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A stabilized discontinuous mortar formulation for elastostatics and elastodynamics problems Part I: abstract framework

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Abstract

In this paper, we first recall the general assumptions and results arising in mortar methods applied to elastostatics [Woh01]. By extension to the curved interfaces case of the ideas from Gopalakrishnan and Brenner [Gop99, Bre03, Bre04], and from the introduction a generalized Scott and Zhang interpolation operator [SZ90], we prove the independence of the coercivity constant of the broken elasticity bilinear form with respect to the number and the size of the subdomains. Moreover, we extend the proof of optimal convergence to the elastodynamic framework. The present results are applied in Part II (discontinuous Lagrange multipliers), in which a stabilized discontinuous formulation is proposed, analyzed and tested.

1 Introduction

In this paper (Part I: abstract framework) and the following (Part II: discontinuous Lagrange multipliers), we introduce, analyze and test a non-conforming formulation using stabilized discontinuous mortar elements to find the vector solution u of linearized elasticity problems such as:

$$\begin{cases} -\,div(\mathbf{E} : \varepsilon(u)) = f, \quad \Omega \subset \mathbb{R}^d, (d = 2, 3) \\ u = 0, \quad \Gamma_D, \\ (\mathbf{E} : \varepsilon(u)) \cdot n = g, \quad \Gamma_N, \end{cases} \tag{1}$$

where the linearized strain tensor is classically given by:

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla^t u\right),$$

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and the fourth order elasticity tensor $\mathbf{E}$ is assumed to be elliptic over the set of symmetric matrices:

$$\exists \alpha > 0, \forall \xi \in \mathbb{R}^{d\times d}, \xi^t = \xi, \quad (\mathbf{E} : \xi) : \xi \geq \alpha\, \xi : \xi.$$

The analysis is also extended to the elastodynamics problem:

$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} - div(\mathbf{E} : \varepsilon(u)) = f, \quad [0,T] \times \Omega, \\ u = 0, \quad [0,T] \times \Gamma_D, \\ (\mathbf{E} : \varepsilon(u)) \cdot n = g, \quad [0,T] \times \Gamma_N, \\ u = u_0, \quad \{0\} \times \Omega, \\ \dfrac{\partial u}{\partial t} = \dot{u}_0, \quad \{0\} \times \Omega, \end{cases} \tag{2}$$

and we consider this analysis as a theoretical background for using discontinuous mortar elements in nonlinear elastodynamics.

Mortar methods have been introduced for the first time in [BMP93, BMP94] as a weak coupling between subdomains with nonconforming meshes, or between subproblems solved with different approximation methods. The main purpose was to overcome the very sub-optimal "$\sqrt{h}$" error estimate obtained with pointwise matching. The analysis of this method as a mixed formulation was first made in [Bel99].

Nevertheless, in spite of the optimal error convergence obtained with the original mortar elements, some numerical difficulties appear. First, the original space of Lagrange multipliers ensuring the weak coupling is rather difficult to build in 3D on the boundary of the interfaces when more than two subdomains have a common intersection (see [BM97, BD98]). Moreover, the original constrained space has a non-local basis on the non-conforming artificial interfaces, which may lead to small spurious oscillations of the approximate solution.

To overcome the first difficulty, one idea is given in [Ses98] when displacements are at least approximated by second order polynomials. The introduced Lagrange multipliers have a lower order, still enabling optimal error estimates, and no special treatment is needed on the boundary of the interfaces. To overcome the second difficulty, dual mortar spaces are proposed in [Woh00, Woh01], enabling the localization of the mortar kinematical constraint. In order to benefit from the advantages of these two approaches, we propose to introduce stabilized low order discontinuous mortar elements. This idea has already been introduced for a first order three-field mortar formulation in [BM00], and we exploit it herein in the two-field framework for first and second order elements when dealing with elastostatics and elastodynamics problems.

Mortar formulations also provide a natural framework for domain decomposition, as observed by [Tal93, AKP95, AMW99, AAKP99, Ste99] and the references therein. A large number of subdomains and their small size is therefore a basic difficulty to overcome. To get an optimal use of such domain decomposition methods, it is then crucial that the constants arising in the analysis of the mortar formulation remain independent (or at least weakly dependent) on the number and the size of the subdomains. One can readily check that the only potential dependence on such parameters is hidden in the coercivity constant of the broken bilinear form associated to the linearized elastostatics problem. In the framework of elliptic scalar problems, both [Gop99, Bre03] and [BM00] have shown the independence of the coercivity constant with respect to the number and the size of the subdomains, respectively when considering two and three-field mortar formulations with plane interfaces. An extension to the vector elasticity case has been proposed by [Bre04]. By definition of a generalized Scott and Zhang [SZ90] interpolation operator, we simplify and extend herein the result to potentially curved interfaces.

In section 2, the fundamental assumptions and results arising in mortar element methods to approximate the solution of the elastostatics problem (1) are recalled. Well-posedness results are recalled in section 3, and we prove the independence of the coercivity constant with respect to the number and the size of the subdomains in section 4. In section 5, we recall the optimal convergence of the method by mesh refinement, and generalize the analysis to the elastodynamics problem (2) in section 6.

The second paper (Part II: discontinuous Lagrange multipliers) proposes the analysis of stabilized discontinuous mortar elements, proving the satisfaction of the fundamental assumptions. Some practical issues are pointed out: the choice of an appropriate penalization term, and the exact integration of the constraint. Numerical tests are also presented to confirm the analysis.

2 Nonconforming setting

2.1 Position of the problem

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$), be an open set partitioned into K subsets $(\Omega_k)_{1\leq k\leq K}$. We denote by $\gamma_{kl} = \overline{\Omega_k} \cap \overline{\Omega_l}$ the interface between Ω_k and Ω_l, and the skeleton of the internal interfaces is denoted by $\mathcal{S} = \bigcup_{k,l\geq 1} \gamma_{kl}$. On the part Γ_D of the boundary $\partial\Omega$, an homogeneous Dirichlet boundary condition is imposed. Concerning the coefficients of the fourth order elasticity tensor $\mathbf{E}$, we assume that the stress tensor is symmetric whatever the deformation is in the material, namely for almost all $x \in \Omega$:

$$\forall \xi \in \mathbb{R}^{d\times d}, \xi^t = \xi, \quad \mathbf{E}(x) : \xi \text{ is a symmetric matrix.}$$

Moreover, in the theoretical analysis, we will suppose that for all $k \geq 1$, there exists two constants c_k and C_k, such that for almost all $x \in \Omega_k$:

$$\forall \xi \in \mathbb{R}^{d\times d}, \xi^t = \xi, \quad c_k\, \xi : \xi \leq (\mathbf{E}(x) : \xi) : \xi \leq C_k\, \xi : \xi. \tag{3}$$

If the material of the subdomain Ω_k has a Young modulus E_k, both c_k and C_k are proportional to E_k.

We introduce the following spaces:

$$H^1_*(\Omega) = \{v \in H^1(\Omega)^d,\ v|_{\Gamma_D} = 0\},$$

$$H^1_*(\Omega_k) = \{v \in H^1(\Omega_k)^d,\ v|_{\Gamma_D\cap\partial\Omega_k} = 0\},$$

$$X = \left\{v \in L^2(\Omega)^d, \quad v_k = v|_{\Omega_k} \in H^1_*(\Omega_k), \forall k\right\} = \prod_{k=0}^{K} H^1_*(\Omega_k),$$

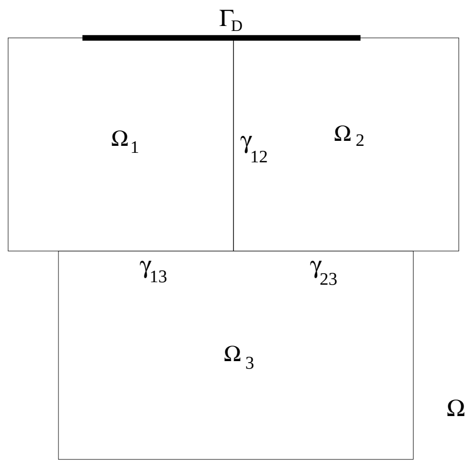


Figure 1: A decomposition of Ω into subdomains.

X being endowed with the H^1 broken norm:

$$\|v\|_X = \left(\sum_{k=0}^{K} \|v\|^2_{H^1(\Omega_k)^d} \right)^{\frac{1}{2}}.$$

Here, in order to be scale independent when dealing with a large number of subdomains, we use a scale invariant definition of the H^1 norm:

$$\|v\|^2_{H^1(\Omega_k)^d} = \frac{1}{(L_k)^2} \|v\|^2_{L^2(\Omega_k)^d} + \|\nabla v\|^2_{L^2(\Omega_k)^{d\times d}},$$

L_k being a characteristic length of Ω_k, for instance its diameter.
We are interested in finding $u \in H^1_*(\Omega)$ such that:

$$a(u, v) = l(v), \quad \forall v \in H^1_*(\Omega), \tag{4}$$

where the continuous coercive bilinear form a is defined by:

$$a(u, v) = \int_\Omega (\mathbf{E} : \varepsilon(u)) : \varepsilon(v), \quad \forall u, v \in H^1_*(\Omega),$$

and the continuous linear form l by:

$$l(v) = \int_\Omega f \cdot v + \int_{\Gamma_N} g \cdot v, \quad \forall v \in H^1_*(\Omega).$$

This problem is classically well-posed by Lax-Milgram lemma, the Korn's inequality (see [DL72]) ensuring the coercivity of the bilinear form a over $H^1_*(\Omega) \times H^1_*(\Omega)$.

2.2 Discretization

We introduce here a non-conforming discretization of the problem (4) using mortar elements to be further defined later on. The discrete problem is proved to be well-posed and error estimates are derived in the mesh-dependent norms already introduced and used in [AT95, Woh99].

2.2.1 The mesh

For each $1 \leq k \leq K$, we consider a family of shape regular *affine meshes* $(\mathcal{T}_{k;h_k})_{h_k>0}$ on the subdomain Ω_k. This means that each element T is the image of a reference element $\hat{T}$ by an affine mapping J_T. For each $T \in \mathcal{T}_{k;h_k}$, we will denote its diameter:

$$h(T) = diam(T),$$

and the local mesh size by:

$$h_k = \sup_{T \in \mathcal{T}_{k;h_k}} h(T).$$

Then, a nonconforming family of domain based meshes $(\mathcal{T}_h)_{h>0}$ over Ω is obtained by:

$$\mathcal{T}_h = \bigcup_{k=1}^{K} \mathcal{T}_{k,h_k}, \quad h = \max_{1 \leq k \leq K} h_k.$$

The skeleton $\mathcal{S} = \bigcup_{k,l \geq 1} \gamma_{kl}$ is partitioned into M interfaces $(\Gamma_m)_{1 \leq m \leq M}$, and can then be decomposed as $\mathcal{S} = \bigcup_{1 \leq m \leq M} \Gamma_m$. Moreover, we assume that for each $1 \leq m \leq M$, there exists at least one domain Ω_k with $k \geq 1$ such that $\Gamma_m \subset \partial\Omega_k$, and denote $k(m) := k$ the name of one of these subdomains, taken once for all for each interface. This side will said to be the non-mortar (or slave) side.

For each $1 \leq m \leq M$, Γ_m inherits a family of meshes $(\mathcal{F}_{m;\delta_m})_{\delta_m>0}$, obtained as the trace of the volumic mesh $(\mathcal{T}_{k(m);h_{k(m)}})_{h_{k(m)}>0}$ of the slave subdomain over Γ_m. We have denoted by:

$$\delta_m = \sup_{F \in \mathcal{F}_{m;\delta_m}} h(F).$$

We also denote by $\overline{\delta}_m$ the size of the mesh on the mortar side:

$$\overline{\delta}_m = \sup_{T \in \mathcal{T}_{l;h_l}, l \neq k(m)} diam(T \cap \Gamma_m).$$

Then, a family of interface meshes $(\mathcal{F}_\delta)_{\delta>0}$ can be defined over $\mathcal{S}$ by:

$$\mathcal{F}_\delta = \bigcup_{m=1}^{M} \mathcal{F}_{m;\delta_m}, \quad \delta = \max_{1 \leq m \leq M} \delta_m.$$

For each $F \in \mathcal{F}_{m;\delta_m}$, we denote by $T(F) \in \mathcal{T}_{k(m);h_{k(m)}}$ the unique element $T \in \mathcal{T}_{k(m);h_{k(m)}}$ such that $T \cap \mathcal{S} = F$.
Moreover, the following assumption is made:

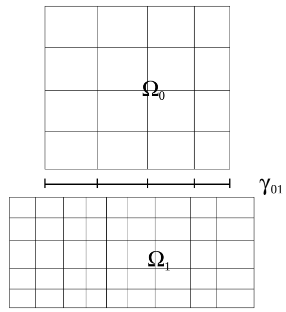


Figure 2: A situation where the mesh $\mathcal{F}_{1;\delta_1}$ of the interface γ_{01} is inherited from the mesh $\mathcal{T}_{0;h_0}$ of Ω_0. The assumption 1 would be violated if at the opposite, Ω_1 were the slave side.

Assumption 1. *$F \in \mathcal{F}_\delta$ is always an entire face of $T(F) \in \mathcal{T}_h$.*

In other words, the construction of the interfaces $(\Gamma_m)_{1\leq m\leq M}$ respects the mesh of the slave sides. An example of situation obeying to assumption 1 is given on figure 2.

Remark 1. *For simplicity, the mesh is assumed to be affine but the following results are still valid for regular quasi-uniform quadrangular meshes, at least in 2D (see [GR86]). In fact, the only assumptions to satisfy are the following standard inequalities:*

$$\begin{cases} |\hat{w}|_{H^m(\hat{K})} \leq C\; diam(K)^m meas(K)^{-\frac{1}{2}} |w|_{H^m(K)}, \\ |w|_{H^m(K)} \leq C\; diam(K)^{-m} meas(K)^{\frac{1}{2}} |\hat{w}|_{H^m(\hat{K})}, \end{cases}$$

between the semi-norms of the function w defined on a mesh-element K and its transformation $\hat{w}$ defined on the corresponding reference element $\hat{K}$.

Remark 2. *In the following sections, C will stand for various constants independent of the discretization.*

2.2.2 Interface mesh-dependent spaces

We define here some mesh-dependent trace spaces, endowed with useful mesh-dependent norms already introduced and used in [AT95, Woh99]. For each $1 \leq m \leq M$, they are defined by:

$$\mathbb{H}_\delta^{1/2}(\Gamma_m) = \{\phi \in L^2(\Gamma_m)^d, \|\phi\|^2_{\delta,\frac{1}{2},m} = \sum_{F\in\mathcal{F}_{m;\delta_m}} \frac{1}{h(F)} \|\phi\|^2_{L^2(F)^d} < +\infty\},$$

$$\mathbb{H}_\delta^{-1/2}(\Gamma_m) = \{\lambda \in L^2(\Gamma_m)^d, \|\lambda\|^2_{\delta,-\frac{1}{2},m} = \sum_{F\in\mathcal{F}_{m;\delta_m}} h(F) \|\lambda\|^2_{L^2(F)^d} < +\infty\},$$

endowed respectively with the norms $\|\cdot\|_{\delta,\frac{1}{2},m}$ and $\|\cdot\|_{\delta,-\frac{1}{2},m}$. The product spaces $\mathbb{W}_\delta = \prod_{k=1}^K \mathbb{H}_\delta^{1/2}(\Gamma_m)$ and $\mathbb{M}_\delta = \prod_{k=1}^K \mathbb{H}_\delta^{-1/2}(\Gamma_m)$, are then respectively endowed with the norms:

$$\|\phi\|_{\delta,\frac{1}{2}} = \left(\sum_{m=1}^{M} \|\phi\|^2_{\delta,\frac{1}{2},m}\right)^{1/2},$$

$$\|\lambda\|_{\delta,-\frac{1}{2}} = \left(\sum_{m=1}^{M} \|\lambda\|^2_{\delta,-\frac{1}{2},m}\right)^{1/2}.$$

They can be viewed as dual spaces by means of the the L^2 inner product:

$$\int_{\mathcal{S}} \phi \cdot \lambda \leq \|\lambda\|_{\delta,-\frac{1}{2}} \|\phi\|_{\delta,\frac{1}{2}}, \quad \forall(\phi,\lambda) \in \mathbb{W}_\delta \times \mathbb{M}_\delta. \tag{5}$$

The advantage of such spaces is that the corresponding norms are easily computable, enabling a posteriori estimates [Woh99] and efficient penalization strategies as shown in the second part of this paper.

2.3 Approximate problem

2.3.1 Nonconforming formulation

Let us define the discrete subspaces of degree q inside each subdomain :

$$X_{k;h_k} = \{p \in H^1_*(\Omega_k) \cap \mathcal{C}^0(\Omega_k)^d, \quad p|_T \in \mathcal{P}_q(T), \forall T \in \mathcal{T}_{k;h_k}\} \oplus \mathcal{B}_{k;h_k},$$

with $\mathcal{P}_q = [\mathbb{P}_q]^d$ or $[\mathbb{Q}_q]^d$. We have denoted by $\mathbb{P}_q$ (resp. $\mathbb{Q}_q$) the space of polynomials of total (resp. partial) degree q, and have introduced the possibility of adding a space $\mathcal{B}_{k;h_k}$ of interface bubble stabilization. Examples of such spaces will be introduced in Part II. The corresponding product space is denoted by:

$$X_h = \prod_{k=0}^{K} X_{k;h_k} \subset X.$$

We introduce the following trace spaces on the non-mortar side:

$$W_{m;\delta_m} = \{p|_{\Gamma_m}, p \in X_{k(m);h_{k(m)}}\}, \qquad W^0_{m;\delta_m} = W_{m;\delta_m} \cap H^1_0(\Gamma_m)^d,$$

and the corresponding product space $W^0_\delta = \prod_{m=1}^{M} W^0_{m;\delta_m}$ endowed with the mesh-dependent norm $\|\cdot\|_{\delta,\frac{1}{2}}$.

In order to formulate the weak continuity constraint, the following spaces of discontinuous Lagrange multipliers are defined:

$$M_{m;\delta_m} = \{p \in L^2(\Gamma_m)^d, \quad p|_F \in \mathcal{P}_{q-1}(F), \forall F \in \mathcal{F}_{m,\delta_m}\}, \tag{6}$$

as well as the product space $M_\delta = \prod_{m=1}^{M} M_{m;\delta_m}$, endowed with the mesh-dependent norm $\|\cdot\|_{\delta,-\frac{1}{2}}$ and $M = \prod_{m=1}^{M} L^2(\Gamma_m)^d$. The following continuous bilinear form is then introduced to impose interface weak continuity:

$$\begin{aligned} b:\ \ & X \times M \to \mathbb{R} \\ & (v,\lambda) \mapsto b(v,\lambda) = \sum_{m=1}^{M} \int_{\Gamma_m} [v]_m \cdot \lambda_m, \end{aligned}$$

with $[v]_m = v_{k(m)} - v_l$ denoting the jump on $\gamma_{k(m)l} \subset \Gamma_m$. Then, the constrained space of discrete unknowns can be defined as:

$$V_h = \{u_h \in X_h, b(u_h, \lambda_h) = 0, \quad \forall \lambda_h \in M_\delta\}.$$

In order to formulate the non-conforming approximate problem, it is standard to consider the broken elliptic form:

$$\begin{aligned} \tilde{a} : \quad & X \times X && \to \mathbb{R} \\ & (u, v) && \mapsto \tilde{a}(u, v) = \sum_{k=1}^{K} a_k(u_k, v_k), \end{aligned}$$

with:

$$a_k(u_k, v_k) = \int_{\Omega_k} (\mathbf{E} : \varepsilon(u_k)) : \varepsilon(v_k).$$

We are then interested in finding $(u_h, \lambda_h) \in X_h \times M_\delta$, such that:

$$\begin{cases} \tilde{a}(u_h, v_h) + b(v_h, \lambda_h) = l(v_h), \quad \forall v_h \in X_h, \\ b(u_h, \mu_h) = 0, \quad \forall \mu_h \in M_\delta. \end{cases} \tag{7}$$

In other words, we solve our variational problem on the product space X_h under the weak kinematic continuity constraint $b(\cdot, \cdot) = 0$.

Remark 3. *The theory proposed in Part I also applies to situations involving continuous Lagrange multipliers defined on spaces like:*

$$M_{m;\delta_m} = \{p \in \mathcal{C}^0(\Gamma_m), \quad p|_F \in \mathcal{P}_q(F), \forall F \in \mathcal{F}_{m;\delta_m}\}.$$

2.3.2 Fundamental assumptions

In order to ensure the well-posedness of the problem (7), some fundamental assumptions have to be made. Concerning the compatibility of X_h and M_δ, we assume:

Assumption 2. *For each interface $1 \leq m \leq M$, there exists an operator:*

$$\pi_m : \mathbb{H}_\delta^{1/2}(\Gamma_m) \to W^0_{m;\delta_m},$$

such that for all $v \in \mathbb{H}_\delta^{1/2}(\Gamma_m)$:

$$\int_{\Gamma_m} (\pi_m v) \cdot \mu = \int_{\Gamma_m} v \cdot \mu, \quad \forall \mu \in M_{m;\delta_m},$$

with:

$$\|\pi_m v_m\|_{\delta, \frac{1}{2}, m} \leq C_m \|v\|_{\delta, \frac{1}{2}, m}.$$

This assumption means that the projection perpendicular to the multiplier space onto the trace space $W^0_{m;\delta_m}$ with zero extension is continuous. This assumption will have to be checked for each choice of discretization. Its major consequence lies in the fact that the weak-continuity constraint is onto, as shown in the next section.

Concerning the coercivity of $\tilde{a}$ over $V \times V$, where V is a constrained subspace of X to be defined in that section, we have to consider Lagrange multipliers spaces which are sufficiently rich on the interfaces to kill local rigid motions, defined on $\omega \subset \Omega$ as the elements of:

$$\mathcal{R}(\omega) = \{x \in \mathbb{R}^d \mapsto t + a \times x; \quad t, a \in \mathbb{R}^d\}.$$

For that purpose, we introduce the following assumption over the Lagrange multipliers spaces:

Assumption 3. *For all $1 \leq m \leq M$, we assume that there exists two integers $1 \leq k, l \leq K$ such that $\Gamma_m = \gamma_{kl}$ and a minimal Lagrange multiplier space M_{kl} such that $M_{kl} \subset M_{m;\delta_m}$ independently of the discretization. Moreover, we assume that for all $v \in X$ which is locally a rigid motion both over the subdomains Ω_k and Ω_l, that is $v|_{\Omega_k} \in \mathcal{R}(\Omega_k)$ and $v|_{\Omega_l} \in \mathcal{R}(\Omega_l)$, we have:*

$$\int_{\gamma_{kl}} [v] \cdot \mu = 0 \quad \forall \mu \in M_{kl} \qquad \Longrightarrow [v]_{kl} = 0, \tag{8}$$

where the jump of v over γ_{kl} is denoted by $[v]_{kl}$.

Under assumption 3, the constrained subspace V of X on which the coercivity of the broken bilinear form $\tilde{a}$ holds, is defined as:

$$V = \{v \in X, \quad \int_{\gamma_{kl}} [v] \cdot \mu = 0, \quad \forall \mu \in M_{kl}, \quad 1 \leq k, l \leq K\}.$$

3 Well-posedness

3.1 Inf-sup condition

The main consequence of assumption 2 is the inf-sup condition satisfied by the bilinear form b expressing the mortar condition, as proved in [Woh01]:

Proposition 1. *Under assumption 2, there exists a constant $\beta > 0$ such that:*

$$\inf_{\lambda_h \in M_\delta \setminus \{0\}} \sup_{u_h \in X_h \setminus \{0\}} \frac{b(u_h, \lambda_h)}{\|\lambda_h\|_{\delta, -\frac{1}{2}} \|u_h\|_X} \geq \beta, \tag{9}$$

where β is of the form:

$$\beta = C \min_{1 \leq m \leq M} \frac{1}{(C_m)^2},$$

where the constant C_m is the stability constant of π_m defined in assumption 2, for all $1 \leq m \leq M$, and C is independent of the discretization and of the number of subdomains.

Remark 4. *In the absence of any triple point on the interface, that is if any function defined on Γ_m has zero trace on all other interfaces Γ_l, $l \neq m$, the previous proposition remains true even if one replaces $W^0_{m;\delta_m}$ by $W_{m;\delta_m}$ in assumption 2.*

3.2 Minimal Lagrange multipliers spaces

For instance, the implication (8) of assumption 3 is true when the traces of first order polynomial displacements over the interfaces belong to the Lagrange multipliers spaces. This result made precise in the following lemma, is detailed in [Hau04]:

Lemma 1. *By chosing M_{kl} as the restriction to γ_{kl} of first order polynomial displacements, i.e:*

$$M_{kl} = M_1(\gamma_{kl}) = \mathbb{P}_1(\Omega)^d\big|_{\gamma_{kl}} := \{v|_{\gamma_{kl}}, \quad v \in \mathbb{P}_1(\Omega)^d\}, \quad 1 \leq k,l \leq K,$$

where $\mathbb{P}_1(\Omega)$ is the space of first order polynomials over Ω, the implication (8) of assumption 3 holds.

Remark 5. *When considering second order approximations for the displacements, first order polynomials must belong to the space of Lagrange multipliers in order to achieve an optimal rate of convergence, as shown in the proof of proposition 5, page 27. The choice of M_{kl} given by lemma 1 is then natural. Nevertheless, when considering first order approximations of the displacements, and when more than two subdomains share a common edge, it is impossible for stability reason to conserve all the affine functions in the spaces of Lagrange multipliers. In particular, the order of Lagrange multipliers should be reduced on the interface elements having a non-empty intersection with the boundary of the interface, as pointed out in [BMP93, BMP94] for the scalar case.*

It is possible to weaken the assumption of lemma 1, for instance by using piecewise constant Lagrange multipliers, at least over interfaces having a tensor product structure, as detailed in [Hau04]:

Lemma 2. *We assume that for all $1 \leq k,l \leq K$ such that Ω_k and Ω_l have a non-empty intersection, the interface $\gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$ between the subdomains is planar. Denoting by G_{kl} its center of gravity defined by:*

$$G_{kl} = \frac{1}{meas(\Gamma_{kl})} \int_{\Gamma_{kl}} x\,dx,$$

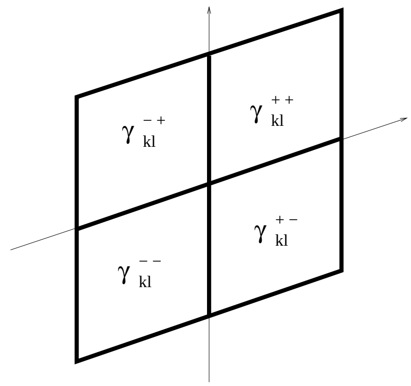
we can characterize γ_{kl} by:

$$\gamma_{kl} = \{x \in \mathbb{R}^3, \quad x - G_{kl} = \xi_1 f_1 + \xi_2 f_2, \quad (\xi_1, \xi_2) \in [-1,1]^2\},$$

where $f_1, f_2 \in \mathbb{R}^3$ are linearly independent. We introduce the following partition over γ_{kl}:

$$\begin{cases} \gamma_{kl}^{++} = \{\xi_1 f_1 + \xi_2 f_2; \quad \xi_1 \in [0;1] \text{ and } \xi_2 \in [0;1]\}, \\ \gamma_{kl}^{+-} = \{\xi_1 f_1 + \xi_2 f_2; \quad \xi_1 \in [0;1] \text{ and } \xi_2 \in [-1;0]\}, \\ \gamma_{kl}^{--} = \{\xi_1 f_1 + \xi_2 f_2; \quad \xi_1 \in [-1;0] \text{ and } \xi_2 \in [-1;0]\}, \\ \gamma_{kl}^{-+} = \{\xi_1 f_1 + \xi_2 f_2; \quad \xi_1 \in [-1;0] \text{ and } \xi_2 \in [0;1]\}, \end{cases}$$

and assume that M_{kl} is made of piecewise constant functions over the sets γ_{kl}^{++}, γ_{kl}^{+-}, γ_{kl}^{--} and γ_{kl}^{-+}. Then, the assertion (8) of assumption 3 holds.



Remark 6. *In the proof of lemma 2, the space of Lagrange multipliers we have used to check the implication (8) of assumption 3, is in fact a subspace of dimension 3 of the proposed space* M_{kl}.

3.3 Standard result of coercivity

We are now ready to recall the standard coercivity result for the bilinear form:

$$\tilde{d}(u,v) := \sum_{k=1}^{K} d_k(u,v), \quad \forall u,v \in V$$

with:

$$d_k(u,v) := \int_{\Omega_k} \varepsilon(u) : \varepsilon(v), \quad \forall u,v \in V.$$

The now standard proof, done by contradiction as in [BMP93] for example in the scalar case, does not guarantee the independence on the number and the size of the subdomains. We recall it nevertheless for completeness.

Proposition 2. *Let* Ω *be a bounded* $\mathcal{C}^1$*connected open set. The assumption 3 is supposed to be satisfied. Then, there exists a constant* $C > 0$ *possibly depending on the number and sizes of subdomains such that for all* $v \in V$*, the following inequality holds:*

$$\sum_{k=1}^{K} \int_{\Omega_k} \varepsilon(v) : \varepsilon(v) \geq C \left(\sum_{k=1}^{K} \frac{1}{diam(\Omega_k)^2} \int_{\Omega_k} v^2 + \int_{\Omega_k} \nabla v : \nabla v \right).$$

4 Uniform coercivity

We improve herein the previous coercivity result by showing the independence of the coercivity constant with respect to the number, the size and the shape of the subdomains. Such a result is known for scalar elliptic problems, when interfaces are plane, as proved in [Gop99, Bre03]. A proof for the vector case is also proposed in a recent publication [Bre04]. The originality of our approach is that it uses a generalization of the Scott and Zhang interpolation [SZ90], and is valid for curved interfaces.

4.1 Fundamental assumptions

Let us introduce the assumptions used in the present section. First, we assume that each subdomain is a “compact deformation” of a reference domain, the reference domains being in finite number. More precisely:

Assumption 4. *It is assumed that:*

1. *there exists a finite collection of reference domains $(\hat{\Omega}_j)_{1\leq j\leq J}$ of unit diameter, of compact sets $(\mathcal{K}_j)_{1\leq j\leq J}$ and of maps $\varphi_j : \hat{\Omega}_j \times \mathcal{K}_j \to \mathbb{R}^d$,$1 \leq j \leq J$ such that for all $1 \leq j \leq J$:*
$$\mathit{diam}\left(\varphi_j(\hat{\Omega}_j, p)\right) = 1, \quad \forall p \in \mathcal{K}_j,$$
and the following application:
$$\begin{array}{ccc} \mathcal{K}_j & \to & W^{1,\infty}(\hat{\Omega}_j)^d, \\ p & \mapsto & \varphi_j(\cdot, p), \end{array}$$
is continuous;
2. *for all $1 \leq j \leq J$, there exists a constant $C_j > 0$ such that:*
$$\det \frac{\partial \varphi_j}{\partial \hat{x}}(\hat{x}, p) \geq C_j, \quad \forall p \in \mathcal{K}_j, \text{ for almost all } \hat{x} \in \hat{\Omega}_j;$$

in other words, for all $p \in \mathcal{K}_j$, $\varphi_j(\cdot, p)$ is a uniform homeomorphism;

3. *for all $(\Omega_k)_{1\leq k\leq K}$ there exists a j with $1 \leq j \leq J$ and an element $p \in \mathcal{K}_j$ such that within a scaling factor:*
$$\frac{1}{\mathit{diam}(\Omega_k)}\, \Omega_k = \varphi_j(\hat{\Omega}_j, p).$$

Moreover, we consider that:

4. *there exists a finite collection of reference interfaces $(\hat{\gamma}_j)_{1\leq j\leq J}$, with $\hat{\gamma}_j \subset \partial\hat{\Omega}_j$, $1 \leq j \leq J$, and that the application:*
$$\begin{array}{ccc} \mathcal{K}_j & \to & W^{1,\infty}(\hat{\gamma}_j)^d, \\ p & \mapsto & \varphi_j(\cdot, p), \end{array}$$
is continuous,
5. *for all $1 \leq j \leq J$, there exists a constant $C_j > 0$ such that:*
$$\det \frac{\partial \varphi_j}{\partial \hat{x}}(\hat{x}, p) \geq C_j, \quad \forall p \in \mathcal{K}_j, \text{ for almost all } \hat{x} \in \hat{\gamma}_j,$$

and when γ is a part of the boundary of $\Omega_k = \varphi_j(\hat{\Omega}_j, p)$, we assume that:

6. $\dfrac{1}{diam(\gamma)}\,\gamma = \varphi_j(\hat{\gamma}_j, p)$.

7. *there exists three constants $\kappa, \kappa', \kappa'' > 0$ such that for all $1 \le k \le K$:*

$$\begin{cases} \rho(\Omega_k) \ge \kappa \; diam(\Omega_k), \\ diam(\gamma_{kl}) \ge \kappa' \; diam(\Omega_k), \quad 1 \le l \le K, \\ |\gamma_{kl}| \ge \kappa'' \; diam(\Omega_k)^{d-1}, \end{cases} \tag{10}$$

   where $\rho(\Omega_k)$ denotes the diameter of the largest ball contained in Ω_k . The constants κ,κ' and κ'' must remain independent of the number and the size of the subdomains. As a consequence of (10), the number of subdomains sharing a common intersection remains bounded by a fixed integer P, independently of the chosen regular decomposition.

The assumptions 1 to 6 are used to show a technical result of shape-independence of the constant in Korn-like inequalities within a proper scaling, as detailed in [Hau04], page 205. Assumption 7 will be used to show our interpolation estimates.

To deal with curved interfaces in the framework of Scott-Zhang like interpolation, we will need the technical assumption 5, page 15, detailed hereafter in the definition of the interpolation operator. The present coercivity result will be shown on the constrained space:

$$V = \{v \in X, \quad \int_{\gamma_{kl}} [v] \cdot \mu = 0, \quad \forall \mu \in \mathbb{P}_1(\gamma_{kl})^d\}$$

In this section, we assume that all these assumptions are satisfied.

4.2 Generalized Korn's inequality

We will use hereafter the two following generalized Korn's inequalities reviewed and detailed in [Hau04], page 205, for domains satisfying the assumptions of section 4.1.

Lemma 3. *There exists a constant C_P such that for all Ω_k and γ_{kl} satisfying the conditions defined in section 4.1, the following inequality holds for all $v \in H^1(\Omega_k)^d$:*

$$\|v\|^2_{H^1(\Omega_k)^d} \le C_P \left(\frac{1}{diam(\Omega_k)} \left(\sup_{\mu \in M_{kl}} \frac{\int_{\gamma_{kl}} v \cdot \mu}{\|\mu\|_{L^2(\gamma_{kl})^d}} \right)^2 + d_k(v,v) \right),$$

where C_P does not depend on Ω_k and γ_{kl}.

Lemma 4. *There exists a constant C_N such that for all Ω_k and γ_{kl} satisfying the conditions defined in section 4.1, the following inequality holds for all $v \in H^1(\Omega_k)^d$:*

$$\|v\|^2_{H^1(\Omega_k)^d} \le C_N \left(\frac{1}{diam(\Omega_k)^2} \left(\sup_{r \in \mathcal{R}(\Omega_k)} \frac{\int_{\Omega_k} v \cdot r}{\|r\|_{L^2(\Omega_k)^d}} \right)^2 + d_k(v,v) \right),$$

where C_N does not depend on Ω_k and γ_{kl}.

Then, we deduce the following trace lemma:

Lemma 5. *There exists a constant C_T such that for all Ω_k and γ_{kl} satisfying the conditions defined in section 4.1, the following inequality holds for all $v \in H^1(\Omega_k)^d$:*

$$\frac{1}{diam(\Omega_k)}\left(\sup_{\mu\in M_{kl}} \frac{\int_{\gamma_{kl}} v\cdot\mu}{\|\mu\|_{L^2(\gamma_{kl})^d}}\right)^2 \le C_T\left(\frac{1}{diam(\Omega_k)^2}\left(\sup_{r\in\mathcal{R}(\Omega_k)} \frac{\int_{\Omega_k} v\cdot r}{\|r\|_{L^2(\Omega_k)^d}}\right)^2 + d_k(v,v)\right),$$

where C_T does not depend on Ω_k and γ_{kl}.

Proof : By using the Cauchy-Schwarz inequality, the Sobolev trace theorem (with proper scaling) and the lemma 4, we get:

$$\begin{aligned}
\frac{1}{diam(\Omega_k)}\left(\sup_{\mu\in M_{kl}} \frac{\int_{\gamma_{kl}} v\cdot\mu}{\|\mu\|_{L^2(\gamma_{kl})^d}}\right)^2 &\le \frac{1}{diam(\Omega_k)}\int_{\gamma_{kl}} v^2 \\
&\le C\left(\frac{1}{diam(\Omega_k)^2}\int_{\Omega_k} v^2 + \int_{\Omega_k}|\nabla v|^2\right) \\
&\le CC_N\left(\frac{1}{diam(\Omega_k)^2}\left(\sup_{r\in\mathcal{R}(\Omega_k)} \frac{\int_{\Omega_k} v\cdot r}{\|r\|_{L^2(\Omega_k)^d}}\right)^2 + d_k(v,v)\right),
\end{aligned}$$

hence the proof. □

4.3 A Scott & Zhang like interpolation operator for mortar methods

The proposed interpolation operator builds a conforming approximation of a non-conforming function defined in the constrained space V of functions whose jump is orthogonal to interface Lagrange multipliers, with the usual stability properties shown in [SZ90], even when considering curved interfaces between the subdomains.

Construction of a coarse conforming basis - Let us introduce a coarse conforming triangulation $\mathcal{T}_H$ of Ω, as shown on figure 3, which satisfies the following conditions:

1. Each $T \in \mathcal{T}_H$ is totally included in a subdomain Ω_k.
2. The tetrahedra in $\mathcal{T}_H$ possibly have curved faces along the skeleton interface $\mathcal{S}$.
3. The tetrahedra $T \in \mathcal{T}_H$ in Ω_k are such that $\rho(T) \ge C diam(\Omega_k)$, with $\rho(T)$ the diameter of the largest ball included in T.

We define on $\mathcal{T}_H$ the following conforming approximation space:

$$X_H = \{v \in H^1(\Omega), \quad v|_T \in \mathbb{P}_1(T), \quad T \in \mathcal{T}_H\},$$

where $\mathbb{P}_1(T)$ denotes the space of affine applications over T. The vertices of the coarse conforming triangulation $\mathcal{T}_H$ are denoted by $(M_i)_{1\le i\le I}$, and the associated nodal basis of X_H by $(\phi_i)_{1\le i\le I}$ such that:

$$\phi_i(M_j) = \delta_{ij},$$

using the Kronecker symbol $\delta_{ij} = 1$ for $i = j$ and 0 otherwise.

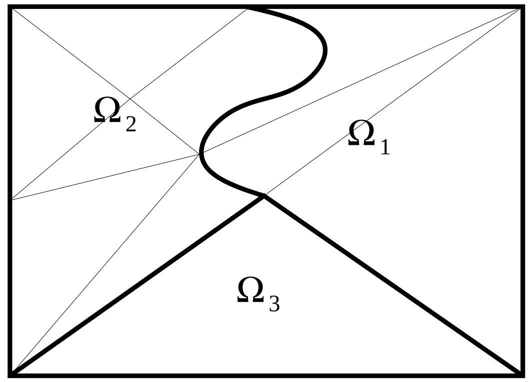


Figure 3: A coarse conforming triangulation $\mathcal{T}_H$ of $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ satisfying conditions 1 and 2.

Set of interfaces - Let us denote by $Z_{\mathcal{S}}$ the set of interfaces γ_{kl} between two adjacent subdomains, and by $\mathring{Z}$ the set of internal faces of the triangulation $\mathcal{T}_H$, that is the faces of the triangles $T \in \mathcal{T}_H$, which are not included in the skeleton interface $\mathcal{S}$. The total set of interfaces is then defined by:

$$Z = Z_{\mathcal{S}} \cup \mathring{Z}.$$

To deal with curved interfaces in the framework of Scott-Zhang like interpolation, we need the following assumption:

Assumption 5. *There exists a constant $C > 0$ such that for each node M_i of the coarse triangulation $\mathcal{T}_H$, there exists an interface $\gamma_i \in Z$ with $M_i \in \gamma_i$ such that for all matrix $B \in \mathbb{R}^{d\times d}$, we have:*

$$\frac{1}{|\gamma_i|} \int_{\gamma_i} |B \cdot (x - G_{\gamma_i})|^2 \, dx \geq C \, \lambda(B, \gamma_i)^2 \, diam(\gamma_i)^2, \tag{11}$$

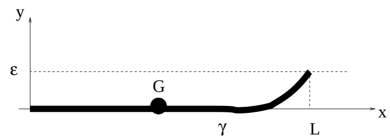
where G_γ is the center of gravity of γ, i.e:

$$G_\gamma = \frac{1}{|\gamma|} \int_\gamma x \, dx,$$

and $\lambda(B,\gamma)$ the maximal singular value of B on γ:

$$\lambda(B,\gamma)^2 = \sup_{x\in\gamma} \frac{|B \cdot (x - G_\gamma)|^2}{|x - G_\gamma|^2}.$$

Remark 7. *The assumption 5 means that for all node M_i of the coarse triangulation $\mathcal{T}_H$, there exists an interface sharing M_i and having a finite "length" along the principal direction of displacement for all affine fields of displacements. As a counter-example, let us consider the curved interface depicted in the following picture:*



and the linear function $v(x,y) = \epsilon^{-1} y = B \cdot \mathbf{x}$. It follows that $\lambda(B,\gamma)^2 \simeq \dfrac{1}{L^2}$ and

$$\frac{1}{|\gamma_i|}\int_{\gamma_i} |B\cdot(x-G_{\gamma_i})|^2\,dx \simeq \frac{1}{L}\int_0^{\epsilon}\left(\epsilon^{-1}y\right)^2\,dy \simeq \frac{\epsilon}{L}.$$

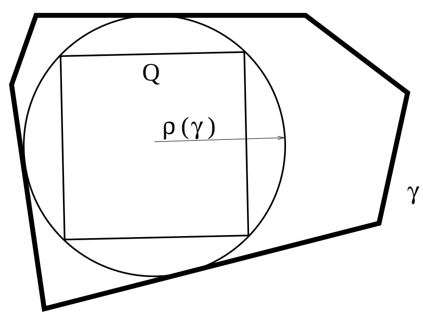
As a consequence, the assertion (11) is not satisfied on γ uniformly in ϵ. The reason is that in this case, γ is nearly orthogonal to the principal direction of displacement.

Nevertheless, the assertion (11) is satisfied for any plane interface γ whatever the matrix $B \in \mathbb{R}^{d\times d}$, as shown in the following lemma:

Lemma 6. *The assumption 5 is satisfied when choosing as γ_i any plane interface sharing the node M_i, provided γ_i is shape regular that is:*

$$\rho(\gamma_i) \geq C diam(\gamma_i).$$

Proof : The present proof is done in three dimensions. Let γ be a plane interface, and Q a square of maximal edge length ($=\rho(\gamma)/\sqrt{2}$) included in the largest ball contained in γ (as shown in the following picture).



We write $x - G_\gamma = x_1 e_1 + x_2 e_2 = J\cdot x$, where e_1 and e_2 are two orthogonal vectors such that $G_\gamma + span\{e_1, e_2\} = \gamma$. As the matrix $J^t \cdot B^t \cdot B \cdot J$ is symmetric semi-definite positive, it can be diagonalized and we still denote by e_1 and e_2 its eigenvectors, associated to the eigenvalues μ_1^2 and μ_2^2 with $\mu_2^2 \geq \mu_1^2$. Finally, we choose among all the possible squares Q, the one whose edges are parallel to the eigenvectors:

$$Q = \{x_1 e_1 + x_2 e_2; \quad x_1 \in [X_1 - a, X_1 + a], x_2 \in [X_2 - a, X_2 + a]\},$$

where the center of the largest ball in γ is $G_\gamma + X_1 e_1 + X_2 e_2$, and $2a = \rho(\gamma)/\sqrt{2}$.

Then, we get:

$$
\begin{aligned}
\frac{1}{|\gamma|}\int_\gamma |B\cdot(x-G_\gamma)|^2 &\geq \frac{1}{|\gamma|}\int_Q |B\cdot(x-G_\gamma)|^2\\
&\geq \frac{1}{|\gamma|}\int_{X_1-a}^{X_1+a}\int_{X_2-a}^{X_2+a}\left(\mu_1^2(x_1)^2+\mu_2^2(x_2)^2\right)\,dx_1dx_2\\
&\geq \frac{2a}{3|\gamma|}\left(\mu_1^2\left((X_1+a)^3-(X_1-a)^3\right)+\mu_2^2\left((X_2+a)^3-(X_2-a)^3\right)\right).
\end{aligned}
$$

Moreover, we have:

$$
\begin{aligned}
(X_1+a)^3-(X_1-a)^3 &= 2a\left((X_1+a)^2+(X_1+a)(X_1-a)+(X_1-a)^2\right)\\
&= 2a\left(3X_1^2+a^2\right)\\
&\geq 2a^3,
\end{aligned}
$$

leading to:

$$
\begin{aligned}
\frac{1}{|\gamma|}\int_\gamma |B\cdot(x-G_\gamma)|^2 &\geq \frac{2a}{3|\gamma|}2a^3(\mu_1^2+\mu_2^2)\\
&\geq \frac{2a}{3|\gamma|}2a^3\mu_2^2.
\end{aligned}
$$

From shape regularity, we have $|\gamma|\leq Cdiam(\gamma)^2\leq C\rho(\gamma)^2=2Ca^2$, and therefore:

$$
\begin{aligned}
\frac{1}{|\gamma|}\int_\gamma |B\cdot(x-G_\gamma)|^2 &\geq Ca^2\mu_2^2\\
&\geq Cdiam(\gamma)^2\mu_2^2,
\end{aligned}
$$

but by definition:

$$\mu_2^2=\lambda(B,\gamma)^2,$$

which ends the proof. □

The main consequence from assumption 5 is the simple:

Lemma 7. *Under assumption 5, there exists a constant $C>0$ such that for all locally affine functions $v\in\mathbb{P}_1(\Omega)^d$, we can find at each node M of the coarse mesh $\mathcal{T}_H$, an interface $\gamma\ni M$ for which:*

$$\|v\|^2_{L^\infty(\gamma)^d}\leq C\frac{1}{|\gamma|}\|v\|^2_{L^2(\gamma)^d}.$$

Proof : Let v be locally in $\mathbb{P}_1(\Omega)^d$. For all $\gamma\in Z$, there exists a vector $v(G_\gamma)\in\mathbb{R}^d$ and a matrix $B\in\mathbb{R}^{d\times d}$ such that:

$$v(x)=v(G_\gamma)+B\cdot(x-G_\gamma),\quad\forall x\in\gamma,$$

the matrix B being independent of the choice of $\gamma \in Z$. From assumption 5, we can always find at each node M_i of the coarse mesh $\mathcal{T}_H$, an interface $\gamma = \gamma_i$ such that (11) is satisfied. Then:

$$\|v\|^2_{L^\infty(\gamma)} \leq 2|v(G_\gamma)|^2 + 2\lambda(B,\gamma)^2 diam(\gamma)^2,$$

and from assumption 5, we deduce:

$$\begin{aligned}
\frac{1}{|\gamma|}\|v\|^2_{L^2(\gamma)} &= v(G_\gamma)^2 + \frac{1}{|\gamma|}\int_\gamma |B\cdot(x-G_\gamma)|^2 \\
&\geq C\left(v(G_\gamma)^2 + \lambda(B,\gamma)^2 diam(\gamma)^2\right) \\
&\geq C\|v\|^2_{L^\infty(\gamma)}.
\end{aligned}$$

□

Conforming approximation - For all functions $v \in X$, we are now ready to define the conforming approximation $\mathbf{P}v \in H^1_*(\Omega)$ by:

$$\mathbf{P}v = \sum_{i\geq 1} p_i v(M_i)\ \phi_i, \tag{12}$$

where:

$$p_i v = \pi_{\gamma_i} v,$$

in which π_γ is the $L^2(\gamma)^d$ projection over $\mathbb{P}_1(\gamma)^d$ (the restrictions to γ of functions in $\mathbb{P}_1(\Omega)^d$), and $\gamma_i \in Z$ is among the interfaces sharing M_i, the one which maximizes:

$$\mathcal{A}(\gamma) = \inf_{B\in\mathbb{R}^{d\times d}} \frac{1}{\lambda(B,\gamma)^2 diam(\gamma)^2}\frac{1}{|\gamma|}\int_\gamma |B\cdot(x-G_\gamma)|^2\ dx.$$

Let us notice that in the expression $\pi_{\gamma_i}v$, we choose arbitrarily the side of γ_i on which the trace of v is taken. When considering $v \in V$ with the constrained space:

$$V = \{v \in X, \int_{\gamma_{kl}} [v]\cdot\mu = 0, \quad \forall\mu \in \mathbb{P}_1(\gamma_{kl})^d, 1 \leq k,l \leq K\},$$

this choice has no influence because:

$$\int_{\gamma_i} v^+\cdot\mu = \int_{\gamma_i} v^-\cdot\mu, \quad \forall\mu\in\mathbb{P}_1(\gamma_i)^d,$$

entailing that $\pi_{\gamma_i}v^+ = \pi_{\gamma_i}v^-$.

Remark 8. *In this section, we use the Lagrange multipliers spaces $M_{kl} = \mathbb{P}_1(\gamma_{kl})^d$. Nevertheless, one can adopt any M_{kl} such that for all $v \in L^2(\gamma_{kl})^d$, there exists a solution $\pi_{\gamma_{kl}}v \in \mathbb{P}_1(\gamma_{kl})^d$ of:*

$$\int_{\gamma_{kl}} \pi_{\gamma_{kl}}v\cdot\mu = \int_{\gamma_{kl}} v\cdot\mu, \quad \forall\mu\in M_{kl},$$

satisfying:

$$\|\pi_{\gamma_{kl}} v\|_{L^2(\gamma_{kl})^d} \leq C \sup_{\mu \in M_{kl}} \frac{\displaystyle\int_{\gamma_{kl}} v \cdot \mu}{\|\mu\|_{L^2(\gamma_{kl})^d}}. \tag{13}$$

Such a statement is true when adopting Lagrange multipliers satisfying the assumption 3, but the constant a priori depends on the shape of the interface γ_{kl}.

Proposition 3. *The interpolation operator $\mathbf{P} : \prod_{k=1}^{K} H^1(\Omega_k)^d \to (X_H)^d$ defined by (12) satisfies the following local inequality for all $1 \leq k \leq K$:*

$$\|v - \mathbf{P}v\|^2_{H^1(\Omega_k)^d} \leq C \left(\sum_{l \in \mathcal{N}(\Omega_k)} d_l(v,v) + \frac{1}{diam(\Omega_k)} \int_{\mathcal{S}_k} (\pi[v])^2 \right), \tag{14}$$

where $\mathcal{N}(\Omega_k)$ denotes the set of indices of the subdomains sharing a vertex with Ω_k, and d_k is the bilinear form over $H^1(\Omega_k)^d \times H^1(\Omega_k)^d$ defined as:

$$d_k(u,v) = \int_{\Omega_k} \varepsilon(u) : \varepsilon(v), \quad \forall u, v \in H^1(\Omega_k)^d.$$

Moreover, we have denoted by $\mathcal{S}_k$ the union of the neighboring interfaces of Ω_k:

$$\mathcal{S}_k = \bigcup_{l,m \in \mathcal{N}(\Omega_k)} \gamma_{lm},$$

and:

$$\pi[v](x) = \pi_\gamma[v](x), \qquad \textit{for all } x \in \gamma, \textit{ with } \gamma \in Z_{\mathcal{S}}.$$

Moreover, when the decomposition into subdomains satisfies the conditions defined in section 4.1, the constant C is independent of the diameter and the shape of the subdomains. The definitions of $\mathcal{N}(\Omega_k)$ and $\mathcal{S}_k$ are illustrated on figure 4.

Proof : The proof is decomposed into 4 parts. For convenience, we will denote by $\mathcal{O}_k$ the neighborhood of Ω_k defined as:

$$\mathcal{O}_k = \bigcup_{l \in \mathcal{N}(\Omega_k)} \Omega_l.$$

1. **Range of $\mathbb{P}_1(\mathcal{O}_k)^d$.**
   Let us consider the affine displacement $v \in \mathbb{P}_1(\mathcal{O}_k)^d$. For all $\gamma \in Z \cap \mathcal{O}_k$, the trace of v over γ belongs to $\mathbb{P}_1(\gamma)^d$ by definition, and therefore:

   $$\pi_\gamma v = v, \quad \text{on } \gamma.$$

   As a consequence, we obtain for all $i \geq 1$ satisfying $M_i \in \overline{\Omega_k}$, that $p_i v = v$, hence:

   $$(\mathbf{P}v)|_{\Omega_k} = \sum_{i \geq 1, M_i \in \overline{\Omega_k}} v(M_i)\phi_i = v|_{\Omega_k},$$

   because $v|_{\Omega_k} \in (X_H)^d$.

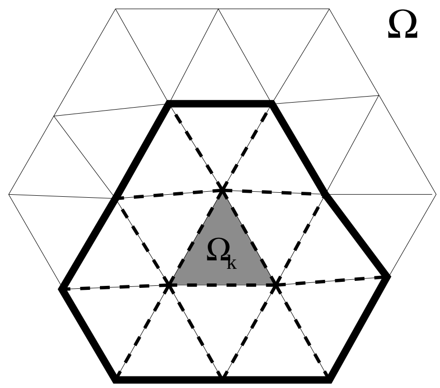


Figure 4: A triangular domain decomposition of $\Omega \subset \mathbb{R}^2$, with illustration of the subdomains $(\Omega_l)_{l\in\mathcal{N}(\Omega_k)}$ sharing a vertex with Ω_k (inside the dark thick line), and of the reunion $\mathcal{S}_k$ of the neighboring interfaces of Ω_k (in dotted lines).

2. **Stability of P in $L^2(\Omega_k)^d$.**
   Let $v \in X$. It is readily obtained from definition (12), that:

$$\begin{aligned}\|\mathbf{P}v\|^2_{L^2(\Omega_k)^d} &\leq \max_{i,M_i\in\overline{\Omega_k}} |p_i v(M_i)|^2 \int_{\Omega_k} \left(\sum_{1\leq i\leq I} |\phi_i|\right)^2 \\ &\leq \max_{i,M_i\in\overline{\Omega_k}} \|\pi_{\gamma_i} v\|^2_{L^\infty(\gamma_i)^d} \int_{\Omega_k} \left(\sum_{1\leq i\leq I} |\phi_i|\right)^2 .\end{aligned}$$

   Under assumption 5, we obtain from lemma 7 that:

$$\|\pi_{\gamma_i} v\|^2_{L^\infty(\gamma_i)^d} \leq C\frac{1}{|\gamma_i|}\,\|\pi_{\gamma_i} v\|^2_{L^2(\gamma_i)^d}\,,$$

   and because π_{γ_i} is the $L^2(\gamma_i)^d$ projection over $\mathbb{P}_1(\gamma_i)^d$, we get:

$$\|\pi_{\gamma_i} v\|^2_{L^2(\gamma_i)^d} \leq \|v\|^2_{L^2(\gamma_i)^d}\,,$$

   resulting in:

$$\|\mathbf{P}v\|^2_{L^2(\Omega_k)^d} \leq \max_{i,M_i\in\overline{\Omega_k}} \frac{1}{|\gamma_i|}\,\|v\|^2_{L^2(\gamma_i)^d} \int_{\Omega_k} \left(\sum_{1\leq i\leq I} |\phi_i|\right)^2 . \tag{15}$$

   As γ_i is a part of the boundary of a domain $\Omega_{l(i)}$ corresponding to the side of γ_i on which the trace of v is taken, we get from the Sobolev trace theorem that:

$$\frac{1}{diam(\Omega_{l(i)})}\,\|v\|^2_{L^2(\gamma_i)^d} \leq C\left(\frac{1}{diam(\Omega_{l(i)})^2}\int_{\Omega_{l(i)}} v^2 + \int_{\Omega_{l(i)}} |\nabla v|^2\right), \tag{16}$$

with C uniformly bounded due to the shape regularity of $\Omega_{l(i)}$. Moreover, we have:

$$\int_{\Omega_k} \left(\sum_{1 \leq i \leq I} |\phi_i| \right)^2 = \int_{\Omega_k} dx = |\Omega_k|, \tag{17}$$

because by construction $\sum_{1 \leq i \leq I} |\phi_i| = 1$. We deduce by exploiting the expressions (16) and (17) in (15) that:

$$\begin{aligned} \|\mathbf{P}v\|^2_{L^2(\Omega_k)^d} &\leq C \max_{i, M_i \in \overline{\Omega_k}} \frac{|\Omega_k|}{|\gamma_i|} diam(\Omega_{l(i)}) \left(\frac{1}{diam(\Omega_{l(i)})^2} \int_{\Omega_{l(i)}} v^2 + \int_{\Omega_{l(i)}} |\nabla v|^2 \right) \\ &\leq C \max_{i, M_i \in \overline{\Omega_k}} \frac{|\Omega_k|}{|\Omega_{l(i)}|} diam(\Omega_{l(i)})^2 \left(\frac{1}{diam(\Omega_{l(i)})^2} \int_{\Omega_{l(i)}} v^2 + \int_{\Omega_{l(i)}} |\nabla v|^2 \right) \end{aligned}$$

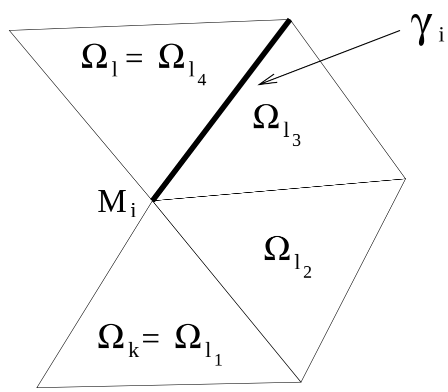
because from the shape regularity conditions (10), we get:

$$\begin{aligned} |\gamma_i| \; diam(\Omega_{l(i)}) &\geq \kappa'' diam(\Omega_{l(i)})^{d-1} \; diam(\Omega_{l(i)}) \\ &= \kappa'' \; diam(\Omega_{l(i)})^d \\ &\geq C \; \kappa'' \; |\Omega_{l(i)}|. \end{aligned}$$

Therefore, there exists a subdomain Ω_l sharing a node with Ω_k such that:

$$\frac{1}{diam(\Omega_l)^2} \|\mathbf{P}v\|^2_{L^2(\Omega_k)^d} \leq C \frac{|\Omega_k|}{|\Omega_l|} \left(\frac{1}{diam(\Omega_l)^2} \int_{\Omega_l} v^2 + \int_{\Omega_l} |\nabla v|^2 \right),$$

after a division of the two sides of the inequality by $diam(\Omega_l)^2$.
Let us show now that $diam(\Omega_l) \leq C diam(\Omega_k)$. From the shape regularity (10) of the decomposition, we can build a sequence of (less than) P adjacent subdomains $(\Omega_{l_m})_{1 \leq m \leq P}$ such that Ω_{l_m} and $\Omega_{l_{m+1}}$ share the interface $\gamma_{l_m l_{m+1}}$ with $\Omega_{l_1} = \Omega_k$ and $\Omega_{l_P} = \Omega_l$, as illustrated on the following figure (for triangular subdomains):



From the shape regularity (10) of the decomposition into subdomains, we then have:

$$\begin{aligned} diam(\Omega_{l_{m+1}}) &\leq \frac{1}{\kappa'} diam(\gamma_{l_m l_{m+1}}) \\ &\leq \frac{1}{\kappa'} diam(\Omega_{l_m}), \end{aligned} \tag{18}$$

and by iteration of (18), we get:

$$diam(\Omega_l) \leq \frac{1}{(\kappa')^P} diam(\Omega_k). \tag{19}$$

Considering that the roles of Ω_k and Ω_l can be swapped in the previous inequality (19), we deduce that $|\Omega_k| \leq C|\Omega_l|$ from the shape regularity (10) of the decomposition because:

$$\begin{aligned} |\Omega_k| &\leq C\, diam(\Omega_k)^d \\ &\leq C \frac{1}{(\kappa')^{dP}} diam(\Omega_l)^d \\ &\leq C \frac{1}{(\kappa')^{dP}} \frac{1}{\kappa^d} \rho(\Omega_l)^d \\ &\leq C \frac{1}{(\kappa')^{dP}} \frac{1}{\kappa^d} |\Omega_l|. \end{aligned}$$

As a consequence, we obtain from (18) with a still generic use of the constant C, that there exists a subdomain Ω_l sharing a node with Ω_k such that:

$$\frac{1}{diam(\Omega_k)^2} \|\mathbf{P}v\|^2_{L^2(\Omega_k)} \leq C \left(\frac{1}{diam(\Omega_l)^2} \int_{\Omega_l} v^2 + \int_{\Omega_l} |\nabla v|^2 \right). \tag{20}$$

3. **Stability of P in $H^1(\Omega_k)^d$.**
Proceeding as previously, we get for all $v \in X$ the following bound on the $H^1(\Omega_k)^d$ semi-norm of the interpolate function $\mathbf{P}v$:

$$|\mathbf{P}v|^2_{H^1(\Omega_k)^d} \leq \max_{1\leq i\leq I} |p_i v(M_i)|^2 \int_{\Omega_k} \left(\sum_{1\leq i\leq I} |\nabla \phi_i| \right)^2$$

$$\leq C \max_{i, M_i \in \overline{\Omega_k}} \frac{diam(\Omega_{l(i)})^2}{|\Omega_{l(i)}|} \left(\frac{1}{diam(\Omega_{l(i)})^2} \int_{\Omega_{l(i)}} v^2 + \int_{\Omega_{l(i)}} |\nabla v|^2 \right) \int_{\Omega_k} \left(\sum_{1\leq i\leq I} |\nabla \phi_i| \right)^2. \tag{21}$$

Moreover, by decomposing the last integral over Ω_k into a sum of integrals over the triangles of the coarse triangulation $\mathcal{T}_H$ belonging to Ω_k:

$$\int_{\Omega_k} \left(\sum_{1\leq i\leq I} |\nabla \phi_i| \right)^2 = \sum_{T\in\mathcal{T}_H, T\subset\Omega_k} \int_T \left(\sum_{1\leq i\leq I} |\nabla \phi_i| \right)^2,$$

and using the fact that for all tetrahedra $T \in \mathcal{T}_H$ belonging to Ω_k, we have the standard result:

$$|\nabla\phi_i| \leq C\frac{1}{\rho(T)} \leq C\frac{1}{diam(\Omega_k)},$$

using the assumption 3- made for the coarse triangulation $\mathcal{T}_H$, we conclude that:

$$\int_{\Omega_k}\left(\sum_{1\leq i\leq I}|\nabla\phi_i|\right)^2 \quad \leq \quad \frac{C}{diam(\Omega_k)^2}|\Omega_k|. \tag{22}$$

Hence from (21) and (22), we get by using the same arguments of shape regularity of the decomposition as in the previous part of the proof that there exists a subdomain Ω_l sharing a node with Ω_k (the same as in (20)) such that:

$$|\mathbf{P}v|^2_{H^1(\Omega_k)} \leq C\left(\frac{1}{diam(\Omega_l)^2}\int_{\Omega_l} v^2 + \int_{\Omega_l}|\nabla v|^2\right).$$

4. **Approximation property**
For all $v \in X$ the interpolation $\mathbf{P}v \in (X_H)^d$ satisfies from the two previous points of the proof, the following stability property:

$$\|\mathbf{P}v\|^2_{H^1(\Omega_k)^d} \leq C\|v\|^2_{H^1(\Omega_l)^d}. \tag{23}$$

For all rigid motion $p \in \mathcal{R}(\mathcal{O}_k)$, which is a fortiori a linear function of $\mathbb{P}_1(\mathcal{O}_k)^d$, we have from point 1 that $\mathbf{P}p = p$ on Ω_k, resulting in the following bounds by using the triangular inequality and the stability estimate (23):

$$\begin{aligned}
\|v-\mathbf{P}v\|^2_{H^1(\Omega_k)^d} &= \|v-p+\mathbf{P}(p-v)\|^2_{H^1(\Omega_k)^d}\\
&\leq 2\|v-p\|^2_{H^1(\Omega_k)^d} + 2\|\mathbf{P}(p-v)\|^2_{H^1(\Omega_k)^d}\\
&\leq C\left(\|v-p\|^2_{H^1(\Omega_k)^d} + \|v-p\|^2_{H^1(\Omega_l)^d}\right)\\
&\leq C\sum_{l\in\mathcal{N}(\Omega_k)}\|v-p\|^2_{H^1(\Omega_l)^d}.
\end{aligned}$$

By taking p as the extension over Ω of the rigid motion projection of v over Ω_k, we get from lemma 8, page 24 that:

$$\|v-\mathbf{P}v\|^2_{H^1(\Omega_k)^d} \leq C\left(\sum_{l\in\mathcal{N}(\Omega_k)} d_l(v,v) + \frac{1}{diam(\Omega_k)}\int_{\mathcal{S}_k}(\pi[v])^2\right),$$

which is exactly (14).

□

In the previous proof, we have used the following lemma which is a generalization to non-conforming vector functions of the Deny-Lions [DL55] or Bramble-Hilbert [BH70] lemma involving the broken elasticity semi-norm.

Lemma 8. *There exists a constant $C > 0$ such that for all $v \in X$:*

$$\sum_{l \in \mathcal{N}(\Omega_k)} \|v - p\|^2_{H^1(\Omega_l)^d} \leq C \left(\sum_{l \in \mathcal{N}(\Omega_k)} d_l(v, v) + \frac{1}{diam(\Omega_k)} \int_{\mathcal{S}_k} (\pi[v])^2 \right), \qquad (24)$$

where $p \in \mathcal{R}(\Omega)$ is the rigid motion satisfying:

$$\int_{\Omega_k} p \cdot w = \int_{\Omega_k} v \cdot w, \quad \forall w \in \mathcal{R}(\Omega).$$

Moreover, provided the decomposition into subdomains satisfy the shape regularity condition defined in section 4.1, the constant C is independent of the size and the shape of the neighbor subdomains.

Proof : We prove herein the announced upper bound for the quantity

$$\sum_{l \in \mathcal{N}(\Omega_k)} \|v - p\|^2_{H^1(\Omega_l)^d},$$

in which the rigid motion $p \in \mathcal{R}(\Omega)$ is defined by:

$$\int_{\Omega_k} p \cdot r = \int_{\Omega_k} v \cdot r, \quad \forall r \in \mathcal{R}(\Omega_k).$$

• First, it follows from lemma 4 that:

$$\|v - p\|^2_{H^1(\Omega_k)^d} \leq C_N \left(\frac{1}{diam(\Omega_k)^2} \left(\sup_{r \in \mathcal{R}(\Omega_k)} \frac{\int_{\Omega_k} (v-p) \cdot r}{\|r\|_{L^2(\Omega_k)^d}} \right)^2 + d_k(v-p, v-p) \right) = C_N \; d_k(v, v),$$

by definition of the local rigid motion projection p.

• If Ω_l shares an interface with Ω_k, we obtain from lemmas 3 and 5 that:

$$\begin{aligned}
& \|v - p\|^2_{H^1(\Omega_l)^d} \\
\leq \;& C_P \left(\frac{1}{diam(\Omega_l)} \left(\sup_{\mu \in M_{kl}} \frac{\int_{\gamma_{kl}} (v-p)|_{\Omega_l} \cdot \mu}{\|\mu\|_{L^2(\gamma_{kl})^d}} \right)^2 + d_l(v, v) \right) \\
\leq \;& 2\, C_P \left(\frac{1}{diam(\Omega_l)} \left(\sup_{\mu \in M_{kl}} \frac{\int_{\gamma_{kl}} (v-p)|_{\Omega_k} \cdot \mu}{\|\mu\|_{L^2(\gamma_{kl})^d}} \right)^2 + d_l(v, v) + \frac{1}{diam(\Omega_l)} \int_{\gamma_{kl}} (\pi_{\gamma_{kl}}[v])^2 \right) \\
\leq \;& 2\, C_P \left(\frac{diam(\Omega_k)}{diam(\Omega_l)} C_T \left(\frac{1}{diam(\Omega_k)^2} \left(\sup_{r \in \mathcal{R}(\Omega_k)} \frac{\int_{\Omega_k} (v-p) \cdot r}{\|r\|_{L^2(\Omega_k)^d}} \right)^2 + d_k(v, v) \right) + d_l(v, v) \right) \\
& + 2\, C_P \frac{1}{diam(\Omega_l)} \int_{\gamma_{kl}} (\pi_{\gamma_{kl}}[v])^2 \\
= \;& 2\, C_P \left(\frac{diam(\Omega_k)}{diam(\Omega_l)} C_T \; d_k(v, v) + d_l(v, v) \right) + 2\, C_P \frac{1}{diam(\Omega_l)} \int_{\gamma_{kl}} (\pi_{\gamma_{kl}}[v])^2 \\
\leq \;& C \left(d_k(v, v) + d_l(v, v) + \frac{1}{diam(\Omega_l)} \int_{\gamma_{kl}} (\pi_{\gamma_{kl}}[v])^2 \right), \qquad (25)
\end{aligned}$$

because we have $diam(\Omega_k) \le C diam(\Omega_l)$ as in the step 2 of the proof of proposition 3.
• For other $l \in \mathcal{N}(\Omega_k)$, we proceed by the same technique used in the step 2 of the proof of proposition 3, by reasonning on a sequence of adjacent subdomains, and obtain as above:

$$
\begin{array}{rl}
& \|v-p\|^2_{H^1(\Omega_{l_{m+1}})^d} \\
\le & C_P \left(\dfrac{1}{diam(\Omega_{l_{m+1}})} \left(\sup\limits_{\mu \in M_{l_m l_{m+1}}} \dfrac{\int_{\gamma_{l_m l_{m+1}}} (v-p)|_{\Omega_{l_{m+1}}} \cdot \mu}{\|\mu\|_{L^2(\gamma_{l_m l_{m+1}})^d}} \right)^2 + d_{l_{m+1}}(v,v) \right) \\
\le & 2\, C_P \left(\dfrac{1}{diam(\Omega_{l_{m+1}})} \left(\sup\limits_{\mu \in M_{l_m l_{m+1}}} \dfrac{\int_{\gamma_{l_m l_{m+1}}} (v-p)|_{\Omega_{l_m}} \cdot \mu}{\|\mu\|_{L^2(\gamma_{l_m l_{m+1}})^d}} \right)^2 + d_{l_{m+1}}(v,v) \right) \\
& +2\, C_P \dfrac{1}{diam(\Omega_{l_{m+1}})} \displaystyle\int_{\gamma_{l_m l_{m+1}}} (\pi_{\gamma_{l_m l_{m+1}}}[v])^2 \\
\le & 2\, C_P \dfrac{diam(\Omega_{l_m})}{diam(\Omega_{l_{m+1}})} C_T \left(\dfrac{1}{diam(\Omega_{l_m})^2} \left(\sup\limits_{r \in \mathcal{R}(\Omega_{l_m})} \dfrac{\int_{\Omega_{l_m}} (v-p) \cdot r}{\|r\|_{L^2(\Omega_{l_m})^d}} \right)^2 + d_{l_m}(v-p, v-p) \right) \\
& +2\, C_P \left(d_{l_{m+1}}(v,v) + \dfrac{1}{diam(\Omega_{l_{m+1}})} \displaystyle\int_{\gamma_{l_m l_{m+1}}} (\pi_{\gamma_{l_m l_{m+1}}}[v])^2 \right) \\
\le & 2\, C_P \left(\dfrac{diam(\Omega_{l_m})}{diam(\Omega_{l_{m+1}})} C_T \, \|v-p\|^2_{H^1(\Omega_{l_m})^d} + d_{l_{m+1}}(v,v) + \dfrac{1}{diam(\Omega_{l_{m+1}})} \displaystyle\int_{\gamma_{l_m l_{m+1}}} (\pi_{\gamma_{l_m l_{m+1}}}[v])^2 \right),
\end{array}
$$

from Cauchy-Schwarz inequality. From the shape regularity (10), it follows that $diam(\Omega_k) \le C diam(\Omega_{l_{m+1}})$ and $diam(\Omega_{l_m}) \le C diam(\Omega_{l_{m+1}})$ as in the step 2 of the proof of proposition 3, and we get:

$$
\begin{array}{rl}
& \|v-p\|^2_{H^1(\Omega_{l_{m+1}})^d} \\
\le & C C_P \left(C_T \, \|v-p\|^2_{H^1(\Omega_{l_m})^d} + d_{l_{m+1}}(v,v) + \dfrac{1}{diam(\Omega_k)} \displaystyle\int_{\gamma_{l_m l_{m+1}}} (\pi_{\gamma_{l_m l_{m+1}}}[v])^2 \right).
\end{array}
$$

By induction on m and from (25), it is then obtained from $\#\mathcal{N}(\Omega_k) \le P$ that:

$$
\|v-p\|^2_{H^1(\Omega_l)^d} \le C (C_P C_T)^P C_N \left(\sum_{j \in \mathcal{N}(\Omega_k)} d_j(v,v) + \frac{1}{diam(\Omega_k)} \int_{\mathcal{S}_k} (\pi[v])^2 \right),
$$

and therefore:

$$
\begin{array}{rcl}
\displaystyle\sum_{l \in \mathcal{N}(\Omega_k)} \|v-p\|^2_{H^1(\Omega_l)^d} & \le & C (C_P C_T)^P C_N \displaystyle\sum_{l \in \mathcal{N}(\Omega_k)} \left(\sum_{j \in \mathcal{N}(\Omega_k)} d_j(v,v) + \frac{1}{diam(\Omega_k)} \int_{\mathcal{S}_k} (\pi[v])^2 \right) \\
& \le & C P (C_P C_T)^P C_N \left(\displaystyle\sum_{j \in \mathcal{N}(\Omega_k)} d_j(v,v) + \frac{1}{diam(\Omega_k)} \int_{\mathcal{S}_k} (\pi[v])^2 \right),
\end{array}
$$

hence the proof. $\square$

Remark 9 (Satisfaction of a Dirichlet homogeneous boundary condition). *If $v \in X$ satisfies a Dirichlet homogeneous boundary condition on the part Γ_D of the boundary of the domain Ω, its interpolation $\mathbf{P}v$ has the same boundary value on Γ_D provided:*

- *$\mathcal{T}_H \cap \Gamma_D$ is a (possibly curved) triangulation of Γ_D,*
- *the nodes $M_i \in \Gamma_D$ are associated to faces $\gamma_i \in Z$ contained in Γ_D.*

4.4 Uniform coercivity result

We improve herein the coercivity result from proposition 2 by showing that the coercivity constant is independent of the number and the size of the subdomains:

Proposition 4. *There exists a constant $C > 0$ independent of any decomposition of Ω into subdomains satisfying the assumptions of section 4.1, such that for all displacements fields $v \in X$:*

$$\|v\|_X^2 \leq C \left(\sum_{k=1}^{K} d_k(v,v) + \sum_{1 \leq k < l \leq K} \frac{1}{diam(\gamma_{kl})} \int_{\gamma_{kl}} (\pi_{\gamma_{kl}}[v])^2 \right). \tag{26}$$

Proof : For all $v \in V$, the conforming interpolate function $\mathbf{P}v \in (X_H)^d \subset H^1(\Omega)^d$ satisfies the same Dirichlet boundary condition as v (see remark 9) resulting in the usual coercivity result, only depending on the shape of Ω:

$$\tilde{d}(\mathbf{P}v, \mathbf{P}v) = d(\mathbf{P}v, \mathbf{P}v) \geq C \|\mathbf{P}v\|_{H^1(\Omega)^d}^2 = C\|\mathbf{P}v\|_X^2. \tag{27}$$

Consequently, we get from (27) and proposition 3 that:

$$\begin{aligned}
\|v\|_X^2 &= \|v - \mathbf{P}v + \mathbf{P}v\|_X^2, \\
&\leq 2\sum_{k=1}^{K} \|v - \mathbf{P}v\|_{H^1(\Omega_k)^d}^2 + 2\sum_{k=1}^{K} \|\mathbf{P}v\|_{H^1(\Omega_k)^d}^2, \\
&\leq C \sum_{k=1}^{K} \left(\sum_{l \in \mathcal{N}(\Omega_k)} d_l(v,v) + \frac{1}{diam(\Omega_k)} \int_{\mathcal{S}_k} (\pi[v])^2 \right) + C\tilde{d}(\mathbf{P}v, \mathbf{P}v).
\end{aligned}$$

Moreover, we obtain by the triangular inequality and the use of proposition 3 that:

$$\begin{aligned}
\tilde{d}(\mathbf{P}v, \mathbf{P}v) &= \tilde{d}(\mathbf{P}v - v + v, \mathbf{P}v - v + v) \\
&\leq 2\tilde{d}(\mathbf{P}v - v, \mathbf{P}v - v) + 2\tilde{d}(v,v) \\
&\leq 2\sum_{k=1}^{K} |\mathbf{P}v - v|_{H^1(\Omega_k)^d}^2 + 2\tilde{d}(v,v) \\
&\leq C \sum_{k=1}^{K} \left(\sum_{l \in \mathcal{N}(\Omega_k)} d_l(v,v) + \frac{1}{diam(\Omega_k)} \int_{\mathcal{S}_k} (\pi[v])^2 \right) + 2\tilde{d}(v,v),
\end{aligned}$$

which leads to the final estimate:

$$\|v\|_X^2 \leq C \left(\sum_{k=1}^{K} d_k(v,v) + \sum_{1\leq k<l\leq K} \frac{1}{diam(\gamma_{kl})} \int_{\gamma_{kl}} (\pi_{\gamma_{kl}}[v])^2 \right), \qquad \forall v \in X, \quad (28)$$

by exploiting the fact that $\#\mathcal{N}(\Omega_k) \leq P$, and $diam(\Omega_k) \geq diam(\gamma_{kl})$. □

4.5 Existence result for problem (7)

From assumption 3, we have $V_h \subset V$ independently of the discretization, and get the uniform coercivity of the bilinear form $\tilde{a}$ over $V_h \times V_h$. Indeed, for all $v_h \in V_h$, we get from (26) that:

$$\begin{aligned}
\tilde{a}(v_h, v_h) &= \sum_{k=1}^{K} \int_{\Omega_k} (\mathbf{E} : \varepsilon(v_h)) : \varepsilon(v_h) \\
&\geq \min_{k\geq 1}(c_k) \sum_{k=1}^{K} \int_{\Omega_k} \varepsilon(v_h) : \varepsilon(v_h) \\
&\geq C \min_{k\geq 1}(c_k) \left(\sum_{k=1}^{K} \frac{1}{diam(\Omega_k)^2} \int_{\Omega_k} (v_h)^2 + \int_{\Omega_k} |\nabla v_h|^2 \right),
\end{aligned}$$

because $\pi[v_h] = 0$ due to the fact that $v_h \in V$. The coercivity of the bilinear form $\tilde{a}$ over $V_h \times V_h$ is then proved, with independence of the coercivity constant $\tilde{\alpha} = C \min_{k\geq 1}(c_k)$ with respect to the number and the size of the subdomains. Let us remark that when the Young moduli of the subdomains are multiplied by a constant, $\tilde{\alpha}$ is multiplicated as well.
Since $\tilde{a}$ is uniformly coercive over $V_h \times V_h$ and since (9) ensures that the weak-continuity constraint b over the interfaces is onto, the discrete problem (7) is well posed by using Babuska and Brezzi's theory of mixed problems [Bre74, Bab73].

5 Error estimates in elastostatics

5.1 Approximation of displacements

We recall now the standard error estimates in elastostatics under the following assumption:

Assumption 6. *For all $1 \leq m \leq M$, the family of interface meshes $(\mathcal{F}_{m;\delta_m})_{\delta_m>0}$ over the non-mortar side is quasi-uniform, and $\overline{\delta}_m/\delta_m$ remains bounded independently of the chosen discretization.*

Error estimates can then be established under assumptions 1 to 6 (proceeding as in [Woh01] for example):

Proposition 5. *If $u \in \prod_{k=1}^{K} H^{q+1}(\Omega_k)^d$ is solution of (4) with $(\mathbf{E} : \varepsilon(u)) \in \prod_{k=1}^{K} H^q(\Omega_k)^{d\times d}$ and $q \geq 1$, and $(u_h, \lambda_h) \in X_h \times M_\delta$ is solution of (7), the following error estimate holds:*

$$\|u - u_h\|_X \leq C \left(1 + \max_{1\leq k\leq K} \frac{C_k}{\tilde{\alpha}}\right) \left(\sum_{k=1}^{K} h_k^{2q} \, |u|^2_{q+1,\mathbf{E},\Omega_k}\right)^{1/2},$$

with:

$$|u|^2_{q+1,\mathbf{E},\Omega_k} = |u|^2_{H^{q+1}(\Omega_k)^d} + \frac{1}{C_k^2} \|\mathbf{E} : \varepsilon(u)\|^2_{H^q(\Omega_k)^{d\times d}}. \tag{29}$$

The constant C is independent of the number, the diameter, the Young moduli and the discretization of the subdomains. The coercivity constant of $\tilde{a}$ over $V_h \times V_h$ is denoted by $\tilde{\alpha}$ and the coefficients $(C_k)_{1\leq k\leq K}$ characterizing the elasticity tensor $\mathbf{E}$ are defined by (3).

Remark 10. *The difference with the original result of Wohlmuth lies in the fact that the constants appearing in the proof do not depend any more on the number of subdomains.*
The result remains true if we replace in (29) q by any integer $1 \leq r \leq q$ because it relies on interpolation results which hold for any $1 \leq r \leq q$ given our choice of finite element.

Remark 11. *It can be noticed by reading precisely this proof, that a better a priori estimate is obtained when the non-mortar side is taken as the coarsest side (to improve the approximation error) and/or the softer one (to improve the consistency error).*

5.2 Approximation of fluxes

The convergence of Lagrange multipliers uses the inf-sup condition (9) and is established (see [Woh01] for example) by the:

Proposition 6. *If $u \in \prod_{k=1}^{K} H^{q+1}(\Omega_k)^d$ is solution of (4) with $(\mathbf{E} : \varepsilon(u)) \in \prod_{k=1}^{K} H^q(\Omega_k)^{d\times d}$ and $q \geq 1$, and $(u_h, \lambda_h) \in X_h \times M_\delta$ is solution of (7), the following error estimate on Lagrange multipliers holds:*

$$\|\lambda - \lambda_h\|_{\delta,-\frac{1}{2}} \leq C \left(\sum_{k=1}^{K} h_k^{2q} \, |u|^2_{q+1,\mathbf{E},\Omega_k}\right)^{1/2},$$

with $\lambda = (\mathbf{E} : \varepsilon(u)) \cdot n$, where n is the normal unit vector on $\mathcal{S}$ which is outward to $\Omega_{k(m)}$ for all $1 \leq m \leq M$. In more details, the constant C has the following dependence:

$$C = C' \max_{1\leq k\leq K} C_k \left(1 + \frac{1}{\beta}\right) + C' \frac{\max\limits_{1\leq k\leq K} C_k}{\beta} \left(1 + \max_{1\leq k\leq K} \frac{C_k}{\tilde{\alpha}}\right),$$

where the various constants denoted by C' do not depend on the number, the diameter, the Young moduli and the discretization of the subdomains.

6 Generalization to elastodynamics.

In this section, we analyze the use of mortar elements to solve the linear elastodynamics problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - div\,(\mathbf{E} : \varepsilon(u)) = f, \quad [0,T] \times \Omega, \\ (\mathbf{E} : \varepsilon(u)) \cdot \nu = g, \quad [0,T] \times \Gamma_N, \\ u = 0, \quad [0,T] \times \Gamma_D, \\ u = u_0, \quad \{0\} \times \Omega, \\ \dfrac{\partial u}{\partial t} = \dot{u}_0, \quad \{0\} \times \Omega, \end{cases} \tag{30}$$

with obvious notation. Let us only notice that the normal outward unit vector over a surface is now denoted by ν instead of n to avoid any possible confusion with the forthcoming numbering of the time steps.

First, the notation of the static case is adapted and a standard result of existence recalled in the elastodynamics framework. In the second subsection, a total approximation in space and time is introduced for the dynamical solution. It uses a mid-point finite difference time integration scheme which is interesting for energy conservation purpose, and a non-conforming finite element space approximation using a mortar weak-continuity constraint over the interfaces. We finally establish the convergence of the approximate solution to the continuous one, which is the main contribution of this section.

Moreover, an important remark has to be done with respect to this analysis. For first order problems in time, Lagrange multipliers are involved in the convergence analysis through the estimation of:

$$\inf_{\mu_h \in M_\delta} \| (\mathbf{E} : \varepsilon(u(t))) \cdot \nu - \mu_h \|_{\delta, -\frac{1}{2}},$$

as shown for example in [BMR01] for an eddy currents model. In the framework of second order problems in time, we underline the idea that the Lagrange multipliers are also involved through the estimation of:

$$\inf_{\mu_h \in M_\delta} \left\| \left(\mathbf{E} : \varepsilon \left(\frac{\partial u}{\partial t}(t) \right) \right) \cdot \nu - \mu_h \right\|_{\delta, -\frac{1}{2}},$$

entailing a higher sensitivity with respect to the choice of Lagrange multipliers. A time discontinuity in interface constraints would lead to a deterioration of convergence.

6.1 Position of the problem.

We formulate here the linear elastodynamics problem, using mainly the same notation as in the static case. The body forces are denoted by $f \in L^2(0,T; L^2(\Omega)^d)$, the density of the material by $\rho \in L^\infty(\Omega)$, which is assumed to be greater than a positive constant, and the initial conditions in displacement by $u_0 \in H^1(\Omega)^d$ and in velocity by $\dot{u}_0 \in L^2(\Omega)^d$. A surfacic force $g \in \mathcal{C}^1(0,T; L^2(\Gamma_N)^d)$ which is regular in time is applied over the part Γ_N of the boundary $\partial\Omega$ and a Dirichlet boundary condition $u = 0$ is imposed on the complementary part $\Gamma_D = \partial\Omega \backslash \Gamma_N$ which can be of zero measure. The elastic properties of the material are the same as in the static case described above.

To give a precise meaning to the system (30), we define a solution as a displacement function:

$$u \in \mathcal{C}^0(0, T; H^1_*(\Omega)) \cap \mathcal{C}^1(0, T; L^2(\Omega)^d),$$

such that in the sense of distributions on $]0, T[$:

$$\frac{\partial^2}{\partial t^2} \int_\Omega \rho u(t) \cdot v + a(u(t), v) = \int_\Omega f(t) \cdot v + \int_{\Gamma_N} g(t) \cdot v, \qquad \forall v \in H^1_*(\Omega). \tag{31}$$

It is now standard that:

Proposition 7. *Under the previous assumptions, there exists a unique displacement field* $u \in \mathcal{C}^0(0, T; H^1_*(\Omega)) \cap \mathcal{C}^1(0, T; L^2(\Omega)^d)$*, such that the equation (31) is satisfied in the sense of distributions on* $]0, T[$*. Moreover, the energy:*

$$\mathcal{E}(t) = \frac{1}{2} \int_\Omega \rho \left(\frac{\partial u}{\partial t}(t) \right)^2 + \frac{1}{2} a(u(t), u(t)),$$

is conserved, that is for all $t \in [0, T]$*:*

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_\Omega f(s) \cdot \frac{\partial u}{\partial t}(s)\, ds + \int_0^t \int_{\Gamma_N} g(s) \cdot \frac{\partial u}{\partial t}(s)\, ds.$$

We refer to [LM72, RT98] for a proof of the proposition.

6.2 A midpoint nonconforming fully discrete approximation.

We introduce here a space non-conforming fully discrete approximation of the solution of (30). First, at each time $t \in [0, T]$ the spaces $H^1_*(\Omega)$ and $L^2(\Omega)^d$ for the displacements and the velocities are replaced by the non-conforming finite element space V_h introduced in section 2.3.1 for the elastostatics problem. We then look for the displacements $u^h \in \mathcal{C}^0(0, T; V_h) \cap \mathcal{C}^1(0, T; V_h)$ such that in the sense of distributions on $]0, T[$:

$$\frac{\partial^2}{\partial t^2} \int_\Omega \rho u^h(t) \cdot v^h + \tilde{a}(u^h(t), v^h) = \int_\Omega f(t) \cdot v^h + \int_{\Gamma_N} g(t) \cdot v^h, \qquad \forall v^h \in V_h. \tag{32}$$

The initial conditions in displacement and velocity take the form:

$$\begin{cases} u^h(0) = \mathcal{P}^1_h u_0 & \in V_h, \\ \dfrac{\partial u^h}{\partial t}(0) = \mathcal{P}^0_h \dot{u}_0 & \in V_h, \end{cases} \tag{33}$$

where $\mathcal{P}^1_h$ (resp. $\mathcal{P}^0_h$) denotes a projection from $H^1_*(\Omega)$ (resp. $L^2(\Omega)^d$) to V_h. Now, let $(t_n)_{n\in\mathbb{N}}$ a sequence of discrete times such that $t_n = n\Delta t$ for $n \in \mathbb{N}$. The use of a constant time step Δt enables the optimal time accuracy order established below. The formal integration of (32) and of the additional relation:

$$\frac{\partial}{\partial t}\int_\Omega u^h(t)\cdot v^h = \int_\Omega \frac{\partial u^h}{\partial t}(t)\cdot v^h, \quad \forall v^h \in V_h,$$

over $t \in [t_n, t_{n+1}]$ by the trapezoidal rule gives the following fully discrete system:

$$\begin{cases} \displaystyle\int_\Omega \rho \frac{\dot{u}^h_{n+1} - \dot{u}^h_n}{\Delta t}\cdot v_h + \tilde{a}\left(\frac{u^h_n + u^h_{n+1}}{2}, v_h\right) = \frac{L_n(v_h) + L_{n+1}(v_h)}{2}, \qquad \forall v_h \in V_h, \\ \displaystyle\frac{u^h_{n+1} - u^h_n}{\Delta t} = \frac{\dot{u}^h_n + \dot{u}^h_{n+1}}{2}. \end{cases} \tag{34}$$

We have introduced the virtual work of the applied forces at the discrete time t_n:

$$L_n(v_h) = \int_\Omega f(t_n)\cdot v_h + \int_{\Gamma_N} g(t_n)\cdot v_h, \quad \forall v_h \in V_h,$$

and have denoted by $u^h_n \in V_h$ (resp. $\dot{u}^h_n \in V_h$) the approximation in time of the space approximation $u^h(t_n) \in V_h$ of the displacement (resp. $\dfrac{\partial u^h}{\partial t}(t_n) \in V_h$ of the velocity), that is the fully discrete approximation of the displacement $u(t_n) \in H^1_*(\Omega)$ (resp. the velocity $\dfrac{\partial u}{\partial t}(t_n) \in L^2(\Omega)^3$). This trapezoidal finite difference scheme in time has been selected for its exact conservation properties with respect to the energy and to the linear momentum (see [ST92]).

The convergence analysis to come could be extended to other time integrators. The system has to be completed with the initial conditions:

$$\begin{cases} u^h_0 = \mathcal{P}^1_h u_0 & \in V_h, \\ \dot{u}^h_0 = \mathcal{P}^0_h \dot{u}_0 & \in V_h. \end{cases} \tag{35}$$

Knowing $u^h_n, \dot{u}^h_n \in V_h$ and after elimination of $\dot{u}^h_{n+1}$ by (34)-2, we can then determine the fully discrete displacement $u^h_{n+1} \in V_h$ at the discrete time $t_{n+1} \in [0, T]$ by solving:

$$\begin{aligned} \int_\Omega \frac{2}{\Delta t^2}\rho\, u^h_{n+1}\cdot v_h + \frac{1}{2}\tilde{a}\left(u^h_{n+1}, v_h\right) \quad = \quad & \int_\Omega \rho\left(\frac{2}{\Delta t^2}u^h_n + \frac{2}{\Delta t}\dot{u}^h_n\right)\cdot v_h \\ & -\frac{1}{2}\tilde{a}\left(u^h_n, v_h\right) \\ & +\frac{L_n(v_h) + L_{n+1}(v_h)}{2}, \qquad \forall v_h \in V_h, \end{aligned}$$

and the velocity $\dot{u}^h_{n+1} \in V_h$ is obtained by the simple computation:

$$\dot{u}^h_{n+1} = \frac{2}{\Delta t}(u^h_{n+1} - u^h_n) - \dot{u}^h_n.$$

The existence of a projection $\mathbb{P}_h$ from $H^1_*(\Omega)$ to V_h is detailed in the following lemma:

Lemma 9. *If Γ_D has a positive measure, there exists a projection operator:*

$$\begin{array}{rlll} \mathbb{P}_h: & H^1_*(\Omega) & \rightarrow & V_h \\ & u & \mapsto & \mathbb{P}_h u, \end{array}$$

such that $\mathbb{P}_h u$ is the unique solution $u_h \in V_h$ of:

$$\tilde{a}(u_h, v_h) = \tilde{a}(u, v_h), \qquad \forall v_h \in V_h.$$

Moreover, for all $u \in H^{r+1}_{\mathbf{E}}(\Omega)$ with $r \geq 1$, we have the following estimates:

$$\|u - \mathbb{P}_h u\|^2_X \leq C \sum_{k=1}^{K} h_k^{2r} |u|^2_{r+1,\mathbf{E},\Omega_k},$$

$$\|u - \mathbb{P}_h u\|^2_{L^2(\Omega)} \leq C \left(\sup_{1 \leq k \leq K} h_k^2 \right) \sum_{k=1}^{K} h_k^{2r} |u|^2_{r+1,\mathbf{E},\Omega_k}.$$

Observation: the last inequality holds within a regularity condition, namely that the solution of all elasticity problems over Ω be in $H^2_{\mathbf{E}}(\Omega)$.

Remark 12. *The constant C in the estimates of proposition 9 is in fact of the form:*

$$C = C' \left(1 + \max_{k \geq 1} \frac{C_k}{\tilde{\alpha}} \right) \max_{k \geq 1} \frac{C_k}{\tilde{\alpha}},$$

where C' is independent of the discretization in space and time, of the number of subdomains, and of the coercivity and continuity constants of the broken bilinear form $\tilde{a}$. Nevertheless, to simplify the present exposition, we will keep the generic notation C.

Proof : The existence of the projection $\mathbb{P}_h$ is a straightforward consequence of the Lax-Milgram lemma. More precisely, for a given function $u \in H^1_*(\Omega)$, let us define the continuous linear form $l \in X'$ by:

$$l(v) = \tilde{a}(u, v), \quad \forall v \in X.$$

The function $u \in H^1_*(\Omega)$ is the unique solution of:

$$a(u, v) = l(v), \quad \forall v \in H^1_*(\Omega),$$

and $\mathbb{P}_h u$ is the unique solution u_h of:

$$\tilde{a}(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h.$$

The error between u_h and u in the broken norm $\| \cdot \|_X$ is given in the proposition 5, resulting in the announced estimate. The estimation in the $L^2(\Omega)^d$ norm can be obtained by a Aubin-Nitsche argument (cf. [Aub87] for example) that we detail here. Let us assume that for all $\phi \in L^2(\Omega)^d$, there exist a solution $\zeta_\phi \in H^2_{\mathbf{E}}(\Omega)$ of:

$$\tilde{a}(v, \zeta_\phi) = \int_\Omega \phi \cdot v, \quad \forall v \in H^1_*(\Omega). \tag{36}$$

Indeed, we have assumed that the solution of all elasticity problems over Ω be in $H^2_{\mathbf{E}}(\Omega)$. First, because the application:

$$\begin{array}{rccl} T: & H^2_{\mathbf{E}}(\Omega) \cap H^1_*(\Omega) & \to & H^1_*(\Omega)' \\ & \zeta & \mapsto & T\zeta; \quad \langle T\zeta, v\rangle_{H^1_*(\Omega)', H^1(\Omega)} = \tilde{a}(v, \zeta), \end{array}$$

is linear, continuous and bijective, the inverse T^{-1} is continuous by the open application theorem [Bré99, Yos65]. As a consequence, the solution $\zeta_\phi \in H^2_{\mathbf{E}}(\Omega)$ of (36) satisfies:

$$\left(\sum_{k=1}^{K} C_k^2 \, |\zeta_\phi|^2_{2,\mathbf{E},\Omega_k} \right)^{1/2} \leq C \, \|\phi\|_{H^1_*(\Omega)'} \leq C \, \|\phi\|_{L^2(\Omega)^d}. \tag{37}$$

Now, let us prove the announced upper bound on $\|u - \mathbb{P}_h u\|_{L^2(\Omega)^d}$, by the Aubin-Nitsche technique. Namely:

$$\begin{aligned} \|u - \mathbb{P}_h u\|_{L^2(\Omega)^d} &= \sup_{\phi \in L^2(\Omega)^d \setminus \{0\}} \frac{\displaystyle\int_\Omega (u - \mathbb{P}_h u) \cdot \phi}{\|\phi\|_{L^2(\Omega)^d}} \\ &= \sup_{\phi \in L^2(\Omega)^d \setminus \{0\}} \frac{\tilde{a}(u - \mathbb{P}_h u, \zeta_\phi)}{\|\phi\|_{L^2(\Omega)^d}}, \end{aligned}$$

and by definition of $\mathbb{P}_h$, $\tilde{a}(u - \mathbb{P}_h u, v_h) = 0$ for all $v_h \in V_h$, resulting in the following expression for all $v_h \in V_h$, and ϕ realizing the supremum in the above inequality:

$$\begin{aligned} \|u - \mathbb{P}_h u\|_{L^2(\Omega)^d} &\leq \frac{\tilde{a}(u - \mathbb{P}_h u, \zeta_\phi - v_h)}{\|\phi\|_{L^2(\Omega)^d}}, \\ &\leq \frac{\tilde{a}(u - \mathbb{P}_h u, u - \mathbb{P}_h u)^{1/2} \, \tilde{a}(\zeta_\phi - v_h, \zeta_\phi - v_h)^{1/2}}{\|\phi\|_{L^2(\Omega)^d}}. \end{aligned}$$

By taking the infimum of the right hand side over $v_h \in V_h$, and by using the approximation property of $\mathbb{P}_h$ in X (proposition 5), and the relation (37), we get:

$$\begin{aligned} \|u - \mathbb{P}_h u\|_{L^2(\Omega)^d} &\leq C \, \frac{\left(\sum_{k=1}^K h_k^{2r} |u|^2_{r+1,\mathbf{E},\Omega_k}\right)^{1/2} \left(\sum_{k=1}^K h_k^2 C_k^2 |\zeta_\phi|^2_{2,\mathbf{E},\Omega_k}\right)^{1/2}}{\|\phi\|_{L^2(\Omega)^d}} \\ &\leq C \left(\sum_{k=1}^{K} h_k^{2r} |u|^2_{r+1,\mathbf{E},\Omega_k} \right)^{1/2} \left(\sup_{1 \leq k \leq K} h_k \right). \end{aligned}$$

□

6.3 Convergence analysis

Now, we prove the convergence of the fully discrete approximation given by (34) to the continuous solution of (31). For that purpose, we introduce the following space:

$$H^{q+1}_{\mathbf{E}}(\Omega) = \{ v \in H^1_*(\Omega); \quad \|v\|_{q+1,\mathbf{E},\Omega} < +\infty \},$$

which is endowed with the following norm:

$$\|v\|^2_{q+1,\mathbf{E},\Omega} = \|v\|^2_{H^1(\Omega)^d} + \sum_{k=1}^{K}\left(\|v\|^2_{H^{q+1}(\Omega_k)^d} + \frac{1}{C_k^2}\|\mathbf{E}:\varepsilon(v)\|^2_{H^q(\Omega_k)^{d\times d}}\right).$$

We also denote as in proposition 5:

$$|v|^2_{q+1,\mathbf{E},\Omega_k} = |v|^2_{H^{q+1}(\Omega_k)^d} + \frac{1}{C_k^2}\|\mathbf{E}:\varepsilon(v)\|^2_{H^q(\Omega_k)^{d\times d}},$$

and state the main result of that section:

Proposition 8 (Error estimate). *If*

$$u \in \mathcal{C}^1(0,T;H^{q+1}_{\mathbf{E}}(\Omega)) \cap \mathcal{C}^2(0,T;\prod_{k=1}^{K} H^{r+1}(\Omega_k)^d) \cap \mathcal{C}^4(0,T;L^2(\Omega)^d)$$

is solution of (31) and $(u_n^h;\dot{u}_n^h)_{n\in\mathbb{N}}$ is the fully discrete solution of (34), then the following error estimate holds:

$$\begin{aligned}
&\left\|\sqrt{\rho}\left(\dot{u}(t_{n+1/2}) - \frac{\dot{u}_n^h + \dot{u}_{n+1}^h}{2}\right)\right\|^2_{L^2(\Omega)^d} + \tilde{\alpha}\left\|u(t_{n+1/2}) - \frac{u_n^h + u_{n+1}^h}{2}\right\|^2_X \\
&\leq C\left(\|\mathcal{P}_h\dot{u}_0 - \dot{u}_0^h\|^2_{L^2(\Omega)^d} + \|\mathcal{P}_h u_0 - u_0^h\|^2_X\right) \\
&+C\Bigg[\left(\frac{\Delta t}{t_0}\right)^4\left\{\tilde{\alpha}\sup_{t\in[0,T]}\|t_0^2\ddot{u}(t)\|^2_X + \sup_{t\in[0,T]}\|\sqrt{\rho}\,t_0^2\,\dddot{u}(t)\|^2_{L^2(\Omega)^d} + \frac{T}{t_0}\sup_{t\in[0,T]}\|\sqrt{\rho}\,t_0^3\,\ddddot{u}(t)\|^2_{L^2(\Omega)^d}\right\} \\
&+h^2\frac{T}{t_0}\sum_{k=1}^{K}h_k^{2r}\left(\sup_{t\in[0,T]}|\sqrt{\rho}\,t_0\,\ddot{u}(t)|^2_{r+1,\mathbf{E},\Omega_k} + \sup_{t\in[0,T]}|\sqrt{\rho}\,\dot{u}(t)|^2_{r+1,\mathbf{E},\Omega_k}\right) + \tilde{\alpha}\sum_{k=1}^{K}h_k^{2q}\sup_{t\in[0,T]}|u(t)|^2_{q+1,\mathbf{E},\Omega_k} \\
&+\sum_{k=1}^{K}h_k^{2q}\frac{C_k^2}{\tilde{\alpha}}\left\{\sup_{t\in[0,T]}|u(t)|^2_{q+1,\mathbf{E},\Omega} + \frac{T}{t_0}\sup_{t\in[0,T]}|t_0\,\dot{u}(t)|^2_{q+1,\mathbf{E},\Omega}\right\}\Bigg]\left(1+\frac{\Delta t}{t_0}\right)^n,
\end{aligned}$$

where C denotes various constants independent of the discretization in space and time, and $t_{n+1/2} = \dfrac{1}{2}(t_n + t_{n+1})$. Moreover, $\mathcal{P}_h$ is the projection $\mathbb{P}_h$ from $H^1_*(\Omega)$ to V_h given in lemma 9 if Γ_D has a positive measure, and is defined by (49) if Γ_D has a null measure, and r is any integer with $1 \leq r \leq q$. Finally, t_0 is a reference length of time.

In order to simplify the exposition of the proof, we assume that Γ_D has a positive measure so that the bilinear form a is coercive over $H^1_*(\Omega) \times H^1_*(\Omega)$. We will enumerate in the remark following the proof the necessary modifications when Γ_D has a null measure. The proof is inspired by the convergence proof introduced in [TM00] for fluid-structure analysis.

Proof : For clarity, the proof is decomposed into six parts. The time derivative of u will be sometimes denoted by $\dot{u}$ to simplify notation.

1. The discrete evolution of error.

Let us define the projection on V_h of the error in displacements at time t_n by:

$$eu_n^h = \mathbb{P}_h u(t_n) - u_n^h,$$

and a new approximation $(V_n^h)_{n\geq 0}$ of velocities by:

$$\frac{1}{2}\left(V_n^h + V_{n+1}^h\right) = \frac{1}{\Delta t}\left(\mathbb{P}_h u(t_{n+1}) - \mathbb{P}_h u(t_n)\right),$$

with the initial condition $V_0^h = \mathbb{P}_h \dot{u}_0$. The gap between the fully discrete velocity $\dot{u}_n^h$ and V_n^h is then defined by:

$$eV_n^h = V_n^h - \dot{u}_n^h.$$

We now establish the equation satisfied by these errors.
To do so, we first show that for all $t \in [0,T]$:

$$\int_\Omega \rho \frac{\partial^2 u}{\partial t^2}(t)\cdot v_h + \tilde{a}(u(t), v_h) = \int_\Omega f(t)\cdot v_h + \int_{\Gamma_N} g(t)\cdot v_h + \int_{\mathcal{S}} \lambda(t)\cdot [v_h], \qquad \forall v_h \in V_h, \tag{38}$$

with $\lambda(t) = (\mathbf{E} : \varepsilon(u(t)))\cdot \nu$, where ν is the normal unit vector on $\mathcal{S}$ which is outward to the non-mortar subdomain.
Due to the assumptions that for all $t \in [0,T]$, $(\mathbf{E} : \varepsilon(u(t))) \in \prod_{k=1}^K H^1(\Omega_k)^{d\times d}$ and that the time derivatives of u have a classical sense, we obtain from (31) that for all $t \in [0,T]$ and all $v \in \mathcal{C}_c^\infty(\Omega)^d$:

$$\int_\Omega \left(\rho \frac{\partial^2 u}{\partial t^2}(t) - div\,(\mathbf{E} : \varepsilon(u(t))) - f(t)\right)\cdot v = 0.$$

By density of $\mathcal{C}_c^\infty(\Omega)^d$ in $L^2(\Omega)^d$ we have then that for all $t \in [0,T]$:

$$\rho \frac{\partial^2 u}{\partial t^2}(t) - div\,(\mathbf{E} : \varepsilon(u(t))) - f(t) = 0, \quad \text{ in } L^2(\Omega)^d. \tag{39}$$

Then, we can obtain some information about the natural boundary conditions. Indeed, we get a fortiori from (39) that:

$$\int_\Omega \left(\rho \frac{\partial^2 u}{\partial t^2}(t) - div\,(\mathbf{E} : \varepsilon(u(t))) - f(t)\right)\cdot v = 0, \quad \forall v \in H_*^1(\Omega), \tag{40}$$

and by substracting the original problem (31) to (40), we obtain for all $v \in H_*^1(\Omega)$:

$$\begin{aligned}\int_{\Gamma_N} g(t)\cdot v \quad &= \quad \int_\Omega (\mathbf{E} : \varepsilon(u(t))) : \nabla v + \int_\Omega div\,(\mathbf{E} : \varepsilon(u(t)))\cdot v \\ &:= \quad \int_{\Gamma_N} ((\mathbf{E} : \varepsilon(u(t)))\cdot \nu)\cdot v.\end{aligned}$$

Obviously, this relation does not depend on $v \in H_*^1(\Omega)$ but only on its trace $v|_{\Gamma_N} \in H_{00}^{1/2}(\Gamma_N)^d$, resulting in:

$$\int_{\Gamma_N} g(t)\cdot \phi = \int_{\Gamma_N} ((\mathbf{E} : \varepsilon(u(t)))\cdot \nu)\cdot \phi, \quad \forall \phi \in H_{00}^{1/2}(\Gamma_N)^d. \tag{41}$$

Now, we can show the relation (38). By exploiting the divergence formula, and the results (39) and (41), we get for all $t \in [0,T]$, and all $v_h \in V_h$:

$$
\begin{aligned}
\tilde{a}(u, v_h) &= \sum_{k=1}^{K} \int_{\Omega_k} (\mathbf{E} : \varepsilon(u(t))) : \varepsilon(v_h) \\
&= -\sum_{k=1}^{K} \int_{\Omega_k} div\,(\mathbf{E} : \varepsilon(u(t))) \cdot v_h + \sum_{k=1}^{K} \int_{\partial\Omega_k} ((\mathbf{E} : \varepsilon(u(t))) \cdot \nu) \cdot v_h \\
&= \int_{\Omega} \left(f(t) - \rho \frac{\partial^2 u}{\partial t^2}(t) \right) \cdot v_h + \int_{\Gamma_N} g(t) \cdot v_h + \int_{\mathcal{S}} \lambda(t) \cdot [v_h],
\end{aligned}
$$

resulting in the announced expression (38).
By computing the half sum of the expressions (38) for $t = t_n$ and $t = t_{n+1}$ and substracting the first line of the system (34), it comes that for all $v_h \in V_h$:

$$
\int_{\Omega} \rho \frac{V_{n+1}^h - V_n^h}{\Delta t} \cdot v_h - \int_{\Omega} \rho \frac{\dot{u}_{n+1}^h - \dot{u}_n^h}{\Delta t} \cdot v_h + \tilde{a}\left(\frac{u(t_n) - u_n^h}{2} + \frac{u(t_{n+1}) - u_{n+1}^h}{2}, v_h \right)
$$

$$
= \int_{\Omega} \rho \frac{V_{n+1}^h - V_n^h}{\Delta t} \cdot v_h - \frac{1}{2} \int_{\Omega} \rho \left(\frac{\partial^2 u}{\partial t^2}(t_n) + \frac{\partial^2 u}{\partial t^2}(t_{n+1}) \right) \cdot v_h + \int_{\mathcal{S}} \frac{\lambda(t_n) + \lambda(t_{n+1})}{2} \cdot [v_h],
$$

where we have added the term $\int_{\Omega} \rho \frac{V_{n+1}^h - V_n^h}{\Delta t} \cdot v_h$ on the both sides of the equality.
From the lemma 9 and the definitions of eu_n^h and eV_n^h, we deduce that for all $v_h \in V_h$:

$$
\int_{\Omega} \rho \frac{eV_{n+1}^h - eV_n^h}{\Delta t} \cdot v_h + \tilde{a}\left(\frac{eu_n^h + eu_{n+1}^h}{2}, v_h \right)
$$

$$
= \int_{\Omega} \rho \frac{V_{n+1}^h - V_n^h}{\Delta t} \cdot v_h - \frac{1}{2} \int_{\Omega} \rho \left(\frac{\partial^2 u}{\partial t^2}(t_n) + \frac{\partial^2 u}{\partial t^2}(t_{n+1}) \right) \cdot v_h + \int_{\mathcal{S}} \frac{\lambda(t_n) + \lambda(t_{n+1})}{2} \cdot [v_h],
$$

that we sum up in the following expression:

$$
\int_{\Omega} \rho \frac{eV_{n+1}^h - eV_n^h}{\Delta t} \cdot v_h + \tilde{a}\left(\frac{eu_n^h + eu_{n+1}^h}{2}, v_h \right) = E_{n+1/2}^a(v_h) + E_{n+1/2}^c(v_h). \tag{42}
$$

We have denoted the approximation error in time and space by:

$$
E_{n+1/2}^a(v_h) = \int_{\Omega} \sqrt{\rho}\, T_{n+1/2} \cdot v_h, \quad \forall v_h \in V_h,
$$

with:

$$
T_{n+1/2} = \sqrt{\rho} \frac{V_{n+1}^h - V_n^h}{\Delta t} - \frac{1}{2} \sqrt{\rho} \left(\frac{\partial^2 u}{\partial t^2}(t_n) + \frac{\partial^2 u}{\partial t^2}(t_{n+1}) \right),
$$

and the consistency error by:

$$
E_{n+1/2}^c(v_h) = \frac{1}{2} \int_{\mathcal{S}} (\lambda(t_n) + \lambda(t_{n+1})) \cdot [v_h], \quad \forall v_h \in V_h.
$$

It will be convenient to have estimations at midtime steps, and this is why we introduce the midtime quantities:

$$eV^h_{n+1/2} = \frac{eV^h_n + eV^h_{n+1}}{2}, \qquad eu^h_{n+1/2} = \frac{eu^h_n + eu^h_{n+1}}{2},$$

whose evolution is given by averaging (42) between two consecutive time steps. We get for all $v_h \in V_h$:

$$\int_\Omega \rho \frac{eV^h_{n+1/2} - eV^h_{n-1/2}}{\Delta t} \cdot v_h + \tilde{a}\left(\frac{eu^h_{n-1/2} + eu^h_{n+1/2}}{2}, v_h\right) = E^a_n(v_h) + E^c_n(v_h), \quad (43)$$

where:

$$E^\square_n(v_h) = \frac{1}{2}\left(E^\square_{n-1/2}(v_h) + E^\square_{n+1/2}(v_h)\right), \quad \forall v_h \in V_h,$$

in which $\square$ stands for "a" or "c". In (43), we choose:

$$v_h = \frac{eu^h_{n+1/2} - eu^h_{n-1/2}}{\Delta t} = \frac{eV^h_{n-1/2} + eV^h_{n+1/2}}{2},$$

by construction of $(V^h_n)_{n\geq 0}$, which gives by summation on all time steps between 1 and n the main estimation of this first step of the proof:

$$\eta^h_{n+1/2} - \eta^h_{1/2} = \Delta t \sum_{i=1}^{n} E^a_i\left(\frac{eV^h_{i-1/2} + eV^h_{i+1/2}}{2}\right) + E^c_i\left(\frac{eu^h_{i+1/2} - eu^h_{i-1/2}}{\Delta t}\right), \quad (44)$$

with:

$$\eta^h_{n+1/2} = \frac{1}{2}\int_\Omega \rho eV^h_{n+1/2} \cdot eV^h_{n+1/2} + \frac{1}{2}\tilde{a}(eu^h_{n+1/2}, eu^h_{n+1/2}).$$

2. An upper bound for $\eta^h_{1/2}$.

We establish here an upper bound for $\eta^h_{1/2}$. By definition of $\eta^h_{1/2}$, we get by using the symmetry of $\tilde{a}$:

$$\begin{aligned}
\eta^h_{1/2} &= \frac{1}{2}\int_\Omega \rho\left(\frac{eV^h_0 + eV^h_1}{2}\right)^2 + \frac{1}{2}\tilde{a}\left(\frac{eu^h_0 + eu^h_1}{2}, \frac{eu^h_0 + eu^h_1}{2}\right) \\
&\leq \frac{1}{4}\int_\Omega \rho\,(eV^h_0)^2 + \frac{1}{4}\int_\Omega \rho\,(eV^h_1)^2 + \frac{1}{4}\tilde{a}(eu^h_0, eu^h_0) + \frac{1}{4}\tilde{a}(eu^h_1, eu^h_1).
\end{aligned}$$

Using (42) with $n = 0$ and:

$$v_h = \frac{eu^h_1 - eu^h_0}{\Delta t} = \frac{eV^h_0 + eV^h_1}{2}$$

by construction, we obtain:

$$\begin{aligned}
\frac{1}{2}\int_\Omega \rho(eV^h_1)^2 + \frac{1}{2}\tilde{a}(eu^h_1, eu^h_1) &= \frac{1}{2}\int_\Omega \rho\,(eV^h_0)^2 + \frac{1}{2}\tilde{a}(eu^h_0, eu^h_0) \\
&\quad + \frac{\Delta t}{2}E^a_{1/2}(eV^h_1) + E^c_{1/2}\left(eu^h_1 - eu^h_0\right).
\end{aligned}$$

The approximation term in the right hand side can be bounded by using the Cauchy-Schwarz inequality:

$$\begin{aligned} \frac{\Delta t}{2} E^a_{1/2}(eV_1^h) &\le \left\|\Delta t\, T_{1/2}\right\|_{L^2(\Omega)^d} \left\| \frac{1}{2}\sqrt{\rho} e V_1^h \right\|_{L^2(\Omega)^d} \\ &\le \frac{1}{2}\left\|\Delta t\, T_{1/2}\right\|^2_{L^2(\Omega)^d} + \frac{1}{8}\int_\Omega \rho (eV_1^h)^2. \end{aligned}$$

Moreover:

$$\begin{aligned} \Delta t T_{1/2} &= \sqrt{\rho}\left(V_1^h - V_0^h - \frac{\Delta t}{2}(\ddot{u}(t_0) + \ddot{u}(t_1))\right) \\ &= \sqrt{\rho}\left(\frac{2}{\Delta t}\left(\mathbb{P}_h u(t_1) - \mathbb{P}_h u(t_0)\right) - 2\mathbb{P}_h \dot{u}(t_0) - \frac{\Delta t}{2}\left(\ddot{u}(t_0) + \ddot{u}(t_1)\right)\right) \\ &= \sqrt{\rho}\left(\frac{2}{\Delta t}\left(u(t_1) - u(t_0)\right) - 2\dot{u}(t_0) - \frac{\Delta t}{2}\left(\ddot{u}(t_0) + \ddot{u}(t_1)\right)\right) \\ &\quad + \sqrt{\rho}(\mathbb{P}_h - id)\left(\frac{2}{\Delta t}\left(u(t_1) - u(t_0)\right) - 2\dot{u}(t_0)\right). \end{aligned} \tag{45}$$

We then use the lemma 9 and a Taylor's expansion with integral remainder to bound the second term in (45) as follows:

$$\begin{aligned} &\left\| \sqrt{\rho}(\mathbb{P}_h - id)\left(\frac{2}{\Delta t}\left(u(t_1) - u(t_0)\right) - 2\dot{u}(t_0)\right)\right\|^2_{L^2(\Omega)^d} \\ &\le C\, h^2 \sum_{k=1}^K h_k^{2r} \left| \frac{2\sqrt{\rho}}{\Delta t}\left(u(t_1) - u(t_0)\right) - 2\dot{u}(t_0)\right|^2_{r+1,\mathbf{E},\Omega_k} \\ &\le C\, \left(\frac{\Delta t}{t_0}\right)^2 h^2 \sum_{k=1}^K h_k^{2r} \sup_{t\in[0,\Delta t]} |\sqrt{\rho}\, t_0\, \ddot{u}(t)|^2_{r+1,\mathbf{E},\Omega_k}. \end{aligned}$$

The first term in (45) is also bounded by the use of a Taylor's expansion with integral remainder, resulting in:

$$\begin{aligned} \Delta t^2 \|T_{1/2}\|^2_{L^2(\Omega)^d} \le C\Bigg(&\left(\frac{\Delta t}{t_0}\right)^4 \sup_{t\in[0,\Delta t]} \|\sqrt{\rho}\, t_0^2\, \dddot{u}(t)\|^2_{L^2(\Omega)^d} \\ &+ \left(\frac{\Delta t}{t_0}\right)^2 h^2 \sum_{k=1}^K h_k^{2r} \sup_{t\in[0,\Delta t]} |\sqrt{\rho}\, t_0\, \ddot{u}(t)|^2_{r+1,\mathbf{E},\Omega_k} \Bigg). \end{aligned}$$

For the consistency term, we use that $eu_1^h - eu_0^h \in V_h$, the Cauchy-Schwarz inequality, and the two following inequalities (classically obtained when analysing the a priori error in the elastostatics framework -see [Woh01] or [Hau04] page 144):

$$\|[v_h]\|_{\delta,\frac{1}{2},m} \le C\|[v_h]\|_{H^{1/2}(\Gamma_m)^d}, \quad \forall v_h \in V_h, \tag{46}$$

$$\inf_{\mu_h\in M_\delta} \|\lambda - \mu_h\|^2_{\delta,-\frac{1}{2}} \le C\sum_{k=1}^K h_k^{2q}\|\mathbf{E} : \varepsilon(u)\|^2_{H^q(\Omega_k)^{d\times d}}, \tag{47}$$

to obtain that:

$$
\begin{aligned}
E^c_{1/2}\left(eu^h_1 - eu^h_0\right) \quad & \leq \max_{i=0,1} \int_{\mathcal{S}} \lambda(t_i)\cdot[eu^h_1 - eu^h_0] \\
& \leq \max_{i=0,1} \int_{\mathcal{S}} (\lambda(t_i) - \mu_h)\cdot[eu^h_1 - eu^h_0], \quad \forall \mu_h \in M_\delta \\
& \leq \theta \max_{i=0,1} \inf_{\mu_h\in M_\delta} \|\lambda(t_i)-\mu_h\|_{\delta,-\frac{1}{2}} \frac{1}{\theta}\|eu^h_1 - eu^h_0\|_X, \quad \forall \theta \in]0,+\infty[, \\
& \leq C\theta^2 \max_{i=0,1} \inf_{\mu_h\in M_\delta} \|\lambda(t_i)-\mu_h\|^2_{\delta,-\frac{1}{2}} + \frac{1}{2\theta^2}\|eu^h_1 - eu^h_0\|^2_X \\
& \leq C\theta^2 \sum_{k=1}^K h_k^{2q} C_k^2 \sup_{t\in[0,\Delta t]} |u(t)|^2_{q+1,\mathbf{E},\Omega_k} + \frac{1}{\theta^2}\|eu^h_1\|^2_X + \frac{1}{\theta^2}\|eu^h_0\|^2_X.
\end{aligned}
$$

As Γ_D has not a null measure, the bilinear form $\tilde{a}$ is coercive over $V_h \times V_h$. Then, we choose $\theta^2 = 8/\tilde{\alpha}$ where $\tilde{\alpha}$ is the coercivity constant of $\tilde{a}$ over $V_h \times V_h$, and obtain the final estimation:

$$
\begin{aligned}
& \frac{3}{8}\int_\Omega \rho(eV^h_1)^2 + \frac{3}{8}\tilde{a}(eu^h_1, eu^h_1) \leq \frac{1}{2}\int_\Omega \rho(eV^h_0)^2 + \frac{5}{8}\tilde{a}(eu^h_0, eu^h_0) \\
& + C\left(\frac{\Delta t}{t_0}\right)^4 \sup_{t\in[0,\Delta t]} \|\sqrt{\rho}\, t_0^2\, \dddot{u}(t)\|^2_{L^2(\Omega)^d} \\
& + C\ \left(\frac{\Delta t}{t_0}\right)^2 h^2 \sum_{k=1}^K h_k^{2r} \sup_{t\in[0,\Delta t]} |\sqrt{\rho}\, t_0\, \ddot{u}(t)|^2_{r+1,\mathbf{E},\Omega_k} \\
& + C\sum_{k=1}^K h_k^{2q} \frac{C_k^2}{\tilde{\alpha}} \sup_{t\in[0,\Delta t]} |u(t)|^2_{q+1,\mathbf{E},\Omega_k},
\end{aligned}
$$

hence:

$$
\begin{aligned}
\eta^h_{1/2} \quad \leq \quad & C\left(\|\sqrt{\rho} eV^h_0\|^2_{L^2(\Omega)^d} + \tilde{a}(eu^h_0, eu^h_0) + \left(\frac{\Delta t}{t_0}\right)^4 \sup_{t\in[0,\Delta t]} \|\sqrt{\rho}\, t_0^2\, \dddot{u}(t)\|^2_{L^2(\Omega)^d}\right) \\
+ \quad & C\ \left(\frac{\Delta t}{t_0}\right)^2 h^2 \sum_{k=1}^K h_k^{2r} \sup_{t\in[0,\Delta t]} |\sqrt{\rho}\, t_0\, \ddot{u}(t)|^2_{r+1,\mathbf{E},\Omega_k} \\
+ \quad & C\sum_{k=1}^K h_k^{2q} \frac{C_k^2}{\tilde{\alpha}} \sup_{t\in[0,\Delta t]} |u(t)|^2_{q+1,\mathbf{E},\Omega_k}.
\end{aligned}
$$

3. Time and space approximation error estimate.
We estimate here the space and time approximation error given by:

$$
A = \Delta t \sum_{i=1}^n E^a_i \left(\frac{eV^h_{i-1/2} + eV^h_{i+1/2}}{2}\right).
$$

By applying the Cauchy-Schwarz inequality, we obtain:

$$
\begin{aligned}
A &\leq \frac{\Delta t}{t_0}\sum_{i=1}^{n}\left\|t_0\,\frac{T_{i-1/2}+T_{i+1/2}}{2}\right\|_{L^2(\Omega)^d}\left\|\sqrt{\rho}\,\frac{eV^h_{i-1/2}+eV^h_{i+1/2}}{2}\right\|_{L^2(\Omega)^d}\\
&\leq \frac{\Delta t}{2t_0}\sum_{i=1}^{n}\left\|t_0\frac{T_{i-1/2}+T_{i+1/2}}{2}\right\|^2_{L^2(\Omega)^d}+\frac{\Delta t}{2t_0}\sum_{i=1}^{n}\left\|\sqrt{\rho}\,\frac{eV^h_{i-1/2}+eV^h_{i+1/2}}{2}\right\|^2_{L^2(\Omega)^d}\\
&\leq \frac{\Delta t}{2t_0}\sum_{i=1}^{n}\left\|t_0\frac{T_{i-1/2}+T_{i+1/2}}{2}\right\|^2_{L^2(\Omega)^d}+\frac{\Delta t}{2t_0}\sum_{i=0}^{n}\left\|\sqrt{\rho}\,eV^h_{i+1/2}\right\|^2_{L^2(\Omega)^d}.
\end{aligned}
$$

Let us remark that:

$$
\begin{aligned}
&T_{i+1/2}+T_{i-1/2}=\sqrt{\rho}\frac{V^h_{i+1}-V^h_{i-1}}{\Delta t}-\sqrt{\rho}\frac{\ddot{u}(t_{i-1})+2\ddot{u}(t_i)+\ddot{u}(t_{i+1})}{2}\\
&=\sqrt{\rho}\frac{V^h_{i+1}+V^h_i-V^h_i-V^h_{i-1}}{\Delta t}-\sqrt{\rho}\frac{\ddot{u}(t_{i-1})+2\ddot{u}(t_i)+\ddot{u}(t_{i+1})}{2}\\
&=2\sqrt{\rho}\frac{\mathbb{P}_h u(t_{i+1})-2\mathbb{P}_h u(t_i)+\mathbb{P}_h u(t_{i-1})}{\Delta t^2}-\sqrt{\rho}\frac{\ddot{u}(t_{i-1})+2\ddot{u}(t_i)+\ddot{u}(t_{i+1})}{2}\\
&=2\sqrt{\rho}\frac{u(t_{i+1})-2u(t_i)+u(t_{i-1})}{\Delta t^2}-\sqrt{\rho}\frac{\ddot{u}(t_{i-1})+2\ddot{u}(t_i)+\ddot{u}(t_{i+1})}{2}\\
&+2\sqrt{\rho}(\mathbb{P}_h-id)\left(\frac{u(t_{i+1})-2u(t_i)+u(t_{i-1})}{\Delta t^2}\right).
\end{aligned}
$$

Proceeding, as in the estimation of the approximation error of the second step of the proof, we use the lemma 9 and Taylor's expansions with integral remainder to obtain:

$$
\begin{aligned}
&\|T_{i+1/2}+T_{i-1/2}\|^2_{L^2(\Omega)^d}\\
\leq\;& C\Bigg(\Delta t^4\sup_{t\in[0,T]}\|\sqrt{\rho}\,\dddot{u}(t)\|^2_{L^2(\Omega)^d}\\
&+h^2\sum_{k=1}^{K}h_k^{2r}\left|\sqrt{\rho}\,\frac{u(t_{i+1})-2u(t_i)+u(t_{i-1})}{\Delta t^2}\right|^2_{r+1,\mathbf{E},\Omega_k}\Bigg)\\
\leq\;& \frac{C}{t_0^2}\left(\left(\frac{\Delta t}{t_0}\right)^4\sup_{t\in[0,T]}\left\|\sqrt{\rho}\,t_0^3\,\dddot{u}(t)\right\|^2_{L^2(\Omega)^d}+h^2\sum_{k=1}^{K}h_k^{2r}\sup_{t\in[0,T]}|\sqrt{\rho}\,t_0\,\ddot{u}(t)|^2_{r+1,\mathbf{E},\Omega_k}\right)
\end{aligned}
$$

Then:

$$
\frac{\Delta t}{2t_0}\sum_{i=1}^{n-1}\left\|t_0\frac{T_{i-1/2}+T_{i+1/2}}{2}\right\|^2_{L^2(\Omega)^d}\leq\frac{T}{8t_0}t_0^2\sup_{i<n}\|T_{i+1/2}+T_{i-1/2}\|^2_{L^2(\Omega)^d}
$$

$$
\leq C\frac{T}{8t_0}\left(\left(\frac{\Delta t}{t_0}\right)^4\sup_{t\in[0,T]}\left\|\sqrt{\rho}\,t_0^3\,\dddot{u}(t)\right\|^2_{L^2(\Omega)^d}+h^2\sum_{k=1}^{K}h_k^{2r}\sup_{t\in[0,T]}|\sqrt{\rho}\,t_0\,\ddot{u}(t)|^2_{r+1,\mathbf{E},\Omega_k}\right).
$$

4. Consistency error

We estimate here the consistency error given by:

$$B = \Delta t \sum_{i=1}^{n} E_i^c \left(\frac{eu^h_{i+1/2} - eu^h_{i-1/2}}{\Delta t} \right).$$

Using a reorganization of the terms (equivalent to a discrete integration by parts in time), we obtain:

$$\begin{aligned}
B &= \Delta t \sum_{i=1}^{n} \int_{\mathcal{S}} \left(\frac{\lambda(t_{i-1}) + 2\lambda(t_i) + \lambda(t_{i+1})}{4} \right) \cdot \left[\frac{eu^h_{i+1/2} - eu^h_{i-1/2}}{\Delta t} \right] \\
&= \Delta t \sum_{i=1}^{n-1} \int_{\mathcal{S}} \left(\frac{\lambda(t_{i-1}) + \lambda(t_i) - \lambda(t_{i+1}) - \lambda(t_{i+2})}{4\Delta t} \right) \cdot \left[eu^h_{i+1/2} \right] \\
&\quad + \int_{\mathcal{S}} \left(\frac{\lambda(t_{n-1}) + 2\lambda(t_n) + \lambda(t_{n+1})}{4} \right) \cdot \left[eu^h_{n+1/2} \right] \\
&\quad - \int_{\mathcal{S}} \left(\frac{\lambda(t_0) + 2\lambda(t_1) + \lambda(t_2)}{4} \right) \cdot \left[eu^h_{1/2} \right] \\
&= \Delta t \, D + E - F.
\end{aligned}$$

Concerning the $\Delta t D$ term, we proceed exactly as in the estimation of the consistency error of the second step of the proof. More precisely, we use that $\left[eu^h_{i+1/2} \right] \in V_h$, the Cauchy-Schwarz inequality and the inequality (46), the estimation (classically obtained when analysing the a priori error in the elastostatics framework -see [Woh01] or [Hau04] page 143):

$$\inf_{\mu_h \in M_\delta} \|\lambda - \mu_h\|^2_{\delta,-\frac{1}{2},m} \leq C \delta_m^{2q} \|\lambda\|^2_{H^{q-\frac{1}{2}}(\Gamma_m)^d}, \tag{48}$$

and a Taylor's expansion to get:

$$
\begin{aligned}
\Delta t\ D &= \frac{\Delta t}{t_0}\sum_{i=1}^{n-1}\int_{\mathcal{S}}\left(t_0\,\frac{\lambda(t_{i-1})+\lambda(t_i)-\lambda(t_{i+1})-\lambda(t_{i+2})}{4\Delta t}-\mu_h\right)\cdot\left[eu^h_{i+1/2}\right],\quad \forall\mu_h\in M_\delta,\\
&\le \frac{\Delta t}{2t_0}\theta^2\sum_{i=1}^{n-1}\left\|t_0\,\frac{\lambda(t_{i-1})+\lambda(t_i)-\lambda(t_{i+1})-\lambda(t_{i+2})}{4\Delta t}-\mu_h\right\|^2_{\delta,-\frac{1}{2}}\\
&\quad+\frac{\Delta t}{2\theta^2 t_0}\sum_{i=1}^{n-1}\|eu_{i+1/2}\|^2_X,\quad \forall\theta\in]0,+\infty[,\forall\mu_h\in M_\delta,\\
&\le \frac{\Delta t}{2t_0}\theta^2\sum_{i=1}^{n-1}\sum_{k=1}^{K}h_k^{2q}\left\|t_0\,\frac{\lambda(t_{i-1})+\lambda(t_i)-\lambda(t_{i+1})-\lambda(t_{i+2})}{4\Delta t}\right\|^2_{H^{q-\frac{1}{2}}(\partial\Omega_k)^d}\\
&\quad+\frac{\Delta t}{2\theta^2 t_0}\sum_{i=1}^{n-1}\|eu_{i+1/2}\|^2_X,\quad \forall\theta\in]0,+\infty[\\
&\le C\frac{\Delta t}{t_0}\theta^2\sum_{i=1}^{n-1}\sum_{k=1}^{K}h_k^{2q}\sup_{t\in[0,T]}\left\|t_0\,\dot{\lambda}(t)\right\|^2_{H^{q-\frac{1}{2}}(\partial\Omega_k)^d}+\frac{\Delta t}{2\theta^2 t_0}\sum_{i=1}^{n-1}\|eu_{i+1/2}\|^2_X,\quad \forall\theta\in]0,+\infty[\\
&\le C\frac{T}{t_0}\theta^2\sum_{k=1}^{K}h_k^{2q}C_k^2\sup_{t\in[0,T]}|t_0\,\dot{u}(t)|^2_{q+1,\mathbf{E},\Omega_k}+\frac{\Delta t}{2\theta^2 t_0}\sum_{i=1}^{n-1}\|eu_{i+1/2}\|^2_X,\quad \forall\theta\in]0,+\infty[,
\end{aligned}
$$

and by choosing $\theta^2 = 1/\tilde{\alpha}$, we obtain:

$$
\Delta t\ D \le C\frac{T}{t_0}\sum_{k=1}^{K}h_k^{2q}\frac{C_k^2}{\tilde{\alpha}}\sup_{t\in[0,T]}|t_0\,\dot{u}(t)|^2_{q+1,\mathbf{E},\Omega_k}+\frac{\Delta t}{2t_0}\sum_{i=1}^{n-1}\tilde{a}(eu_{i+1/2},eu_{i+1/2}).
$$

The terms E and F are easily bounded by using the same technique:

$$
E \le C\sum_{k=1}^{K}h_k^{2q}\frac{C_k^2}{\tilde{\alpha}}\sup_{t\in[0,T]}|u(t)|^2_{q+1,\mathbf{E},\Omega}+\frac{1}{4}\tilde{a}(eu^h_{n+1/2},eu^h_{n+1/2}),
$$

$$
F \le C\left(\sum_{k=1}^{K}h_k^{2q}\frac{C_k^2}{\tilde{\alpha}}\sup_{t\in[0,T]}|u(t)|^2_{q+1,\mathbf{E},\Omega}+\tilde{a}(eu^h_{1/2},eu^h_{1/2})\right).
$$

Moreover, the second term $\tilde{a}(eu^h_{1/2},eu^h_{1/2})$ in the upper bound of F can be bounded optimally by the second point of the present proof.

5. Estimate on $\eta^h_{n+1/2}$.

Putting together the estimations from the previous points, we obtain that:

$$
\begin{aligned}
&\frac{1}{2}\left(1-\frac{\Delta t}{t_0}\right)\int_\Omega \rho(eV^h_{n+1/2})^2 + \frac{1}{4}\tilde{a}(eu^h_{n+1/2}, eu^h_{n+1/2})\\
&\leq C\left(\int_\Omega \rho(eV^h_0)^2 + \tilde{a}(eu^h_0, eu^h_0)\right)\\
&+C\left(\frac{\Delta t}{t_0}\right)^4\left\{\sup_{t\in[0,T]}\|\sqrt{\rho}\,t_0^2\,\dddot{u}(t)\|^2_{L^2(\Omega)^d} + \frac{T}{t_0}\sup_{t\in[0,T]}\|\sqrt{\rho}\,t_0^3\,\ddddot{u}(t)\|^2_{L^2(\Omega)^d}\right\}\\
&+C\left(\max_{1\leq k\leq K} C_k\right)\, h^2\left(\frac{T}{t_0}+\left(\frac{\Delta t}{t_0}\right)^2\right)\sum_{k=1}^K h_k^{2r}\sup_{t\in[0,T]}|\sqrt{\rho}\,t_0\,\ddot{u}(t)|^2_{r+1,\mathbf{E},\Omega_k}\\
&+C\sum_{k=1}^K h_k^{2q}\frac{C_k^2}{\tilde{\alpha}}\left\{\sup_{t\in[0,T]}|u(t)|^2_{q+1,\mathbf{E},\Omega} + \frac{T}{t_0}\sup_{t\in[0,T]}|t_0\,\dot{u}(t)|^2_{q+1,\mathbf{E},\Omega}\right\}\\
&+\frac{\Delta t}{2t_0}\sum_{i=0}^{n-1}\|\sqrt{\rho}\;eV^h_{i+1/2}\|^2_{L^2(\Omega)^d} + \frac{\Delta t}{2t_0}\sum_{i=1}^{n-1}\tilde{a}(eu^h_{i+1/2}, eu^h_{i+1/2}).
\end{aligned}
$$

We deduce by applying the discrete Gronwall's lemma, and for sufficiently small time steps ($\Delta t \leq t_0/2$) that:

$$
\begin{aligned}
&\int_\Omega \rho(eV^h_{n+1/2})^2 + \tilde{a}(eu^h_{n+1/2}, eu^h_{n+1/2}) \leq C\left(\|eV^h_0\|^2_{L^2(\Omega)^d} + \tilde{a}(eu^h_0, eu^h_0)\right)\\
&+\Bigg[C\left(\frac{\Delta t}{t_0}\right)^4\left\{\sup_{t\in[0,T]}\|\sqrt{\rho}\,t_0^2\,\dddot{u}(t)\|^2_{L^2(\Omega)^d} + \frac{T}{t_0}\sup_{t\in[0,T]}\|\sqrt{\rho}\,t_0^3\,\ddddot{u}(t)\|^2_{L^2(\Omega)^d}\right\}\\
&+C\,h^2\frac{T}{t_0}\sum_{k=1}^K h_k^{2r}\sup_{t\in[0,T]}|\sqrt{\rho}\,t_0\,\ddot{u}(t)|^2_{r+1,\mathbf{E},\Omega_k}\\
&+C\sum_{k=1}^K h_k^{2q}\frac{C_k^2}{\tilde{\alpha}}\left\{\sup_{t\in[0,T]}|u(t)|^2_{q+1,\mathbf{E},\Omega} + \frac{T}{t_0}\sup_{t\in[0,T]}|t_0\,\dot{u}(t)|^2_{q+1,\mathbf{E},\Omega}\right\}\Bigg]\left(1+\frac{\Delta t}{t_0}\right)^n.
\end{aligned}
$$

6. Conclusion.
We end this proof by establishing the announced error estimates on velocities and displacements. Concerning the estimate on velocities, let us remark that:

$$
\dot{u}(t_{n+1/2}) - \frac{\dot{u}^h_n + \dot{u}^h_{n+1}}{2} = \dot{u}(t_{n+1/2}) - \frac{V^h_n + V^h_{n+1}}{2} + eV^h_{n+1/2}.
$$

We have by definition:

$$
\begin{aligned}
\dot{u}(t_{n+1/2}) - \frac{V^h_n + V^h_{n+1}}{2} \quad &= \quad \dot{u}(t_{n+1/2}) - \frac{\mathbb{P}_h u(t_{n+1}) - \mathbb{P}_h u(t_n)}{\Delta t},\\
&= \quad \dot{u}(t_{n+1/2}) - \frac{u(t_{n+1}) - u(t_n)}{\Delta t} + (id - \mathbb{P}_h)\left(\frac{u(t_{n+1}) - u(t_n)}{\Delta t}\right),
\end{aligned}
$$

which entails that:

$$\left\|\sqrt{\rho}\,\left(\dot u(t_{n+1/2})-\frac{V_n^h+V_{n+1}^h}{2}\right)\right\|^2_{L^2(\Omega)^d}$$
$$\leq C\left(\left(\frac{\Delta t}{t_0}\right)^4\sup_{t\in[0,T]}\|\sqrt{\rho}\,t_0^2\,\dddot u(t)\|^2_{L^2(\Omega)^d}+h^2\sum_{k=1}^K h_k^{2r}\sup_{t\in[0,T]}|\sqrt{\rho}\,\dot u(t)|^2_{r+1,\mathbf{E},\Omega_k}\right).$$

Therefore, we deduce the final estimate on velocities by the triangular inequality:

$$\left\|\sqrt{\rho}\,\left(\dot u(t_{n+1/2})-\frac{\dot u_n^h+\dot u_{n+1}^h}{2}\right)\right\|^2_{L^2(\Omega)^d}\leq C\left(\|\sqrt{\rho}\,eV_0^h\|^2_{L^2(\Omega)^d}+\tilde a(eu_0^h,eu_0^h)\right)$$
$$+\Bigg[C\left(\frac{\Delta t}{t_0}\right)^4\left\{\sup_{t\in[0,T]}\|\sqrt{\rho}\,t_0^2\,\dddot u(t)\|^2_{L^2(\Omega)^d}+\frac{T}{t_0}\sup_{t\in[0,T]}\|\sqrt{\rho}\,t_0^3\,\ddddot u(t)\|^2_{L^2(\Omega)^d}\right\}$$
$$+C\,h^2\frac{T}{t_0}\sum_{k=1}^K h_k^{2r}\left(\sup_{t\in[0,T]}|\sqrt{\rho}\,t_0\,\ddot u(t)|^2_{r+1,\mathbf{E},\Omega_k}+\sup_{t\in[0,T]}|\sqrt{\rho}\,\dot u(t)|^2_{r+1,\mathbf{E},\Omega_k}\right)$$
$$+C\sum_{k=1}^K h_k^{2q}\frac{C_k^2}{\tilde\alpha}\left\{\sup_{t\in[0,T]}|u(t)|^2_{q+1,\mathbf{E},\Omega}+\frac{T}{t_0}\sup_{t\in[0,T]}|t_0\,\dot u(t)|^2_{q+1,\mathbf{E},\Omega}\right\}\Bigg]\left(1+\frac{\Delta t}{t_0}\right)^n.$$

We end by the estimate on displacements. We remark that:

$$u(t_{n+1/2})-\frac{u_n^h+u_{n+1}^h}{2}=u(t_{n+1/2})-\frac{\mathbb{P}_hu(t_n)+\mathbb{P}_hu(t_{n+1})}{2}+eu_{n+1/2}^h.$$

Moreover, we notice that:

$$\begin{aligned}u(t_{n+1/2})-\frac{\mathbb{P}_hu(t_n)+\mathbb{P}_hu(t_{n+1})}{2} &= u(t_{n+1/2})-\frac{u(t_n)+u(t_{n+1})}{2}\\ &\quad+(id-\mathbb{P}_h)\left(\frac{u(t_n)+u(t_{n+1})}{2}\right),\end{aligned}$$

resulting in:

$$\left\|u(t_{n+1/2})-\frac{\mathbb{P}_hu(t_n)+\mathbb{P}_hu(t_{n+1})}{2}\right\|^2_X$$
$$\leq C\left(\left(\frac{\Delta t}{t_0}\right)^4\sup_{t\in[0,T]}\|t_0^2\,\ddot u(t)\|^2_X+\sum_{k=1}^K h_k^{2q}\sup_{t\in[0,T]}|u(t)|^2_{q+1,\mathbf{E},\Omega_k}\right),$$

and we conclude by the triangular inequality that:

$$
\begin{aligned}
&\tilde{\alpha} \left\| u(t_{n+1/2}) - \frac{u_n^h + u_{n+1}^h}{2} \right\|_X^2 \\
&\leq C \left(\|\sqrt{\rho}\, eV_0^h\|^2_{L^2(\Omega)^d} + \tilde{\alpha}(eu_0^h, eu_0^h) \right) \\
&+ C \Bigg[\left(\frac{\Delta t}{t_0} \right)^4 \left\{ \tilde{\alpha} \sup_{t\in[0,T]} \|t_0^2 \ddot{u}(t)\|_X^2 + \sup_{t\in[0,T]} \|\sqrt{\rho}\, t_0^2 \dddot{u}(t)\|^2_{L^2(\Omega)^d} + \frac{T}{t_0} \sup_{t\in[0,T]} \|\sqrt{\rho}\, t_0^3 \ddddot{u}(t)\|^2_{L^2(\Omega)^d} \right\} \\
&+ \left(\max_{1\leq k\leq K} C_k \right) h^2 \frac{T}{t_0} \sum_{k=1}^K h_k^{2r} \sup_{t\in[0,T]} |\sqrt{\rho}\, t_0\, \ddot{u}(t)|^2_{r+1,\mathbf{E},\Omega_k} + \tilde{\alpha} \sum_{k=1}^K h_k^{2q} \sup_{t\in[0,T]} |u(t)|^2_{q+1,\mathbf{E},\Omega_k} \\
&+ \sum_{k=1}^K h_k^{2q} \frac{C_k^2}{\tilde{\alpha}} \left\{ \sup_{t\in[0,T]} |u(t)|^2_{q+1,\mathbf{E},\Omega} + \frac{T}{t_0} \sup_{t\in[0,T]} |t_0\, \dot{u}(t)|^2_{q+1,\mathbf{E},\Omega} \right\} \Bigg] \left(1 + \frac{\Delta t}{t_0} \right)^n .
\end{aligned}
$$

The proof is complete. □

Remark 13. *The proof of the convergence has been done in the case where the measure of Γ_D was positive. Let us mention the necessary modifications of the proof when it is not the case. The displacements have to be decomposed in the space of rigid motions:*

$$\mathcal{R} = \{v \in H^1(\Omega)^d, \quad a(v, w) = 0, \forall w \in H^1(\Omega)\},$$

and in the complementary:

$$\mathcal{V} = \{v \in H^1(\Omega)^d, \quad \int_\Omega v \cdot r = 0, \forall r \in \mathcal{R}\},$$

such that $H^1(\Omega)^d = \mathcal{R} \oplus \mathcal{V}$. The solution u of (31) can then be decomposed into $u = \overline{u} + u'$, with $\overline{u} \in \mathcal{C}^0(0,T;\mathcal{R}) \cap \mathcal{C}^1(0,T;\mathcal{R})$ such that in the sense of distributions over $]0,T[$:

$$\frac{\partial^2}{\partial t^2} \int_\Omega \rho \overline{u}(t) \cdot \overline{v} = \int_\Omega f(t) \cdot \overline{v} + \int_{\Gamma_N} g(t) \cdot \overline{v}, \quad \forall \overline{v} \in \mathcal{R},$$

and $u' \in \mathcal{C}^0(0,T;\mathcal{V}) \cap \mathcal{C}^1(0,T;\mathcal{W})$ such that in the sense of distributions over $]0,T[$:

$$\frac{\partial^2}{\partial t^2} \int_\Omega \rho u'(t) \cdot v' + a(u', v') = \int_\Omega f(t) \cdot v' + \int_{\Gamma_N} g(t) \cdot v', \quad \forall v' \in \mathcal{V},$$

with $\mathcal{W} = \{v \in L^2(\Omega)^d, \quad \int_\Omega v \cdot r = 0, \forall r \in \mathcal{R}\}$. The fully discrete approximation of u at time t_n is $u_n^h = \overline{u}_n^h + u_n'^h$ in displacements and $\dot{u}_n^h = \dot{\overline{u}}_n^h + \dot{u}_n'^h$ in velocities. To find $(u_n'^h; \dot{u}_n'^h)_{n\geq 1}$, one has to replace V_h by:

$$V_h' = \{v_h \in V_h; \quad \int_\Omega v_h \cdot r = 0, \forall r \in \mathcal{R}\}$$

in (34). The previous proof gives an upper bound for:

$$\left\| \frac{\partial u'}{\partial t}(t_{n+1/2}) - \frac{\dot{u}'^h_n + \dot{u}'^h_{n+1}}{2} \right\|^2_{L^2(\Omega)^d} + \left\| u'(t_{n+1/2}) - \frac{u'^h_n + u'^h_{n+1}}{2} \right\|^2_{H^1(\Omega)^d},$$

because $\tilde{a}$ is coercive over $V'_h \times V'_h$. To find $(\overline{u}^h_n; \dot{\overline{u}}^h_n)_{n\geq 1}$, one has to replace V_h by $\mathcal{R}$ in (34). An upper bound on:

$$\left\| \frac{\partial \overline{u}}{\partial t}(t_{n+1/2}) - \frac{\dot{\overline{u}}^h_n + \dot{\overline{u}}^h_{n+1}}{2} \right\|^2_{L^2(\Omega)^d},$$

is then obtained by the previous proof, which still applies. Indeed, it is noticeable that there is no consistency error because $\mathcal{R} \subset V_h$, and then, no need of coercivity. Putting together the estimates concerning the rigid motion part of the solution and the complementary part, the announced estimate then remains the same when Γ_D has a null measure.

In this case, the projection $\mathcal{P}_h$ of the proposition 8 can be constructed as follows. For all $u \in H^1(\Omega)^d$, we can build the decomposition $u = \overline{u} + u'$, with $\overline{u} \in \mathcal{R}$ and $u' \in \mathcal{V}$. The projection $\mathcal{P}_h u$ of $u \in H^1(\Omega)^d$ is then defined by:

$$\mathcal{P}_h u = \overline{u} + u'_h, \tag{49}$$

where $u'_h \in V'_h$ is such that:

$$\tilde{a}(u'_h, v'_h) = \tilde{a}(u', v'_h), \quad \forall v' \in V'_h.$$

7 Conclusion

This abstract framework has now to be completed with concrete choices of Lagrange multipliers spaces. In the second part of this paper, we introduce a stabilized discontinuous formulation, more local than usual formulations, which will be analyzed and tested.

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