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Two-scale Dirichlet-Neumann preconditioners for elliptic problems with small disjoint geometric refinements on the boundary

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Abstract

We propose, analyze and test herein two simple Dirichlet-Neumann preconditioners to solve a non-conforming mortar formulation of elasticity problems presenting small disjoint geometric refinements on the boundary. In particular, we show a two-scale property, that is the independence of the condition number of the preconditioned system in the number and the size of the small details on the boundary. On the other hand, we introduce for one of the preconditioners, a coarse space counterbalancing the effect of essential boundary conditions on the small details. Finally, a quasi-Newton method inspired by these preconditioners is proposed when dealing with nonlinear elasticity.

1 Introduction

The present paper is devoted to the construction of efficient numerical procedures to solve vector elliptic problems with small geometric details on the boundary of the domain, that is where a *localized fine scale* behavior of the solution is expected. In particular, the solution in displacements $u \in \mathbb{R}^d$ of the linearized elastostatics problem will be considered, that is for $d = 2, 3$ the solution of:

$$\begin{cases} -\, div(\mathbf{E} : \varepsilon(u)) = f, \quad \Omega \subset \mathbb{R}^d, \\ u = 0, \quad \Gamma_D, \\ (\mathbf{E} : \varepsilon(u)) \cdot n = g, \quad \Gamma_N, \end{cases} \tag{1}$$

where the linearized strain tensor is denoted by:

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla^t u\right),$$

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and the fourth order tensor $\mathbf{E}$ is assumed to be elliptic over the set of symmetric matrices:

$$\exists \alpha > 0, \forall \xi \in \mathbb{R}^{d\times d}, \xi^t = \xi, \quad (\mathbf{E} : \xi) : \xi \geq \alpha\, \xi : \xi.$$

In our framework, we consider that inside the disjoint subsets $(\Omega_k)_{1\leq k\leq K}$ of Ω, the solution rapidly varies. In applications like tires development, one could think of geometric refinements or sculptures on the boundary $\partial\Omega$. At the opposite, u slowly varies in $\Omega_0 = \Omega \setminus \overline{(\cup_{1\leq k\leq K}\Omega_k)}$.

The strategy proposed in this paper consists in using a non-conforming mortar formulation for (1) in order to decompose the physical domain into coarse and fine zones. Then, simple Dirichlet-Neumann preconditioners are proposed in order to solve the obtained linear system for the approximate cost of inversion of the coarse system, that is the problem set over Ω_0. To do so, we assume that the computational cost of the solution over each $(\Omega_k)_{1\leq k\leq K}$ knowing the solution over Ω_0 is reasonably low when compared to the resolution over Ω_0. For the proposed strategies, we then show a two-scale property in the sense that the condition number of the preconditioned system remains independent of the number and the size of the small subdomains.

Mortar methods have been introduced for the first time in [BMP93, BMP94] as a weak coupling between subdomains with non-conforming meshes, or between subproblems solved with different approximation methods. The main purpose was to overcome the very sub-optimal "$\sqrt{h}$" error estimate obtained with pointwise matching. The analysis of this method as a mixed formulation can be found in [Bel99]. For the present purpose, various Lagrange multipliers spaces can be indifferently adopted. For example, one can use the original formulation from [BMP93]. It is worth noticing that because of the disjoint character of the small subdomains, no modification of Lagrange multipliers is necessary on the boundary of the interfaces. Indeed, interfaces are only shared by two subdomains: the coarse one, and a fine one. The dual variant from [Woh00] can present the advantage of making the weak continuity constraint diagonal, at least in the case of plane interfaces. It is always nearly diagonal when using discontinuous stabilized Lagrange multipliers as described in [Hau04]. In the case of a second order approximation in displacements, one can also adopt the proposal from [Ses98], opting for affine Lagrange multipliers.
Moreover, the independence of the coercivity constant of the broken elastostatics bilinear form with respect to the number and the size of the subdomains has been proved in [Bre04, Hau04]. There is then no limitation in considering here a high number of small subdomains. Indeed, the error estimates remain optimal. A brief review on the non-conforming formulation adopted to discretize (1) is done in section 2.

The challenge is then to develop a solver which efficiently handles such situations. In the present framework, the disymmetric roles played by the coarse subdomain and fine ones give greater importance to Dirichlet-Neumann preconditioners (see [QV99, Woh01]), rather than symmetric strategies such as Neumann-Neumann [TRV91] or FETI [FR91], studied in the mortar framework in the references [Tal93, AKP95, AMW99, AAKP99, Ste99]. In section 4, we begin by proposing a basic Dirichlet-Neumann preconditioner and prove that its quality is independent of the number and of the size of the refinements of the boundary. In this sense, we can talk of two-scale preconditioning. Nevertheless, the quality of this first preconditioner deteriorates when an essential boundary condition is imposed on such a boundary refinement. This inconvenient is overcome by considering a special coarse space taking interface rigid motions into account. An enhanced Dirichlet-Neumann preconditioner insensitive to essential boundary conditions is then obtained and analyzed. These preconditioners are tested in section 4 to confirm the previous analysis.

When considering nonlinear problems with soft geometric refinements on the boundary, it is illustrated in section 6 that such preconditioners can be used to build efficient quasi-Newton methods.

2 A mortar formulation

2.1 Continuous problem

Let $\Omega \subset \mathbb{R}^d$, be an open set partitioned into $K+1$ subsets $(\Omega_k)_{0\le k\le K}$ satisfying $\overline{\Omega} = \cup_{i=0}^{K}\overline{\Omega_k}$ and $\Omega_k \cap \Omega_l = \emptyset$ if $k,l \ge 1$. We denote by $\Gamma_{0k} = \overline{\Omega_0} \cap \overline{\Omega_k}$ the interface between Ω_0 and Ω_k, and the skeleton of the internal interfaces is denoted by $\mathcal{S} = \cup_{k=1}^{K}\Gamma_{0k}$. For the understanding of the situation, let us say that Ω_0 has slowly varying physical properties whereas the disjoint subsets $(\Omega_k)_{1\le k\le K}$ have rapidly varying ones or complex geometries. Moreover, the subdomain Ω_0 has a non-empty intersection with all the subdomains $(\Omega_k)_{1\le k\le K}$. We will also assume as a simplification that the intersection between two local subdomains Ω_k, $k \ge 1$ is empty. In other words, for the time being, the inclusions are disconnected. On the part Γ_D of the boundary $\partial\Omega$, an homogeneous Dirichlet boundary condition is imposed. Concerning the coefficients of the fourth order elasticity tensor $\mathbf{E}$, we assume that the stress tensor is symmetric whatever the deformation in the material, namely for almost all $x \in \Omega$:

$$\forall \xi \in \mathbb{R}^{d\times d}, \xi^t = \xi, \quad \mathbf{E}(x) : \xi \text{ is a symmetric matrix.}$$

Moreover, the different materials are spectrally isotropic, namely for all $k \ge 1$, there exists two constants c_k and C_k, such that for almost all $x \in \Omega_k$:

$$\forall \xi \in \mathbb{R}^{d\times d}, \xi^t = \xi, \quad c_k\, \xi : \xi \le (\mathbf{E}(x) : \xi) : \xi \le C_k\, \xi : \xi. \tag{2}$$

For homogeneous isotropic materials, if E_k stands for the Young modulus of the material used in Ω_k, both c_k and C_k are proportional to E_k within a shape dependent constant. We introduce the following spaces:

$$H^1_*(\Omega) = \{v \in H^1(\Omega)^d,\ v|_{\Gamma_D} = 0\},$$

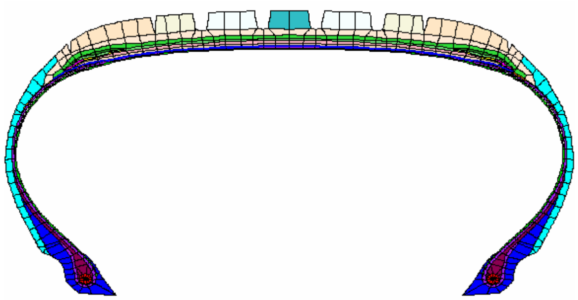

Figure 1: Example of a structure presenting small geometric refinements on its boundary.

$$H^1_*(\Omega_k) = \{v \in H^1(\Omega_k)^d,\ v|_{\Gamma_D \cap \partial\Omega_k} = 0\},$$

$$X = \left\{v \in L^2(\Omega)^d, \quad v_k = v|_{\Omega_k} \in H^1_*(\Omega_k), \forall k\right\} = \prod_{k=0}^{K} H^1_*(\Omega_k),$$

X being endowed with the H^1 broken norm:

$$\|v\|_X = \left(\sum_{k=0}^{K} \|v\|^2_{H^1(\Omega_k)^d}\right)^{\frac{1}{2}},$$

and:

$$M = \prod_{k=1}^{K} L^2(\Gamma_{0k})^d.$$

In the whole paper, for homogeneity reason, the H^1 norm is rescaled, that is:

$$\|v\|^2_{H^1(\Omega_k)^d} = \frac{1}{(L_k)^2}\|v\|^2_{L^2(\Omega_k)^d} + \|\nabla v\|^2_{L^2(\Omega_k)^d},$$

where L_k denotes the diameter of Ω_k.
We are interested in finding $u \in H^1_*(\Omega)$ such that:

$$a(u, v) = l(v), \quad \forall v \in H^1_*(\Omega), \tag{3}$$

where the continuous coercive bilinear form a is defined as:

$$a(u, v) = \int_\Omega (\mathbf{E} : \varepsilon(u)) : \varepsilon(v), \quad \forall u, v \in H^1_*(\Omega),$$

and the continuous linear form l as:

$$l(v) = \int_{\Omega} f \cdot v + \int_{\Gamma_N} g \cdot v, \quad \forall v \in H^1_*(\Omega),$$

with $f \in L^2(\Omega)^d$ and $g \in L^2(\Gamma_N)^d$. This problem is well-posed from Lax-Milgram lemma, by using the Korn's inequality (see [DL72]) to prove the coercivity of the bilinear form a.

2.2 Discretization

We introduce here a domain based non-conforming discretization of the problem using mortar elements. Under standard assumptions, well-posedness results and error estimates are reviewed below.

2.2.1 The mesh

For each $0 \le k \le K$, let us consider a family of shape regular meshes $(\mathcal{T}_{k;h_k})_{h_k>0}$ defined over each domain Ω_k, and denote:

$$h_k = \sup_{T \in \mathcal{T}_{k;h_k}} diam(T).$$

The mesh $\mathcal{T}_{0;h_0}$ defined on Ω_0 is the coarsest, i.e $h_0 > h_k$, for all $1 \le k \le K$, and a non-conforming family of meshes $(\mathcal{T}_h)_{h>0}$ over Ω is obtained by:

$$\mathcal{T}_h = \cup_{k=0}^{K} \mathcal{T}_{k,h_k}, \quad h = \max_{0 \le k \le K} h_k.$$

For each $1 \le k \le K$, Γ_{0k} inherits from the family of meshes $(\mathcal{F}_{k;\delta_k})_{\delta_k>0}$, obtained as the trace of the fine mesh $(\mathcal{T}_{k;h_k})_{h_k>0}$ over Γ_{0k}. We have adopted the notation:

$$\delta_k = \sup_{F \in \mathcal{F}_{k;\delta_k}} h(F).$$

Then, the family of meshes $(\mathcal{F}_\delta)_{\delta>0}$ can be defined over the skeleton $\mathcal{S}$ by:

$$\mathcal{F}_\delta = \cup_{k=1}^{K} \mathcal{F}_{k;\delta_k}, \quad \delta = \max_{1 \le k \le K} \delta_k.$$

Moreover, the following assumption is made (Figure 2).

Assumption 1. *$F \in \mathcal{F}_\delta$ is always an entire face of an element $T \in \mathcal{T}_h$.*

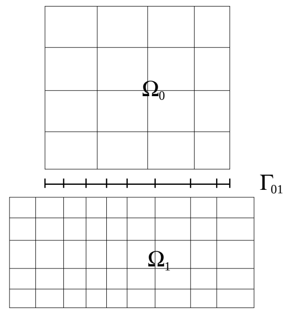

Figure 2: A situation where the mesh $\mathcal{F}_{1;\delta_1}$ of the interface Γ_{01} is inherited from the mesh $\mathcal{T}_{1;h_1}$ of Ω_1, and where assumption 1 is violated.

2.2.2 Mesh-dependent spaces

We define here some mesh-dependent spaces, endowed with useful mesh-dependent norms already proposed and used in [AT95, Woh99]. For each $1 \leq k \leq K$, they are defined by:

$$\mathbb{H}_\delta^{1/2}(\Gamma_{0k}) = \{\phi \in L^2(\Gamma_{0k})^d, \|\phi\|^2_{\delta,\frac{1}{2},k} = \sum_{F \in \mathcal{F}_{k;\delta_k}} \frac{1}{h(F)} \|\phi\|^2_{L^2(F)^d} < +\infty\},$$

$$\mathbb{H}_\delta^{-1/2}(\Gamma_{0k}) = \{\lambda \in L^2(\Gamma_{0k})^d, \|\lambda\|^2_{\delta,-\frac{1}{2},k} = \sum_{F \in \mathcal{F}_{k;\delta_k}} h(F) \|\lambda\|^2_{L^2(F)^d} < +\infty\},$$

endowed respectively with the norms $\|\cdot\|_{\delta,\frac{1}{2},k}$ and $\|\cdot\|_{\delta,-\frac{1}{2},k}$. The product spaces $\mathbb{W}_\delta = \prod_{k=1}^K \mathbb{H}_\delta^{1/2}(\Gamma_{0k})$ and $\mathbb{M}_\delta = \prod_{k=1}^K \mathbb{H}_\delta^{-1/2}(\Gamma_{0k})$, are then respectively endowed with the norms:

$$\|\phi\|_{\delta,\frac{1}{2}} = \left(\sum_{k=1}^K \|\phi\|^2_{\delta,\frac{1}{2},k} \right)^{1/2},$$

$$\|\lambda\|_{\delta,-\frac{1}{2}} = \left(\sum_{k=1}^K \|\lambda\|^2_{\delta,-\frac{1}{2},k} \right)^{1/2}.$$

They can be viewed as dual spaces by means of the the L^2 inner product:

$$\int_{\mathcal{S}} \phi \cdot \lambda \leq \|\lambda\|_{\delta,-\frac{1}{2}} \|\phi\|_{\delta,\frac{1}{2}}, \quad \forall (\phi, \lambda) \in \mathbb{W}_\delta \times \mathbb{M}_\delta.$$

Remark 1. *The use of such mesh-dependent spaces instead of $H_{00}^{1/2}(\Gamma_{0k})^d$ and its dual $H^{-1/2}(\Gamma_{0k})^d = \left(H_{00}^{1/2}(\Gamma_{0k})^d\right)'$ for example, has several advantages. First, these mesh-dependent norms are computable, which make easier a posteriori estimations (see [Woh99])* *and penalized formulations (see [Hau04]). Moreover, their use enables to avoid some technical difficulties for 3D problems.*

2.2.3 Non-conforming approximation

Let us introduce the discrete subspaces of degree q inside each subdomain:

$$X_{k;h_k} = \{p \in H^1_*(\Omega_k) \cap \mathcal{C}^0(\Omega_k)^d, \quad p|_T \in \mathcal{P}_q(T), \forall T \in \mathcal{T}_{k;h_k}\} \oplus \mathcal{B}_{k;h_k},$$

with $\mathcal{P}_q = [\mathbb{P}_q]^d$ or $[\mathbb{Q}_q]^d$, where $\mathbb{P}_q$ (resp. $\mathbb{Q}_q$) is the space of polynomials of total (resp. partial) degree q, and where we have introduced a possible stabilization space $\mathcal{B}_{k;h_k}$ built with bubbles on the interface as in [BM00, Hau04]. The corresponding product space is denoted by:

$$X_h = \prod_{k=0}^{K} X_{k;h_k} \subset X.$$

Let us define the following trace spaces on the non-mortar side (small subdomain side herein):

$$W_{k;\delta_k} = \{p|_{\Gamma_{0k}}, p \in X_{k;h_k}\}, \qquad W^0_{k;\delta_k} = W_{k;\delta_k} \cap H^1_0(\Gamma_{0k})^d,$$

endowed with the mesh-dependent norm $\|\cdot\|_{\delta,\frac{1}{2},k}$.

In order to formulate the weak continuity constraint, we introduce the spaces $M_{k;\delta_k}$ of (possibly discontinuous) Lagrange multipliers defined on the meshes $\mathcal{F}_{k;\delta_k}$. In order to achieve optimal approximation, they must contain all polynomials $[\mathbb{P}_{q-1}]^d$ of degree $q-1$. The product space $M_\delta = \prod_{k=1}^{K} M_{k;\delta_k}$ is endowed with the mesh-dependent norm $\|\cdot\|_{\delta,-\frac{1}{2}}$. The following bilinear form is then introduced to express the constraint on the jump of the displacements on the non-conforming interfaces:

$$\begin{aligned} b: \quad & X \times M && \to \mathbb{R} \\ & (v,\lambda) && \mapsto b(v,\lambda) = \sum_{k=1}^{K} \int_{\Gamma_{0k}} [v_k] \cdot \lambda_k, \end{aligned}$$

with $[v_k] = v_0 - v_k$, on Γ_{0k}. We denote:

$$\begin{aligned} b(v,\lambda) &= \sum_{k=1}^{K} \int_{\Gamma_{0k}} v_0 \cdot \lambda_k - \sum_{k=1}^{K} \int_{\Gamma_{0k}} v_k \cdot \lambda_k \\ &:= \sum_{k=1}^{K} b_{0k}(v_0,\lambda_k) - \sum_{k=1}^{K} b_k(v_k,\lambda_k). \end{aligned}$$

Then, the constrained space of admissible displacements can be defined as:

$$V_h = \{u_h \in X_h, \quad b(u_h,\lambda_h) = 0, \quad \forall \lambda_h \in M_\delta\}.$$

They are continuous "in average" across the interfaces $(\Gamma_{0k})_{1\leq k\leq K}$. In order to formulate the approximate problem, the broken elliptic form $\tilde{a}$ is defined as:

$$\begin{aligned}\tilde{a}: \quad & X\times X && \to \mathbb{R}\\ & (u,v) && \mapsto \tilde{a}(u,v)=\sum_{k=0}^{K} a_k(u_k,v_k),\end{aligned}$$

with:

$$a_k(u_k,v_k)=\int_{\Omega_k}(\mathbf{E}:\varepsilon(u_k)):\varepsilon(v_k).$$

We are then interested in finding $(u_h,\lambda_h)\in X_h\times M_\delta$, such that:

$$\begin{cases}\tilde{a}(u_h,v_h)+b(v_h,\lambda_h)=l(v_h), \quad \forall v_h\in X_h,\\ b(u_h,\mu_h)=0, \quad \forall \mu_h\in M_\delta.\end{cases} \tag{4}$$

In other words, we solve our variational problem on the product space X_h under the kinematic continuity constraint $b(\cdot,\cdot)=0$.

2.2.4 Fundamental assumptions and error estimates

In order to ensure the well-posedness of the problem (4), some fundamental assumptions have to be made. Concerning the compatibility of X_h and M_δ, we assume (cf. [Woh01, Hau04]):

Assumption 2. *For each $1\leq k\leq K$, there exists an operator:*

$$\pi_k:\mathbb{H}_\delta^{1/2}(\Gamma_{0k})\to W_{k;\delta_k},$$

such that for all $v\in\mathbb{H}_\delta^{1/2}(\Gamma_{0k})$:

$$\int_{\Gamma_{0k}}(\pi_k v)\cdot\mu=\int_{\Gamma_{0k}} v\cdot\mu, \quad \forall\mu\in M_{k;\delta_k},$$

with:

$$\|\pi_k v\|_{\delta,\frac{1}{2},k}\leq C\|v\|_{\delta,\frac{1}{2},k}.$$

This assumption means that the projection perpendicular to the multiplier space onto the trace space $W_{k;\delta_k}$ is continuous. This implies a limitation on the size of M_δ with respect to X_h. If more than two subdomains had a common intersection, the range $W_{k;\delta_k}$ of π_k in assumption 2 would be replaced by $W^0_{k;\delta_k}$, in order to enable independent projections on each interface.

The coercivity of $\tilde{a}$ over $V_h\times V_h$ is obtained under the following assumption (cf. [Woh01, Hau04]):

Assumption 3. *For all $1 \leq k \leq K$, we assume that there exists a subspace $\tilde{M}_k$ of the Lagrange multipliers space $M_{k;\delta_k}$ such that $\tilde{M}_k \subset M_{k;\delta_k}$ independently of δ_k. Moreover, we assume that for all $v \in X$ which is locally a rigid motion over all the $(\Omega_k)_{k\geq 1}$ in the sense that:*

$$\tilde{a}(v, w) = 0, \quad \forall w \in X,$$

and satisfying:

$$\int_{\Gamma_{0k}} [v] \cdot \mu = 0, \quad \forall \mu \in \tilde{M}_k, \quad k = 1, .., K,$$

then $v = 0$.

Various pairs of spaces $X_h \times M_\delta$ can be chosen to satisfy the assumptions 2 and 3:

- The initial formulation from [BMP93, BMP94] proposes discrete displacements of degree q without stabilization, i.e. $\mathcal{B}_{k;h_k} = \emptyset$, and continuous Lagrange multipliers of degree q. In our framework, no modification of the Lagrange multipliers is necessary on the boundaries of the interfaces $(\partial\Gamma_{0k})_{1\leq k\leq K}$ because they are disjoint. Therefore, with this choice, the displacements trace spaces over the fine subdomains interfaces coincides with Lagrange multipliers spaces, that is $M_{k;\delta_k} = W_{k;\delta_k}$ for all $1 \leq k \leq K$.
- In order to make the mortar weak continuity constraint diagonal, one can adopt the dual Lagrange multipliers from Wohlmuth [Woh00], again without special treatment on the boundaries of the interfaces.
- As shown in [Ses98] for second order approximations of the displacements ($q \geq 2$), the formulation from [BMP93, BMP94] can be modified by using only continuous Lagrange multipliers of degree $q - 1$.
- Discrete displacements of degree q with a proper stabilization are compatible with discontinuous Lagrange multipliers of degree $q - 1$, as proved in [Hau04]. Such a formulation has been first developed for three-field matching formulations in [BM00].

In this framework, we recall the following optimal approximation result (cf. [Woh01, Hau04]):

Proposition 1. *Under assumptions 2 and 3, the problem (4) is well-posed. Moreover, if $u \in \prod_{k=0}^{K} H^{q+1}(\Omega_k)^d$ is solution of (3) with $(\mathbf{E} : \varepsilon(u)) \in \prod_{k=0}^{K} H^q(\Omega_k)^{d\times d}$ in which $q \geq 1$, and $(u_h, \lambda_h) \in X_h \times M_\delta$ is solution of (4), the following error estimates hold:*

$$\|u - u_h\|_X \leq C \left(\sum_{k=1}^{K} h_k^{2q} \, |u|^2_{q+1,\mathbf{E},\Omega_k} \right)^{1/2},$$

$$\|\lambda - \lambda_h\|_{\delta,-\frac{1}{2}} \leq C \left(\sum_{k=0}^{K} h_k^{2q} \, |u|^2_{q+1,\mathbf{E},\Omega_k} \right)^{1/2},$$

with:

$$|u|^2_{q+1,\mathbf{E},\Omega_k} = |u|^2_{H^{q+1}(\Omega_k)^d} + \frac{1}{C_k^2}\|\mathbf{E} : \varepsilon(u)\|^2_{H^q(\Omega_k)^{d\times d}}.$$

We have denoted the flux over the artificial interfaces by $\lambda = (\mathbf{E} : \varepsilon(u)) \cdot n$, where the normal outward unit vector on $\partial\Omega_0$ is denoted by n. C denotes various constant independent of the decomposition into subdomains, and of the discretization.

Remark 2 (Choice of the non-mortar side). *In this discretization, as confirmed by assumption 2, we have chosen the non-mortar side defining the multipliers as the fine scale side of the interface $\mathcal{S}$. The main motivation is that in the preconditioners to be defined later, it is crucial to get a stable extension operator over the small scale subdomains, which is the case with the present choice while compatible with the standard assumption 1.*

3 Two-scale preconditioners.

The previous discretization leads to a well-posed linear discrete problem with optimal error estimates. In this section, we propose and analyze preconditioners to solve this linear system for the approximate computational cost of the coarse scale problem on Ω_0, provided the solution of the problem over each $(\Omega_k)_{1\leq k\leq K}$ be at a reasonnably low-cost. That is why we have assumed that the $(\Omega_k)_{1\leq k\leq K}$ were small and disjoint. Then, the inversions of the fine scale problems on the boundary can be parallelized and are relatively cheap in terms of computation.
Some notation and remarks must first be introduced:

- In this section, all quantities live in finite dimensional spaces. If a is a bilinear form, then $\mathbf{A}$ represents the matrix of a in the discrete space. If u is a function, then U is the vector of its nodal degrees of freedom in the chosen discrete space.
- For all $0 \leq k \leq K$, the bilinear form $a_k(\cdot,\cdot)$ is continuous in $H^1(\Omega_k)^d \times H^1(\Omega_k)^d$ and its continuity constant is C_k, already defined in (2).
- When $\Gamma_D \cap \partial\Omega_0$ has a positive measure, $a_0(\cdot,\cdot)$ is coercive in $H^1_*(\Omega_0) \times H^1_*(\Omega_0)$. We denote by α_0 its constant of coercivity, which is proportional to c_0 defined in (2), within a shape dependent constant.
- For all $1 \leq k \leq K$ such that Ω_k is fixed on a part of its boundary, the bilinear form $a_k(\cdot,\cdot)$ is coercive over $H^1_*(\Omega_k)\times H^1_*(\Omega_k)$ and its coercivity constant is denoted by α_k. It is proportional to c_k defined in (2), within a constant which depends continuously on the shape of Ω_k but not of its size because a_k and the scaled norm of H^1 have the same dependence with respect to a change of scale.

3.1 Introduction

With obvious notation, the discrete problem (4) leads to the following linear system to solve:

$$\begin{cases} \mathbf{A}_0 U_0 + \displaystyle\sum_{k=1}^{K} \mathbf{B}_{0k}^t \Lambda_k = F_0, \\ \mathbf{A}_k U_k - \mathbf{B}_k^t \Lambda_k = F_k, \quad 1 \leq k \leq K, \\ \mathbf{B}_{0k} U_0 - \mathbf{B}_k U_k = 0, \quad 1 \leq k \leq K. \end{cases} \tag{5}$$

Defining the local extended stiffness matrix of the k-th $(k \geq 1)$ subproblem by:

$$\mathbf{K}_k = \begin{pmatrix} \mathbf{A}_k & -\mathbf{B}_k^t \\ -\mathbf{B}_k & 0 \end{pmatrix},$$

the problem (5) can be rewritten as:

$$\begin{cases} \mathbf{A}_0 U_0 + \displaystyle\sum_{k=1}^{K} \mathbf{B}_{0k}^t \Lambda_k = F_0, \\ \mathbf{K}_k \begin{pmatrix} U_k \\ \Lambda_k \end{pmatrix} = \begin{pmatrix} F_k \\ -\mathbf{B}_{0k} U_0 \end{pmatrix}, \quad 1 \leq k \leq K. \end{cases} \tag{6}$$

The operator R_k of matrix $\left(0, I_{M_{k;\delta_k}}\right)$ is defined as the canonical restriction from $X_{k;h_k} \times M_{k;\delta_k}$ to $M_{k;\delta_k}$, and therefore, from (6), we can obtain Λ_k as a function of U_0 as:

$$\Lambda_k = R_k \mathbf{K}_k^{-1} \begin{pmatrix} F_k \\ -\mathbf{B}_{0k} U_0 \end{pmatrix} = R_k \mathbf{K}_k^{-1} \begin{pmatrix} F_k \\ 0 \end{pmatrix} - R_k \mathbf{K}_k^{-1} R_k^t \mathbf{B}_{0k} U_0.$$

Then, by elimination of Λ_k in the coarse scale problem, (6) becomes:

$$\begin{cases} \left(\mathbf{A}_0 - \displaystyle\sum_{k=1}^{K} \mathbf{B}_{0k}^t R_k \mathbf{K}_k^{-1} R_k^t \mathbf{B}_{0k} \right) U_0 = F_0 - \displaystyle\sum_{k=1}^{K} \mathbf{B}_{0k}^t R_k \mathbf{K}_k^{-1} \begin{pmatrix} F_k \\ 0 \end{pmatrix}, \\ \mathbf{K}_k \begin{pmatrix} U_k \\ \Lambda_k \end{pmatrix} = \begin{pmatrix} F_k \\ -\mathbf{B}_{0k} U_0 \end{pmatrix}, \quad 1 \leq k \leq K, \end{cases} \tag{7}$$

which can be re-written as:

$$\begin{cases} \mathbf{D}_0 U_0 = \overline{F_0}, \\ \mathbf{K}_k \begin{pmatrix} U_k \\ \Lambda_k \end{pmatrix} = \begin{pmatrix} F_k \\ -\mathbf{B}_{0k} U_0 \end{pmatrix}, \quad 1 \leq k \leq K. \end{cases} \tag{8}$$

Here, $\mathbf{D}_0 = \mathbf{A}_0 - \sum_{k=1}^{K} \mathbf{B}_{0k}^t R_k \mathbf{K}_k^{-1} R_k^t \mathbf{B}_{0k}$ is the Schur complement matrix. The problem is now split into a coarse problem defined on Ω_0, and into fine problems defined on $(\Omega_k)_{1 \leq k \leq K}$

using the coarse solution U_0. It seems that the calculus on the subdomains are now separated, but the price to pay is in the building of the coarse Schur complement $\mathbf{D}_0$. Our aim is to obtain a good preconditioner for this problem, using an approximate coarse operator $\hat{\mathbf{D}}_0$. In other terms, we need to construct an approximate solution $(\tilde{u}, \tilde{\lambda}) \in X_h \times M_\delta$ of (8) by:

$$\begin{cases} \tilde{U}_0 = \hat{\mathbf{D}}_0^{-1}\overline{F_0}, \\ \mathbf{K}_k \begin{pmatrix} \tilde{U}_k \\ \tilde{\Lambda}_k \end{pmatrix} = \begin{pmatrix} F_k \\ -\mathbf{B}_{0k}\tilde{U}_0 \end{pmatrix}, \quad 1 \leq k \leq K, \end{cases} \tag{9}$$

and the main issue is to build an appropriate definition of the Schur inverse $\hat{\mathbf{D}}_0^{-1}$.

3.2 Two possible definitions for $\hat{\mathbf{D}}_0$

3.2.1 A symmetrized Dirichlet-Neumann preconditioner

The simplest idea consists in replacing the Schur complement $\mathbf{D}_0$ by the stiffness of the coarse problem:

$$\hat{\mathbf{D}}_0 = \mathbf{A}_0, \tag{10}$$

which reduces the proposed preconditioning to a symmetrized Dirichlet-Neumann iteration. Indeed, solving (9) with $\hat{\mathbf{D}}_0 = \mathbf{A}_0$ then amounts to solving:

1. Dirichlet problems on the $(\Omega_k)_{1\leq k\leq K}$ with zero weak trace on the interface to obtain

$$\overline{F_0} = F_0 - \sum_{k=1}^{K} \mathbf{B}_{0k}^t R_k \mathbf{K}_k^{-1} \begin{pmatrix} F_k \\ 0 \end{pmatrix},$$

2. a Neumann problem on Ω_0 with the sollicitation $\overline{F_0}$ to compute U_0,
3. Dirichlet problems on the $(\Omega_k)_{1\leq k\leq K}$ to compute the $(U_k)_{1\leq k\leq K}$ with right-hand sides

$$\begin{pmatrix} F_k \\ -\mathbf{B}_{0k}U_0 \end{pmatrix}.$$

In section 3.3, we prove that the condition number of the associated preconditioned system is independent of the number and of the size of the fine scale subdomains $(\Omega_k)_{k\geq 1}$. We also prove that the method is efficient when Ω_0 has not a small stiffness in comparison with the $(\Omega_k)_{k\geq 1}$, and when the small subdomains are not fixed on a part of their boundary.

3.2.2 An enhanced symmetrized Dirichlet-Neumannn preconditioner

The previous simplest choice of preconditioner may lack of efficiency in two simple situations:

- the substructure Ω_k is of small size and is fixed on a part of its boundary. In this situation, because of its size, the substructure will have a rather large stiffness to interface rigid body displacements.
- the substructure Ω_k may have other privileged directions of large stiffness to interface motions (rigid links, incompressibility).

Assuming that these directions of interface localized stiffness be in very small number N_k (this is indeed the case for interface rigid body motions), we propose a modification of the previous preconditioner enabling to correct such a lack of efficiency.
For all $k \geq 1$ such that Ω_k is fixed on a part of its boundary, we denote by $(e_k^i)_{1\leq i\leq N_k}$ (with $N_k = 6$ in general) the interface rigid motions of Γ_{0k} or rigid links and introduce:

$$\mathring{W}_k = span\{e_k^i, i = 1, .., N_k\}.$$

To each interface rigid body motion e_k^i, we introduce its local a_k-harmonic extension $(u_k^i, \lambda_k^i) \in X_{k;h_k} \times M_{k;\delta_k}$ solution of:

$$\begin{cases} a_k(v, u_k^i) - \displaystyle\int_{\Gamma_{0k}} v \cdot \lambda_k^i = 0, \quad \forall v \in X_{k;h_k}, \\ -\displaystyle\int_{\Gamma_{0k}} u_k^i \cdot \mu = -\int_{\Gamma_{0k}} e_k^i \cdot \mu, \quad \forall \mu \in M_{k;\delta_k}. \end{cases} \tag{11}$$

These solutions span two small local spaces:

$$\mathring{X}_k = span\{u_k^i, i = 1, .., N_k\} \subset X_{k;h_k},$$

$$\mathring{M}_k = span\{\lambda_k^i, i = 1, .., N_k\} \subset M_{k;\delta_k}.$$

If $k \geq 1$ is such that Ω_k is not fixed on its boundary, we adopt:

$$\mathring{W}_k = \mathring{M}_k = \{0\}.$$

Then, instead of finding U_0 such that $\mathbf{D}_0 U_0 = \overline{F_0}$, we propose to compute $u_0 \in X_{0;h_0}$, $(u_k) \in (\mathring{X}_k)_{1\leq k\leq K}$, $(\lambda_k) \in (\mathring{M}_k)_{1\leq k\leq K}$ solution of the coupled problem:

$$\begin{cases} a_0(u_0, v_0) + \displaystyle\sum_{k=1}^{K} \int_{\Gamma_{0k}} v_0 \cdot \lambda_k = \overline{l_0}(v_0), \quad \forall v_0 \in X_{0;h_0}, \\ a_k(u_k, v_k) - \displaystyle\int_{\Gamma_{0k}} v_k \cdot \lambda_k = 0, \quad \forall v_k \in \mathring{X}_k, \quad 1 \leq k \leq K, \\ -\displaystyle\int_{\Gamma_{0k}} u_k \cdot \mu_k = -\int_{\Gamma_{0k}} u_0 \cdot \mu_k, \quad \forall \mu_k \in \mathring{M}_k, \quad 1 \leq k \leq K, \end{cases} \tag{12}$$

where $\overline{l_0}$ is the linear form associated to the coarse sollicitation $\overline{F_0}$.

We introduce the matrix $\mathbf{I}_{0k} \in \mathbb{R}^{N_k \times dim X_{0;h_0}}$ defined for all $v_0 \in X_{0;h_0}$ by:

$$(\mathbf{I}_{0k} V_0)_i = \int_{\Gamma_{0k}} v_0 \cdot \lambda_k^i = \left\langle \mathbf{B}_{0k} V_0, \Lambda_k^i \right\rangle, \quad \forall i = 1, .., N_k,$$

that is $\mathbf{I}_{0k} = \left[\Lambda_k^1, .., \Lambda_k^{N_k}\right]^t \mathbf{B}_{0k} = \mathbf{\Lambda}_k^t \mathbf{B}_{0k}$, and the restriction $\mathring{\mathbf{A}}_k$ of the displacement stiffness matrix $\mathbf{A}_k$ on the local space $\mathring{X}_k$. Thus:

$$\left(\mathring{\mathbf{A}}_k\right)_{ij} = (U_k^i)^t \mathbf{A}_k U_k^j = a_k(u_k^j, u_k^i) = \int_{\Gamma_{0k}} u_k^j \cdot \lambda_k^i.$$

From (11)-1, the system (12) can be rewritten as:

$$\begin{cases} \mathbf{A}_0 U_0 + \sum_{k=1}^K \mathbf{I}_{0k}^t \Theta_k = \overline{F_0}, \\ \mathring{\mathbf{A}}_k Z_k - \mathring{\mathbf{A}}_k^t \Theta_k = 0, \\ -\mathring{\mathbf{A}}_k Z_k = -\mathbf{I}_{0k} U_0, \quad 1 \leq k \leq K. \end{cases} \tag{13}$$

The new vector Θ_k (resp. Z_k) denotes the component of λ_k (resp. u_k) in $\mathring{M}_k$ (resp. $\mathring{W}_k$) appearing in (12). From the elimination of Θ_k and Z_k in (13), it follows that:

$$\hat{\mathbf{D}}_0 U_0 = \overline{F_0}, \tag{14}$$

with a new approximate Schur complement given by:

$$\begin{aligned} \hat{\mathbf{D}}_0 &= \mathbf{A}_0 + \sum_{k=1}^K \mathbf{I}_{0k}^t \mathring{\mathbf{A}}_k^{-t} \mathbf{I}_{0k} \qquad (15) \\ &= \mathbf{A}_0 + \sum_{k=1}^K \mathbf{B}_{0k}^t \mathbf{\Lambda}_k \mathring{\mathbf{A}}_k^{-t} \mathbf{\Lambda}_k^t \mathbf{B}_{0k}. \end{aligned}$$

Its complexity is much smaller than (7) because the local problem (13)-2,(13)-3 for the subproblem $k \geq 1$ used in the construction of $\hat{\mathbf{D}}_0$, is of dimension N_k.

For analysis purpose, this enhanced Dirichlet-Neumann preconditioner corresponds to a Dirichlet-Neumann decomposition where the Dirichlet substructures are defined by:

$$\mathring{X}_{k;h_k}^{\perp} = \{u_k \in X_{k;h_k}, \quad \int_{\Gamma_{0k}} u_k \cdot \mu = 0, \quad \forall \mu \in \mathring{M}_k\}, \quad 1 \leq k \leq K,$$

and where the Neumann substructure is defined by:

$$\mathring{X}_h = \{u \in X_h, \quad b(u, \mu) = 0, \quad \forall \mu \in \mathring{M}_k\}.$$

The analysis of this preconditioner is done in section 4.3, proving now an independence with respect to essential boundary conditions imposed over the small subdomains $(\Omega_k)_{k\geq 1}$. For further analysis, we introduce:

Definition 1. *For any $v_0 \in X_{0;h_0}$, its "rigid body projection" over Ω_k denoted by $\mathring{\pi}_k v_0 \in \mathring{X}_k$ is defined as the solution of (13)-2,(13)-3 for the subproblem k. More precisely $(\mathring{\pi}_k v_0, \mathring{\lambda}_k) \in \mathring{X}_k \times \mathring{M}_k$ is such that:*

$$\begin{cases} a_k(\mathring{\pi}_k v_0, v_k) - \int_{\Gamma_{0k}} \mathring{\lambda}_k \cdot v_k = 0, \quad \forall v_k \in \mathring{X}_k, \\ - \int_{\Gamma_{0k}} \mathring{\pi}_k v_0 \cdot \mu_k = - \int_{\Gamma_{0k}} v_0 \cdot \mu_k, \quad \forall \mu_k \in \mathring{M}_k. \end{cases} \tag{16}$$

In matricial form, we have $\mathring{\pi}_k v_k = \sum_{j=1}^{N_k} z_j u_k^j$ with:

$$-\mathring{\mathbf{A}}_k Z = -\mathbf{I}_{0k} V_0,$$

yielding:

$$\mathring{\Pi}_k V_0 = \left[U_k^1, .., U_k^{N_k} \right] \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k} V_0 = \mathring{\mathbf{U}}_k \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k} V_0,$$

that is:

$$\mathring{\Pi}_k = \mathring{\mathbf{U}}_k \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k}.$$

We then have by construction of $\mathring{\mathbf{A}}_k$:

$$\begin{aligned} \mathring{\Pi}_k^t \mathbf{A}_k \mathring{\Pi}_k &= \mathbf{I}_{0k}^t \mathring{\mathbf{A}}_k^{-t} \mathring{\mathbf{U}}_k^t \mathbf{A}_k \mathring{\mathbf{U}}_k \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k} \\ &= \mathbf{I}_{0k}^t \mathring{\mathbf{A}}_k^{-t} \mathring{\mathbf{A}}_k \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k} \\ &= \mathbf{I}_{0k}^t \mathring{\mathbf{A}}_k^{-t} \mathbf{I}_{0k}, \end{aligned}$$

and therefore the new preconditioner (15) takes the form:

$$\hat{\mathbf{D}}_0 = \mathbf{A}_0 + \sum_{k=1}^{K} \mathring{\Pi}_k^t \mathbf{A}_k \mathring{\Pi}_k. \tag{17}$$

Also observe from (11) that when a_k is symmetric, we have:

$$\mathring{\pi}_k e_k^i = u_k^i, \quad 1 \le i \le N_k. \tag{18}$$

4 Condition number analysis

In this section, we establish upper bounds on the condition number of the preconditioned systems based on the two symmetrized Dirichlet-Neumann preconditioners respectively defined in the subsections 3.2.1 and 3.2.2. First, the same factorized form for the original linear system and the preconditioner is introduced. Then, we show the spectral equivalence between $\hat{\mathbf{D}}_0$ and $\mathbf{D}_0$, detailing the dependence of the constants on the size of the domains, the stiffness of the materials, and on the mesh sizes, and deduce estimates on the condition number of the preconditioned system.

4.1 Factorization

The original system to solve is:

$$\mathbf{A}\begin{pmatrix} U_0 \\ U_1 \\ \Lambda_1 \\ \vdots \\ U_K \\ \Lambda_K \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ 0 \\ \vdots \\ F_K \\ 0 \end{pmatrix},$$

with:

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_0 & 0 & \mathbf{B}_{01}^t & \dots & 0 & \mathbf{B}_{0K}^t \\ 0 & \mathbf{A}_1 & -\mathbf{B}_1^t & & & \\ \mathbf{B}_{01} & -\mathbf{B}_1 & 0 & & & \\ \vdots & & & \ddots & & \\ 0 & & & & \mathbf{A}_K & -\mathbf{B}_K^t \\ \mathbf{B}_{0K} & & & & -\mathbf{B}_K & 0 \end{pmatrix}.$$

Now, let us factorize the expression of $\mathbf{A}$. Introducing the triangular matrix:

$$T = \begin{pmatrix} I & 0 & \dots & 0 \\ \mathbf{K}_1^{-1} R_1^t \mathbf{B}_{01} & I & & \\ \vdots & & \ddots & \\ \mathbf{K}_K^{-1} R_K^t \mathbf{B}_{0K} & 0 & \dots & I \end{pmatrix},$$

and the block diagonal matrix:

$$H = \begin{pmatrix} \mathbf{D}_0 & 0 & \dots & 0 \\ 0 & \mathbf{K}_1 & & \\ \vdots & & \ddots & \\ 0 & & & \mathbf{K}_K \end{pmatrix},$$

it is straightforward to check that $\mathbf{A} = T^t H T$.

The matrix of our preconditioner can be written under the similar form $\mathbf{C} = T^t \hat{H} T$, with the block diagonal matrix:

$$\hat{H} = \begin{pmatrix} \hat{\mathbf{D}}_0 & 0 & \dots & 0 \\ 0 & \mathbf{K}_1 & & \\ \vdots & & \ddots & \\ 0 & & & \mathbf{K}_K \end{pmatrix}.$$

We have then:

$$\mathbf{C}\begin{pmatrix} \tilde{U}_0 \\ \tilde{U}_1 \\ \tilde{\Lambda}_1 \\ \vdots \\ \tilde{U}_K \\ \tilde{\Lambda}_K \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ 0 \\ \vdots \\ F_K \\ 0 \end{pmatrix}.$$

The matrices **A** and **C** are not positive, and we introduce the kernel on which the following results hold, and in which **A** and **C** are definite positive:

$$E = \{U = (U_0, U_1, \Lambda_1, .., U_K, \Lambda_K)^t; \mathbf{B}_{0k}U_0 = \mathbf{B}_k U_k, 1 \leq k \leq K\}.$$

Our aim is to bound the condition number $\kappa_{\mathbf{A},E}(\mathbf{C}^{-1}\mathbf{A})$ in **A**-norm on E.

4.2 Spectral equivalence for the simple Dirichlet-Neumann

We show herein the spectral equivalence between the Schur complement $\mathbf{D}_0$ and its approximation $\hat{\mathbf{D}}_0$ for the symmetrized Dirichlet-Neumann preconditioner presented in subsection 3.2.1. For the choice $\hat{\mathbf{D}}_0 = \mathbf{A}_0$ made in section 3.2.1 and corresponding to the simple symmetrized Dirichlet-Neumann preconditioner, we obtain:

Proposition 2. *Assuming that $\mathbf{A}_0$ is invertible that is $\Gamma_D \cap \partial\Omega_0$ has a positive measure, the following spectral equivalence holds for all U_0:*

$$W_{1,h}\langle \mathbf{D}_0 U_0, U_0\rangle \leq \langle \mathbf{A}_0 U_0, U_0\rangle \leq \langle \mathbf{D}_0 U_0, U_0\rangle,$$

with:

$$\frac{1}{W_{1,h}} = 1 + C\left(\max_{k\in I_1}\frac{C_k}{c_0} + \max_{k\in I_2}\frac{C_k L_0}{\alpha_0 L_k}\right),$$

where I_1 (resp. I_2) is the set of indices $k \geq 1$ such that Ω_k is not fixed on its boundary (resp. is fixed on a part of its boundary). The constant C is independent of the number K and the size of the subdomains.

Observe that the condition number deteriorates for a small fixed subdomain $L_k << L_0$, $k \in I_2$, and for very stiff subdomains $C_k >> \alpha_0$.
The following lemma is needed in the proof:

Lemma 1. *Let us assume that Γ_{0k} is of class $\mathcal{C}^1$. Then, there exists an open set $\Omega'_k \subset \Omega_0$ which is the restriction of a neighborhood of Γ_{0k} to Ω_0, and a linear extension operator:*

$$\mathcal{D}_k : H^1(\Omega'_k)^d \to H^1(\Omega_k)^d,$$

such that for all $u \in H^1(\Omega_0)^d$, $\mathcal{D}_k u = u$ on Γ_{0k}, and:

$$\int_{\Omega_k} (\mathcal{D}_k u)^2 \leq C \int_{\Omega'_k} u^2,$$

$$\int_{\Omega_k} (\nabla \mathcal{D}_k u)^2 \leq C \int_{\Omega'_k} (\nabla u)^2 ,$$

where the constant C does not depend on Ω_k.

The proof of this lemma is rather standard in functional analysis, and the existence of such an extension operator can be found in ([Bré99], page 158) for example. Now, we can prove the proposition.

Proof : [of the proposition] Let U_0 be given. For all $k \geq 1$, let us define (U_k, Λ_k) such that:

$$\begin{pmatrix} \mathbf{A}_k & -\mathbf{B}_k^t \\ -\mathbf{B}_k & 0 \end{pmatrix} \begin{pmatrix} U_k \\ \Lambda_k \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathbf{B}_{0k} U_0 \end{pmatrix}.$$

In other words, we have:

$$\Lambda_k = -R_k \mathbf{K}_k^{-1} R_k^t \mathbf{B}_{0k} U_0,$$

and then by construction of U_k and Λ_k:

$$\begin{aligned} -\left\langle \mathbf{B}_{0k}^t R_k \mathbf{K}_k^{-1} R_k^t \mathbf{B}_{0k} U_0, U_0 \right\rangle &= \left\langle \mathbf{B}_{0k}^t \Lambda_k, U_0 \right\rangle \\ &= \left\langle \Lambda_k, \mathbf{B}_k U_k \right\rangle \\ &= \left\langle \mathbf{A}_k U_k, U_k \right\rangle \\ &\geq 0. \end{aligned}$$

We deduce by addition that:

$$\begin{aligned} \left\langle \mathbf{D}_0 U_0, U_0 \right\rangle &= \left\langle \mathbf{A}_0 U_0, U_0 \right\rangle - \sum_{k=1}^{K} \left\langle \mathbf{B}_{0k}^t R_k \mathbf{K}_k^{-1} R_k^t \mathbf{B}_{0k} U_0, U_0 \right\rangle \\ &\geq \left\langle \mathbf{A}_0 U_0, U_0 \right\rangle . \end{aligned}$$

Hence the inequality:

$$\left\langle \mathbf{A}_0 U_0, U_0 \right\rangle \leq \left\langle \mathbf{D}_0 U_0, U_0 \right\rangle , \quad \forall U_0.$$

Let us now bound $\mathbf{A}_0$ from below. Let $u_0 \in H^1_*(\Omega)$ be given. For all $k \geq 1$, such that Ω_k has an empty intersection with Γ_D (we denote $k \in I_1$), we decompose u_0 on Ω'_k (as defined in lemma 1) into:

$$u_0 = r_k + w_k, \quad \text{on } \Omega'_k,$$

where r_k belongs to the space $\mathcal{R}(\Omega'_k)$ of rigid motions over Ω'_k, and:

$$\int_{\Omega'_k} w_k \cdot r = 0, \quad \forall r \in \mathcal{R}(\Omega'_k). \tag{19}$$

We define the function:

$$u_k = r_k + u'_k, \quad \text{ on } \Omega_k,$$

where $r_k \in \mathcal{R}(\Omega_k)$ is the natural extension to Ω_k of $r_k \in \mathcal{R}(\Omega'_k)$ (thus $r_k \in \mathcal{R}(\Omega_k \cup \Omega'_k)$), and:

$$u'_k = \mathcal{I}_{k;h_k}\mathcal{D}_k w_k + \mathcal{R}_{k;\delta_k}\pi_k(w_k - \mathcal{I}_{k;h_k}\mathcal{D}_k w_k),$$

where $\mathcal{I}_{k;h_k}$ denotes the Scott-Zhang [SZ90] interpolation over $X_{k;h_k}$, and $\mathcal{R}_{k;\delta_k}$ is the extension by zero operator over the grid points of Ω_k. By construction, the mortar condition is satisfied:

$$\int_{\Gamma_{0k}} u_k \cdot \mu = \int_{\Gamma_{0k}} u_0 \cdot \mu, \quad \forall \mu \in M_{k;\delta_k}.$$

Moreover, by using the stability of the extension operator $\mathcal{R}_{k;\delta_k}$ from $W_{k;\delta_k}$ to $H^1(\Omega_k)^d$, the assumption 2, the stability of $\mathcal{I}_{k;h_k}$ from $H^1(\Omega_k)^d$ to $H^1(\Omega_k)^d$, the classical estimation (see [SZ90]):

$$\|u - \mathcal{I}_{k;h_k} u\|_{\delta,\frac{1}{2},k} \leq C|u|_{H^1(\Omega_k)^d},$$

and the stability property of $\mathcal{D}_k$ in lemma 1, we obtain:

$$\begin{aligned}
a_k(u_k,u_k) &\leq C_k \int_{\Omega_k} |\nabla u'_k|^2 = C_k \left|u'_k\right|^2_{H^1(\Omega_k)^d} \\
&\leq 2C_k \left|\mathcal{I}_{k;h_k}\mathcal{D}_k w_k\right|^2_{H^1(\Omega_k)^d} + 2C_k \left|\mathcal{R}_{k;\delta_k}\pi_k(\mathcal{D}_k w_k - \mathcal{I}_{k;h_k}\mathcal{D}_k w_k)\right|^2_{H^1(\Omega_k)^d} \\
&\leq 2C_k \left|\mathcal{I}_{k;h_k}\mathcal{D}_k w_k\right|^2_{H^1(\Omega_k)^d} + 2CC_k \|\pi_k(\mathcal{D}_k w_k - \mathcal{I}_{k;h_k}\mathcal{D}_k w_k)\|^2_{\delta,\frac{1}{2},k} \\
&\leq 2C_k \left|\mathcal{I}_{k;h_k}\mathcal{D}_k w_k\right|^2_{H^1(\Omega_k)^d} + 2CC_k \|\mathcal{D}_k w_k - \mathcal{I}_{k;h_k}\mathcal{D}_k w_k\|^2_{\delta,\frac{1}{2},k} \\
&\leq CC_k \left|\mathcal{D}_k w_k\right|^2_{H^1(\Omega_k)^d} \leq CC_k \left|w_k\right|^2_{H^1(\Omega'_k)^d}.
\end{aligned} \tag{20}$$

Moreover, the following inequality holds for all $v \in H^1(\Omega'_k)^d$:

$$|v|^2_{H^1(\Omega'_k)^d} \leq C_{\Omega'_k} \left(\int_{\Omega'_k} \varepsilon(v) : \varepsilon(v) + \frac{1}{diam(\Omega'_k)^2} \left(\sup_{\substack{r \in \mathcal{R}(\Omega'_k), \\ \int_{\Omega'_k} r = 0}} \frac{\int_{\Omega'_k} v \cdot r}{\|r\|_{L^2(\Omega'_k)^d}} \right)^2 \right), \tag{21}$$

with a constant $C_{\Omega'_k}$ independent of the size of Ω'_k from the adopted scaling of the norms, but possibly depending on its shape. The shape independence of this constant is insured for polyhedral shape regular domains in [Bre04], or in [Hau04] for slightly less restrictive assumptions. Therefore, we have from (19) by definition of w_k:

$$|w_k|^2_{H^1(\Omega'_k)^d} \leq C_{\Omega'_k} \int_{\Omega'_k} \varepsilon(w_k) : \varepsilon(w_k).$$

By summing over $k \in I_1$, we get from (20) that:

$$\sum_{k \in I_1} a_k(u_k, u_k) \leq C \sum_{k \in I_1} C_k \int_{\Omega'_k} \varepsilon(u_0) : \varepsilon(u_0), \tag{22}$$

with a constant C independent of the size of the subdomains. Since by construction $\cup_{k \in I_1} \Omega'_k \subset \Omega_0$, and since there is a bounded number of domains Ω'_k overlapping at a given point, we deduce:

$$\sum_{k \in I_1} a_k(u_k, u_k) \leq C \max_{k \in I_1}(C_k) \int_{\Omega_0} \varepsilon(u_0) : \varepsilon(u_0) \leq \frac{C}{c_0} \max_{k \in I_1}(C_k) a_0(u_0, u_0).$$

For all $k \geq 1$ such that Γ_D is fixed on a part of its boundary (that is $k \in I_2$), we cannot use the extension operator $\mathcal{D}_k$ because it will not satisfy the Dirichlet boundary condition on Γ_D. But, the Sobolev lifting theorem proves the existence of a function $\tilde{u}_k$ whose trace is u_0 on Γ_{0k} and such that:

$$\frac{1}{(L_k)^2} \int_{\Omega_k} |\tilde{u}_k|^2 + \int_{\Omega_k} |\nabla \tilde{u}_k|^2 \leq C \left(\frac{1}{L_k} \int_{\Gamma_{0k}} \langle u_0 \rangle_k^2 + |u_0|^2_{H^{1/2}(\Gamma_{0k})^d} \right).$$

Here, $\langle u_0 \rangle_k$ denotes the average

$$\langle u_0 \rangle_k = \frac{1}{meas(\Gamma_{0k})} \int_{\Gamma_{0k}} u_0$$

of u_0 on Γ_{0k} and C is a constant which is independent of the size of Ω_k but which depends on the ratio between L_k and the distance from Γ_{0k} to Γ_D. We then modify $\tilde{u}_k$ to obtain a discrete function satisfying the weak-continuity constraint on Γ_{0k}, and define using our previous notation:

$$u_k = \mathcal{I}_{k;h_k} \tilde{u}_k + \mathcal{R}_{k;\delta_k} \pi_k(\tilde{u}_k - \mathcal{I}_{k;h_k} \tilde{u}_k).$$

By construction, the mortar condition is satisfied:

$$\begin{aligned} \int_{\Gamma_{0k}} u_k \cdot \mu &= \int_{\Gamma_{0k}} (\mathcal{I}_{k;h_k} \tilde{u}_k + \tilde{u}_k - \mathcal{I}_{k;h_k} \tilde{u}_k) \cdot \mu \\ &= \int_{\Gamma_{0k}} u_0 \cdot \mu, \quad \forall \mu \in M_{k;\delta_k}. \end{aligned}$$

From the same argument as in the case $k \in I_1$, we get:

$$\begin{aligned} a_k(u_k, u_k) &\leq CC_k \int_{\Omega_k} |\nabla \tilde{u}_k|^2 \\ &\leq CC_k \left(\frac{1}{L_k} \int_{\Gamma_{0k}} \langle u_0 \rangle_k^2 + |u_0|^2_{H^{1/2}(\Gamma_{0k})^d} \right) \\ &\leq CC_k \frac{L_0}{L_k} \left(\frac{1}{L_0} \int_{\Gamma_{0k}} \langle u_0 \rangle_k^2 + |u_0|^2_{H^1(\Omega'_k)^d} \right). \end{aligned}$$

By summation, we have:

$$\begin{aligned} \sum_k \int_{\Gamma_{0k}} \langle u_0 \rangle_k^2 &= \sum_k meas(\Gamma_{0k}) \langle u_0 \rangle_k^2 \\ &= \sum_k meas(\Gamma_{0k})^{-1} \left(\int_{\Gamma_{0k}} 1 u_0 \right)^2 \\ &\leq \sum_k meas(\Gamma_{0k})^{-1} \int_{\Gamma_{0k}} u_0^2 \int_{\Gamma_{0k}} 1 \\ &\leq \sum_k \int_{\Gamma_{0k}} u_0^2 = \int_{\Gamma_0} u_0^2. \end{aligned} \tag{23}$$

By summing over $k \in I_2$, we get as before:

$$\begin{aligned} \sum_{k \in I_2} a_k(u_k, u_k) &\leq C \max_{k \in I_2} \left(C_k \frac{L_0}{L_k} \right) \left(\frac{1}{L_0} \int_{\Gamma_0} u_0^2 + |u_0|^2_{H^1(\partial\Omega_0)^d} \right) \\ &\leq C \max_{k \in I_2} \left(C_k \frac{L_0}{L_k} \right) \|u_0\|^2_{H^1(\Omega_0)^d} \\ &\leq C \max_{k \in I_2} \frac{C_k L_0}{\alpha_0 L_k} \, a_0(u_0, u_0). \end{aligned}$$

As a consequence, with this choice of u_k:

$$\langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \langle \mathbf{A}_k U_k, U_k \rangle \leq \left(1 + C \max_{k \in I_1} \frac{C_k}{c_0} + C \max_{k \in I_2} \frac{C_k L_0}{\alpha_0 L_k} \right) \langle \mathbf{A}_0 U_0, U_0 \rangle .$$

Now, let us show that for all $(V_k)_{k \geq 1}$ such that $\mathbf{B}_k V_k = \mathbf{B}_{0k} U_0$, we have:

$$\langle \mathbf{D}_0 U_0, U_0 \rangle \leq \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \langle \mathbf{A}_k V_k, V_k \rangle . \tag{24}$$

For all $k \geq 1$, we decompose V_k into $V_k = U_k^* + \delta U_k$, where:

$$\begin{pmatrix} \mathbf{A}_k & -\mathbf{B}_k^t \\ -\mathbf{B}_k & 0 \end{pmatrix} \begin{pmatrix} U_k^* \\ \Lambda_k^* \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathbf{B}_{0k} U_0 \end{pmatrix},$$

and $\mathbf{B}_k \delta U_k = 0$. Then, since by construction:

$$\langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \langle \mathbf{A}_k U_k^*, U_k^* \rangle = \langle \mathbf{D}_0 U_0, U_0 \rangle \, ,$$

we obtain by symmetry of $\mathbf{A}_k$:

$$\langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \langle \mathbf{A}_k V_k, V_k \rangle = \langle \mathbf{D}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} 2 \langle \mathbf{A}_k U_k^*, \delta U_k \rangle + \langle \mathbf{A}_k \delta U_k, \delta U_k \rangle \, .$$

Moreover:

$$\langle \mathbf{A}_k U_k^*, \delta U_k \rangle = \left\langle \mathbf{B}_k^t \Lambda_k^*, \delta U_k \right\rangle = \langle \Lambda_k^*, \mathbf{B}_k \delta U_k \rangle = 0,$$

resulting in:

$$\begin{aligned} \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \langle \mathbf{A}_k V_k, V_k \rangle &= \langle \mathbf{D}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \langle \mathbf{A}_k \delta U_k, \delta U_k \rangle \\ &\geq \langle \mathbf{D}_0 U_0, U_0 \rangle \, . \end{aligned}$$

In particular, we can take for all $k \geq 1$, $V_k = U_k$ where U_k has been built above. We conclude that:

$$\langle \mathbf{D}_0 U_0, U_0 \rangle \leq \left(1 + C \max_{k \in I_1} \frac{C_k}{c_0} + C \max_{k \in I_2} \frac{C_k L_0}{\alpha_0 L_k} \right) \langle \mathbf{A}_0 U_0, U_0 \rangle \, ,$$

which ends the proof. □

4.3 Spectral equivalence for the enhanced Dirichlet Neumann

For the enhanced Dirichlet-Neumann preconditioner presented in section 3.2.2, we prove that:

Proposition 3. *For all U_0, the following spectral equivalence holds:*

$$W_{1,h} \langle \mathbf{D}_0 U_0, U_0 \rangle \leq \left\langle \hat{\mathbf{D}}_0 U_0, U_0 \right\rangle \leq \langle \mathbf{D}_0 U_0, U_0 \rangle \, ,$$

with:

$$\frac{1}{W_{1,h}} = C \left(1 + \max_{k \in I_1 \cup I_2} \frac{C_k}{c_0} \right),$$

where I_1 (resp. I_2) is the set of indices $k \geq 1$ such that Ω_k is not fixed on its boundary (resp. is fixed on a part of its boundary). The constant C is independent of the number K and the size of the subdomains.

Proof : Let U_0 be given. We proceed as in the last part of the previous proof, and introduce (U_k^*, Λ_k^*) satisfying:

$$\begin{pmatrix} \mathbf{A}_k & -\mathbf{B}_k^t \\ -\mathbf{B}_k & 0 \end{pmatrix} \begin{pmatrix} U_k^* \\ \Lambda_k^* \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathbf{B}_{0k} U_0 \end{pmatrix}. \tag{25}$$

We introduce the decomposition $U_k^* = \mathring{U}_k^* + W_k^*$ with $\mathring{U}_k^* = \mathring{\Pi}_k U_0$, and by construction of U_k^*, we get:

$$\begin{aligned}
\langle \mathbf{D}_0 U_0, U_0 \rangle &= \left\langle \left(\mathbf{A}_0 - \sum_{k=1}^{K} \mathbf{B}_{0k}^t R_k \mathbf{K}_k^{-1} R_k^t \mathbf{B}_{0k} \right) U_0, U_0 \right\rangle \\
&= \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \langle \mathbf{A}_k U_k^*, U_k^* \rangle \\
&\geq \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \left\langle \mathbf{A}_k \mathring{U}_k^*, \mathring{U}_k^* \right\rangle + 2 \left\langle \mathbf{A}_k \mathring{U}_k^*, W_k^* \right\rangle .
\end{aligned}$$

But decomposing $\mathring{U}_k^* = \mathring{\Pi}_k U_0 = \sum_{j=1}^{N_k} z_j U_k^j$ we have:

$$\begin{aligned}
\left\langle \mathbf{A}_k W_k^*, \mathring{U}_k^* \right\rangle &= \sum_{j=1}^{N_k} z_j a_k(w_k^*, u_k^j) \\
&= \sum_{j=1}^{N_k} z_j \int_{\Gamma_{0k}} \lambda_k^j \cdot w_k^*, \quad \text{from (11).1,} \\
&= \sum_{j=1}^{N_k} z_j \int_{\Gamma_{0k}} (u_k^* - \mathring{u}_k^*) \cdot \lambda_k^j, \quad \text{by construction of } w_k^*, \\
&= \sum_{j=1}^{N_k} z_j \left[\int_{\Gamma_{0k}} u_k^* \cdot \lambda_k^j - \int_{\Gamma_{0k}} \mathring{u}_k^* \cdot \lambda_k^j \right], \quad \text{from (25) and (16).2,} \\
&= 0.
\end{aligned}$$

This gives:

$$
\begin{aligned}
\langle \mathbf{D}_0 U_0, U_0 \rangle &\geq \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \left\langle \mathbf{A}_k \mathring{U}_k^*, \mathring{U}_k^* \right\rangle \\
&= \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \left\langle \mathbf{A}_k \mathring{\Pi}_k U_0, \mathring{\Pi}_k U_0 \right\rangle \\
&= \left\langle \left(\mathbf{A}_0 + \sum_{k=1}^{K} \mathring{\Pi}_k^t \mathbf{A}_k \mathring{\Pi}_k \right) U_0, U_0 \right\rangle \\
&= \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \left\langle \mathbf{A}_k \mathring{U}_k^*, \mathring{U}_k^* \right\rangle, \quad \text{from (17).}
\end{aligned}
$$

Let us prove now a lower bound for $\hat{\mathbf{D}}_0$. For all $1 \leq k \leq K$, as in the proof of the previous proposition, we build a particular function $u_k \in W_{k;\delta_k}$ satisfying the weak continuity constraint on the interface Γ_{0k}. When Ω_k is not fixed on a part of its boundary, which we have denoted by $k \in I_1$, we take the u_k defined in the previous proof by "reflexion" with respect to Γ_{0k}. When Ω_k is fixed on a part of its boundary, namely $k \in I_2$, we proceed differently, and define here $\langle u_0 \rangle_k \in \mathcal{R}(\Gamma_{0k})$ (the trace over Γ_{0k} of a rigid motion) such that:

$$
\int_{\Gamma_{0k}} \langle u_0 \rangle_k \cdot r = \int_{\Gamma_{0k}} u_0 \cdot r, \quad \forall r \in \mathcal{R}(\Gamma_{0k}).
$$

Then, we introduce:

$$
u_k = \mathcal{I}_{k;h_k} \tilde{u}_k + \mathcal{R}_{k;\delta_k} \pi_k \left[\tilde{u}_k - \mathcal{I}_{k;h_k} \tilde{u}_k \right] + \mathring{\pi}_k \langle u_0 \rangle_k ,
$$

where $\tilde{u}_k$ is a function whose trace is zero on Γ_D and is $u_0 - \langle u_0 \rangle_k$ on Γ_{0k} satisfying from the Sobolev lifting theorem:

$$
\begin{aligned}
\int_{\Omega_k} |\nabla \tilde{u}_k|^2 &\leq C \left[\frac{1}{L_k} \langle u_0 - \langle u_0 \rangle_k \rangle_k + |u_0 - \langle u_0 \rangle_k|^2_{H^{1/2}(\Gamma_{0k})^d} \right] \\
&= C |u_0 - \langle u_0 \rangle_k|^2_{H^{1/2}(\Gamma_{0k})^d}, \quad \text{by construction of } \langle u_0 \rangle_k. \qquad (26)
\end{aligned}
$$

The mortar condition is indeed satisfied because:

$$
\begin{aligned}
\int_{\Gamma_{0k}} u_k \cdot \mu &= \int_{\Gamma_{0k}} (\mathcal{I}_{k;h_k} \tilde{u}_k + \tilde{u}_k - \mathcal{I}_{k;h_k} \tilde{u}_k) \cdot \mu + \int_{\Gamma_{0k}} \mathring{\pi}_k \langle u_0 \rangle_k \cdot \mu \\
&= \int_{\Gamma_{0k}} \tilde{u}_k \cdot \mu + \int_{\Gamma_{0k}} \mathring{\pi}_k \langle u_0 \rangle_k \cdot \mu \\
&= \int_{\Gamma_{0k}} (u_0 - \langle u_0 \rangle_k + \mathring{\pi}_k \langle u_0 \rangle_k) \cdot \mu, \quad \forall \mu \in M_{k;\delta_k},
\end{aligned}
$$

and because, since $\langle u_0 \rangle_k$ is a linear combination of rigid body motions e_k^i, we have from (18):

$$\int_{\Gamma_{0k}} \left(\langle u_0 \rangle_k - \mathring{\pi}_k \langle u_0 \rangle_k \right) \cdot \mu = 0, \quad \forall \mu \in M_{k;\delta_k}.$$

On the other hand, we have for $k \in I_2$:

$$\begin{aligned} a_k(u_k, u_k) &\leq 2a_k \left(u_k - \mathring{\pi}_k \langle u_0 \rangle_k , u_k - \mathring{\pi}_k \langle u_0 \rangle_k \right) \\ &\quad +2a_k \left(\mathring{\pi}_k \langle u_0 \rangle_k , \mathring{\pi}_k \langle u_0 \rangle_k \right). \end{aligned} \tag{27}$$

Using the same argument as in (20), we get by construction of u_k:

$$\begin{aligned} a_k \left(u_k - \mathring{\pi}_k \langle u_0 \rangle_k , u_k - \mathring{\pi}_k \langle u_0 \rangle_k \right) &\leq CC_k \int_{\Omega_k} \left| \nabla \tilde{u}_k \right|^2 \\ &\leq CC_k \left| u_0 - \langle u_0 \rangle_k \right|^2_{H^{1/2}(\Gamma_{0k})^d}, \quad \text{from (26)}, \\ &\leq CC_k \int_{\Omega_k'} \varepsilon(u_0) : \varepsilon(u_0), \end{aligned} \tag{28}$$

from the Sobolev trace theorem and the inequality (21). On the other hand, we have from lemma 2:

$$\begin{aligned} a_k \left(\mathring{\pi}_k \langle u_0 \rangle_k , \mathring{\pi}_k \langle u_0 \rangle_k \right) &\leq 2a_k \left(\mathring{\pi}_k \left(u_0 - \langle u_0 \rangle_k \right), \mathring{\pi}_k \left(u_0 - \langle u_0 \rangle_k \right) \right) \\ &\quad +2a_k \left(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0 \right) \\ &\leq CC_k \left| u_0 - \langle u_0 \rangle_k \right|^2_{H^{1/2}(\Gamma_{0k})^d} + 2a_k \left(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0 \right) \\ &\leq CC_k \int_{\Omega_k'} \varepsilon(u_0) : \varepsilon(u_0) + 2a_k \left(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0 \right). \end{aligned}$$

We then deduce from (22),(27) and (28):

$$\begin{aligned} a_0(u_0, u_0) + \sum_{k=1}^{K} a_k(u_k, u_k) &\leq a_0(u_0, u_0) + C \sum_{k=1}^{K} C_k \int_{\Omega_k'} \varepsilon(u_0) : \varepsilon(u_0) \\ &\quad +4a_k \left(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0 \right) \\ &\leq \left(4 + \frac{C}{c_0} \max_{k \geq 1}(C_k) \right) \left[a_0(u_0, u_0) + \sum_{k \in I_2} a_k \left(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0 \right) \right] \\ &= \left(4 + \frac{C}{c_0} \max_{k \geq 1}(C_k) \right) \left\langle \hat{\mathbf{D}}_0 U_0, U_0 \right\rangle. \end{aligned}$$

We deduce from (24) and from the mortar conditions satisfied by the $(u_k)_{k \geq 1}$, that:

$$\langle \mathbf{D}_0 U_0, U_0 \rangle \leq \left(4 + \frac{C}{c_0} \max_{k \geq 1}(C_k) \right) \left\langle \hat{\mathbf{D}}_0 U_0, U_0 \right\rangle.$$

□

In the above proof, we have used the following lemma:

Lemma 2. *If a_k is symmetric, the projection operator $\mathring{\pi}_k$ satisfies:*

$$a_k\left(\mathring{\pi}_k w, \mathring{\pi}_k w\right) \leq C C_k \left[\frac{1}{L_k}\int_{\Gamma_{0k}} \langle w\rangle_k^2 + |w|^2_{H^{1/2}(\Gamma_{0k})^d}\right]$$

Proof : Let $\tilde{w}$ be a lifting function of w with zero trace on Γ_D, with $\tilde{w} = w$ on Γ_{0k} and satisfying the Sobolev lifting theorem:

$$\int_{\Omega_k} |\nabla \tilde{w}|^2 \leq C\left[\frac{1}{L_k}\int_{\Gamma_{0k}} \langle w\rangle_k^2 + |w|^2_{H^{1/2}(\Gamma_{0k})^d}\right].$$

Let us define as before $\tilde{w}_k = \mathcal{I}_{k;h_k}\tilde{w} + \mathcal{R}_{k;\delta_k}\pi_k(\tilde{w} - \mathcal{I}_{k;h_k}\tilde{w})$ which belongs to $X_{k;h_k}$ and which satisfies by construction:

$$\int_{\Gamma_{0k}} \tilde{w}_k \cdot \mu = \int_{\Gamma_{0k}} \tilde{w}\cdot\mu, \quad \forall \mu \in M_{k;\delta_k}. \tag{29}$$

We then have on one hand:

$$\begin{aligned} a_k(\tilde{w}_k, \tilde{w}_k) &= a_k(\mathring{\pi}_k w, \mathring{\pi}_k w) + a_k(\mathring{\pi}_k w - \tilde{w}_k, \mathring{\pi}_k w - \tilde{w}_k) \\ &\quad +2a_k(\mathring{\pi}_k w, \mathring{\pi}_k w - \tilde{w}_k). \end{aligned} \tag{30}$$

Developing $\mathring{\pi}_k w_k$ into $\mathring{\pi}_k w_k = \sum_{j=1}^{N_k} z_j u_k^j$, we have from (11).1

$$\begin{aligned} a_k(\mathring{\pi}_k w_k - \tilde{w}_k, \mathring{\pi}_k w_k) &= \sum_{j=1}^{N_k} z_j a_k(\mathring{\pi}_k w - \tilde{w}_k, u_k^j) \\ &= \sum_{j=1}^{N_k} z_j \int_{\Gamma_{0k}} (\mathring{\pi}_k w - \tilde{w}_k)\cdot \lambda_k^j \\ &= \sum_{j=1}^{N_k} z_j \left[\int_{\Gamma_{0k}} w\cdot\lambda_k^j - \int_{\Gamma_{0k}} w \cdot \lambda_k^j\right], \quad \text{from (16).2 and (29)} \\ &= 0. \end{aligned}$$

Plugged back in (30), this implies:

$$a_k(\mathring{\pi}_k w, \mathring{\pi}_k w) \leq a_k(\tilde{w}_k, \tilde{w}_k).$$

But on the other hand, proceeding as in (20), we have:

$$a_k(\tilde{w}_k, \tilde{w}_k) \leq C C_k \int_{\Omega_k} |\nabla\tilde{w}|^2 \leq C C_k \left[\frac{1}{L_k}\int_{\Gamma_{0k}} \langle w\rangle_k^2 + |w|^2_{H^{1/2}(\Gamma_{0k})^d}\right]$$

the last inequality coming from the Sobolev lifting theorem. This concludes the proof. □

4.3.1 Bound on condition number

We prove now a classical result, using for example the technique from the Matsokin-Nepomniaschik [MN85] framework :

Proposition 4. *Let us assume that there exist two positive quantities $W_{1,h}, W_{2,h}$ such that for all U_0 :*

$$W_{1,h} \langle \mathbf{D}_0 U_0, U_0 \rangle \leq \left\langle \hat{\mathbf{D}}_0 U_0, U_0 \right\rangle \leq W_{2,h} \langle \mathbf{D}_0 U_0, U_0 \rangle . \tag{31}$$

Then, the condition number of $\mathbf{C}^{-1}\mathbf{A}$ in $\mathbf{A}$-norm on E admits the following upper bound:

$$\kappa_{\mathbf{A},E}(\mathbf{C}^{-1}\mathbf{A}) \leq \frac{\max(1, W_{2,h})}{\min(1, W_{1,h})}.$$

We conclude by the main result of that section, which gives an upper bound on the condition number of the preconditioned systems:

Proposition 5. *For the symmetrized Dirichlet-Neumann preconditioner given in section 3.2.1, we have:*

$$\kappa_{\mathbf{A},E}(\mathbf{C}^{-1}\mathbf{A}) \leq 1 + C \left(\max_{k \in I_1} \frac{C_k}{c_0} + \max_{k \in I_2} \frac{C_k L_0}{\alpha_0 L_k} \right),$$

and for the enhanced Dirichlet-Neumann preconditioner given in section 3.2.2:

$$\kappa_{\mathbf{A},E}(\mathbf{C}^{-1}\mathbf{A}) \leq C \left(1 + \max_{k \in I_1 \cup I_2} \frac{C_k}{c_0} \right).$$

Both condition numbers are independent of the number K of fine scale subdomains and of their sizes. In that sense, we can reasonnably talk of two-scale preconditioners. The simplest symmetrized Dirichlet-Neumann preconditioner, which imposes the invertibility of $\mathbf{A}_0$ (i.e. a Dirichlet boundary condition on Ω_0 for example), is strongly affected by the presence of small subdomains that are fixed on a part of their boundary, through the ratio L_0/L_k. The enhanced symmetrized Dirichlet-Neumann preconditioner avoids efficiently this dependence, and its use is not limited to the case where $\Gamma_D \cap \partial\Omega_0$ has a positive measure. Nevertheless, both condition numbers are affected by the presence of stiff fine subdomains in comparison with the coarse domain, through the presence of the ratio C_k/α_0 because C_k (resp. α_0) is proportional to the Young modulus E_k (resp E_0) of the material in Ω_k (resp. Ω_0).

5 Algorithm

Before testing these two preconditioners, we summarize herein the algorithm coming from their application. The action of a preconditioner on a right hand side

$$\begin{pmatrix} F_0 \\ F_1 \\ 0 \\ \vdots \\ F_K \\ 0 \end{pmatrix}$$

in the dual of E leads to the following sequence of operations:

1. Compute the equivalent coarse scale sollicitation on Ω_0:

$$\overline{F_0} = F_0 - \sum_{k=1}^{K} \mathbf{B}_{0k}^t R_k \mathbf{K}_k^{-1} \begin{pmatrix} F_k \\ 0 \end{pmatrix},$$

   by solving in parallel one Dirichlet problem by small subdomain.

2. Use the equivalent coarse scale operator $\hat{\mathbf{D}}_0$ to determine:

$$\tilde{U}_0 = \hat{\mathbf{D}}_0^{-1} \overline{F_0}.$$

3. Solve the local problems for $1 \leq k \leq K$:

$$\mathbf{K}_k \begin{pmatrix} \tilde{U}_k \\ \tilde{\Lambda}_k \end{pmatrix} = \begin{pmatrix} F_k \\ -\mathbf{B}_0 \tilde{U}_0 \end{pmatrix}.$$

If the computational cost of $\mathbf{A}_k^{-1}$ for $k \geq 1$ is low with respect to the one of $\mathbf{A}_0^{-1}$, the calculation cost is concentrated in the step 2.

Remark 3. *This preconditioner is multiplicative, in the sense that the two scales cannot be solved simultaneously. Nevertheless, the solutions over the small details can be performed simultaneously in parallel.*

6 Numerical tests

6.1 A basic two-scale model

Let us consider a two-scale linear model beam whose tips are clamped. We impose a negative constant pressure on the lower face of the small details. A $\mathbb{Q}_1$ approximation is adopted for displacements, and an example of the resulting deformed configuration of our model is represented on figure 3. The Young modulus and the Poisson coefficient are taken constant over the coarse (E_0, ν_0) and the fine (E', ν') subdomains. As assumed above, the non-mortar side is taken as the fine side of the interface and Lagrange multipliers are taken piecewise constant, together with an interface bubble stabilization for the displacements (see [Hau04]). Moreover, the weak-continuity constraint is ensured by a penalization strategy and the associated penalization coefficient is taken as:

$$\frac{1}{\eta} = 10^6 E'.$$

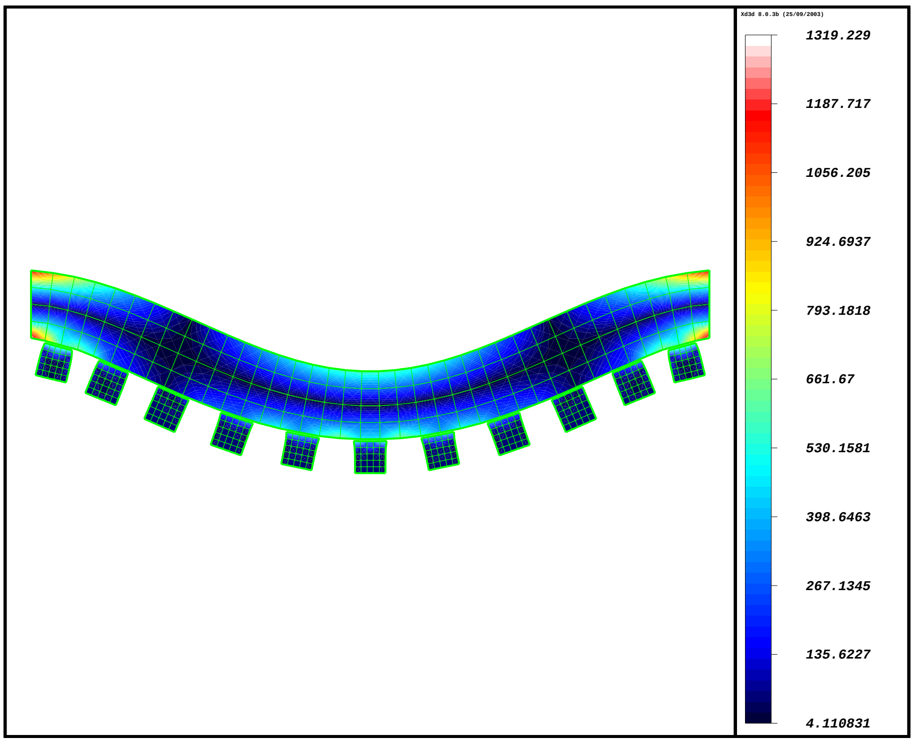

Figure 3: Maximal stress distribution on a deformed configuration of our two-scale model problem ($E_0 = E'$, $\nu_0 = \nu'$, 497 elements mesh).

On this model, we use the first symmetrized Dirichlet-Neumann preconditioner in a standard Conjugate Gradient algorithm, and the L^2 norm of the successive increments on Lagrange multipliers along the iterations is illustrated on figure 4 for different values of the ratio $r = E'/E_0$. Conversely the number of iterations necessary to obtain a 10^{-9} convergence, estimated in terms of the L^2 norm of the current increment on the Lagrange multiplier, is represented on figure 5. The degradation of the performance as r grows is in conformity with our predictions.

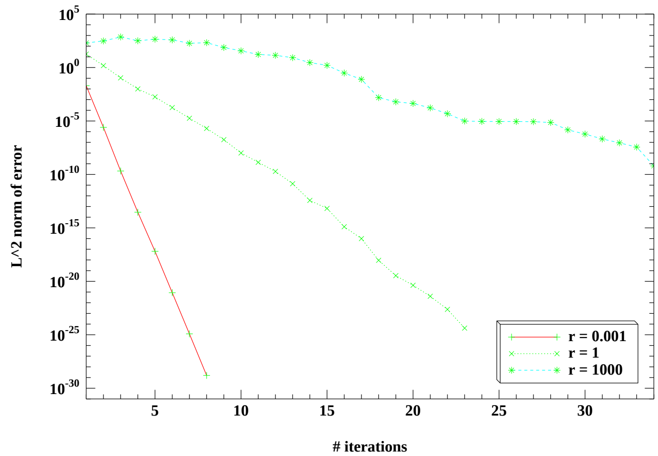

Figure 4: L^2 norm of the successive increments on Lagrange multipliers along the iterations.

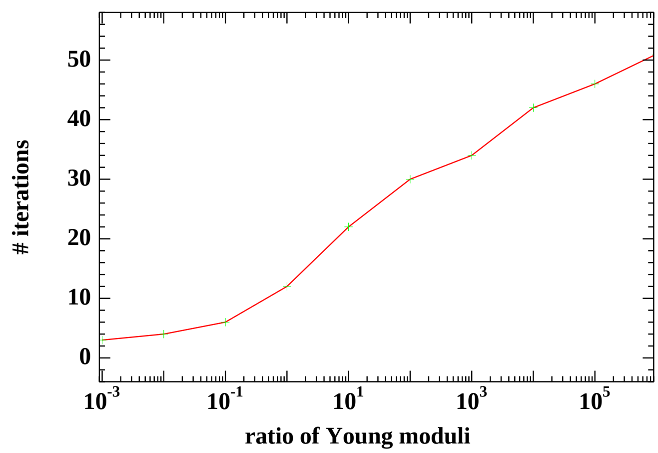

Figure 5: Number of iterations necessary to obtain a 10^{-9} convergence of the simple Dirichlet-Neumann preconditioned Conjugate Gradient, estimated in terms of the L^2 norm of the current increment on the Lagrange multiplier, as a function of the ratio $r = E'/E_0$.

Let us assume now, that two of the details are clamped on their lower face, leading under the same load to the new deformed configuration illustrated on figure 6. The convergence of simple and enhanced Dirichlet-Neumann algorithms are then compared on figure 7 for the ratios $r = 10$, 100, 1000, 10^6. Conversely, the number of iterations necessary to reach a 10^{-9} convergence as a function of r is represented on figure 8 both for simple and enhanced

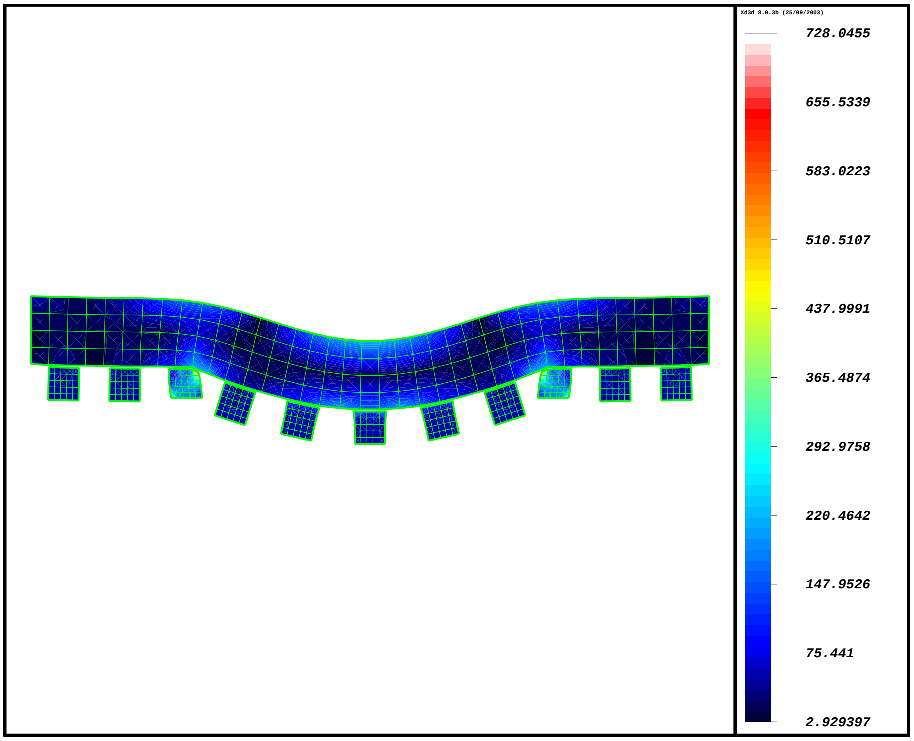

Figure 6: Maximal stress distribution on a deformed configuration of our two-scale model problem where two of the details are clamped on their lower face ($E_0 = E'$, $\nu_0 = \nu'$, 497 elements mesh).

Dirichlet-Neumann algorithms. We observe a much better performance of the enhanced preconditioner, the number of iterations being typically divided by 3 for an additional computational cost of 6 additional degrees of freedom on the coarse part of the model. Indeed, 3 rigid motions per clamped small structure have been added to the coarse model. The resulting overcost per iteration in terms of computation is negligible.

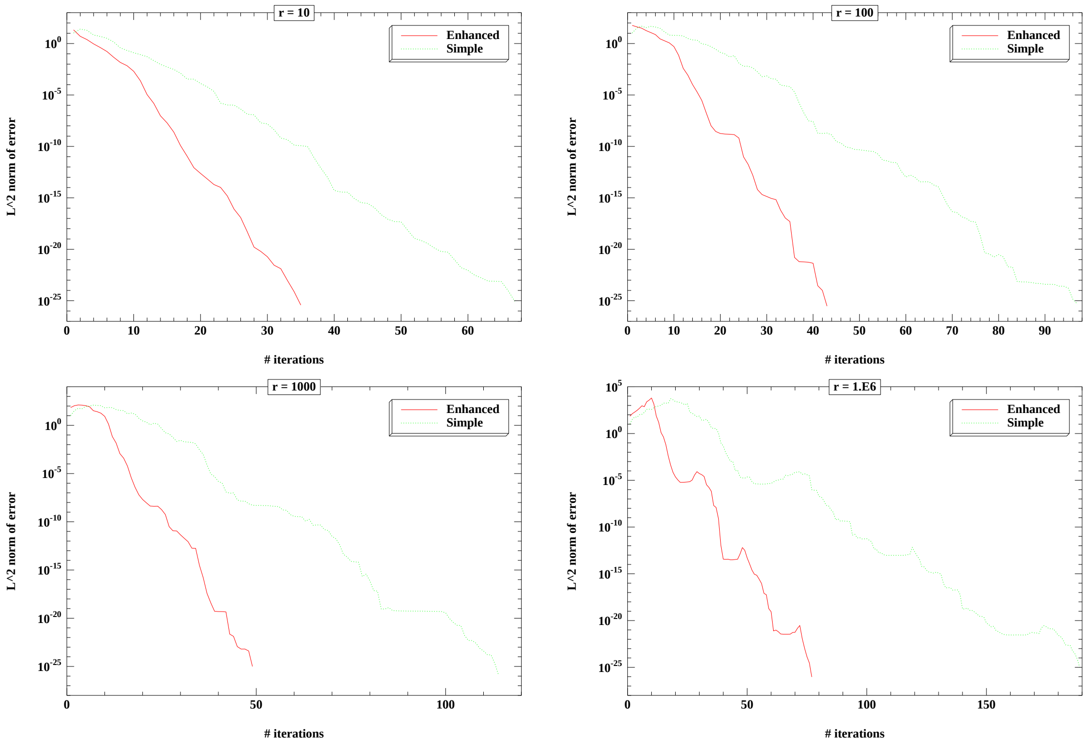

Figure 7: Convergence of the simple and enhanced Dirichlet-Neumann algorithms for different values of the ratio r of Young moduli.

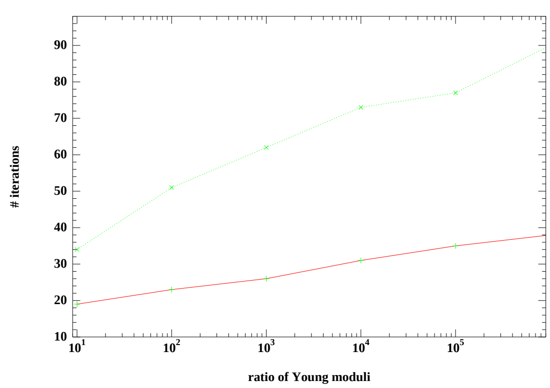

Figure 8: Number of iterations necessary to obtain a 10^{-9} convergence of the simple and the enhanced Dirichlet-Neumann preconditioned Conjugate Gradient, estimated in terms of the L^2 norm of the current increment on the Lagrange multiplier, as a function of the ratio $r = E'/E_0$.

6.2 Extension to a quasi-Newton method

When considering nonlinear problems with soft fine geometrical details on the boundary, the previous preconditioners can be successfully applied to quasi-Newton methods. Instead of solving each tangent problem by a preconditioned Conjugate Gradient method, the idea is to replace the tangent problems by the preconditioning problems. From the implementation point of view, it is no more necessary to keep in memory the non-inverted matrix of the tangent problem. Moreover, the numerical tests show that this strategy entails almost no overcost in terms of iterations of the Newton method.

For example, let us consider the following elastostatics problem:

$$\begin{cases} -\, div\, \dfrac{\partial \hat{\mathcal{W}}}{\partial F}(id + \nabla u) = f, \quad \Omega, \\ u = 0, \quad \Gamma_D, \\ \dfrac{\partial \hat{\mathcal{W}}}{\partial F}(id + \nabla u) \cdot n = g, \quad \Gamma_N. \end{cases}$$

Let us assume that the potential $\hat{\mathcal{W}}$ is given by the Saint-Venant-Kirchhoff constitutive law defined by:

$$\hat{\mathcal{W}}(F) = \frac{\lambda}{4} \left[tr(F^t \cdot F - id) \right]^2 + \frac{\mu}{8}\, tr \left[(F^t \cdot F - id)^2 \right].$$

After a non-conforming finite element discretization, we have then to solve a nonlinear discrete problem of the form:

$$\begin{cases} \mathcal{F}_0(U_0) + \sum_{k=1}^K \mathbf{B}_{0k}\Lambda_k = F_0, \\ \mathcal{F}_k(U_k) - \mathbf{B}_k^t \Lambda_k = F_k, \quad 1 \le k \le K, \\ \mathbf{B}_{0k} U_0 - \mathbf{B}_k U_k = 0, \quad 1 \le k \le K. \end{cases}$$

A standard Newton algorithm would build two sequences $(U^n)_n$ and $(\Lambda^n)_n$ such that:

$$\begin{cases} U^{n+1} = U^n + \delta U^n, \\ \Lambda^{n+1} = \Lambda^n + \delta\Lambda^n, \end{cases}$$

with:

$$\begin{cases} \partial_{U_0}\mathcal{F}_0(U_0^n) \cdot \delta U_0^n + \displaystyle\sum_{k=1}^K \mathbf{B}_{0k}\delta\Lambda_k^n = F_0 - \mathcal{F}_0(U_0^n) - \sum_{k=1}^K \mathbf{B}_{0k}\Lambda_k^n, \\ \partial_{U_k}\mathcal{F}_k(U_k^n) \cdot \delta U_k^n - \mathbf{B}_k^t \Lambda_k^n = F_k - \mathcal{F}_k(U_k^n) + \mathbf{B}_k^t \Lambda_k^n, \quad 1 \le k \le K, \\ \mathbf{B}_{0k}\delta U_0^n - \mathbf{B}_k \delta U_k^n = 0, \quad 1 \le k \le K. \end{cases}$$

At iteration n, this linear system can then be written as follows:

$$\mathbf{A}\begin{pmatrix} \delta U_0^n \\ \delta U_1^n \\ \delta \Lambda_1^n \\ \vdots \\ \delta U_K^n \\ \delta \Lambda_K^n \end{pmatrix} = \begin{pmatrix} F_0^n \\ F_1^n \\ 0 \\ \vdots \\ F_K^n \\ 0 \end{pmatrix}.$$

We propose to define the new increments $\delta\tilde{U}^n$ and $\delta\tilde{\Lambda}^n$ as the solutions of:

$$\mathbf{C}\begin{pmatrix} \delta \tilde{U}_0^n \\ \delta \tilde{U}_1^n \\ \delta \tilde{\Lambda}_1^n \\ \vdots \\ \delta \tilde{U}_K^n \\ \delta \tilde{\Lambda}_K^n \end{pmatrix} = \begin{pmatrix} F_0^n \\ F_1^n \\ 0 \\ \vdots \\ F_K^n \\ 0 \end{pmatrix},$$

with the same notations used in section 2. Our two-scale quasi-Newton method is then defined by:

$$\begin{cases} U^{n+1} = U^n + \delta\tilde{U}^n, \\ \Lambda^{n+1} = \Lambda^n + \delta\tilde{\Lambda}^n. \end{cases}$$

Let us consider the same model problem as in the previous section, under a dead pressure of $p = 100Pa$. We have adopted the following Lamé coefficients:

$$\lambda_0 = E_0 \frac{\nu_0}{(1+\nu_0)(1-2\nu_0)} = 1389Pa, \quad \mu_0 = \frac{E_0}{2(1+\nu_0)} = 2083Pa,$$

$$\lambda' = r\lambda_0, \qquad \mu' = r\mu_0 Pa,$$

respectively for the coarse and the fine subdomains, characterized by the stiffness ratio:

$$r = \frac{E_0}{E'} = \frac{\lambda_0}{\lambda'} = \frac{\mu_0}{\mu'}.$$

The solution remains unchanged when p, λ_0, E_0, λ' and μ' are multiplied by the same coefficient. We have observed numerically that for $r \geq 10$, the quasi-Newton method does not converge well, as shown on the table on figure 9. Whereas, the convergence becomes extremely slow with $r = 1$, the method does not converge any more with $r = 100$. The convergence of the Newton-Raphson method is represented as a comparison. Nevertheless, when the ratio r remains sufficiently small, the proposed quasi-Newton method appears to be interesting, enven though the convergence is no more quadratic. The overcost in terms

	$r = 1$		$r = 100$	
it.	quasi-Newton	Newton	quasi-Newton	Newton
1	0.6193E+01	0.5839E+01	0.6187E+01	0.5224E+01
2	0.1904E+01	0.1649E+01	0.1380E+02	0.1401E+01
3	0.1013E+01	0.9821E+00	0.6958E+03	0.7683E+00
4	0.6684E+00	0.6221E+00	0.1283E+04	0.4046E+00
5	0.3309E+00	0.3032E+00	0.3672E+04	0.2419E+00
6	0.8885E-01	0.8811E-01	0.1847E+04	0.1454E+00
7	0.4654E-02	0.8719E-02	0.9162E+03	0.1096E+00
8	0.5162E-02	0.1591E-03	0.6159E+03	0.5302E-01
9	0.4352E-02	0.8287E-07	0.1027E+04	0.5350E-01
10	0.3714E-02		0.6719E+03	0.7019E-02
11	0.3155E-02		0.8720E+03	0.3277E-02
12	0.2716E-02		0.5561E+03	0.3023E-04
13	0.2334E-02		0.6285E+03	0.3357E-07
14	0.2023E-02		0.8873E+03	
15	0.1753E-02		0.5120E+03	
16	0.1528E-02		0.5499E+03	
17	0.1333E-02		0.6496E+03	
18	0.1167E-02		0.9376E+03	
19	0.1023E-02		0.3581E+03	
20	0.8981E-03		0.3805E+03	
21	0.7895E-03		0.5372E+03	
22	0.6950E-03		0.8865E+03	
23	0.6123E-03		0.7312E+03	
24	0.5399E-03		0.7739E+03	
25	0.4764E-03		0.7279E+03	

Figure 9: Slow convergence of the method for $r = 1$, and lack of convergence for $r = 100$.

of iterations compared with a Newton-Raphson method is low, as shown in the table, on figure 10. Finally, we represent on figure 11 the different evolutions of the L^2 norm of the residual for the proposed quasi-Newton method along the iterations, depending on the value of the ratio r.

This kind of quasi-Newton Dirichlet-Neumann strategy has been recently used with success in fluid-structure interactions problems and specially hemodynamics, as developped in [GV03] where a simplified model for the fluid is adopted in the preconditioner.

it.	L^2 norm of the residual with Newton algorithm	two-scale quasi-Newton
1	0.6192E+01	0.6249E+01
2	0.1775E+01	0.1811E+01
3	0.1061E+01	0.1075E+01
4	0.6671E+00	0.6747E+00
5	0.3254E+00	0.3292E+00
6	0.8096E-01	0.7836E-01
7	0.5036E-02	0.3414E-02
8	0.2010E-04	0.8871E-05
9	0.3750E-09	0.5000E-06
10	converged	0.1387E-07
11	converged	0.3629E-08

Figure 10: Convergence of the exact Newton and two-scale quasi-Newton algorithm using the preconditioner (10). We have chosen $E_0/E' = 10$ and the convergence criterion is that the L^2 norm of the residual become $\leq 10^{-9}$.

7 Conclusion

In this paper, we have introduced, analyzed and tested two symmetrized Dirichlet-Neumann preconditioners that can be used efficiently together with a non-conforming mortar formulation to solve elliptic problems with small geometrical details on the boundary. This method is well-adapted to the case where the details are localized enough to make their resolution relatively cheap. In the case where the small structures would not be so localized to satisfy this assumption, one can imagine a Neumann-Neumann domain decomposition approach [TRV91] to solve the Dirichlet part of the present Dirichlet-Neumann method. Finally, we have deduced a quasi-Newton method which is well-adapted for soft details in the framework of nonlinear problems.

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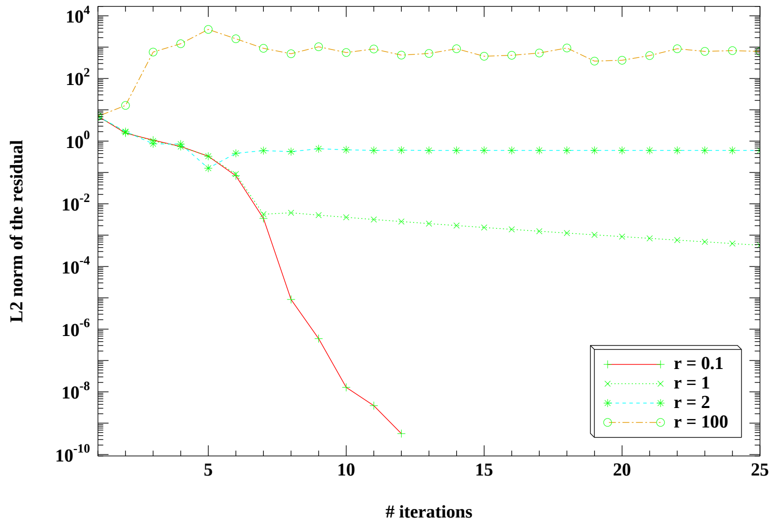

Figure 11: Evolutions of the L^2 norm of the residual for the proposed quasi-Newton method along the iterations, depending on the value of the ratio r.

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