrices $(\tilde{Z}_n(s)_{s \in I\!R^+})$ and we want to associate to the process $x(u)_{u \in I\!R^+}$ a process of diagonal matrices with real coefficients. We construct now this process.

Lemma 4.1 *Let N be a commutative C^* algebra with a finite trace τ. There is a family of homomorphisms H_n from N to Δ_n (the set of diagonal matrices with complex coefficients) such that for all $x \in N$,*

$$\tau(x) = \lim_{n \to \infty} \frac{1}{n} Tr_n(H_n(x))$$

Proof:

From the chapter 2 of [2] the set of states on N is the weak* closed convex hull of the set of pure states on N. Furthermore as N is commutative, the pure states are the characters. Denote $\mathcal{X}$ the set of the characters of N. Denote $\overline{S}$ the weak* closure of

$$S = \{\frac{1}{n}\sum_{1\leq i\leq n}\chi_i; n\in \mathbb{N}^*; \chi_i \in \mathcal{X}\}$$

Using the density of $\{\frac{k}{n}; n\in\mathbb{N}^*; 1\leq k\leq n\}$ in $[0,1]$, it is easy to verify that $\overline{S}$ is a convex set; so $\overline{S}$ is equal to the set of all the states on N. It follows that there is a sequence S_n of elements of S such that the limit of S_n for the weak* topology of N is equal to the trace τ. i.e. for all x in N,$S_n(x)\to\tau(x)$ as $n\to\infty$

$$S_n = \frac{1}{n}\sum_{1\leq i\leq n}\chi_{i,n}$$

Define now the homomorphism H_n from N to Δ_n by:

$$H_n(x) = \begin{pmatrix} \chi_{1,n}(x) & 0 & \cdots & 0 \\ 0 & \chi_{2,n}(x) & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \chi_{n,n}(x) \end{pmatrix}$$

Then

$$||H_n(x)|| \leq \sup(|\chi_{i,n}(x)|) \leq ||x||$$

and for every $x\in N$,

$$\frac{1}{n}Tr_n(H_n(x)) = \frac{1}{n}\sum_{1\leq i\leq n}\chi_{i,n}(x) = S_n(x)$$

So

$$\lim_{n\to\infty}\frac{1}{n}Tr_n(H_n(x)) = \tau(x)$$

q.e.d.

For all $u\in\mathbb{R}^+$ denote $D_n(u)$ the diagonal matrix $D_n(u) = H_n(x(u))$. It is a real matrix because $x(u)$ is self adjoint.

Definition 4.2 *The asymptotic system associated to $(\sigma_s, x(u))_{s,u\in\mathbb{R}^+}$ is the system of random matrices $(\tilde{Z}_n(s), D_n(u))_{s,u\in\mathbb{R}^+}$ in $\mathcal{M}_n(L^\infty[0,1]*L)$ where $(\tilde{Z}_n(s))_{s\in\mathbb{R}^+}$ is the process of random matrices defined in theorem 3.10 and $(D_n(u))_{u\in\mathbb{R}^+}$ is the process of diagonal matrices defined above.*

Two traces on $\mathcal{M}_n(L^\infty[0,1] * L)$

In the preceding section we have associated to $(\sigma_s, x(u))_{s,u\in I\!R^+}$ an asymptotic system in $\mathcal{M}_n(L^\infty[0,1] * L)$. Define now two traces Ψ_n and $\tilde{\Psi}_n$ on $\mathcal{M}_n(L^\infty[0,1] * L)$ such that their asymptotic limits will give the two traces τ and $\tilde{\tau}$.

Denote

$$h_n = \exp -[\int_0^\infty \sum_{1\leq i\leq n} \frac{1}{\sqrt{n}} D_n(u)_{i,i} dW_{i,i,n}(u)$$

$$+\int_0^\infty \sum_{1\leq i\leq n} \frac{D_n(u)_{i,i}^2}{2n} du]$$

Lemma 4.3 *For all $n \in I\!N^*$ $\phi_0(h_n) = 1$ For all $p \geq 2$ $\sup_{n\in I\!N^*} \phi_0(h_n^p) < \infty$, $\lim_{n\to\infty} \phi_0(h_n^p) = exp[\frac{p^2-p}{2} \int_0^\infty \tau(x(u)^2) du]$ and the family*

$$([W_{(i,j,n)}(s) + \int_0^s \frac{1}{\sqrt{n}} (D_n(u))_{i,i} \delta_{i,j} du]_{s\in I\!R^+})_{1\leq i\leq j\leq n}$$

is a family of independent Brownian motions for $\phi_0(h_n .)$

Proof:

$$\phi_0(h_n^p) = E(exp{-p}[\int_0^\infty \frac{1}{\sqrt{n}} \sum_{1\leq i\leq n} (D_n(u))_{i,i} dW_{i,i,n}(u) + \frac{1}{2n} \int_0^\infty \sum_{1\leq i\leq n} (D_n(u))_{i,i}^2] du]$$

$$= exp[\frac{1}{2} \int_0^\infty \sum_{1\leq i\leq n} \frac{(D_n(u))_{i,i}^2 p^2}{n} du - \frac{p}{2n} \int_0^\infty \sum_{1\leq i\leq n} (D_n(u))_{i,i}^2 du]$$

$$= exp[\frac{p^2-p}{2} \int_0^\infty \frac{1}{n} Tr_n(D_n(u)^2) du]$$

$D_n(u) = H_n(x(u))$ so for all n $\frac{1}{n} Tr_n(D_n(u)^2) \leq ||x(u)||^2$ and from lemma 4.1, $\lim_{n\to\infty} \frac{1}{n} Tr_n(D_n(u)^2) = \tau(x(u)^2)$ so we get the result for $\phi_0(h_n^p)$ applying the dominated convergence theorem of Lebesgue.

As the Brownian motions $(W_{i,j,n}(s))_{1\leq i\leq j\leq n}$ are independent and for all $i \leq n$,

$$\int_0^\infty H_n(x(u))_{i,i}^2 du < \infty$$

it results from the usual Girsanov theorem that

$$([W_{(i,j,n)}(s) + \int_0^s \frac{1}{\sqrt{n}}(D_n(u))_{i,i}\delta_{i,j}du]_{s\in I\!R^+})_{1\leq i\leq j\leq n}$$

are independent Brownian motions for $\phi_0(h_n.)$.

Q.E.D.

Definition 4.4 *Define now the traces Ψ_n and $\tilde{\Psi}_n$ on $\mathcal{M}_n(L^\infty([0,1]) *\ L)$ by*

$$\Psi_n\left(\sum_{1\leq i,j\leq n} x_{ij}e(i,j,n)\right) = \frac{1}{n}\sum_{1\leq i\leq n}(\phi * \phi_0)(x_{ii})$$

and

$$\tilde{\Psi}_n\left(\sum_{1\leq i,j\leq n} x_{ij}e(i,j,n)\right) = \frac{1}{n}\sum_{1\leq i\leq n}(\phi * \phi_0(h_n.))(x_{ii})$$

For simplicity we will denote

$$\Psi_n = \frac{1}{n}Tr_n(\phi * \phi_0)$$

and

$$\tilde{\Psi}_n = \frac{1}{n}Tr_n(\phi * \phi_0(h_n.))$$

Ψ_n is the same state as in section 3.

Proposition 4.5 *The joint distribution of $((\tilde{Z}_n(s)_{s\in I\!R^+}, H_n(x)_{x\in N})$ for Ψ_n is the same as the joint distribution of $((\tilde{Z}_n(s) + \int_0^s D_n(u)du)_{s\in I\!R^+}, H_n(x)_{x\in N})$ for $\tilde{\Psi}_n$*

Proof:

$$(\tilde{Z}_n(s) + \int_0^s D_n(u)du)_{i,j} =$$

$$\sum_{k=1}^{n^2} q_{k,n}\sqrt{n}W_{(i,j,n)}(s)\ q_{k,n}$$

$$+ \int_0^s (D_n(u))_{i,i}\delta_{i,j}du$$

$$= \sum_{k=1}^{n^2} q_{k,n}[\sqrt{n}W_{(i,j,n)}(s) + \int_0^s (D_n(u))_{i,i}\delta_{i,j}du]\ q_{k,n}$$

To compute the joint distribution of $((\tilde{Z}_n(s)+\int_0^s D_n(u)du)_{s\in I\!R^+}, H_n(x)_{x\in N})$ for $\tilde{\Psi}_n$, it is enough, as N is a unital C^*-algebra, to compute for all $p \in I\!N$, $s_i \geq 0$ and $x_i \in N$

$$\tilde{\Psi}_n(H_n(x_1)(\tilde{Z}_n(s_1) + \int_0^{s_1} D_n(u)du)H_n(x_2)...H_n(x_p)(\tilde{Z}_n(s_p) + \int_0^{s_p} D_n(u)du))$$

$$= \frac{1}{n}\sum_{1\leq k\leq n^2}\sum_{1\leq i_1\leq n}\sum_{1\leq j_1\leq n}...\sum_{1\leq j_{p-1}\leq n}(\phi * \phi_0(h_n.))(H_n(x_1))_{i_1,i_1}(q_{k,n}$$

$$[\sqrt{n}W_{(i_1,j_1,n)}(s_1) + \int_0^{s_1} (D_n(u))_{i_1,i_1}\delta_{i_1,j_1}du]q_{k,n})$$

$$(H_n(x_2))_{j_1,j_1}(q_{k,n}[\sqrt{n}W_{(j_1,j_2,n)}(s_2) + \int_0^{s_2} (D_n(u))_{j_1,j_1}\delta_{j_1,j_2}du])...$$

$$(H_n(x_p))_{j_{p-1},j_{p-1}}q_{k,n}[\sqrt{n}W_{(j_{p-1},i_1,n)}(s_p) + \int_0^{s_p} (D_n(u))_{j_{p-1},j_{p-1}}\delta_{j_{p-1},i_1}du]q_{k,n})$$

Remark now that the $q_{k,n}$ are free from L for $\phi * \phi_0(h_n.)$ and also for $\phi * \phi_0$. Furthermore from lemma 4.3, the joint distribution of

$$[\sqrt{n}W_{(i,j,n)}(s) + \int_0^s (D_n(u))_{i,i}\delta_{i,j}du]$$

for $\phi_0(h_n.)$ is equal to the joint distribution of $[\sqrt{n}W_{(i,j,n)}(s)]$ for ϕ_0.

We then get that the preeceding sum is equal to

$$= \frac{1}{n}\sum_{1\leq k\leq n^2}\sum_{1\leq i_1\leq n}\sum_{1\leq j_1\leq n}\cdots\sum_{1\leq j_{p-1}\leq n}(\phi * \phi_0)((H_n(x_1))_{i_1,i_1}$$
$$(q_{k,n}[\sqrt{n}W_{(i_1,j_1,n)}(s_1)]q_{k,n})$$
$$(H_n(x_2))_{j_1,j_1}(q_{k,n}[\sqrt{n}W_{(j_1,j_2,n)}(s_2)]q_{k,n})\ldots$$
$$(H_n(x_p))_{j_{p-1},j_{p-1}}q_{k,n}[\sqrt{n}W_{(j_{p-1},i_1,n)}(s_p)]q_{k,n})$$
$$= \Psi_n(H_n(x_1)\tilde{Z}_n(s_1)H_n(x_2)\tilde{Z}_n(s_2)\ldots H_n(x_p)\tilde{Z}_n(s_p))$$

This ends the proof of the proposition 4.5

Main result

We can now prove our main result: a free Girsanov property for the free Brownian motion.

Theorem 4.6 *Let $(\sigma_s)_{s\in\mathbb{R}^+}$,be a free Brownian motion in (M,τ). Let N be a commutative C^*-subalgebra of M such that N is free from $(\sigma_s)_{s\in\mathbb{R}^+}$. Let $x:\mathbb{R}^+\to N$ measurable such that $x(u)=x(u)^*$ for all u and $\int_0^\infty ||(x(u)||^2du<\infty$. Let $\tilde{\sigma}_s=\sigma_s+\int_0^s x(u)du$.*

Then there is a trace $\tilde{\tau}$ on the free product algebra $N*\mathbb{C}[(\sigma_s)_{s\in\mathbb{R}^+}]$ such that the joint distribution of $((\tilde{\sigma}_s)_{s\in\mathbb{R}^+}, x(u)_{u\in\mathbb{R}^+})$ for $\tilde{\tau}$ is the same as the joint distribution of $((\sigma_s)_{s\in\mathbb{R}^+}, x(u)_{u\in\mathbb{R}^+})$ for τ (in particular, $(\tilde{\sigma}_s)_{s\in\mathbb{R}^+}$ is a free Brownian motion for the new trace $\tilde{\tau}$). Furthermore the two traces are asymptotically equivalent in the following sense: There is a family $\tilde{Z}_n(s)$ of random matrices in $\mathcal{M}_n(L^\infty[0,1]*L)$ and a family $D_n(u)$ in Δ_n such that

$$(\mathbb{C}[\sigma_s,x(t)]_{s,t\in\mathbb{R}^+},\tau)=\lim_{n\to\infty}(\mathbb{C}[\tilde{Z}_n(s),D_n(t)]_{s,t\in\mathbb{R}^+},\Psi_n)\quad (I)$$

$$(\mathbb{C}[\sigma_s, x(t)]_{s,t\in\mathbb{R}^+}, \tilde{\tau}) = \lim_{n\to\infty} (\mathbb{C}[\tilde{Z}_n(s), D_n(t)]_{s,t\in\mathbb{R}^+}, \tilde{\Psi}_n) \quad (II)$$

where $\tilde{\Psi}_n$ is obtained from Ψ_n by a change of probability with exponential density h_n

$$\Psi_n = \frac{1}{n} Tr_n(\phi * \phi_0) \quad \tilde{\Psi}_n = \frac{1}{n} Tr_n(\phi_* \phi_0(h_n .))$$

Furthermore for all p: $\sup_{n\in\mathbb{N}} \phi_0(h_n^p) < \infty$

Proof:

$\tilde{\sigma}_s = \sigma_s + \int_0^s x(u)du$ Denote $y(s) = \int_0^s x(u)du$ $y(s)$ is an element of the C^*-algebra N and $H_n(y(s)) = \int_0^s D_n(u)du$

From proposition 4.5, the joint distribution of $((\tilde{Z}_n(s)+H_n(y(s))_{s\in\mathbb{R}^+}, H_n(x)_{x\in N})$ for $\tilde{\Psi}_n$ is the same as the joint distribution of $(\tilde{Z}_n(s)_{s\in\mathbb{R}^+}, H_n(x)_{x\in N})$ for Ψ_n.

Hence for every non commutative polynomial P,

$$\tilde{\Psi}_n(P(\tilde{Z}_n(s_i) + H_n(y(s_i)), H_n(x_j)) = \Psi_n(P(\tilde{Z}_n(s_i), H_n(x_j))$$

From theorem 3.10, and lemma 4.1 this last quantity has a limit as n tends to ∞ and this limit is equal to $\tau(P(\sigma_{s_i}, x_j))$. So this gives (I).

It follows also that there is a trace $\tilde{\tau}$ well defined on $\mathbb{C}[\tilde{\sigma}_s] * N$ by

$$\tilde{\tau}(P(\tilde{\sigma}_{s_i}, x_j)) = \lim_{n\to\infty} \tilde{\Psi}_n(P(\tilde{Z}_n(s_i) + H_n(y(s_i)), H_n(x_j))$$

and that the joint distribution of $((\tilde{\sigma}_s)_{s\in\mathbb{R}^+}, x_{x\in N})$ for $\tilde{\tau}$ is the same as the joint distribution $((\sigma_s)_{s\in\mathbb{R}^+}, x_{x\in N})$ for τ.

This gives also the equality (II).

Q.E.D.

Now we finish by the following remark: if we replace in the preceding theorem the random process $\tilde{Z}_n(s)$ by the random process $B_{n,s} = (\frac{1}{\sqrt{n}} W_{n,i,j,s})_{1\leq i,j\leq n}$ considered by Voiculescu (cf section 2),the asymptotic limits for Ψ_n and $\tilde{\Psi}_n$ give both the trace τ. This is why we were obliged to construct a matrix random process with values in a free product algebra. More precisely we have the following result

Proposition 4.7 *Let $B_{n,s}$ be the matrix random process $B_{n,s} = (\frac{1}{\sqrt{n}} W_{n,i,j,s})_{1\leq i,j\leq n}$ where $(W_{n,i,j,s})_{1\leq i\leq j\leq n}$ are independent Brownian motions. Let Ψ_n and $\tilde{\Psi}_n$ be the traces of theorem 4.6.Then*

$$(\mathbb{C}[\sigma_s, x(t)]_{s,t\in \mathbb{R}^+}, \tau) = \lim_{n\to\infty} (\mathbb{C}[B_{n,s}, D_n(t)]_{s,t\in \mathbb{R}^+}, \Psi_n) \quad (I')$$

$$(\mathbb{C}[\sigma_s, x(t)]_{s,t\in \mathbb{R}^+}, \tau) = \lim_{n\to\infty} (\mathbb{C}[B_{n,s}, D_n(t)]_{s,t\in \mathbb{R}^+}, \tilde{\Psi}_n) \quad (II')$$

Proof:

i) The equality (I') results from the theorem 2.2 of Voiculescu [7] as it is recalled in section 2 and from the lemma 4.1

Notice that $B_{n,s}$ and $D_n(t) = H_n(x(t))$ are matrices with coefficients in L so here Ψ_n resp.$\tilde{\Psi}_n$ are simply equal to $\frac{1}{n}Tr_n(\phi_0)$ resp. $\frac{1}{n}Tr_n(\phi_0(h_n \cdot))$; Ψ_n restricted to $\mathcal{M}_n(L)$ is equal to the trace ϕ_n of section 2. As in the proof of theorem 4.6 denote $y(s) = \int\limits_0^s x(u)du$.

ii) From lemma 4.3, the joint distribution of $(B_{n,s} + \frac{1}{n}H_n(y(s)), H_n(x(t)))$ for $\tilde{\Psi}_n$ is the same as the joint distribution of $(B_{n,s}, H_n(x(t)))$ for Ψ_n.

iii) Let P be a non commutative polynomial. Compute now:

$$\Psi_n([B_{n,s_1} + \frac{1}{n}H_n(y(s_1))]^{\alpha_1} H_n(x_1)$$

$$[B_{n,s_2} + \frac{1}{n}H_n(y(s_2))]^{\alpha_2} H_n(x_2)...$$

$$[B_{n,s_m} + \frac{1}{n}H_n(y(s_m))]^{\alpha_m} H_n(x_m)) =$$

$$\Psi_n((B_{n,s_1})^{\alpha_1} H_n(x_1)(B_{n,s_2})^{\alpha_2} H_n(x_2)...(B_{n,s_m})^{\alpha_1} H_n(x_m))$$

$$+ \sum_{i=1}^{\alpha_1+...\alpha_m} (\frac{1}{n})^i (\Psi_n(Q_i(B_{n,s_1}, .., B_{n,s_m}, H_n(x_1), ..., H_n(x_m))$$

where Q_i is a non commutative polynomial.

From the theorem of Voiculescu recalled in section 2 , for all i

$$\lim_{n\to\infty} \Psi_n(Q_i(B_{n,s_1}, .., B_{n,s_m}, H_n(x_1), ..., H_n(x_m)) = \tau(Q_i(\sigma_{s_1}, ..., \sigma_{s_m}, x_1, ..., x_m))$$

i.e. for every non commutative polynomial P

$$\Psi_n(P(B_{n,s_i} + \frac{1}{n}H_n(y(s_i)), H_n(x_j))) - \Psi_n(P(B_{n,s_i}, H_n(x_j)))$$

tends to zero as n tends to ∞.

and

$$|\tilde{\Psi}_n((P(B_{n,s_i}+\frac{1}{n}H_n(y(s_i)),H_n(x_j)))-\tilde{\Psi}_n(P(B_{n,s_i},H_n(x_j)))|\leq$$

$$\phi_0(h_n^2)^{\frac{1}{2}}\Psi_n([P(B_{n,s_i}+\frac{1}{n}H_n(y(s_i)),H_n(x_j))-P(B_{n,s_i},H_n(x_j))]^*$$
$$[P(B_{n,s_i}+\frac{1}{n}H_n(y(s_i)),H_n(x_j))-P(B_{n,s_i},H_n(x_j))])^{\frac{1}{2}}$$

and we know from lemma 4.3 that $\lim_{n\to\infty}\phi_0(h_n^2)=exp(\tau(a^2))$

It follows that the limit joint distribution of $(B_{n,s},H_n(x(t)))$ for $\tilde{\Psi}_n$ is the same as the limit joint distribution of $(B_{n,s}+\frac{1}{n}H_n(y(s)),H_n(x(t)))$ for $\tilde{\Psi}_n$. Applying now ii) and i) it follows that this limit is equal to the joint distribution of $(\sigma_s,x(t))$ for τ.

So we get (II').

The generalization of this Girsanov property (theorem 4.6) to the case where the process x is adapted to the free Brownian motion (σ_s) is a work in progress.

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