ECOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES APPLIQUÉES
UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 41 50. Fax: 01 69 33 30 11

http://www.cmap.polytechnique.fr/

Integro-differential equations for option prices in exponential Lévy models.

Rama Cont and Ekaterina Voltchkova

R.I. N^0 547 *Juillet 2004*

Integro-differential equations for option prices in exponential Lévy models*

Rama Cont and Ekaterina Voltchkova
Centre de Mathématiques Appliquées
Ecole Polytechnique F 91128 Palaiseau, France.

Email: voltchko@cmapx.polytechnique.fr

Abstract

We derive the partial integro-differential equations (PIDEs) verified by the values of European and barrier options in exponential Lévy models. We discuss the conditions under which options prices are classical solutions of the PIDEs. Since these conditions may fail in general, we consider the notion of continuous viscosity solution. We give sufficient conditions on the Lévy triplet for the option price to be continuous; in this case we show that it is the unique viscosity solution of the PIDE.

Keywords: parabolic integro-differential equations, Lévy process, jump-diffusion models, option pricing, viscosity solutions.

*This work was presented at the Bachelier seminar (Paris, 2003), Workshop on Lévy processes and Partial integro-differential equations (Palaiseau, 2003), IFIP 2003 (Nice), the Workshop on Semimartingale theory and applications in finance (Banff) and the Workshop on Computational Finance (Zürich, 2003). We thank Yves Achdou, Mariko Arisawa, Daniel Gabay, Huyen Pham and Peter Tankov for helpful discussions.

Contents

The shortcomings of diffusion models in representing the risk related to large market movements have led to the development of various option pricing models with jumps, where large returns are represented as discontinuities of prices as a function of time. Models with jumps allow for more realistic representation of price dynamics and a greater flexibility in modelling and have been the focus of much recent work. A review of financial modelling with jump processes may be found in [9].

Exponential Lévy models, where the market price of an asset is represented as the exponential $S_t = \exp(rt + X_t)$ of a Lévy process X_t, offer analytically tractable examples of positive processes with jumps which are simple enough to allow a detailed study both in terms of statistical properties and as models for risk-neutral dynamics i.e. option pricing models. Option pricing with exponential Lévy models is discussed in [9, 14, 16, 19]. The flexibility of choice of the Lévy process X allows to calibrate the model to market prices of options and reproduce a wide variety of implied volatility skews/smiles. The Markov property of the price allows us to express option prices as solutions of partial integro-differential equations (PIDEs) which involve, in addition to a (possibly degenerate) second-order differential operator, a non-local integral term which requires specific treatment both at the theoretical and numerical level.

Such partial integro-differential equations (PIDEs) have been used by several authors to price options in models with jumps [3, 8, 22, 13] but the derivation of these equations is omitted in these works. We explore in this paper the precise link between option prices in exponential Lévy models and the related partial integro-differential equations (PIDEs) in the case of European and barrier options in exponential Lévy models. We first discuss the conditions under which options prices are classical solutions of the PIDEs and show that these conditions may fail in pure jump models. The notion of continuous viscosity solution allows to cover this case: we give sufficient conditions on the Lévy triplet for the option price to be continuous and show that in this case it is the unique viscosity solution of the PIDE.

Section 1 recalls some basic facts about Lévy processes and exponential Lévy models. Section 2 derives the PIDE verified by option prices in a heuristic manner and discusses sufficient conditions for this derivation to hold. Section 3 gives two examples illustrating the lack of smoothness with respect to the underlying in pure jump models and gives sufficient conditions on the Lévy triplet for option prices to be continuous. In Section 4 we define the notion of viscosity solutions for PIDEs and give a characterization of option prices in terms of viscosity solution to a PIDE. Section 5 concludes by discussing relations with previous work, possible extensions and applications.

1 Exponential Lévy models

We consider here the class of models where the risk neutral dynamics of the underlying asset is given by $S_t = \exp(rt + X_t)$ where X_t is a Lévy process.

1.1 Lévy processes: definitions

A Lévy process is a stochastic process X_t with stationary independent increments which is continuous in probability (but may have discontinuous trajectories). Without loss of generality we assume $X_0 = 0$. The characteristic function of X_t has the following Lévy-Khinchin representation [27]:

$$E[e^{izX_t}] = \exp t\phi(z), \quad \phi(z) = -\frac{\sigma^2 z^2}{2} + i\gamma z + \int_{-\infty}^{\infty} (e^{izx} - 1 - izx1_{|x|\leq 1})\nu(dx),$$

where $\sigma \geq 0$ and γ are real constants and ν is a positive Radon measure on $\mathbb{R} - \{0\}$ verifying

$$\int_{-1}^{+1} x^2 \nu(dx) < \infty, \qquad \int_{|x|>1} \nu(dx) < \infty.$$

The random process X can be interpreted as the independent superposition of a Brownian motion with drift and an infinite superposition of independent (compensated) Poisson processes with various jump sizes x, $\nu(dx)$ being the intensity of jumps of size x.

In general ν is not a finite measure: $\int \nu(dx)$ need not be finite. In the case where $\lambda = \int \nu(dx) < +\infty$, the measure ν can be normalized to define a *probability measure* μ which can now be interpreted as the distribution of jump sizes:

$$\mu(dx) \quad = \quad \frac{\nu(dx)}{\lambda}.$$

The jumps of X are then described by a *compound Poisson* process with λ as jump intensity (average number of jumps per unit time) and jump size distribution $\mu(.)$. More generally, if $\int |x|\nu(dx) < \infty$, the (possibly infinite) sum of jumps is absolutely convergent with probability 1 and X_t can be represented as a pathwise sum of a Brownian motion plus jumps:

$$X_t \quad = \quad \sigma W_t + \gamma_0 t + \sum_{0<s\leq t} \Delta X_t \tag{1.1}$$

where $\gamma_0 = \gamma - \int_{|x|\leq 1} x\nu(dx)$. In this case the compensation of small jumps is not needed and the Lévy-Khinchin representation reduces to:

$$\phi(z) = -\frac{\sigma^2 z^2}{2} + i\gamma_0 z + \int_{-\infty}^{\infty} (e^{izx} - 1)\nu(dx).$$

In the case where $\int |x|\nu(dx) = \infty$ the jumps have infinite variation and small jumps need to be compensated.

A Lévy process is a (strong) Markov process; the associated semigroup is a convolution semigroup and its infinitesimal generator $L : f \to Lf$ is an integro-differential operator given by:

$$\begin{aligned} Lf(x) &= \lim_{t\to 0}\frac{E[f(x+X_t)]-f(x)}{t} = \\ &= \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}+\gamma\frac{\partial f}{\partial x}+\int \nu(dy)[f(x+y)-f(x)-y1_{\{|y|\leq 1\}}\frac{\partial f}{\partial x}(x)] \end{aligned} \quad (1.2)$$

which is well defined for $f \in C^2(\mathbb{R})$ with compact support.

1.2 Exponential Lévy models

Let $(S_t)_{t\in[0,T]}$ be the price of a financial asset modelled as a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. $\mathcal{F}_t$ is usually taken to be the price history up to t. Under the hypothesis of absence of arbitrage there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which the discounted prices of all financial assets are $\mathbb{Q}$- martingales; in particular the discounted underlying $(e^{-rt}S_t)$ is a martingale under $\mathbb{Q}$.

In *exponential Lévy* models, the (risk-neutral) dynamics of S_t under $\mathbb{Q}$ is represented as the exponential of a Lévy process:

$$S_t = S_0 e^{rt+X_t}.$$

Here X_t is a Lévy process (under $\mathbb{Q}$) with characteristic triplet (σ,γ,ν), and the interest rate r is included for ease of notation. The absence of arbitrage then imposes that $\hat{S}_t = S_t e^{-rt} = \exp X_t$ is a martingale, which is equivalent to the following conditions on the triplet (σ,γ,ν):

$$\int_{|y|>1}\nu(dy)e^y < \infty, \qquad \gamma = \gamma(\sigma,\nu) = -\frac{\sigma^2}{2} - \int (e^y - 1 - y1_{|y|\leq 1})\nu(dy). \quad (1.3)$$

We will assume this relation holds in the sequel. The infinitesimal generator L then becomes:

$$Lf(x) = \frac{\sigma^2}{2}[\frac{\partial^2 f}{\partial x^2}-\frac{\partial f}{\partial x}]+\int_{-\infty}^{\infty}\nu(dy)\ [f(x+y)-f(x)-(e^y-1)\frac{\partial f}{\partial x}(x)]. \quad (1.4)$$

The risk-neutral dynamics of S_t is given by

$$S_t = S_0 + \int_0^t rS_{u-}du + \int_0^t S_{u-}\sigma dW_u + \int_0^t\int_{-\infty}^{\infty}(e^x-1)S_{u-}\tilde{J}_X(du\ dx), \quad (1.5)$$

where $\tilde{J}_X$ is the compensated random measure describing the jumps of X [25].

(S_t) is also a Markov process with state space $(0,\infty)$ and infinitesimal generator:

$$L^S f(x) = rx\frac{\partial f}{\partial x}(x)+\frac{\sigma^2 x^2}{2}\frac{\partial^2 f}{\partial x^2}(x)+\int \nu(dy)[f(xe^y)-f(x)-x(e^y-1)\frac{\partial f}{\partial x}(x)]. \quad (1.6)$$

While in principle one can have both a non-zero diffusion component $\sigma \neq 0$ *and* an infinite activity jump component, in practice the models encountered in the financial literature are of two types: either we combine a non-zero diffusion part $\sigma > 0$ with a finite activity jump process (in this case one speaks of a *jump-diffusion* model) or one totally suppresses the diffusion part, in which case frequent small jumps are needed to generate realistic trajectories: these are *infinite activity pure jump models.* Different exponential Lévy models proposed in the financial modelling literature simply correspond to different choices for the Lévy measure ν, see [9, Chap. 3] for a review.

2 Partial integro-differential equations for option prices

The value of a European option is defined as a discounted conditional expectation of its terminal payoff $H(S_T)$ under risk-neutral probability $\mathbb{Q}$:

$$C_t = E[e^{-r(T-t)} H(S_T)|\mathcal{F}_t].$$

From the Markov property, $C_t = C(t, S)$ where

$$C(t, S) \quad = \quad E[e^{-r(T-t)} H(S_T)|S_t = S]. \tag{2.1}$$

Introducing the change of variable $\tau = T - t$, $x = \ln(S/S_0)$, and defining: $h(x) = H(S_0 e^x)$ and $f(\tau, x) = e^{r\tau} C(T - \tau, S_0 e^x)$, then

$$f(\tau, x) = E[h(x + r\tau + X_\tau)]. \tag{2.2}$$

If h is in the domain of the infinitesimal generator L of X given by (1.4), then we can differentiate with respect to τ to obtain the following integro-differential equation:

$$\frac{\partial f}{\partial \tau} = Lf + r\frac{\partial f}{\partial x}, \quad \text{on } (0, T] \times \mathbb{R}; \qquad f(0, x) = h(x), \quad x \in \mathbb{R}. \tag{2.3}$$

Similarly, if f is smooth then using a change of variable we obtain a similar equation for $C(t, S)$:

$$\frac{\partial C}{\partial t}(t, S) + L^S C(t, S) - rC(t, S) = 0; \qquad C(T, S) = H(S). \tag{2.4}$$

This equation is similar to the Black-Scholes partial differential equation, except that the second-order differential operator is replaced by the integro-differential operators L^S.

However, the above reasoning is heuristic: the payoff function h is usually not in the domain of L and in fact it is usually not even differentiable. For example $h(x) = (K - S_0 e^x)^+$ for a put option and $h(x) = 1_{x \geq x_0}$ for a binary option.

If f is a smooth solution of (2.3), by applying the Ito formula to $f(t, X_t)$ between 0 and T one can show [6, 23] that f has the probabilistic representation (2.2):

Proposition 1 (Feynman–Kac representation for Lévy processes). *Assume $\sigma > 0$ or $\exists a > 0$ such that $\int_{|x|>1} \exp(a|x|)\nu(dx) < \infty$. If $f \in C^{1,2}$ is a classical solution of (2.3) and its derivatives are bounded by a polynomial function of x, uniformly in $t \in [0,T]$, then f has the probabilistic representation (2.2).*

The case $\sigma > 0$ is shown in [6, Chap. 4]; the pure jump case is treated in [23]. This type of result is sometimes called a *verification theorem*: f is *assumed* to be smooth and its derivatives assumed to verify some integrability conditions. The conditions on f and ν ensure that $f(t, X_t)$ can be represented as a martingale plus a finite variation process. However, it is readily seen that such conditions are *never* verified in option pricing applications.[1] For instance, even for a European put option, the second derivative (Gamma of the option) is certainly not uniformly bounded in t!

These assumptions can be weakened in various ways, see [6, 26]. In the next section we will give some sufficient conditions for this smoothness to be verified. Under these conditions the value of European options $f(\tau, x)$, $C(t, S)$ defined above are classical solutions of the partial integro-differential equations (2.3), (2.4). However, as we will see in section 3, these conditions are not always verified, especially in pure jump models. This will lead us to consider the notion of viscosity solution; we show in section 4 that under more general conditions, values of European or barrier options can be expressed as viscosity solutions of appropriate PIDEs.

2.1 Classical solutions

Consider a European option with maturity T and payoff $H(S_T)$. Assume that the payoff function H is Lipschitz:

$$|H(x) - H(y)| \leq c|x - y| \tag{2.5}$$

for some $c > 0$. This condition is of course verified by call and put options with $c = 1$. The value C_t of such an option is given by $C_t = C(t, S_t)$ where

$$C(t, S) = e^{-r(T-t)}E[H(S_T)|S_t = S] = e^{-r(T-t)}E[H(Se^{r(T-t)+X_{T-t}})].$$

We will furthermore assume, throughout this section, that

$$\int_{|y|>1} e^{2y}\nu(dy) < \infty. \tag{2.6}$$

This condition is equivalent to the existence of a second moment for the price process S_t. Then $\hat{S}_t = \exp X_t$ is a square integrable martingale:

$$\frac{d\hat{S}_t}{\hat{S}_{t-}} = \sigma dW_t + \int_{-\infty}^{\infty} (e^x - 1)\tilde{J}_X(dt\ dx), \qquad \sup_{t\in[0,T]} E[\hat{S}_t^2] < \infty.$$

[1] In particular, the hypotheses given in [23] do not apply to the example of a call or put option.

Proposition 2 (Backward PIDE for European options). *Consider the exponential Lévy model $S_t = S_0 \exp(rt + X_t)$ where X is a Lévy process verifying (2.6). If*

$$\sigma > 0 \quad or \quad \exists \beta \in (0,2), \qquad \liminf_{\epsilon \downarrow 0} \epsilon^{-\beta} \int_{-\epsilon}^{\epsilon} |x|^2 \nu(dx) > 0 \tag{2.7}$$

then the value of a European option with terminal payoff $H(S_T)$ is given by $C(t,S)$ where:

$$\begin{aligned} C : [0,T] \times [0,\infty) &\to \mathbb{R} \\ (t,S) &\mapsto C(t,S) = e^{-r(T-t)} E[H(S_T)|S_t = S] \end{aligned}$$

is continuous on $[0,T] \times [0,\infty)$, $C^{1,2}$ on $(0,T) \times (0,\infty)$, and verifies the partial integro-differential equation:

$$\begin{aligned} &\frac{\partial C}{\partial t}(t,S) + rS\frac{\partial C}{\partial S}(t,S) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}(t,S) - rC(t,S) + \\ &\qquad + \int \nu(dx)[C(t,Se^x) - C(t,S) - S(e^x - 1)\frac{\partial C}{\partial S}(t,S)] = 0 \end{aligned} \tag{2.8}$$

on $[0,T) \times (0,\infty)$ with the terminal condition:

$$\forall S > 0, \qquad C(T,S) = H(S). \tag{2.9}$$

Proof. The proof involves, as in the Black-Scholes case, applying the Itô formula to the martingale $\hat{C}(t,S_t) = e^{r(T-t)}C(t,S_t)$, identifying the drift component and setting it to zero.

Condition (2.7) implies that X_t has a smooth C^2 density with derivatives vanishing at infinity (see [27, proposition 28.3]); $C(t,S)$ is then a smooth function of S. Smoothness in time can be shown by Fourier methods, see [9, Proposition 12.2].

By construction, $\hat{C}_t = E[H|\mathcal{F}_t]$ is a martingale. Applying the Itô formula to $\hat{C}_t = e^{r(T-t)}C(t,S_t)$ and using equation (1.5) we obtain:

$$\begin{aligned} d\hat{C}_t = e^{r(T-t)}[-rC_t + \frac{\partial C}{\partial t}(t,S_{t-}) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 C}{\partial S^2}(t,S_{t-})]dt + e^{r(T-t)}\frac{\partial C}{\partial S}(t,S_{t-})dS_t + \\ + e^{r(T-t)}[C(t,S_{t-}e^{\Delta X_t}) - C(t,S_{t-}) - S_{t-}(e^{\Delta X_t} - 1)\frac{\partial C}{\partial S}(t,S_{t-})] = \\ = a(t)dt + dM_t \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} a(t) = e^{r(T-t)}[-rC(t,S_{t-}) + \frac{\partial C}{\partial t}(t,S_{t-}) + \frac{\sigma^2 S_{t-}^2}{2}\frac{\partial^2 C}{\partial S^2}(t,S_{t-}) + rS_{t-}\frac{\partial C}{\partial S}(t,S_{t-})] \\ + \int_{-\infty}^{\infty} \nu(dx) e^{r(T-t)}[C(t,S_{t-}e^x) - C(t,S_{t-}) - S_{t-}(e^x - 1)\frac{\partial C}{\partial S}(t,S_{t-})], \end{aligned}$$

and

$$dM_t = e^{r(T-t)}\frac{\partial C}{\partial S}(t,S_{t-})\sigma S_{t-}dW_t+\int_{\mathbb{R}} e^{r(T-t)}[C(t,S_{t-}e^x)-C(t,S_{t-})]\tilde{J}_X(dt\,dx).$$

Let us now show that M_t is a martingale. Since the payoff function H is Lipschitz, C is also Lipschitz with respect to the second variable:

$$\begin{aligned}|C(t,S_1)-C(t,S_2)| &= e^{-r(T-t)}|E[H(S_1e^{r(T-t)+X_{T-t}})]-E[H(S_2e^{r(T-t)+X_{T-t}})]| \\ &\leq c|S_1-S_2|E[e^{X_{T-t}}] = c|S_1-S_2| \end{aligned} \tag{2.11}$$

since e^{X_t} is a martingale. Therefore the predictable random function $\psi(t,x) = C(t,S_{t-}e^x)-C(t,S_{t-})$ verifies

$$\begin{aligned} E[\int_0^T dt\int_{\mathbb{R}}\nu(dx)|\psi(t,x)|^2] &= E[\int_0^T dt\int_{\mathbb{R}}\nu(dx)|C(t,S_{t-}e^x)-C(t,S_{t-})|^2\ ] \\ &\leq E[\int_0^T dt\int_{\mathbb{R}} c^2(e^{2x}+1)S_{t-}^2\nu(dx)] \\ \text{using (2.6)} &\leq c^2\int_{\mathbb{R}}(e^{2x}+1)\nu(dx)\ E[\int_0^T S_{t-}^2dt]<\infty, \end{aligned}$$

so the compensated Poisson integral

$$\int_0^t\int_{-\infty}^{\infty} e^{r(T-t)}[C(t,S_{t-}e^x)-C(t,S_{t-})]\ \tilde{J}_X(dt\ dx)$$

is a square integrable martingale. Also, since C is Lipschitz, $\partial C/\partial S\ \in L^\infty$ and

$$||\frac{\partial C}{\partial S}(t,.)||_{L^\infty}\leq c, \quad \text{so} \quad E[\int_0^T S_{t-}^2|\frac{\partial C}{\partial S}(t,S_{t-})|^2dt]\leq c^2E[\int_0^T S_{t-}^2dt]<\infty.$$

Using the isometry relation for Wiener integrals, we obtain that $\int_0^t \sigma S_t\frac{\partial C}{\partial S}(t,S_{t-})dW_t$ is also a square integrable martingale. Therefore M_t is a square integrable martingale. $\hat{C}_t-M_t$ is thus a (square integrable) martingale; but $\hat{C}_t-M_t=\int_0^t a(s)ds$ is also a *continuous* process with finite variation so, by [17, Theorem 4.13-4.50], we must have $a(t)=0$ $\mathbb{Q}$-almost surely which yields the PIDE (2.8). □

The condition (2.7) holds for all jump diffusion models with non-zero diffusion component as well as for Lévy densities behaving near zero as $\nu(x)\sim c/x^{1+\beta}$ with $\beta>0$ such as the tempered stable model, but not for the Variance Gamma model [21]. In the case of the Variance Gamma model, the PIDE reduces to a first order equation for which only C^1 smoothness is required but, as we shall observed in Section 3, even this condition may fail.

2.2 Barrier options

Barrier options can be similarly represented in terms of solutions to PIDEs. Consider for instance an up-and-out call option with maturity T, strike K, and (upper) barrier $U > S_0$. The terminal payoff is given by

$$H_T \quad = \quad (S_T - K)^+ 1_{T<\theta},$$

where $\theta = \inf\{t \geq 0 \mid S_t \geq U\}$, the first moment when the barrier is crossed.

The value of the barrier option at time t is defined as the discounted expectation of it's terminal payoff: $C_t = e^{-r(T-t)} E[H_T|\mathcal{F}_t]$. By construction, $e^{r(T-t)} C_t$ is a martingale.

Due to the Markov property of Lévy processes, it is possible to express the price C_t as a deterministic function of time t and current stock value S_t before the barrier is crossed. Namely, for any $(t, S) \in [0, T] \times (0, \infty)$ we can define

$$C_b(t, S) \quad = \quad e^{-r(T-t)} \; E[H(Se^{Y_{T-t}}) 1_{T<\theta_t}], \tag{2.12}$$

where $H(S) = (S - K)^+$, $\{Y_{s-t}, \; s \geq t\}$ is a Lévy process, and $\theta_t = \inf\{s \geq t \mid Se^{Y_{s-t}} \geq U\}$, the first exit time after t. Then,

$$C_t = C_b(t, \; S_t) 1_{t \leq \theta} \tag{2.13}$$

for all $t \leq T$. Note that outside of the set $\{t \leq \theta\}$ the objects C_t and $C_b(t, S_t)$ are different: if the barrier has already been crossed, C_t will always be zero, but $C_b(t, S_t)$ may become positive if the stock returns to the region below the barrier.

As in the European case, by going to the log variables we define

$$f_b(\tau, x) = e^{r\tau} C_b(T - \tau, S_0 e^x). \tag{2.14}$$

Again, if f_b is smooth the Itô formula can be used to show that f_b is a solution of the following initial-boundary-value problem:

$$\frac{\partial f}{\partial \tau} = Lf + r\frac{\partial f}{\partial x}, \qquad \text{on } (0, T] \times (-\infty, \log(U/S_0)), \tag{2.15}$$

$$f(0, x) = h(x), \qquad x < \log(U/S_0),$$

$$f(\tau, x) = 0, \qquad x \geq \log(U/S_0). \tag{2.16}$$

The main difference between this equation and the analogous PDEs for diffusion models is in the "boundary condition": (2.16) not only specifies the behavior of the solution *at* the barrier $S = U$ but also *beyond* the barrier ($S > U$). This is necessary because of the non-local nature of the operator L: to compute the integral term we need the function $f(\tau, .)$ on $(-\infty, \infty)$ and (2.16) extends the function beyond the barrier by zero. In the case of a rebate, the function would be replaced by the value of the rebate in the knock-out region $S \geq U$. Similar results and corresponding Feynman-Kac formulae hold — in the case $\sigma > 0$ — when the boundary condition is not zero but given by a function

$g(\tau, x)$ where $g \in W_p^{1,2}([0,T] \times \mathbb{R})$ with $p > 3$, see [26]. More generally, if $f_b(.,.)$ defined by (2.14) can be shown to be $C^{1,2}$ (or simply $C^{1,1}$ in the case of finite variation models) then following the proof of Proposition 2, one can show that f_b is a solution of the PIDE (2.15). However, as we shall see below (Example 2) in the case of pure jump models where $\sigma = 0$ such smoothness with respect to the underlying does not hold in general.

3 Smoothness with respect to the underlying

In the case where the log-price X_t has a non-degenerate diffusion component, it is known [6, 15] that the fundamental solution of the pricing PIDE, which corresponds to the density of X_t, is in fact a smooth C^∞ function. As a consequence, the option price $u(t,x)$ depends smoothly on the underlying and results such as Proposition 1 allow to use the solution of the PIDE to compute the option price. In pure jump models, this property may fail. In this section we present examples where smoothness fails; we then give sufficient conditions under which the option prices is continuous as a function of the underlying. This minimal regularity will be required later to show that the option price is a generalized (viscosity) solution of the PIDE.

3.1 Lack of smoothness in pure jump models

In the case of processes with a degenerate diffusion component, such as pure jump models, the smoothness of the conditional expectation as a function of the initial (spot) value of the underlying S does not always hold, as the following example shows.

Example 1 (Variance Gamma process). The Variance Gamma process, introduced by Madan & Milne [20], is a pure jump finite variation process with infinite activity, popular in financial modelling. Its Lévy measure has a density given by:

$$\nu(x) = \frac{1}{\kappa|x|} e^{Ax - B|x|} \qquad \text{with} \quad A = \frac{\theta}{\sigma^2} \quad \text{and} \quad B = \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2}. \qquad (3.1)$$

The characteristic function of X_t, the Fourier transform of its distribution, is given by:

$$\Phi_t(u) \quad = \quad (1 + \frac{u^2\sigma^2\kappa}{2} - i\theta\kappa u)^{-\frac{t}{\kappa}} \qquad (3.2)$$

$\Phi_t(.)$ decays as $|u|^{-2t/\kappa}$ when $|u| \to \infty$: the decay exponent increases with t. The fundamental solution $\rho(t,x)$ of the PIDE therefore has a degree of regularity which increases *gradually* with t: for $t \in (p\kappa/2, (p+1)\kappa/2)$, the fundamental solution $\rho(t,.)$ is in $C^{p-1}(\mathbb{R})$ but not $C^p(\mathbb{R})$. For $t < \kappa/2$, $\rho(t,.)$ is not even locally bounded. Consider now the value of a European binary option defined by the payoff $h(x) = 1_{x \geq x_0}$: its value is shown in figure 1. Being the primitive of $\rho(t,.)$, its value is continuous but not differentiable in x for $t < \kappa/2$.

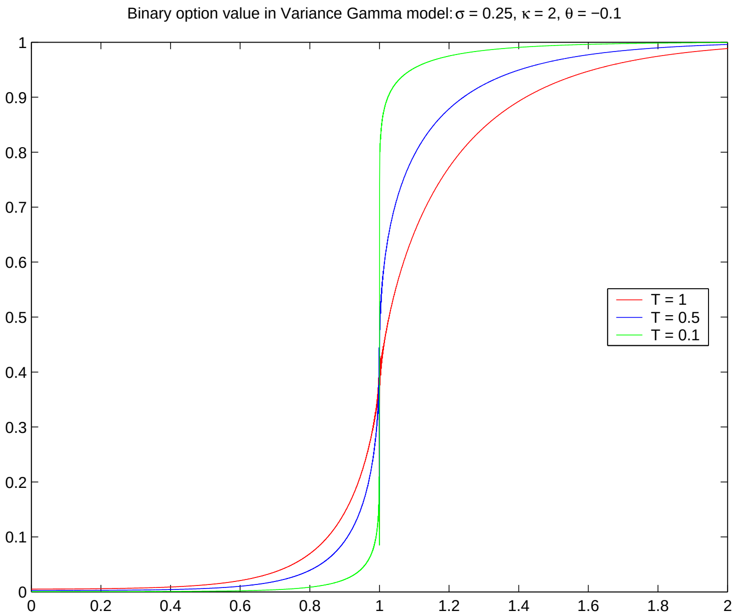

Figure 1: Value of a binary option in the Variance Gamma model, as a function of the underlying.

The case of barrier options is even less regular. As the following example illustrates, if no restriction is imposed on the Lévy process, the value of a barrier option — which is formally the solution of the Dirichlet problem with zero boundary conditions — can even turn out to be discontinuous at all times:

Example 2. Consider $X_t = N_t^1 - N_t^2$ where N_t^i are independent Poisson processes with jump intensities λ_1 and λ_2. Let, for simplicity, $r = 0$. If $\lambda_2 = \lambda_1 e$ then the corresponding price process $S_t = S_0 e^{X_t}$ is a martingale.

Consider now a knock-out option which pays 1 at time T if S_t has not crossed the barrier $U > S_0$ before T, and 0 otherwise:

$$H_T = 1_{T<\theta(S_0)},$$

where $\theta(S) = \inf\{t \geq 0 \mid Se^{X_t} \geq U\}$ is the first exit time if the process starts from S. Let us show that the initial option value

$$C(0, S) = \mathbb{E}[H_T | S_0 = S] = \mathbb{E}[1_{T<\theta(S)}]$$

is not continuous at $S^* = U/e$.

Let $0 < \varepsilon < U - S^*$. By definition, $\theta(S^* + \varepsilon) \leq \theta(S^* - \varepsilon)$. Therefore,

$$\begin{aligned} C(0, S^* - \varepsilon) - C(0, S^* + \varepsilon) &= \mathbb{E}[1_{\{\theta(S^*+\varepsilon) \leq T < \theta(S^*-\varepsilon)\}}] \\ &= \mathbb{Q}\left(\theta(S^* + \varepsilon) \leq T < \theta(S^* - \varepsilon)\right). \end{aligned}$$

Consider the following possibility: $N_T^1 = 1$ et $N_T^2 = 0$, that is, there was one positive and no negative jumps. In this case, if S_t starts from $S^* - \varepsilon$ it stays below U, while starting from $S^* + \varepsilon$ it crosses the barrier. This means that $\theta(S^* + \varepsilon) \leq T < \theta(S^* - \varepsilon)$. So,

$$C(0, S^* - \varepsilon) - C(0, S^* + \varepsilon) \;\geq\; \mathbb{Q}(N_T^1 = 1 \;\&\; N_T^2 = 0) \;=\; e^{-\lambda_1 T(e+1)}\lambda_1 T > 0.$$

Thus $S \mapsto C(0, S)$ is discontinuous at $S = S^*$.

This example is a finite activity process without diffusion component. As noted above, this case is not the interesting one in financial modelling. In the next section, we show that in fact, in most cases of interest, the option price is a continuous function of the underlying.

3.2 Continuity with respect to the underlying

We start with showing that the value of a European option is a continuous function of its arguments, under the Lipschitz condition on the payoff and without any additional restriction on the Lévy density. Next, we give sufficient conditions on the Lévy triplet of X which guarantee the continuity of the barrier options.

Proposition 3. (Continuity of European options)
If H satisfies the Lipschitz condition (2.5) then the forward value of a European option defined by (2.2): $f(\tau, x) = \mathbb{E}[H(S_0 e^{x+r\tau+X_\tau})]$ is continuous on $[0, T] \times \mathbb{R}$.

Proof. The continuity in x is straigthforward:

$$\begin{aligned} |f(\tau, x+\Delta x) - f(\tau, x)| &= |\mathbb{E}[H(S_0 e^{x+\Delta x + r\tau + X_\tau})] - \mathbb{E}[H(S_0 e^{x+r\tau+X_\tau})]| \\ &\leq cS_0 e^{x+r\tau}|e^{\Delta x} - 1|\mathbb{E}[e^{X_\tau}] \to 0 \quad \text{as } |\Delta x| \to 0, \end{aligned}$$

since $\mathbb{E}e^{X_\tau} = 1$ by the martingale condition.

Let us show the continuity in time. Let $t \geq s \geq 0$ (the case $s \geq t$ is symmetrical). Then, $X_t \overset{d}{=} X_s + X_{t-s}$, $X_{t-s} \perp\!\!\!\perp X_s$, and we obtain

$$\begin{aligned} |f(t,x) - f(s,x)| &\leq \mathbb{E}[|H(S_0 e^{x+rt+X_t}) - H(S_0 e^{x+rs+X_s})|] \\ &\leq cS_0 e^{x+rs}\mathbb{E}|e^{r(t-s)+X_{t-s}} - 1|. \end{aligned}$$

So, we need to show that $\mathbb{E}|e^{r\tau+X_\tau} - 1| \to 0$ as $\tau \to 0$. First, we remark that the martingale condition implies:

$$\mathbb{E}|e^{r\tau+X_\tau} - 1| = e^{r\tau} - 1 + 2\mathbb{E}[(1 - e^{r\tau+X_\tau})^+]. \tag{3.3}$$

Let $C_0(\mathbb{R})$ be the set of continuous functions vanishing at infinity. By the Feller property (see, for example, [9], Section 3.8), for any $g \in C_0(\mathbb{R})$, we have

$$P_\tau g(0) \equiv \mathbb{E}g(r\tau + X_\tau) \xrightarrow{\tau\downarrow 0} g(0), \tag{3.4}$$

where P_τ is the semigroup of the process $\{r\tau + X_\tau\}$.

Since $g(x) = (1 - e^x)^+$ is not in $C_0(\mathbb{R})$, we approximate it with a function $\tilde{g}(x)$, such that

$$\begin{aligned} &\tilde{g}(x) = g(x), \text{ if } x \geq -1, \\ &\tilde{g}(x) = 0, \text{ if } x \leq -2, \\ &0 \leq \tilde{g}(x) \leq g(x), \end{aligned}$$

and $\tilde{g}(x)$ is continuously interpolated between -2 and -1. By construction, $\tilde{g} \in C_0(\mathbb{R})$. We obtain

$$\begin{aligned} \mathbb{E}[(1 - e^{r\tau+X_\tau})^+] &= |P_\tau g(0)| \;\leq\; |P_\tau g(0) - P_\tau \tilde{g}(0)| + |P_\tau \tilde{g}(0)| \\ &= \mathbb{E}[g(r\tau + X_\tau) - \tilde{g}(r\tau + X_\tau)] + |P_\tau \tilde{g}(0)| \\ &= \mathbb{E}[(g(r\tau + X_\tau) - \tilde{g}(r\tau + X_\tau))1_{\{r\tau + X_\tau < -1\}}] + |P_\tau \tilde{g}(0)| \\ &\leq \mathbb{Q}[r\tau + X_\tau < -1] + |P_\tau \tilde{g}(0)| \;\leq\; \mathbb{Q}[X_\tau \leq -1] + |P_\tau \tilde{g}(0)|, \end{aligned}$$

since $g(x) \leq 1$. By (3.4), we have $|P_\tau \tilde{g}(0)| \to 0$ as $\tau \to 0$. So, the last point to show is that $\mathbb{Q}[X_\tau \leq -1] \to 0$ as $\tau \to 0$.

Defining $M_\tau^- = \sup_{0\leq s\leq \tau}(-X_s)$, we have $\mathbb{Q}[X_\tau \leq -1] \leq \mathbb{Q}[M_\tau^- \geq 1]$. Let us take a sequence $\tau_n \downarrow 0$ and define $\Omega_n = \{\omega \in \Omega \mid M_{\tau_n}^-(\omega) \geq 1\}$. The sequence $\{\Omega_n\}$ is decreasing, and

$$\bigcap_{n>0} \Omega_n = \{\omega \in \Omega \mid \forall n, M_{\tau_n}^-(\omega) \geq 1\} \subseteq \{\omega \in \Omega \mid M_0^-(\omega) \geq 1\},$$

by the right-continuity of X_τ. Since $M_0^- = X_0 = 0$, we obtain

$$\mathbb{Q}[M_{\tau_n}^- \geq 1] = \mathbb{Q}(\Omega_n) \overset{\tau_n \downarrow 0}{\longrightarrow} \mathbb{Q}(\bigcap \Omega_n) = 0,$$

which implies $\mathbb{Q}[M_\tau^- \geq 1] \to 0$, since $\{\tau_n\}$ is arbitrary. Therefore, $\mathbb{Q}[X_\tau \leq -1] \to 0$, and the proof is completed. □

Consider the Lévy process Y_t. To study the continuity of barrier options we extensively use the properties of the *first passage time process*. Following the notation of [27], we define

$$\begin{aligned} R_x &= \inf\{s \geq 0 \mid Y_s > x\}, \\ R''_x &= \inf\{s \geq 0 \mid Y_s \vee Y_{s-} \geq x\}. \end{aligned}$$

Note that $\{R_x, x \geq 0\}$ is a process with non-decreasing paths, so we can define $R_{x-}(\omega) = \lim_{\varepsilon\downarrow 0} R_{x-\varepsilon}(\omega)$. Since Y_t is right-continuous, the process R_x is also right-continuous in x.

To classify Lévy processes we use the following terminology.

Definition 1. Let $\{Y_t\}$ be a Lévy process on $\mathbb{R}$ with generating triplet (σ, γ, ν). It is said to be of

type A, if $\sigma = 0$ and $\nu(\mathbb{R}) < \infty$

type B, if $\sigma = 0$, $\nu(\mathbb{R}) = \infty$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$

type C, if $\sigma > 0$ or $\int_{|x|\leq 1} |x|\nu(dx) = \infty$.

Type A corresponds to compound Poisson processes; type B corresponds to finite variation processes with infinite intensity and type C corresponds to infinite variation Lévy processes.

Now we give some properties of the process $\{R_x\}$ which are essential to prove the continuity of option values.

Lemma 1. *If $\{Y_t\}$ is of type B or C then:*

$$\forall x > 0, \qquad \mathbb{Q}[R_{x-} = R_x] = 1. \tag{3.5}$$

Proof. For a fixed $x > 0$, we introduce two subsets of Ω:

$$\Omega_1 = \{\omega \mid R_{x-}(\omega) < R_x(\omega)\}, \qquad \Omega_2 = \{\omega \mid R_x(\omega) = R''_x(\omega)\}.$$

By the Lemma 49.6 of [27], $\mathbb{Q}(\Omega_2) = 1$. So, it suffices to show that $\Omega_1 \cap \Omega_2 = \emptyset$, since, by definition, $R_{x-} \leq R_x$.

If there exists $\omega \in \Omega_1 \cap \Omega_2$ then for the sample path $R_x = R_x(\omega)$ we have $R_{x-} < R''_x$. Therefore,

$$\exists u \geq 0,\ R_{x-} = u, \tag{3.6}$$
$$\exists t > u,\ R''_x = t. \tag{3.7}$$

The definition of R_{x-} together with (3.6) implies that

$$\forall \varepsilon > 0,\ \forall \delta > 0,\ \exists s < u + \delta :\ Y_s > x - \varepsilon.$$

Take $\varepsilon_n = \delta_n = 1/n$. Then, there exists a sequence $\{s_n\}$, such that $\forall n$,

$$s_n < u + 1/n, \qquad Y_{s_n} > x - 1/n.$$

Since $\{s_n\}$ is bounded, one can extract a convergent subsequence $s_n \uparrow s_0$ or $s_n \downarrow s_0$, with $s_0 \le u < t$. In the first case we obtain $Y_{s_0 -} \ge x$, and in the second, $Y_{s_0} \ge x$. So, in all cases, $Y_{s_0 -} \vee Y_{s_0} \ge x$. But (3.7) implies

$$\forall s < t, \qquad Y_{s-} \vee Y_s < x.$$

This contradiction proves that $\Omega_1 \cap \Omega_2 = \emptyset$, hence $\mathbb{Q}(\Omega_1) = 0$, and the proof is completed. $\square$

An important property of $\{R_x\}$ is whether $R_0 = 0$ a.s.. Table 1, which is a consequence of Theorem 47.5 of [27], relates this property to properties of the Lévy triplet for different types of Lévy processes.

Define now the *supremum process* of Y:

$$M_t \quad = \quad \sup_{0 \le s \le t} Y_s.$$

M_t is non-decreasing and *càdlàg*, since Y_t is *càdlàg*. We use it to prove the next lemma.

Lemma 2. *If $\{Y_t\}$ is of type B with $R_0 = 0$ a.s., or of type C, then:*

$$\forall x > 0, \quad \forall t \ge 0, \qquad \mathbb{Q}[R_x = t] = 0.$$

Proof. By the definition of R_x,

$$\begin{aligned} \mathbb{Q}[R_x = t] &= \mathbb{Q}[\forall s < t,\ Y_s \le x;\ \exists s_n \downarrow t,\ Y_{s_n} > x] \\ &\le \mathbb{Q}[M_{t-} \le x;\ M_t \ge x]. \end{aligned}$$

Lemma 49.3 of [27] states that, in our hypotheses, $\forall t > 0$, $\forall x \ge 0$,

$$\mathbb{Q}[M_t = x] = 0.$$

Therefore: $\forall x \ge 0$, $\forall t > 0$,

$$\mathbb{Q}[R_x = t] \quad \le \quad \mathbb{Q}[M_{t-} \le x < M_t] \ \le\ \mathbb{Q}[M_{t-} \ne M_t] \ \le\ \mathbb{Q}[Y_{t-} \ne Y_t] = 0,$$

since a Lévy process has, almost surely, no fixed times of discontinuity. For the same reason, $\forall x > 0$, $\mathbb{Q}[R_x = 0] = 0$, which completes the proof. $\square$

Lemma 3. *If $\{Y_t\}$ is of type B or C, then $\forall t \geq 0$, $\forall x > 0$,*

$$\mathbb{Q}[R_x \leq t < R_{x+\varepsilon}] \to 0, \tag{3.8}$$
$$\mathbb{Q}[R_{x-\varepsilon} \leq t < R_x] \to 0, \tag{3.9}$$

as $\varepsilon \downarrow 0$. If, in addition, $R_0 = 0$ almost surely, then (3.8) is also satisfied for $x = 0$, $t > 0$.

Proof. For all fixed $t \geq 0$, $x \geq 0$, the sequence $\Omega_\varepsilon = \{\omega \in \Omega \mid R_x(\omega) \leq t < R_{x+\varepsilon}(\omega)\}$ is decreasing, and $\bigcap \Omega_\varepsilon = \{\omega \in \Omega \mid R_x(\omega) = t\}$, by the right-continuity of R_x. Therefore,

$$\mathbb{Q}[R_x \leq t < R_{x+\varepsilon}] \quad \to \quad \mathbb{Q}[R_x = t].$$

If $x > 0$, this probability is zero by the Lemma 2. If $R_0 = 0$ a.s. then, $\forall t > 0$, $\mathbb{Q}[R_0 = t] = 0$.

Similarly, for all $t \geq 0$, $x > 0$,

$$\mathbb{Q}[R_{x-\varepsilon} \leq t < R_x] \quad \to \quad \mathbb{Q}[R_{x-} \leq t < R_x] \;\leq\; \mathbb{Q}[R_{x-} \neq R_x] \;=\; 0,$$

by the Lemma 1. □

Define $Y_t = rt + X_t$. The Lévy triplet of $\{Y_t\}$ is $(\sigma, r + \gamma, \nu)$, where γ is determined by the martingale condition (1.3). So, $\{Y_t\}$ is of the same type as $\{X_t\}$, in the sense of the Definition 1. But the property of R_0, which in the finite-variation case depends on the drift, is not necessary the same for the two processes. Therefore, it is worth noting that $\{R_x\}$ will be always defined with respect to $\{Y_t\}$.

Now we are in a position to consider the continuity of value functions for barrier options. We start with the case of a single upper barrier $U > S_0$.

Proposition 4. (Continuity of up-and-out options)
Let Y_t be of type B or C, and $R_0 = 0$ a.s. Assume that $H : (0, U) \to [0, \infty)$ is Lipschitz:

$$\forall S_1, S_2 \in (0, U), \quad |H(S_1) - H(S_2)| \leq c|S_1 - S_2|,$$

with $c > 0$, and denote $u = \log(U/S_0)$. Then, the function

$$f_u(\tau, x) = \begin{cases} \mathbb{E}[H(S_0 e^{x+Y_\tau}) 1_{\{\tau < R_{u-x}\}}], & x < u, \\ 0, & x \geq u, \end{cases} \tag{3.10}$$

is continuous on $(0, T] \times \mathbb{R}$.

Remark 1. One can verify directly that $C(t, S) = e^{-r(T-t)} f_u(T - t, \log(S/S_0))$ is just a different representation of the function defined by (2.12). Recall that it gives the value of an up-and-out option with the payoff $H(S_T) 1_{T < \inf\{t \geq 0,\, S_t \geq U\}}$ at time t when the stock price is S, if S_t has not yet crossed the barrier (see (2.13)).

Proof. Since H is Lipschitz, it is bounded on $(0, U)$. Let $M = \sup_{(0,U)} H(S)$.

We first show the continuity in x, for all $\tau > 0$. If $x < u$ and $\varepsilon \in (0, u-x)$, we have

$$\begin{aligned}|f_u(\tau, x+\varepsilon) - f_u(\tau, x)| &= |\mathbb{E}[H(S_0 e^{x+\varepsilon+Y_\tau}) 1_{\{\tau < R_{u-x-\varepsilon}\}}] - \mathbb{E}[H(S_0 e^{x+Y_\tau}) 1_{\{\tau < R_{u-x}\}}]| \\ &\leq |\mathbb{E}[(H(S_0 e^{x+\varepsilon+Y_\tau}) - H(S_0 e^{x+Y_\tau})) 1_{\{\tau < R_{u-x-\varepsilon}\}}]| + \\ &\quad + |\mathbb{E}[H(S_0 e^{x+Y_\tau}) 1_{\{R_{u-x-\varepsilon} \leq \tau < R_{u-x}\}}]| \\ &\leq c S_0 e^{x+r\tau}(e^{\varepsilon} - 1) + M\mathbb{Q}[R_{u-x-\varepsilon} \leq \tau < R_{u-x}] \xrightarrow{\varepsilon \downarrow 0} 0,\end{aligned}$$

by (3.9). We have used the martingale condition $\mathbb{E}e^{Y_\tau} = e^{r\tau}$ and the fact that $\tau < R_{u-x}$ implies $Y_\tau \leq u - x$, which is equivalent to $S_0 e^{x+Y_\tau} \leq U$.

Similarly, for all $x < u$,

$$\begin{aligned}|f_u(\tau, x-\varepsilon) - f_u(\tau, x)| &= |\mathbb{E}[H(S_0 e^{x-\varepsilon+Y_\tau}) 1_{\{\tau < R_{u-x+\varepsilon}\}}] - \mathbb{E}[H(S_0 e^{x+Y_\tau}) 1_{\{\tau < R_{u-x}\}}]| \\ &\leq |\mathbb{E}[(H(S_0 e^{x-\varepsilon+Y_\tau}) - H(S_0 e^{x+Y_\tau})) 1_{\{\tau < R_{u-x}\}}]| + \\ &\quad + \mathbb{E}[H(S_0 e^{x-\varepsilon+Y_\tau}) 1_{\{R_{u-x} \leq \tau < R_{u-x+\varepsilon}\}}] \\ &\leq c S_0 e^{x+r\tau}(1 - e^{-\varepsilon}) + M\mathbb{Q}[R_{u-x} \leq \tau < R_{u-x+\varepsilon}] \xrightarrow{\varepsilon \downarrow 0} 0,\end{aligned}$$

by (3.8). This proves the continuity of $f_u(\tau, \cdot)$ for all $x \neq u$.

The right continuity at $x = u$ is straightforward, since $f = 0$ if $x \geq u$. It remains to verify the left continuity. For all $\tau > 0$,

$$|f_u(\tau, u-\varepsilon) - f_u(\tau, u)| = |\mathbb{E}[H(S_0 e^{u-\varepsilon+Y_\tau}) 1_{\{\tau < R_\varepsilon\}}]| \leq M\mathbb{Q}[R_\varepsilon > \tau] \xrightarrow{\varepsilon \downarrow 0} 0,$$

since $R_0 = 0$ almost surely. In consequence, $\forall \tau > 0$, $f_u(\tau, \cdot)$ is continuous on $\mathbb{R}$.

Let us now show the continuity in time. For a fixed $x < u$, and $t \geq s \geq 0$, we obtain:

$$\begin{aligned}|f_u(t,x) - f_u(s,x)| &= |\mathbb{E}[H(S_0 e^{x+Y_t}) 1_{\{t < R_{u-x}\}}] - \mathbb{E}[H(S_0 e^{x+Y_s}) 1_{\{s < R_{u-x}\}}]| \\ &= |\mathbb{E}[(H(S_0 e^{x+Y_t}) - H(S_0 e^{x+Y_s})) 1_{\{t < R_{u-x}\}}] - \mathbb{E}[H(S_0 e^{x+Y_s}) 1_{\{s < R_{u-x} \leq t\}}]| \\ &\leq c\mathbb{E}[S_0 e^{x+Y_s} |e^{Y_{t-s}} - 1| \, 1_{\{t < R_{u-x}\}}] + M\mathbb{Q}[s < R_{u-x} \leq t] \\ &\leq c S_0 e^{x+rs} \, \mathbb{E}|e^{Y_{t-s}} - 1| + M\mathbb{Q}[s < R_{u-x} \leq t].\end{aligned}$$

The convergence of the first term to zero as $t \to s$ was already proven in the Proposition 3. Let $t_n \downarrow s$ be an arbitrary sequence, and denote $\Omega_n = \{\omega \in \Omega \mid s < R_{u-x}(\omega) \leq t_n\}$. Then, $\{\Omega_n\}$ is decreasing as $n \to \infty$, and

$$\bigcap_{n>0} \Omega_n = \{\omega \in \Omega \mid \forall n, s < R_{u-x}(\omega) \leq t_n\} \subseteq \{\omega \in \Omega \mid s < R_{u-x}(\omega) \leq s\} = \emptyset.$$

So, $\mathbb{Q}[s < R_{u-x} \leq t] \to 0$ as $t \to s$, and the proof is completed. □

Remark 2. As the proof shows, if $\{Y_t\}$ is of type B or C, $f_u(\tau, x)$ is continuous on $(0, T] \times \mathbb{R} \setminus \{u\}$. If the condition $R_0 = 0$ a.s. is not satisfied, f_u may be discontinuous at the barrier.

In order to study down-and-out options, let us define the process $\{R^-_x,\ x \geq 0\}$ of the fist passage below a negative level:

$$R^-_x \quad = \quad \inf\{s \geq 0 | Y_s < -x\} = \inf\{s \geq 0 \mid -Y_s > x\}.$$

It is clear that Lemmas 1–3 apply to R^-_x provided $\{-Y_t\}$ (the *dual process* of $\{Y_t\}$) satisfies the corresponding conditions. The generating triplet of $\{-Y_t\}$ being $(\sigma, -(r+\gamma), \nu(-dx))$, the dual process has the same type as Y_t (in the sense of the Definition 1). However, note that $R_0 = 0$ a.s. does not imply $R^-_0 = 0$ a.s., as shows Table 1.

Proposition 5. (Continuity of down-and-out options) *Let $\{Y_t\}$ be of type B or C, and $R^-_0 = 0$ a.s. Assume that $H : (L, \infty) \to [0, \infty)$ is Lipschitz:*

$$\forall S_1, S_2 \in (L, \infty), \quad |H(S_1) - H(S_2)| \leq c|S_1 - S_2|,$$

with $L < S_0$, $c > 0$, and denote $l = \log(L/S_0)$. Then, the function

$$f_l(\tau, x) = \begin{cases} \mathbb{E}[H(S_0 e^{x+Y_\tau}) 1_{\{\tau < R^-_{x-l}\}}], & x > l, \\ 0, & x \leq l, \end{cases} \tag{3.11}$$

is continuous on $(0, T] \times \mathbb{R}$ (f_l represents the forward value of a down-and-out option with the payoff $H(S_T) 1_{T < \inf\{t \geq 0,\ S_t \leq L\}}$).

Proof. The proof is similar to the one of the Proposition 4. The main difference is that H may be unbounded, so we need to refine certain estimates.

To show the continuity of $f(\tau, \cdot)$ at $x > l$ (for a fixed $\tau > 0$), we write:

$$\begin{aligned} |f_l(\tau, x+\varepsilon) - f_l(\tau, x)| &= |\mathbb{E}[H(S_0 e^{x+\varepsilon+Y_\tau}) 1_{\{\tau < R^-_{x+\varepsilon-l}\}}] - \mathbb{E}[H(S_0 e^{x+Y_\tau}) 1_{\{\tau < R^-_{x-l}\}}]| \\ &\leq |\mathbb{E}[(H(S_0 e^{x+\varepsilon+Y_\tau}) - H(S_0 e^{x+Y_\tau})) 1_{\{\tau < R^-_{x-l}\}}]| + \\ &\qquad + \mathbb{E}[H(S_0 e^{x+\varepsilon+Y_\tau}) 1_{\{R^-_{x-l} \leq \tau < R^-_{x+\varepsilon-l}\}}]. \end{aligned}$$

The first term may be estimated as previously, and goes to zero as $\varepsilon \downarrow 0$. For the second term we obtain:

$$\begin{aligned} \mathbb{E}[H(S_0 e^{x+\varepsilon+Y_\tau}) 1_{\{R^-_{x-l} \leq \tau < R^-_{x+\varepsilon-l}\}}] &\leq \mathbb{E}[C(1 + S_0 e^{x+\varepsilon+Y_\tau}) 1_{\{R^-_{x-l} \leq \tau < R^-_{x+\varepsilon-l}\}}] \\ = C\mathbb{Q}[R^-_{x-l} \leq \tau < R^-_{x+\varepsilon-l}] &+ C S_0 e^{x+\varepsilon+r\tau} \mathbb{E}[e^{Y_\tau} 1_{\{R^-_{x-l} \leq \tau < R^-_{x+\varepsilon-l}\}}]. \end{aligned} \tag{3.12}$$

The quantity $e^{Y_\tau} 1_{\{R^-_{x-l} \leq \tau < R^-_{x+\varepsilon-l}\}}$ is bounded by an integrable variable e^{Y_τ} and converges to 0 in probability, since

$$\forall \sigma > 0, \quad \mathbb{Q}[e^{Y_\tau} 1_{\{R^-_{x-l} \leq \tau < R^-_{x+\varepsilon-l}\}} > \sigma] \leq \mathbb{Q}[R^-_{x-l} \leq \tau < R^-_{x+\varepsilon-l}] \xrightarrow{\varepsilon \downarrow 0},$$

by (3.8). Therefore, the dominated convergence theorem implies

$$\mathbb{E}[e^{Y_\tau} 1_{\{R^-_{x-l} \leq \tau < R^-_{x+\varepsilon-l}\}}] \xrightarrow{\varepsilon \downarrow 0}, \tag{3.13}$$

and the whole expression in (3.12) tends to zero as $\varepsilon \downarrow 0$.

Using the same technique, one can show that $|f_l(\tau, x-\varepsilon) - f_l(\tau, x)| \to 0$, $\forall x > l$, and $|f_l(\tau, l-\varepsilon)| \to 0$, as $\varepsilon \downarrow 0$, which proves the continuity of f_l in x.

Similarly, to show the continuity in time, we write for a fixed $x > l$ and $t \geq s \geq 0$:

$$
\begin{aligned}
&|f_l(t,x) - f_l(s,x)| = |\mathbb{E}[H(S_0 e^{x+Y_t}) 1_{\{t<R^-_{x-l}\}}] - \mathbb{E}[H(S_0 e^{x+Y_s}) 1_{\{s<R^-_{x-l}\}}]| \\
&\quad = |\mathbb{E}[(H(S_0 e^{x+Y_t}) - H(S_0 e^{x+Y_s})) 1_{\{t<R^-_{x-l}\}}] - \mathbb{E}[H(S_0 e^{x+Y_s}) 1_{\{s<R^-_{x-l}\leq t\}}]| \\
&\qquad \leq c S_0 e^{x+rs} \mathbb{E}|e^{Y_{t-s}} - 1| + C\mathbb{Q}[s < R^-_{x-l} \leq t] + C S_0 e^x \mathbb{E}[e^{Y_s} 1_{\{s<R^-_{x-l}\leq t\}}].
\end{aligned}
$$

The convergence of the first two terms to zero, as $t \to s$, has already been proved, and the last term can be treated in the same way as (3.13). $\square$

Finally, we present a continuity result for double-barrier options. For $L < S_0 < U$, denote, as previously, $l = \log(L/S_0)$ and $u = \log(U/S_0)$.

Proposition 6. (Continuity of double-barrier options) *Let $\{Y_t\}$ be of type B or C, with $R_0 = 0$ and $R^-_0 = 0$ a.s. Assume that $H : (L,U) \to [0,\infty)$ is Lipschitz:*

$$\forall S_1, S_2 \in (L,U), \quad |H(S_1) - H(S_2)| \leq c|S_1 - S_2|.$$

Then, the forward value of a double-barrier option with the payoff $H(S_T) 1_{T<\inf\{t\geq 0,\ S_t \notin (L,U)\}}$, defined by

$$
f_d(\tau, x) = \begin{cases} \mathbb{E}[H(S_0 e^{x+Y_\tau}) 1_{\{\tau<R_{u-x}\wedge R^-_{x-l}\}}], & x \in (l,u), \\ 0, & x \notin (l,u), \end{cases} \tag{3.14}
$$

is continuous on $(0,T] \times \mathbb{R}$.

Proof. Let $M = \sup_{(L,U)} H(S)$. As in the two preceding propositions, we show the right and left continuity of f_d at every point $x \in [l,u]$ using the Lemma 3. For example, $\forall \tau > 0$, $\forall x \in (l,u)$,

$$
\begin{aligned}
&|f_d(\tau, x+\varepsilon) - f_d(\tau, x)| = \\
&\quad = |\mathbb{E}[H(S_0 e^{x+\varepsilon+Y_\tau}) 1_{\{\tau<R_{u-x-\varepsilon}\wedge R^-_{x+\varepsilon-l}\}}] - \mathbb{E}[H(S_0 e^{x+Y_\tau}) 1_{\{\tau<R_{u-x}\wedge R^-_{x-l}\}}]| \\
&\qquad \leq |\mathbb{E}[(H(S_0 e^{x+\varepsilon+Y_\tau}) - H(S_0 e^{x+Y_\tau})) 1_{\{\tau<R_{u-x-\varepsilon}\wedge R^-_{x-l}\}}]| + \\
&\qquad\quad + \mathbb{E}[H(S_0 e^{x+\varepsilon+Y_\tau}) 1_{\{\tau<R_{u-x-\varepsilon}\}} 1_{\{R^-_{x-l}\leq\tau<R^-_{x+\varepsilon-l}\}}] + \\
&\qquad\quad + \mathbb{E}[H(S_0 e^{x+Y_\tau}) 1_{\{\tau<R^-_{x-l}\}} 1_{\{R_{u-x-\varepsilon}\leq\tau<R_{u-x}\}}] \\
&\leq c S_0 e^{x+r\tau}(e^\varepsilon - 1) + M\mathbb{Q}[R^-_{x-l} \leq \tau < R^-_{x+\varepsilon-l}] + M\mathbb{Q}[R_{u-x-\varepsilon} \leq \tau < R_{u-x}] \xrightarrow{\varepsilon\downarrow 0} 0.
\end{aligned}
$$

We do not give in detail the whole proof because it repeates almost literally the proofs of the Propositions 4 and 5. $\square$

Type of $Y_t = rt + X_t$			$R_0 = 0$ a.s.	$R_0^- = 0$ a.s.	Continuity		
					f_h	f_l	f_d
A	$\gamma_0 > 0$		**yes**	*no*			
A	$\gamma_0 < 0$		*no*	**yes**			
A	$\gamma_0 = 0$		*no*	*no*			
B	$\gamma_0 > 0$		**yes**	*no*	**yes**		
B	$\gamma_0 < 0$		*no*	**yes**		**yes**	
B	$\gamma_0 = 0$	$\nu(-\infty, 0) < \infty$	**yes**	*no*	**yes**		
B	$\gamma_0 = 0$	$\nu(0, \infty) < \infty$	*no*	**yes**		**yes**	
B	$\gamma_0 = 0$	$\nu(-\infty, 0) = \infty$, $\nu(0, \infty) = \infty$,	Depends on the further properties of $\{Y_t\}$				
C			**yes**	**yes**	**yes**	**yes**	**yes**

Table 1: This table shows the properties of R_0 and R_0^- for different types of Lévy processes and summarizes our results on the continuity of the barrier options. (An empty box does not mean the function is necessary discontinuous but there is no continuity result for this case). In the finite-variation case, $\gamma_0 = r - \int (e^y - 1)\nu(dy)$ is the drift.

Table 1 summarizes the results of this section. Interestingly, while investigating a different issue — the validity of smooth pasting conditions for American options in exp-Lévy models — Alili & Kyprianou [1] arrive at conditions similar to the ones given in Propositions 4-5.

As shown by the examples above, in general one cannot hope for more than Lipschitz continuity with respect to the underlying. In particular, uniform bounds on derivatives, such as the ones required in [23], do not hold in cases of interest in finance — such as call or put options — where the payoff function H is not smooth. In these cases, verification theorems such as the Proposition 1 do not apply and the option pricing function should be seen as a *viscosity* solution of the PIDE (2.3).

4 Option prices as viscosity solutions of PIDEs

Existence and uniqueness of (classical) solutions for the PIDEs considered above in Sobolev / Hölder spaces have been studied in [6, 15] in the case where the diffusion component is non-degenerate: for a Lévy process this simply means $\sigma > 0$ but more generally these results apply to jump diffusion where the diffusion coefficient is bounded away from zero. However many of the models in the financial modelling literature are pure jump models with $\sigma = 0$, for which such results are not available. A notion of solution which yields both existence and uniqueness in this case is the notion of viscosity solution, introduced by Crandall & Lions for PDEs (see e.g. [12] for a review) and extended to integro-differential equations of the type considered here in [2, 5, 24, 28, 29].[2]

[2]Definitions of viscosity solutions in these papers vary in the choice of test functions; we present here a version which is suitable for option pricing applications.

4.1 Viscosity solutions for PIDEs

Denote by USC (respectively LSC) the class of upper semicontinuous (respectively lower semicontinuous) functions $v : [0,T) \times \mathbb{R} \to \mathbb{R}$ and by $C_p^+([0,T] \times \mathbb{R})$ the set of measurable functions on $[0,T] \times \mathbb{R}$ with polynomial growth of degree p at plus infinity and bounded on $[0,T] \times \mathbb{R}^-$:

$$\varphi \in C_p^+([0,T] \times \mathbb{R}) \iff \exists C > 0,\ |\varphi(t,x)| \le C(1 + |x|^p \, 1_{x>0}). \tag{4.1}$$

Under a polynomial decay condition on the right tail of the Lévy density, $L\varphi$ can be defined for $\varphi \in C^2([0,T] \times \mathbb{R}) \cap C_p^+([0,T] \times \mathbb{R})$:

$$L\varphi(x) = A\varphi(x) + \int_{|y|\le 1} \nu(dy)[\varphi(x+y) - \varphi(x) - y\frac{\partial \varphi}{\partial x}(x)] \tag{4.2}$$

$$+ \int_{|y|>1} \nu(dy)[\varphi(x+y) - \varphi(x)], \tag{4.3}$$

where A is a differential operator. The terms in (4.2) are well defined for $\varphi \in C^2([0,T] \times \mathbb{R})$ since

$$|\varphi(\tau, x+y) - \varphi(\tau,x) - y\frac{\partial \varphi}{\partial x}(\tau,x)| \le y^2 \sup_{B(x,1)} |\varphi''(\tau,\cdot)| \qquad \text{for } |y| \le 1,$$

while the term in (4.3) is well defined for $\varphi \in C_p^+([0,T] \times \mathbb{R})$ if

$$\int_{y>1} y^p \nu(dy) < +\infty,$$

which is satisfied due to the martingale condition (1.3).

Let $O = (l,u) \subseteq \mathbb{R}$ be an open interval, $\partial O = \{l,u\}$ its boundary, and $g \in C_p^+([0,T] \times\ \mathbb{R} \setminus O)$ a continuous function. Consider the following initial-boundary value problem on $[0,T] \times \mathbb{R}$:

$$\frac{\partial f}{\partial \tau} = Lf + r\frac{\partial f}{\partial x}, \qquad \text{on } (0,T] \times O, \tag{4.4}$$

$$f(0,x) = h(x), \quad x \in O; \qquad f(\tau,x) = g(\tau,x), \quad x \notin O. \tag{4.5}$$

Definition 2 (Viscosity solution). A function $v \in USC$ is a *viscosity subsolution* of (4.4)–(4.5) if for any test function $\varphi \in C^2([0,T]\times\mathbb{R}) \cap C_p^+([0,T]\times\mathbb{R})$ and any global maximum point $(\tau,x) \in [0,T] \times \mathbb{R}$ of $v - \varphi$, the following properties are verified:

$$\text{if } (\tau,x) \in (0,T] \times O, \qquad \left(\frac{\partial \varphi}{\partial \tau} - L\varphi - r\frac{\partial \varphi}{\partial x}\right)(\tau,x) \le 0, \tag{4.6}$$

$$\text{if } \tau = 0,\ x \in \overline{O}, \qquad \min\{\left(\frac{\partial \varphi}{\partial \tau} - L\varphi - r\frac{\partial \varphi}{\partial x}\right)(\tau,x),\ v(\tau,x) - h(x)\} \le 0,$$

$$\text{if } \tau \in (0,T],\ x \in \partial O, \qquad \min\{\left(\frac{\partial \varphi}{\partial \tau} - L\varphi - r\frac{\partial \varphi}{\partial x}\right)(\tau,x),\ v(\tau,x) - g(\tau,x)\} \le 0,$$

$$\text{if } x \notin \overline{O}, \qquad v(\tau,x) \le g(\tau,x). \tag{4.7}$$

A function $v \in LSC$ is a *viscosity supersolution* of (4.4)–(4.5) if for any test function $\varphi \in C^2([0,T]\times\mathbb{R}) \cap C_p^+([0,T]\times\mathbb{R})$ and any global minimum point $(\tau,x) \in [0,T]\times\mathbb{R}$ of $v-\varphi$, we have:

$$\begin{aligned}
\text{if } (\tau,x) \in (0,T]\times O, &\qquad \left(\frac{\partial\varphi}{\partial\tau} - L\varphi - r\frac{\partial\varphi}{\partial x}\right)(\tau,x) \geq 0,\\
\text{if } \tau = 0,\ x \in \overline{O}, &\qquad \max\{\left(\frac{\partial\varphi}{\partial\tau} - L\varphi - r\frac{\partial\varphi}{\partial x}\right)(\tau,x),\ v(\tau,x)-h(x)\} \geq 0,\\
\text{if } \tau \in (0,T],\ x \in \partial O, &\qquad \max\{\left(\frac{\partial\varphi}{\partial\tau} - L\varphi - r\frac{\partial\varphi}{\partial x}\right)(\tau,x),\ v(\tau,x)-g(\tau,x)\} \geq 0,\\
\text{if } x \notin \overline{O}, &\qquad v(\tau,x) \geq g(\tau,x).
\end{aligned}$$

A function $v \in C_p^+([0,T]\times\mathbb{R})$ is called a *viscosity solution* of (4.4)–(4.5) if it is both a subsolution and a supersolution. This function is then continuous on $(0,T]\times\mathbb{R}$.

Note that a subsolution/supersolution need not be continuous and the initial and boundary conditions are verified in a viscosity sense. The definition also includes the case of initial value problems: $O = \mathbb{R}$.

Several variations on this definition can be found in the articles cited above. First, one can restrict the maximum/mimimum of $v-\varphi$ to be equal to zero:

Lemma 4. *Function $v \in USC$ is a viscosity subsolution of (4.4)–(4.5) if and only if for any $(\tau,x) \in [0,T]\times\mathbb{R}$ and any $\varphi \in C^2([0,T]\times\mathbb{R}) \cap C_p^+[0,T]\times\mathbb{R}([0,T]\times\mathbb{R})$, properties*

$$v(\tau,x) = \varphi(\tau,x), \qquad \textit{and} \qquad \forall (t,y) \in [0,T]\times\mathbb{R},\quad v(t,y) \leq \varphi(t,y) \tag{4.8}$$

imply (4.6)–(4.7).

Proof. Clearly, (4.8) means in particular that (τ,x) is a global maximum point of $v-\varphi$. Therefore, if v is a subsolution then, by definition, (4.8) implies (4.6)–(4.7).

Conversely, if v satisfies the condition of the lemma, we take a test function φ and a global maximum point (τ,x) of $v-\varphi$, i.e.

$$\forall (t,y) \in [0,T]\times\mathbb{R},\quad v(t,y)-\varphi(t,y) \leq v(\tau,x)-\varphi(\tau,x),$$

and need to show that φ verifies (4.6)–(4.7). Let us define a new function ψ by adding a constant to φ:

$$\psi(t,y) = \varphi(t,y) + [v(\tau,x)-\varphi(\tau,x)].$$

This function satisfies (4.8), so, by the condition, (4.6)–(4.7) are verified for ψ. But, $\forall (t,y) \in [0,T]\times\mathbb{R}$,

$$\left(\frac{\partial\psi}{\partial\tau} - L\psi - r\frac{\partial\psi}{\partial x}\right)(t,y) \quad = \quad \left(\frac{\partial\varphi}{\partial\tau} - L\varphi - r\frac{\partial\varphi}{\partial x}\right)(t,y) \tag{4.9}$$

which implies that the same properties are verified by φ, hence v is a viscosity subsolution. $\square$

A similar result holds for supersolutions. Also, as shown in [5], one can replace "maximum" by "strict maximum". Finally, one can require the test functions to be $C^{1,2}$ or C^∞ with bounded derivatives instead of C^2. The growth condition at infinity $\varphi \in C_p^+$ on test functions is essential for $L\varphi$ to make sense. It may be replaced by other growth conditions under stronger hypotheses on the decay of the Lévy density.

Since L verifies a maximum principle [7], one can show that a classical solution $u \in C^{1,2}([0,T] \times \mathbb{R}) \cap C_p^+([0,T] \times \mathbb{R})$ is also a viscosity solution. However, since the definition above only involves applying derivatives to the test functions φ, a viscosity solution need not to be smooth: it is simply required to be continuous on $(0,T] \times \mathbb{R}$.

Remark 3 (Boundary conditions). We noted above that, for classical solutions, "boundary" conditions have to be imposed on $\mathbb{R} \setminus O$ and not only on the boundary $\partial O = \{l, u\}$. This seems not to be the case here since the non-local integral term only involves the test function and not the solution itself, so one can be led to think that conditions *on* the boundary are enough (see remark in [24, Sec. 5.1.]). However note that the test functions have to verify $\varphi \geq v$ (resp. $\varphi \leq v$) on $[0,T] \times \mathbb{R}$ and not only on $[0,T] \times O$, which requires specifying v outside O.

4.2 Option prices as viscosity solutions of PIDE

Existence and uniqueness of viscosity solutions for such parabolic integro-differential equations are discussed in [2] in the case where ν is a finite measure and in [5] and [24] for general Lévy measures. Growth conditions other than $u \in C_p^+$ can be considered (see e.g. [2, 5]) with additional conditions on the Lévy measure ν. The main tool for showing uniqueness is the comparison principle: if u, v are viscosity solutions and $u(0,x) \geq v(0,x)$ then $\forall \tau \in [0,T]$, $u(\tau,x) \geq v(\tau,x)$. This property has been extended to subsolutions and supersolutions in [2] for the case where ν is a bounded measure; the case of a general Lévy measure has been recently treated in [18].

The uniqueness result is available in the literature for viscosity solutions with the polynomial growth at infinity. In the context of option prices, this restricts the choice of the payoff functions. We will give a sufficient condition on the payoff for the price to be in C_p^+ (see (4.1) for the definition).

Lemma 5. *For every $p \geq 0$ and $n \geq 1$, there exists $c > 0$, such that*

$$\forall x_1 \dots x_n \geq 0, \qquad (\sum_{i=1}^n x_i)^p \leq c \sum_{i=1}^n x_i^p. \tag{4.10}$$

Proof. If $p \geq 1$, it is Jensen's inequality with $g(x) = x^p$, and $c = n^{p-1}$. If $0 \leq p < 1$, it is easily verified by induction with $c = 1$. □

Proposition 7. (Polynomial growth)
If $H : (0,\infty) \to [0,\infty)$ is Lipschitz: $|H(S_1) - H(S_2)| \leq C|S_1 - S_2|$, and there

exists $p > 0$, such that:

$$H(S_0 e^x) \leq C_1(1 + |x|^p), \tag{4.11}$$

then $f(\tau, x) = \mathbb{E}[H(S_0 e^{x + r\tau + X_\tau})]$ belongs to $C_p^+([0,T] \times \mathbb{R})$.

Proof. We first show that

$$\mathbb{E}[(X_\tau)^p \, 1_{X_\tau > 0}] < \infty. \tag{4.12}$$

Theorem 25.3 of [27] states that if $g : \mathbb{R} \to \mathbb{R}$ is a submultiplicative, locally bounded function, then $\mathbb{E} g(X_\tau) < \infty$ for all $\tau > 0$ if and only if $\int_{|x|>1} g(x)\nu(dx) < \infty$.

A function $g(x) \geq 0$ is called *submultiplicative* if there exists $a > 0$, such that

$$\forall x, y \in \mathbb{R}, \;\; g(x+y) \;\; \leq \;\; a g(x) g(y).$$

A function is *locally bounded* if it is bounded on each compact.

For all $p > 0$, the function $x^p \vee 1$ is submultiplicative (see Proposition 25.4, [27]), and

$$x^p \vee 1 \;\; \leq \;\; x^p 1_{x>0} + 1 \; \leq \; 2(x^p \vee 1).$$

In consequence, for all $x, y \in \mathbb{R}$, we have

$$\begin{aligned} (x+y)^p \, 1_{x+y>0} + 1 \; &\leq \; 2((x+y)^p \vee 1) \; \leq \\ &\leq \; 2a(x^p \vee 1)(y^p \vee 1) \; \leq \; 2a(x^p \, 1_{x>0} + 1)(y^p \, 1_{y>0} + 1), \end{aligned}$$

so, $g(x) = x^p \, 1_{x>0} + 1$ is submultiplicative and locally bounded. By the theorem cited above, we obtain

$$\mathbb{E}[(X_\tau)^p \, 1_{X_\tau>0} + 1] < \infty \quad \Longleftrightarrow \quad \int_{|x|>1} (x^p \, 1_{x>0} + 1)\nu(dx) < \infty.$$

Since ν is integrable on $|x| > 1$, this clearly implies

$$\mathbb{E}[(X_\tau)^p \, 1_{X_\tau>0}] < \infty \quad \Longleftrightarrow \quad \int_{x>1} x^p \nu(dx) < \infty. \tag{4.13}$$

Thanks to the martingale condition, we have $\int_{x>1} e^x \nu(dx) < \infty$, and the condition on ν in (4.13) is satisfied all the more. So, (4.12) is also satisfied.

Since H is Lipschitz, there exists $\tilde{c} > 0$, such that $H(S) \leq \tilde{c}(1 + S)$. Thus, for all $x \in \mathbb{R}$, we have

$$f(\tau, x) \;\; \leq \;\; \tilde{c}\,\mathbb{E}[1 + S_0 e^{x + r\tau + X_\tau}] = \tilde{c}(1 + S_0 e^{x + r\tau}). \tag{4.14}$$

For the negative values of x, f is bounded by the constant $\tilde{c}(1 + S_0 e^{rT})$. Let us study the growth of f as $x \to +\infty$.

Let $x > 0$. We can estimate $f(\tau, x)$ in the following way:

$$
\begin{aligned}
&f(\tau, x) = \mathbb{E}[H(S_0 e^{x+r\tau+X_\tau}) 1_{X_\tau < -x}] + \mathbb{E}[H(S_0 e^{x+r\tau+X_\tau}) 1_{X_\tau \geq -x}] \\
&\leq \tilde{c}\,\mathbb{E}[(1 + S_0 e^{x+r\tau+X_\tau}) 1_{X_\tau < -x}] + C_1 \mathbb{E}[(1 + (x + r\tau + X_\tau)^p) 1_{X_\tau \geq -x}]. \quad (4.15)
\end{aligned}
$$

The first term is bounded by $\tilde{c}(1 + S_0 e^{rT})$, as previously. For the second, we obtain, using the Lemma 5:

$$
\begin{aligned}
&\mathbb{E}[(x + r\tau + X_\tau)^p\, 1_{X_\tau \geq -x}] = \\
&\qquad = \mathbb{E}[(x + r\tau + X_\tau)^p\, 1_{|X_\tau| \leq x}] + \mathbb{E}[(x + r\tau + X_\tau)^p\, 1_{X_\tau > x}] \\
&\qquad \leq c(2x^p + (rT)^p) + c(x^p + (rT)^p + \mathbb{E}[(X_\tau)^p\, 1_{X_\tau \geq 0}]) \leq C_2(1 + x^p).
\end{aligned}
$$

Putting this estimate into (4.15) gives

$$
f(\tau, x) \;\leq\; \tilde{c}(1 + S_0 e^{rT}) + C_1(1 + C_2(1 + x^p)) \;\leq\; C_3(1 + x^p),
$$

and the proof is completed. □

Corollary 1. If $H : (L, \infty) \to [0, \infty)$ is Lipschitz and satisfies the polynomial growth condition (4.11), then $f_l(\tau, x)$ defined by (3.11) is $C_p^+([0, T] \times \mathbb{R})$.

Proof. Let us extend H continuously on $(0, \infty)$ by a suitable constant:

$$
\tilde{H}(S) = \begin{cases} H(S), & S > L \\ \lim_{S \to L} H(S), & S \leq L. \end{cases}
$$

Then, $\tilde{H}$ satisfies the conditions of the Proposition 7, hence

$$
f_l(\tau, x) \quad \leq \quad \mathbb{E}[\tilde{H}(S_0 e^{x+r\tau+X_\tau})] \;\leq\; C(1 + x^p 1_{x \geq 0}).
$$

□

The following result shows that values of European and barrier options can be expressed as viscosity solutions of (4.4)–(4.5):

Proposition 8 (Option prices as viscosity solutions). *Let the payoff function H verify the Lipschitz condition (2.5) and let $h(x) = H(S_0 e^x)$ have polynomial growth at infinity. Then:*

- *The forward value of a European option $f_e(\tau, x)$ defined by (2.2) is the unique viscosity solution of the Cauchy problem (2.3) (that is (4.4)–(4.5) with $O = \mathbb{R}$).*
- *Let $f_b(\tau, x)$ be the forward value of a knockout (single or double) barrier option defined by (3.10), (3.11) or (3.14). If $f_b(\tau, x)$ is continuous then it is the unique viscosity solution of (4.4)–(4.5) (with $g \equiv 0$).*

Proof. $f_e(\tau, x)$ is continuous by the Proposition 3 and is $C_p^+([0,T] \times \mathbb{R})$ by the Proposition 7. The functions f_u and f_d are bounded on $[0,T] \times \mathbb{R}$, and f_l is $C_p^+([0,T] \times \mathbb{R})$ by the Corollary 1.

We will denote f_e, f_u, f_l, and f_d by a generic name f, and O will stand respectively for $\mathbb{R}$, $(-\infty, u)$, (l, ∞) or (l, u). So, f is continuous and $C_p^+([0,T] \times \mathbb{R})$, which is required in the definition of a viscosity solution.

Let us now show that f is a subsolution of (4.4)–(4.5). From the definition of f it is easily seen that $f(0,x) = h(x)$ and $f(\tau, x) = 0$ if $x \notin O$. Consider $(\tau_0, x_0) \in (0,T] \times O$ and a test function $\varphi \in C^2([0,T] \times \mathbb{R}) \cap C_p^+([0,T] \times \mathbb{R})$ such that $\varphi(\tau_0, x_0) = f(\tau_0, x_0)$, and

$$\varphi(\tau, x) \geq f(\tau, x) \qquad \text{on } [0,T] \times \mathbb{R}. \tag{4.16}$$

As noticed in Section 4, we can suppose that $|\frac{\partial \varphi}{\partial \tau}|$, $|\frac{\partial \varphi}{\partial x}|$, and $|\frac{\partial^2 \varphi}{\partial x^2}|$ are bounded by a constant C. Our goal is to show that φ satisfies (4.6) at (τ_0, x_0).

For $t \in [0, \tau_0]$, let $\theta_t = \inf\{s \geq t \mid x_0 + Y_s \notin O\}$ where $Y_s = rs + X_s$. Define

$$M_t = \mathbb{E}[H(S_0 e^{x_0 + Y_{\tau_0}}) 1_{\tau_0 \leq \theta_0} \mid \mathcal{F}_t].$$

Note that $1_{\tau_0 \leq \theta_0} = 1_{\tau_0 \leq \theta_t} 1_{t \leq \theta_0}$, and $1_{t \leq \theta_0} \in \mathcal{F}_t$. Since $Y_s \stackrel{d}{=} Y_t + Z_{s-t}$, $\forall s \geq t$, where Z is a Lévy process independent of Y and identically distributed, we can rewrite M_t in the following way:

$$\begin{aligned} M_t &= 1_{t \leq \theta_0} \mathbb{E}[H(S_0 e^{(x_0 + Y_t) + Z_{\tau_0 - t}}) 1_{\tau_0 - t \leq \inf\{s \geq 0,\ (x_0 + Y_t) + Z_s \notin O\}} \mid \mathcal{F}_t] \\ &= 1_{t \leq \theta_0} f(\tau_0 - t, x_0 + Y_t) \quad \text{a.s.} \end{aligned}$$

By construction, M_t is a martingale. So, by the sampling theorem,

$$f(\tau_0, x_0) = M_0 = \mathbb{E}[M_{t \wedge \theta_0}] = \mathbb{E}[f(\tau_0 - t \wedge \theta_0, x_0 + Y_{t \wedge \theta_0})],$$

since $0 \leq \theta_0$ and $t \wedge \theta_0 \leq \theta_0$. Then (4.16) implies, for all $t \in [0, \tau_0]$,

$$f(\tau_0, x_0) \leq \mathbb{E}[\varphi(\tau_0 - t \wedge \theta_0, x_0 + Y_{t \wedge \theta_0})]. \tag{4.17}$$

Applying the Itô formula to the smooth function $\varphi(\tau_0 - t, x_0 + Y_t)$ between 0 and $t \wedge \theta_0$ gives:

$$\begin{aligned} f(\tau_0, x_0) \ \leq\ & \varphi(\tau_0, x_0) + \mathbb{E}[\int_0^{t \wedge \theta_0} (-\frac{\partial \varphi}{\partial \tau} + L\varphi + r\frac{\partial \varphi}{\partial x})(\tau_0 - u, x_0 + Y_{u_-}) du] \\ & + \mathbb{E}[\int_0^{t \wedge \theta_0} \frac{\partial \varphi}{\partial x}(\tau_0 - u, x_0 + Y_{u_-}) \sigma dW_u + \\ & + \int_0^{t \wedge \theta_0} \int_{-\infty}^{\infty} (\varphi(\tau_0 - u, x_0 + Y_{u-} + y) - \varphi(\tau_0 - u, x_0 + Y_{u-})\ )\ \tilde{J_X}(du\ dy)], \end{aligned} \tag{4.18}$$

where $\tilde{J_X}$ is the compensated jump measure of X.

The stochastic integral in (4.18) is a martingale (with zero expectation) if $\mathbb{E}[A_{t\wedge\theta_0}] < \infty$, where

$$A_t = \int_0^t |\frac{\partial\varphi}{\partial x}(\tau_0 - u, x_0 + Y_{u_-})|^2 du$$
$$+ \int_0^t du \int_{-\infty}^{\infty} \nu(dy)|\varphi(\tau_0 - u, x_0 + Y_{u-} + y) - \varphi(\tau_0 - u, x_0 + Y_{u-})\ |^2.$$

Since φ has bounded derivatives, $\mathbb{E}[A_{t\wedge\theta_0}]$ is bounded:

$$\mathbb{E}[A_{t\wedge\theta_0}] \leq \mathbb{E}[\int_0^t |\frac{\partial\varphi}{\partial x}(\tau_0 - u, x_0 + Y_{u_-})|^2 du +$$
$$+ \int_0^t du \int_{-\infty}^{\infty} \nu(dy) y^2 |\frac{\partial\varphi}{\partial x}(\tau_0 - u, x_0 + Y_{u-} + \xi(y))|^2] \leq C^2 t(1 + \int_{-\infty}^{\infty} y^2 \nu(dy)).$$

Therefore, taking into account that $f(\tau_0, x_0) = \varphi(\tau_0, x_0)$, we derive from (4.18):

$$\mathbb{E}[\int_0^t 1_{u\leq\theta_0} (\frac{\partial\varphi}{\partial\tau} - L\varphi - r\frac{\partial\varphi}{\partial x})(\tau_0 - u, x_0 + Y_{u_-}) du] \quad \leq \quad 0. \qquad (4.19)$$

It is easily seen that the integrand is bounded, again by the boundedness of the derivatives of φ. Dividing (4.19) by t, taking the limit $t \to 0$, and applying the dominated convergence theorem, we finally obtain

$$(\frac{\partial\varphi}{\partial\tau} - L\varphi - r\frac{\partial\varphi}{\partial x})(\tau_0, x_0) \leq 0.$$

Hence, f is a subsolution.

Similarly, if $\varphi \in C^2([0,T] \times \mathbb{R}) \cap C_p^+([0,T] \times \mathbb{R})$ and $(\tau_0, x_0) \in (0,T] \times O$ are such that $\varphi(\tau_0, x_0) = f(\tau_0, x_0)$ and $\varphi \leq f$ on $[0,T] \times \mathbb{R}$, one can show that

$$(\frac{\partial\varphi}{\partial\tau} - L\varphi - r\frac{\partial\varphi}{\partial x})(\tau_0, x_0) \geq 0$$

which implies that f is a supersolution. Uniqueness of the solution follows from the comparison principle. □

The hypotheses above on the payoff function apply to put options, single-barrier knockout puts, double barrier knockout options and also to the log-contract. One can then retrieve call options by put-call parity. For barrier options with rebate, the zero boundary condition has to be replaced by the value of the rebate, as in the case of diffusion models.

5 Conclusion

The characterization of option prices in terms of solutions of partial integro-differential equations allows to use efficient numerical methods for pricing options on a single asset in presence of jumps. This relation has already been used by several authors to develop numerical methods for pricing options in models with jumps. In this paper we have shown that this characterization is less obvious in exponential Lévy model than in diffusion models, because of the possible lack of smoothness of option values with respect to the underlying in pure jump models. This lack of smoothness prevents the value function from being a classical solution of the pricing PIDE: we are led to use a notion of generalized solution. Using the notion of *viscosity solution* we have characterized in Proposition 8 the precise relation between PIDEs and prices of European or barrier options in exponential Lévy models. Such results are straightforward to extend to the case of time-dependent characteristics (additive processes) (see [9, Chapter 14]). From the mathematical point of view one could also consider the case of state-dependent coefficients i.e. a general Markov process ("local volatility models with jumps") such as in [3] . However, as shown in [10], the addition of a local volatility component generates features which are redundant with the small jumps of the Lévy process and leads to an identification problem when calibrating the model to option prices. The gain from generality is therefore not clear and we have refrained from venturing in this direction.

The notion of viscosity solution discussed in this work turns out to be convenient for analyzing the convergence of finite difference schemes, without requiring smoothness with respect to the underlying. Such numerical methods are discussed in a companion paper [11] and make a key use of the comparison principle for semicontinuous solutions [2]. The use of viscosity solutions allows to obtain pointwise convergence of option prices [11], which is more relevant for approximating option prices than L^2-type convergence obtained using the notion of weak solution in Sobolev spaces [22].

Let us note that, in principle, one can also define the notion of discontinuous viscosity solution (see e.g. [4]) for PIDEs . However, the comparison principle fails to hold in this case and thus one is not able to exhibit convergent numerical schemes for computing such solutions (at least, using finite difference methods).

References

[1] L. Alili and A. Kyprianou, *Some remarks on first passage of Lévy processes, the American put and pasting principles.* 2004.

[2] O. Alvarez and A. Tourin, *Viscosity solutions of non-linear integro-differential equations*, Annales de l'Institut Henri Poincaré, 13 (1996), pp. 293–317.

[3] L. Andersen and J. Andreasen, *Jump-diffusion models: Volatility smile fitting and numerical methods for pricing*, Rev. Derivatives Research, 4 (2000), pp. 231–262.

[4] G. Barles, *Solutions de viscosité des équations de Hamilton-Jacobi*, vol. 17 of Mathématiques & Applications, Springer-Verlag, Paris, 1994.

[5] G. Barles, R. Buckdahn, and E. Pardoux, *Backward stochastic differential equations and integral-partial differential equations*, Stochastics and Stochastic Reports, 60 (1997), pp. 57–83.

[6] A. Bensoussan and J.-L. Lions, *Contrôle Impulsionnel et Inéquations Quasi-Variationnelles*, Dunod, Paris, 1982.

[7] J. Bony, *Problème de Dirichlet et semi-groupe fortement fellerien associé à un opérateur integro-différentiel*, Comptes Rendus de l'Académie des Sciences, 265 (1967), pp. 361–364.

[8] T. Chan, *Pricing contingent claims on stocks driven by Lévy processes*, Ann. Appl. Probab., 9 (1999), pp. 504–528.

[9] R. Cont and P. Tankov, *Financial modelling with jump processes*, Chapman & Hall / CRC Press, 2003.

[10] R. Cont and P. Tankov, *Nonparametric calibration of jump-diffusion option pricing models*, Journal of Computational Finance, 7 (2004), pp. 1–49.

[11] R. Cont and E. Voltchkova, *Finite difference methods for option pricing in jump-diffusion and exponential Lévy models*, Rapport Interne 513, CMAP, Ecole Polytechnique, 2003.

[12] M. Crandall, H. Ishii, and P. Lions, *Users guide to viscosity solutions of second order partial differential equations*, Bulletin of the American Mathematical Society, 27 (1992), pp. 1–42.

[13] Y. d'Halluin, P. Forsyth, and G. Labahn, *A penalty method for American options with jump-diffusion processes*, working paper, University of Waterloo, March 2003.

[14] E. Eberlein, *Applications of generalized hyperbolic Lévy motion to Finance*, in Lévy Processes—Theory and Applications, O. Barndorff-Nielsen, T. Mikosch, and S. Resnick, eds., Birkhäuser, Boston, 2001, pp. 319–336.

[15] M. Garroni and J. Menaldi, *Second Order Elliptic Integro-Differential Problems*, CRC Press, Boca Raton, FL, 2001.

[16] H. Geman, *Pure jump Lévy processes for asset price modeling*, Journal of Banking and Finance, 26 (2002), pp. 1297–1316.

[17] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, Berlin, 2nd ed., 2002.

[18] E. Jakobsen and K. Karlsen, *A maximum principle for semicontinuous functions applicable to integro-partial differential equations*, working paper, Dept. of Mathematics, University of Oslo, 2003.

[19] D. Madan, *Financial modeling with discontinuous price processes*, in Lévy Processes—Theory and Applications, O. Barndorff-Nielsen, T. Mikosch, and S. Resnick, eds., Birkhäuser, Boston, 2001.

[20] D. Madan and F. Milne, *Option pricing with variance gamma martingale components*, Math. Finance, 1 (1991), pp. 39–55.

[21] D. Madan and E. Seneta, *Simulation of estimates using the empirical characteristic function*, Internat. Statist. Rev., 55 (1987), pp. 153–161.

[22] A.-M. Matache, T. von Petersdorff, and C. Schwab, *Fast deterministic pricing of options on Lévy driven assets*, M2AN Math. Model. Numer. Anal., 38 (2004), pp. 37–71.

[23] D. Nualart and W. Schoutens, *Backward stochastic differential equations and Feynman-Kac formula for Lévy processes, with applications in finance*, Bernoulli, 7 (2001), pp. 761–776.

[24] H. Pham, *Optimal stopping of controlled jump-diffusion processes: A viscosity solution approach*, Journal of Mathematical Systems, 8 (1998), pp. 1–27.

[25] P. Protter, *Stochastic integration and differential equations*, Springer, Berlin, 1990.

[26] S. Rong, *On solutions of backward stochastic differential equations with jumps and applications*, Stochastic Process. Appl., 66 (1997), pp. 209–236.

[27] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, UK, 1999.

[28] A. Sayah, *Equations d'Hamilton Jacobi du premier ordre avec termes integro-differentiels*, Comm. Partial Differential Equations, 16 (1991), pp. 1057–1093.

[29] H. Soner, *Jump Markov Processes and Viscosity Solutions*, vol. 10 of IMA Volumes in mathematics and applications, Springer Verlag, New York, 1986, pp. 501–511.