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Conditional risk measure and robust representation of convex conditional risk measures.

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Abstract

In this paper we introduce a new notion, that of "conditional risk measure", which has properties reminiscent of the conditional expectation, in order to deal with situations of partial information, or asymmetric information. We do so in a general context of uncertainty, where there is not necessarily an a priori given probability.

Important results of this paper are theorems of representation. We study in detail a particular example important in finance, that of a conditional risk measure associated to a loss function.

Introduction

In recent years there has been an increasing interest in methods defining the risk of a financial position. Artzner, Delbaen, Eber and Heath [1] have introduced the concept of coherent risk measures on a probability space. More recently Föllmer and Schied [10] [11] and [12] have addressed a more general issue, defining the notion of monetary measure of risk, not necessarily coherent and in a more general context, where no probability measure is given a priori.

In this paper we introduce a new notion, that of "conditional risk measure",which has properties reminiscent of the conditional expectation, in order to deal with situations of partial information, or asymmetric information. We do so in a general context of uncertainty, where there is not necessarily an a priori given probability.

Partial information means here that the set of financial positions is a linear space $\mathcal{X}$ of bounded maps on a space Ω, and that the investor has not access to all the maps defined on Ω, but only to the measurable maps relative to a σ-algebra $\mathcal{F}$.

In the first section we assume that $(\Omega, \mathcal{F}, P)$ is a probability space and we introduce the notion of risk measure conditional to a probability space as follows: a risk measure conditional to a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, $\rho_{\mathcal{F}}$, associates to each financial position in $\mathcal{X}$ a $(\Omega, \mathcal{F})$ bounded measurable map defined P $a.s.$ and, satisfies axioms of monotonicity, translation invariance by $(\Omega, \mathcal{F})$ bounded measurable maps, and multiplicative invariance by 1_A where A is $\in \mathcal{F}$. This risk measure of a position X conditional to a σ-algebra $\mathcal{F}$ can be viewed as the "minimal" $\mathcal{F}$-measurable map which added to the initial position X makes this position acceptable.

We also consider this notion of conditional risk measure in a case of complete uncertainty, that is in the case where one does not know which probability is "the good one" even on the σ-algebra $\mathcal{F}$ (recall that $\mathcal{F}$ represents all the accessible information). This can be the point of view of a supervising agency.

This new notion of conditional risk measure generalizes the notion of monetary risk measure introduced by H. Föllmer and A. Schied [10] and [11]. It generalizes as well the notion of conditional g-expectation defined by S. Peng [14] and [15] or dynamic risk measure given by E. Rosazza Gianin [16]. Indeed, $\rho_{\mathcal{F}_t}(X) = E_g(-X|\mathcal{F}_t)$ is a risk measure conditional to the Brownian σ-algebra $\mathcal{F}_t$, when $Y_t = E_g(-X|\mathcal{F}_t)$ is the solution of the backward stochastic differential equation $-dY_t = g(t, Z_t)dt - Z_t^* dB_t$; $Y_T = -X$ (see also El Karoui [8], El Karoui, Peng and Quenez [9], and Pardoux and Peng [13] for the backward stochastic differential equations).

An important other example in finance is the conditional risk measure associated to a loss function (which generalizes the monetary risk measure associated lo a loss function introduced by H. Föllmer and A.

Schied [10] and [11]). We assume that an investor has only access to partial information represented by a σ-algebra $\mathcal{F}$. This investor has chosen a convex increasing loss function l ($l(x) = -u(-x)$ where u is his utility function). In that case it is natural to define the conditional risk measure of a position X as the minimal $\mathcal{F}$-measurable map $\rho(X)$ such that $Y = X + \rho(X)$ is acceptable in the following sense: $E(l(-Y)|\mathcal{F}) \leq g \;\; P \; a.s.$ where g is a bounded $\mathcal{F}$-measurable map. When the loss function is exponential we obtain the conditional entropic risk measure. The conditional entropic risk measure generalizes the entropic risk measure studied by H. Föllmer and A. Schied ([10] and [11]), Delbaen *et al* [6], P. Barrieu and N. El Karoui [3]. We study in section 5 the conditional risk measure associated to a loss function and completely describe it.

K. Detlefsen [7] has independently defined and studied a notion of conditional risk measure but only in the particular case where the space of financial positions is a probability space. He assumes that an a priori probability measure is given on the whole space and not only on the space representing the accessible information.

Important results of this paper are the theorems of representation, in the same vein as in Föllmer and Schied [10] and [11], that we obtain in a more technical way, using tools of measure theory and also of convex analysis (section 4).

These main results are as follows. Assume that $\mathcal{X}$ is the set of all bounded measurable functions on a measurable space $(\Omega, \mathcal{G})$. Let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{G}$. Then each convex risk measure continuous from below defined on $\mathcal{X}$ conditional to the σ algebra $\mathcal{F}$ can be represented in terms of conditional expectations for a class M_1 of probability measures on (Ω, G): For all $Q \in M_1$ and $X \in \mathcal{X}$,

$$\rho_{\mathcal{F}}(X) \geq E_Q(-X|\mathcal{F}) - \alpha(Q) \;\; Q \; a.s.$$

$\alpha(Q) = ess\ sup_{\{Y \in \mathcal{A}_{\rho_{\mathcal{F}}}\}}(E_Q(-Y|\mathcal{F}))$ is called the penality function of Q ($\mathcal{A}_{\rho_{\mathcal{F}}}$ is the set of acceptable positions).

In addition for all $X \in \mathcal{X}$ for every probability measure P on $\mathcal{F}$ there is a probability measure $Q_X \in M_1$ whose restriction to $\mathcal{F}$ is equal to P such that

$$\rho_{\mathcal{F}}(X) = E_{Q_X}(-X|\mathcal{F}) - \alpha(Q_X) \;\; P \; a.s.$$

In the case of a risk measure conditional to a probability space $(\Omega, \mathcal{F}.P)$, the preceding representation can be expressed in terms of conditional expectations for a class $M_1(P)$ of probability measures Q on $(\Omega, \mathcal{G})$ such that the restriction of Q to $\mathcal{F}$ is equal to P:

$$\rho_{\mathcal{F}}(X) = ess\ max_{Q \in M_1(P)}((E_Q(-X|\mathcal{F}) - \alpha(Q))$$

Remark: if we don't assume continuity from above, we still get a representation theorem. However it is expressed, as for monetary risk measures, in terms of finitely additive set functions on $\mathcal{G}$ instead of probability measures.

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1 Risk measure conditional to a probability space

1.1 Definition of conditional risk measures

Let Ω be a set. A financial position is described by a bounded map defined on the set Ω of scenarios. We consider a linear space $\mathcal{X}$ of financial positions.

We consider also a σ-algebra $\mathcal{F}$ on the space Ω. Then $(\Omega, \mathcal{F})$ is a measurable space. We denote $\mathcal{E}_{\mathcal{F}}$ the set of all bounded real valued $(\Omega, \mathcal{F})$ measurable maps.

We assume in this first section, in order to be in a well known context, that a probability measure P is given on the σ-algebra $\mathcal{F}$. It is a case of partial uncertainty. It is very relevant in finance. Indeed we assume that the investor has access to partial information represented by the σ-algebra $\mathcal{F}$ and it is natural to study the case where a probability measure is given on this σ-algebra $\mathcal{F}$. We want to define a notion of risk measure conditional to $(\Omega, \mathcal{F}, \mathcal{P})$ which generalizes the notion of monetary risk measure and which has several properties similar to those of the conditional expextation (but not the linearity).

Hence we give the following definition:

Definition 1:

A mapping

$$\rho_{\mathcal{F}} : \mathcal{X} \to L^{\infty}(\Omega, \mathcal{F}, \mathcal{P})$$

is called a risk measure conditional to the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ if it satisfies the following conditions:

- Conditions similar to the properties satisfied by monetary risk measures.

i) monotonicity: for all X,$Y \in \mathcal{X}$ if $X \leq Y$ then $\rho_{\mathcal{F}}(Y) \leq \rho_{\mathcal{F}}(X)$ P $a.s.$

ii) translation invariance: for all $Y \in \mathcal{E}_{\mathcal{F}}$, for all $X \in \mathcal{X}$,

$$\rho_{\mathcal{F}}(X + Y) = \rho_{\mathcal{F}}(X) - Y \ \ P \ a.s.$$

- A new property reminiscent of conditional expectations.

iii) multiplicative invariance: for all $X \in \mathcal{X}$, for all $\mathcal{A} \in \mathcal{F}$,

$$\rho_{\mathcal{F}}(X 1_{\mathcal{A}}) = 1_{\mathcal{A}} \rho_{\mathcal{F}}(X) \ \ P \ a.s.$$

In some cases we will require that the conditional risk measure satisfies additional properties.

Definition 2:

i) A risk measure defined on $\mathcal{X}$ conditional to the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is called convex if for all X,$Y \in \mathcal{X}$, for all $0 \leq \lambda \leq 1$,

$$\rho_{\mathcal{F}}(\lambda X + (1 - \lambda)Y) \leq \lambda \rho_{\mathcal{F}}(X) + (1 - \lambda)\rho_{\mathcal{F}}(Y) \ \ P \ a.s.$$

ii) A convex conditional risk measure is called coherent if it satisfies the positive homogeneity: for all $X \in \mathcal{X}$, for all $\lambda \geq 0$,

$$\rho_{\mathcal{F}}(\lambda X) = \lambda \rho_{\mathcal{F}}(X) \ \ P \ a.s.$$

iii) A convex conditional risk measure is continuous from below if:

For all increasing sequence X_n of elements of $\mathcal{X}$ converging to X, the decreasing sequence $\rho_{\mathcal{F}}(X_n)$ converges to $\rho_{\mathcal{F}}(X)$ P $a.s.$.

We now give some immediate but usefull properties of conditional risk measures.

We consider the supremum norm $|| \; ||$ on $\mathcal{X}$ and the usual norm on $L^\infty(\Omega, \mathcal{F}, \mathcal{P})$.

Lemma 1:

1) Any conditional risk measure is Lipschitz continuous with Lipschitz constant equal 1 from$(\mathcal{X}, || \; ||)$ to $L^\infty(\Omega, \mathcal{F}, \mathcal{P})$.

2) The restriction to $\mathcal{E}_\mathcal{F}$ of any risk measure conditional to $(\Omega, \mathcal{F}, \mathcal{P})$ is equal to $-id \;\; P \; a.s.$

Proof:

1) Let X,$Y \in \mathcal{X}$, X and Y are bounded. Then $X \leq Y + ||X - Y||$. From properties of monotonicity and translation invariance, it follows that $\rho_\mathcal{F}(Y) - \rho_\mathcal{F}(X) \leq ||X - Y||$. Exchanging the roles of X and Y, we obtain 1).

2) is an easy consequence of the properties of translation invariance and multiplicative invariance.

Q.E.D.

1.2 Conditional risk measures and their acceptance sets

Definition 3:

The $\mathcal{F}$-acceptance set of the risk measure $\rho_\mathcal{F}$ conditional to the probability space $(\Omega, \mathcal{F})$ is

$$\mathcal{A}_{\rho_\mathcal{F}} = \{X \in \mathcal{X} \; / \; \rho_\mathcal{F}(X) \leq 0 \;\; P \; a.s.\}$$

We list now the characteristic properties of an acceptance set.

Proposition 1: Let $\rho_\mathcal{F}$ a risk measure conditional to the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with acceptance set $\mathcal{A} = \mathcal{A}_{\rho_\mathcal{F}}$. Then

- As in the monetary case $\mathcal{A}$ satisfies properties independent of the σ-algebra $\mathcal{F}$

1) $\mathcal{A}$ is non empty, closed with respect to the supremum norm and has

hederitary property: for all $X \in \mathcal{A}$, for all $Y \in \mathcal{X}$, if $Y \geq X$, then Y is in $\mathcal{A}$.

- $\mathcal{A}$ satisfies two new properties dependent on $\mathcal{F}$:

2) Bifurcation property: for all X_1,$X_2 \in \mathcal{A}$, for all B_1,B_2 disjoint sets $\in \mathcal{F}$, $X = X_1 1_{|B_1} + X_2 1_{|B_2}$ is in $\mathcal{A}$.

3) Positivity: Every element $\mathcal{F}$-measurable of $\mathcal{A}$ is positive P $a.s.$

4) Furthermore $\rho_{\mathcal{F}}$ can be recovered from $\mathcal{A}$

$$\rho_{\mathcal{F}}(X) = ess\ inf\{Y \in \mathcal{E}_{\mathcal{F}}\ /\ X + Y \in \mathcal{A}\}$$

Proof:

1) $\mathcal{A}$ is non empty because $0 \in \mathcal{A}$.

Let $X \in \mathcal{X} - \mathcal{A}$. Let $\epsilon > 0$ such that $P(\{\omega \in \Omega\ /\ \rho_{\mathcal{F}}(X)(\omega) > \epsilon\}) > 0$. Let $O = \{Y \in \mathcal{X}\ /\ ||\rho_{\mathcal{F}}(Y) - \rho_{\mathcal{F}}(X)|| < \epsilon\}$. O is an open subset of $(\mathcal{X}, ||\ ||)$ from lemma 1 and it is clear that it is contained in $\mathcal{X} - \mathcal{A}$. So $\mathcal{X} - \mathcal{A}$ is open and $\mathcal{A}$ is closed in $\mathcal{X}$ with respect to $||\ ||$.

The other property is an easy consequence of monotonicity.

2) From multiplicative invariance,

$$\rho_{\mathcal{F}}(X_i 1_{|B_i}) = \rho_{\mathcal{F}}(X) 1_{|B_i}$$

and $\rho_{\mathcal{F}}(0) = 0$.

This proves the bifurcation property.

3) For all X $\mathcal{F}$-measurable, from lemma 1, $\rho_{\mathcal{F}}(X) = -X$ P $a.s.$ so the result follows from the definition of $\mathcal{A}$.

4) Let $X \in \mathcal{X}$. Denote $B_X = \{f \in \mathcal{E}_{\mathcal{F}} \mid X + f \in \mathcal{A}\}$.

From translation invariance, it follows that $\rho_{\mathcal{F}}(X) \in B_X$.

On the other hand, for every f in B_X,

$$\rho_{\mathcal{F}}(X + f) = \rho_{\mathcal{F}}(X) - f \leq 0\ \ P\ a.s.$$

So

$$\rho_{\mathcal{F}}(X) = ess\ inf\{f \in \mathcal{E}_{\mathcal{F}}\ /\ f + X \in \mathcal{A}\}$$

Q.E.D.

The properties of the conditional risk measure can be viewed on its acceptance set:

Proposition 2:

i) $\rho_{\mathcal{F}}$ is convex if and only if)$\mathcal{A}_{\rho_{\mathcal{F}}}$ is convex.

ii) $\rho_{\mathcal{F}}$ is coherent if and only if)$\mathcal{A}_{\rho_{\mathcal{F}}}$ is a convex cone.

Proof:

i) The convexity of $\rho_{\mathcal{F}}$ trivially implies the convexity of $\mathcal{A}_{\rho_{\mathcal{F}}}$.

Conversely assume the convexity of $\mathcal{A}_{\rho_{\mathcal{F}}}$.

Let X,$Y \in \mathcal{X}$. Then $X + \rho_{\mathcal{F}}(X) \in \mathcal{A}_{\rho_{\mathcal{F}}}$ and $Y + \rho_{\mathcal{F}}(Y) \in \mathcal{A}_{\rho_{\mathcal{F}}}$. From the convexity of $\mathcal{A}_{\rho_{\mathcal{F}}}$, it follows that

$$\forall \lambda \in [0,1],\ \ \lambda(X + \rho_{\mathcal{F}}(X)) + (1-\lambda)(Y + \rho_{\mathcal{F}}(Y)) \in \mathcal{A}_{\rho_{\mathcal{F}}}.$$

Applying 4) of proposition 1, this gives the convexity of $\rho_{\mathcal{F}}$.

ii) If $\rho_{\mathcal{F}}$ is coherent, $\mathcal{A}_{\rho_{\mathcal{F}}}$ is clearly a convex cone.

Conversely if $\mathcal{A}_{\rho_{\mathcal{F}}}$ is a convex cone, we know from i) that $\rho_{\mathcal{F}}$ is convex.

Also from 4) of Proposition 1, let $\lambda > 0$,

$\rho_{\mathcal{F}}(\lambda X) = \inf\{f \in \mathcal{E}_{\mathcal{F}}; f + \lambda X \in \mathcal{A}_{\rho_{\mathcal{F}}}\}$.

As $\mathcal{A}_{\rho_{\mathcal{F}}}$ is a cone, this is equal to $\lambda\ \inf\{g \in \mathcal{E}_{\mathcal{F}}; g + X \in \mathcal{A}_{\rho_{\mathcal{F}}}\} = \lambda\rho_{\mathcal{F}}(X)$. Q.E.D.

We can now give the definition of a $\mathcal{F}$-acceptance set.

Definition 4:

A subset $\mathcal{A}$ of positions is a $\mathcal{F}$-acceptance set if it satisfies the properties 1) 2) and 3) of proposition 1.

We prove now that conversely, given a $\mathcal{F}$-acceptance set, we can construct a risk measure conditional to the probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Proposition 3:

Let $\mathcal{A}$ a $\mathcal{F}$-acceptance set. For all $X \in \mathcal{X}$ consider the set

$$B_X = \{Y \in \mathcal{E}_{\mathcal{F}}, X + Y \in \mathcal{A}\}$$

Define now

$$\rho_{\mathcal{F}}(X) = ess\ inf B_X$$

($ess\ inf$ relative to P). Then $\rho_{\mathcal{F}}$ is a risk measure conditional to the probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Hence a risk measure conditional to a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ can be defined either directly or from a $\mathcal{F}$-acceptance set.

Proof:

B_X is non empty

Indeed, $\mathcal{A}$ is non empty. Let $f \in \mathcal{A}$. Let $X \in \mathcal{X}$. There is a real number m such that $X + m \geq f$ so $m \in B_X$

$\rho_{\mathcal{F}}(X) = ess\ inf B_X$ is then a $\mathcal{F}$-measurable map. The property of monotonicity of $\rho_{\mathcal{F}}$ follows from the property 1) of $\mathcal{A}$. Property of translation invariance follows from the definition of $\rho_{\mathcal{F}}$.

Let $X \in \mathcal{X}$ and $B \in \mathcal{F}$. If $Y \in \mathcal{E}_{\mathcal{F}}$, and $Y + X \in \mathcal{A}$ then $(Y + X)1_B \in \mathcal{A}$ from property 2) so $\rho_{\mathcal{F}}(X1_B) \leq \rho_{\mathcal{F}}(X)1_B$.

On the other end, if $Z \in \mathcal{E}_{\mathcal{F}}$,and $Z + X1_B \in \mathcal{A}$, then $Z1_{\Omega-B} \in \mathcal{E}_{\mathcal{F}} \cap \mathcal{A}$. It follows from property 3) that $Z1_{\Omega-B}$ is positive P almost surely. So we get the multiplicative invariance.

Q.E.D.

1.3 Risk measure defined on a probability space

In this subsection we assume that the set of financial positions is $L^\infty(\Omega, \mathcal{G}, P)$. We are in a situation where we assume that we have only access to partial information represented by a sub-σ-algebra $\mathcal{F}$ but however we assume that a probability measure is given on the whole set of financial positions.

Definition 5: A conditional risk measure on $L^\infty(\Omega, \mathcal{G}, P)$ is a risk measure $\rho_{\mathcal{F}}$ defined on the set of all bounded $\mathcal{G}$-measurable functions conditional to the probability space $(\Omega, \mathcal{F}, P)$ such

$$\rho_{\mathcal{F}}(X) = \rho_{\mathcal{F}}(Y)\ \ P\ a.s.\ \ if\ \ X = Y\ \ P\ a.s.$$

1.4 Dynamic risk measures

Assume that the set of financial positions is $\mathcal{X} = L^\infty(\Omega, \mathcal{G}, P)$

Assume that a filtration $(\mathcal{F}_t)_{0 \leq t \leq \infty}$ is given ; i.e. a family of increasing sub-σ-algebras of $\mathcal{G}$

Definition 6:

A family $(\rho_{s,t})_{s \leq t}$ is a dynamic risk measure if:

$$\rho_{s,t} : L^\infty(\Omega, \mathcal{F}_t, P) \to L^\infty(\Omega, \mathcal{F}_s, P)$$

is for all (s,t) a conditional risk measure and if

$$\forall (r,s,t) \quad \rho_{r,t} = \rho_{r,s}(-\rho_{s,t}) \ \ if \ r \leq s \leq t$$

2 Examples of conditional risk measures

2.1 Monetary risk measures

The general notion of monetary risk measures was introduced by Föllmer and Schied([10] and [11]). Consider the trivial σ-algebra $\mathcal{F}_0 = \{\Omega, \emptyset\}$

Then the risk measures conditional to $\mathcal{F}_0$ are exactly the normalized monetary risk measures i.e. the monetary risk measures ρ such that $\rho(0) = 0$

A particular example of monetary risk measure is the entropic risk measure defined on a probability space. (for this we refer to Föllmer and Schied [11] and also to Barrieu and El Karoui [3]). The entropic risk measure is defined by its acceptance set. Here we assume that $\mathcal{X}$ is a probability space.($\mathcal{X} = L^\infty(\Omega, \mathcal{G}, P)$). We assume that the investor has an exponential utility function and that the set of acceptable positions is defined by

$$\mathcal{A} = \{Y \in \mathcal{X} \ \ /E_P(e^{(-\alpha Y)}) \leq 1\}$$

Then for all $X \in \mathcal{X}$, $\rho_{\mathcal{A}}(X) = \frac{1}{\alpha} Ln[E_P(e^{-\alpha X})]$

2.2 Conditional risk measure associated to a loss function

The preceding example can be generalized. Assume that the set of financial positions is the set of all bounded $\mathcal{G}$-measurable maps. Assume that the investor has only access to partial information represented by the events $\mathcal{F}$ measurable.And assume that the investor has choosen a convex loss function (i.e. an increasing convex non constant function) 1) Assume first that the investor has an a priori probability measure on the whole σ-algebra $\mathcal{G}$.

$$\mathcal{A}_P = \{Y \in \mathcal{X} \ /E_P(l(-Y)|\mathcal{F}) \leq l(0) \ \ P \ a.s.\}$$

$\mathcal{A}$ is a $\mathcal{F}$-acceptance set. We will study the corresponding conditional risk measure ρ_P in section 5 and completely describe it in terms of conditional expectations. When $l(x) = e^{\alpha x}$ we get the conditional entropic risk measure.

2)Consider now a more uncertain case. Assume now that there is no a priori probability measure given on $(\Omega, \mathcal{G})$. But we assume that the investor knows a probability measure on $(\Omega, \mathcal{F})$ (or at least he knows which events of the σ-algebra $\mathcal{F}$ are of null probability).

Consider now a set $\mathcal{Q}$ of probability measures on $(\Omega, \mathcal{G})$ such that for all $Q \in \mathcal{Q}$, the restriction of Q to $\mathcal{F}$ is equivalent to P

$\underset{Q\in\mathcal{Q}}{\cap} \mathcal{A}_Q$ is a $\mathcal{F}$-acceptance set

and the corresponding risk measure ρ is a risk measure conditional to the probability space $(\Omega, \mathcal{F}, P)$

$$\rho(X) = ess \ sup_{Q\in\mathcal{Q}} \ \rho_Q(X) \ \ P \ a.s.$$

2.3 Dynamic risk measures,conditional g-expectations anf backward stochastic differential equations

F. Coquet, Y. Hu, J. Memin and S. Peng [5] introduced the notion of filtration-consistent non linear expectations and of conditional g-expectations; see also [14]. In these cases a probability space $(\Omega, \mathcal{G}, \mathcal{P})$ is given and also the augmented filtration associated to an X_t d-dimensional process. In the case of the conditional g-expectation, $\mathcal{F}_t$ is the filtration associated to a Brownian motion B_t and the conditional g-expectation with respect to $\mathcal{F}_t$ is defined for all $X \in L^2(\Omega, \mathcal{F}_T, P)$ by $E_g(X|\mathcal{F}_t) = y_t$ where y_t is the unique solution of the following backward stochastic differential equation:

$$-dy_t = g(t, y_t, z_t)dt - z_t^* dB_t$$

$$y_T = X$$

If g is independent of y and if $g(s,0)=0$ then the map $\rho_t : X \rightarrow E_g(-X|\mathcal{F}_t)$ satisfies the properties of monotonicity,translation invariance and multiplicative invariance([14], [15] and [9]) so it is a risk measure conditional to $\mathcal{F}_t$. Furthermore if for all t $g(t,.)$ is convex the conditional risk measure is convex.

E.Rosazza Gianin [12] has given a notion of dynamic risk measure [16] $(\rho_t)_{0\leq t\leq T}$. In her definition the filtration considered is also the augmented filtration of a Brownian motion and $\mathcal{X} = L^2(\Omega,\mathcal{F}_T,P)$.

If the dynamic risk measure has the properties of monotonicity, translation invariance and multiplicative invariance, then the restriction of ρ_t to $L^\infty(\Omega,\mathcal{F}_T,P)$ is for each t a risk measure conditional to the probability space $(\Omega,\mathcal{F}_t,P)$.

2.4 Conditional maximum

Barron, Cardaliaguet and Jensen have introduced in [4] the notion of conditional maximum relative to a sub-σ-algebra $\mathcal{F}$. Let $(\Omega,\mathcal{G},\mathcal{P})$ a probability space. Let $X \in L^\infty(\Omega,\mathcal{G},\mathcal{P})$. Let $\mathcal{F}$ a sub-σ-algebra of $\mathcal{G}$.

$$\mathcal{M}(X|\mathcal{F}) = \; ess\ inf\ \{Y \in \mathcal{E}_\mathcal{F} \mid Y \geq X \;\; P\ a.s.\}$$

Denote

$$\rho^M_\mathcal{F}(X) = \mathcal{M}(-X|\mathcal{F})$$

and

$$\rho^m_\mathcal{F}(X) = -\mathcal{M}(X|\mathcal{F})$$

It is very easy to verify that $\rho^M_\mathcal{F}$ and $\rho^m_\mathcal{F}$ are risk measures conditional to the probability space $(\Omega,\mathcal{F},P)$ Furthermore from the property of monotonicity and the fact that the restriction to $\mathcal{E}_\mathcal{F}$ of every risk measure conditional to $\mathcal{F}$ is equal to $-Id$ (lemma 1) it follows easily that $\rho^M_\mathcal{F}$ is maximal in the set of risk measure conditional to $\mathcal{F}$ and that $\rho^m_\mathcal{F}$ is minimal.

$$\forall X\ \forall \rho_\mathcal{F}\ \rho^m_\mathcal{F}(X) \leq \rho_\mathcal{F}(X) \leq \rho^M_\mathcal{F}(X)$$

3 Conditional risk measure in a context of uncertainty

In this section we want to extend the notion of conditional risk measure to the case of uncertainty; i.e.to the case where no probability is given.

As in the first section, a financial position is described by a bounded map defined on the set Ω of scenarios. We consider a linear space $\mathcal{X}$ of financial positions. We consider also a σ-algebra $\mathcal{F}$ on the space Ω. Then $(\Omega, \mathcal{F})$ is a measurable space. We denote $\mathcal{E}_{\mathcal{F}}$ the set of all bounded real valued $(\Omega, \mathcal{F})$ measurable maps.

But now we don't assume that a probability measure is given nor that there is any consensus on which $\mathcal{F}$-measurable sets should be null sets.

We will refer to that case as to the case of complete uncertainty.

A risk measure conditional to the σ-algebra $\mathcal{F}$ is a mapping defined on $\mathcal{X}$ with values in $\mathcal{E}_{\mathcal{F}}$ which satisfies the conditions of monotonicity, translation invariance and multiplicative invariance in each point. The precise definition is:

Definition 7:

A mapping

$$\rho_F : \mathcal{X} \to \mathcal{E}_{\mathcal{F}}$$

is called a risk measure conditional to the σ-algebra $\mathcal{F}$ if it satisfies the following conditions:

i) monotonicity: for all X,$Y \in \mathcal{X}$ if $X \leq Y$ then $\rho_{\mathcal{F}}(Y) \leq \rho_{\mathcal{F}}(X)$

ii) translation invariance: for all $Y \in \mathcal{E}_{\mathcal{F}}$, for all $X \in \mathcal{X}$,

$$\rho_{\mathcal{F}}(X+Y) = \rho_{\mathcal{F}}(X) - Y$$

iii) multiplicative invariance: for all $X \in \mathcal{X}$,for all $A \in \mathcal{F}$,

$$\rho_{\mathcal{F}}(X 1_A) = 1_A \rho_{\mathcal{F}}(X)$$

The acceptance set of the conditional risk measure $\rho_{\mathcal{F}}$ is

$$\mathcal{A}_{\rho_{\mathcal{F}}} = \{X \in \mathcal{X} \;/\; \rho_{\mathcal{F}}(X) \leq 0\}.$$

The characteristic properties of an acceptance set are summerized in the following proposition:

Proposition 4: The acceptance set $\mathcal{A} = \mathcal{A}_{\rho_{\mathcal{F}}}$ of a conditional risk measure $\rho_{\mathcal{F}}$ satisfies the properties 1) and 2) of Proposition 1 without any change ($\mathcal{A}$ is closed non empty and satisfies the hereditary and the bifurcation properties).

3) Every $\mathcal{F}$-measurable element of $\mathcal{A}$ is positive.

4) $\rho_{\mathcal{F}}$ can be recovered from $\mathcal{A}$

$$\rho_{\mathcal{F}}(X) = inf\{Y \in \mathcal{E}_{\mathcal{F}} \ / \ X + Y \in \mathcal{A}\}.$$

Proof:

The proof of properties 1) 2) and 3) of proposition 4 is similar to that of proposition 1.

We prove now 4). For all $X \in \mathcal{X}$ denote $B_X = \{Y \in \mathcal{E}_{\mathcal{F}} \ / \ X+Y \in \mathcal{A}\}$

$$\rho_{\mathcal{F}}(X + \rho_{\mathcal{F}}(X)) = 0$$

. So $X + \rho_{\mathcal{F}}(X) \in \mathcal{A}$ and $\rho_{\mathcal{F}}(X) \in B_X$.

Furthermore for every Y in B_X, $\rho_{\mathcal{F}}(X + Y) \leq 0$ i.e. $\rho_{\mathcal{F}}(X) \leq Y$.

This proves that B_X has a minimal element which is equal to $\rho_{\mathcal{F}}(X)$.

Q.E.D.

4 Representation of convex conditional risk measures

In all this section we assume that there is a σ-algebra $\mathcal{G}$ such that $\mathcal{X}$ is the set of all bounded measurable functions on the measurable space $(\Omega, \mathcal{G})$. Then $(\mathcal{X}, || \ ||)$ is a Banach space.

Let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{G}$. Denote $M_{1,f}$ the set of all finitely additive set functions $Q : \mathcal{G} \to [0, 1]$ such that $Q(\Omega) = 1$.

We study first the case of complete uncertainty and then the case where a probability measure is given on the sub-σ-algebra $\mathcal{F}$.

4.1 Case of complete uncertainty

We prove the following result:

Theorem 1:

Let $\rho_{\mathcal{F}}$ a convex risk measure conditional to $\mathcal{F}$.

For all $X \in \mathcal{X}$ there is Q_X in $M_{1,f}$ such that for all $B \in \mathcal{F}$

$$E_{Q_X}(\rho_{\mathcal{F}}(X)1_B) = E_{Q_X}(-X1_B) - \sup_{Y \in \mathcal{A}_{\rho_{\mathcal{F}}}} E_{Q_X}(-Y1_B) \quad (I)$$

For all $X \in \mathcal{X}$ for all Q in $M_{1,f}$ for all $B \in \mathcal{F}$

$$E_Q(\rho_{\mathcal{F}}(X)1_B) \geq E_Q(-X1_B) - \sup_{Y \in \mathcal{A}_{\rho_F}} E_Q(-Y1_B) \quad (II)$$

Proof:

For all $X \in \mathcal{X}$, $(\rho_{\mathcal{F}}(\rho_{\mathcal{F}}(X) + X)) = \rho_{\mathcal{F}}(X) - \rho_{\mathcal{F}}(X) = 0$.
So $\rho_{\mathcal{F}}(X) + X \in \mathcal{A}_{\rho_{\mathcal{F}}}$.
So for all $Q \in M_{1,f}$, for all $B \in \mathcal{F}$, we get:

$$sup_{Y \in \mathcal{A}_{\rho_{\mathcal{F}}}} E_Q(-Y1_B) \geq E_Q(-X1_B) - E_Q(\rho_{\mathcal{F}}(X)1_B)$$

This proves the inequality (II).

In order to prove the equality (I), it is enough to prove it in the case where $\rho_{\mathcal{F}}(X) = 0$.

Indeed when $\rho_{\mathcal{F}}(X) \neq 0$,we get the result, replacing X by $X + \rho_{\mathcal{F}}(X)$.

Consider now the convex hull $\mathcal{C}$ of $\{(Y-X)1_B;\ \rho_{\mathcal{F}}(Y) < 0 \ and\ B \in \mathcal{F}\}$.

1) Step 1:

Prove that $\mathcal{C} \cap \{0\} = \emptyset$.

Indeed assume that there are $\lambda_i \geq 0$; $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{i=1}^n \lambda_i(Y_i - X)1_{B_i} = 0$.

Choose $J \subset \{1, 2, ...n\}$ such that $\tilde{B} = \underset{i \in J}{\cap} B_i \neq \emptyset$ and such that $\forall j \in \{1, 2, ...n\} - J$, $\tilde{B} \cap B_j = \emptyset$

$$\sum_{i \in J} \lambda_i(Y_i - X)1_{\tilde{B}} = 0.$$

Let $\tilde{Y} = \frac{\sum_{i \in J} \lambda_i(Y_i)}{\sum_{i \in J} \lambda_i}$.

From the convexity of $\rho_{\mathcal{F}}$, it follows that $\rho_{\mathcal{F}}(\tilde{Y}) < 0$.
But $\tilde{Y}1_{\tilde{B}} = X1_{\tilde{B}}$ So $\rho_F(\tilde{Y}1_{\tilde{B}}) = \rho_F(X1_{\tilde{B}}) = 0$.
And this equality applied to elements of $\tilde{B}$ gives a contradiction.
This ends the first step of the proof.

2) Step 2

$\mathcal{C}$ contains the open ball

$$B_1(1-X) = \{Y \in \mathcal{X}\ ;\ ||Y-(1-X)|| < 1\}.$$

Indeed if Y is in $B_1(1-X)$, $Y = Z - X$ with $Z \in B_1(1)$ so from lemma 1, $\rho_{\mathcal{F}}(Z) < 0$.
This ends step 2.

3) Step 3

We prove the existence of $Q_X \in M_{1,f}$ such that $\forall B \in \mathcal{F}, E_{Q_X}(-X1_B) = \sup_{Y \in \mathcal{A}_{\rho_{\mathcal{F}}}} E_{Q_X}(-Y1_B)$.

From the first step, 0 doesn't belong to the convex set $\mathcal{C}$, and from step 2 the interior of $\mathcal{C}$ is non empty so there is a non-zero continuous linear form L on $\mathcal{X}$ such that

$0 = L(0) \leq L(Z)$ for all $Z \in \mathcal{C}$

$0 \leq L((Y-X)1_B)$ for all Y such that $\rho_{\mathcal{F}}(Y) < 0$ and for all $B \in \mathcal{F}$.

Now $\forall Y \in \mathcal{A}_{\rho_{\mathcal{F}}}$, $\forall \epsilon > 0$, $\rho_{\mathcal{F}}(Y+\epsilon) < 0$.

Hence by continuity of L,

$$0 \leq L((Y-X)1_B) \quad \forall Y \in \mathcal{A}_{\rho_{\mathcal{F}}}. \quad (III)$$

Now $\forall Y \geq 0, \forall \lambda > 0, \rho_{\mathcal{F}}(1+\lambda Y) < 0)$, so $(1+\lambda Y - X) \in C$ and $L(1) + \lambda L(Y) - L(X) \geq 0$.

This implies that $\forall Y \geq 0$, $L(Y) \geq 0$; i.e. L is a positive linear form.

From this it follows that $L(1) > 0$.

And so there is a unique $Q_X \in M_{1,f}$ defined by $E_{Q_X}(Y) = \frac{L(Y)}{L(1)}$ for all Y in $\mathcal{X}$.

And $\forall B \in \mathcal{F}, \forall Y \in \mathcal{A}_{\rho_{\mathcal{F}}}$, it follows from (III) that $E_{Q_X}(-X1_B) \geq E_{Q_X}(-Y1_B)$.

Using the inequality (II), this ends the proof of step 3 and also the proof of the theorem. Remark: This theorem contains as a special case the theorem of representation of the monetary risk measures when the σ-algebra $\mathcal{F}$ is the trivialσ-algebra.

In fact we can prove that for each $X \in \mathcal{X}$, the restriction to $\mathcal{F}$ of the finitely additive set function Q_X of theorem 1 can be choosen arbitrarily.

Lemma 2:

Let P a finitely additive set function on $\mathcal{F}$; $P : \mathcal{F} \rightarrow [0,1]$ such that $P(\Omega) = 1$. For each $X \in \mathcal{X}$ there is a finitely additive set function Q_X on $\mathcal{G}$ such that the equality (I) is

satisfied and such that the restriction of Q_X to $\mathcal{F}$ is equal to P.

Proof:

Define $\tilde{\rho}(X) = P(\rho_{\mathcal{F}}(X))$. $\tilde{\rho}$ is a convex risk measure. So for all $X \in \mathcal{X}$ there is Q_X in $M_{1,f}$ such that

$\tilde{\rho}(X) = E_{Q_X}(-X) - \sup_{Y \in \mathcal{A}_{\tilde{\rho}}} E_{Q_X}(-Y)$ $(\tilde{I})$

and for all $Z \in \mathcal{X}$

$\tilde{\rho}(Z) \geq E_{Q_X}(-Y) - \sup_{Y \in \mathcal{A}_{\tilde{\rho}}} E_{Q_X}(-Y)$ $(\tilde{I}I)$.

From the equality $(\tilde{I})$, as X is bounded, it follows that $\sup_{Y \in \mathcal{A}_{\tilde{\rho}}} E_{Q_X}(-Y)$ is a real number. Denote it $\alpha(Q_X)$.

Apply the inequality $(\tilde{I}I)$ to $Z = \beta 1_B$ for all $\beta \in I\!R$ and $B \in \mathcal{F}$. We get $\beta(P(B) - Q_X(B)) \geq \alpha(Q_X)$ for all β in $I\!R$. So necessarily $P(B) = Q_X(B)$ for all B i.e. the restriction of Q_X to $\mathcal{F}$ is equal to P. $\tilde{I}$ can then be written

$$E_{Q_X}(\rho_{\mathcal{F}}(X)) = E_{Q_X}(-X) - \sup_{Y \in \mathcal{A}_{\tilde{\rho}}} E_{Q_X}(-Y)$$

As $\mathcal{A}_{\rho_{\mathcal{F}}}$is contained in $\mathcal{A}_{\tilde{\rho}}$ it follows that

$$E_{Q_X}(\rho_{\mathcal{F}}(X)) \leq E_{Q_X}(-X) - \sup_{Y \in \mathcal{A}_{\rho_{\mathcal{F}}}} E_{Q_X}(-Y)$$

But the converse inequality is always true so Q_X satisfies the equality

$E_{Q_X}(\rho_{\mathcal{F}}(X)) = E_{Q_X}(-X) - \sup_{Y \in \mathcal{A}_{\rho_{\mathcal{F}}}} E_{Q_X}(-Y)$ (E).

Assume now that there is one $\mathcal{F}$ measurable set B such that the inequality (II) of theorem 1 for Q_X is strict. There is $Y_0 \in \mathcal{A}_{\rho_{\mathcal{F}}}$ such that

$$E_{Q_X}(\rho_{\mathcal{F}}(X)1_B) > E_{Q_X}(-X1_B) - E_{Q_X}(-Y_0 1_B).$$

Let $Y = Y_0 1_B + (X + \rho_{\mathcal{F}}(X))1_{\Omega - B}$. From the bifurcation property of $\mathcal{A}_{\rho_{\mathcal{F}}}$ it follows that $Y \in \mathcal{A}_{\rho_{\mathcal{F}}}$ and

$$E_{Q_X}(\rho_{\mathcal{F}}(X)) > E_{Q_X}(-X1_B) - E_{Q_X}(-Y1_B) + E_{Q_X}(-X1_{\Omega-B}) - E_{Q_X}(-Y1_{\Omega-B})$$

This contradicts the equality (E).

So Q_X satisfies the equality (I) of theorem 1 for all $B \in \mathcal{F}$and the restriction of Q_X to $\mathcal{F}$ is equal to P.

Q.E.D.

Now we can prove the theorem of representation, in terms of conditional expectations, for the convex conditional risk measures continuous from below.

Theorem 2:
Let $\rho_{\mathcal{F}}$ be a convex risk measure conditional to $\mathcal{F}$. Assume that $\rho_{\mathcal{F}}$ is continuous from below then:

1) For all $X \in \mathcal{X}$ for every probability measure Q on $(\Omega, \mathcal{G})$

$$\rho_{\mathcal{F}}(X) \geq E_Q(-X|\mathcal{F}) - \alpha(Q) \;\; Q \; a.s.$$

$$\alpha(Q) = ess \; sup_{\{Y \in \mathcal{A}_{\rho_{\mathcal{F}}}\}}(E_Q(-Y|\mathcal{F})) \;\; Q \; a.s.$$

2) For all $X \in \mathcal{X}$, for every probability measure P on $(\Omega, \mathcal{F})$ there is Q_X in $M_1(\mathcal{G}, \mathcal{F}, P)$ such that

$$\rho_{\mathcal{F}}(X) = E_{Q_X}(-X|\mathcal{F}) - \alpha(Q_X) \;\; P \; a.s.$$

(where $M_1(\mathcal{G}, \mathcal{F}, P)$ is the set of all probability measures Q on $(\Omega, \mathcal{G})$ such that the restriction of Q to $\mathcal{F}$ is equal to P).

3) For all $X \in \mathcal{X}$,

$$\rho_{\mathcal{F}}(X) = inf\{g \in \mathcal{E}_{\mathcal{F}}; \forall Q \in M_1(\Omega, \mathcal{G}), g \geq (E_Q(-X|\mathcal{F}) - ess \; sup_{\{Y \in \mathcal{A}_{\rho_{\mathcal{F}}}\}}(E_Q(-Y|\mathcal{F})) \; Q \; a.s.\}$$

Proof:
1) Let Q a probability measure on $(\Omega, \mathcal{G})$. The inequality (II) of Theorem 1 applied to Q gives 1).

2) Let P a probability measure on $(\Omega, \mathcal{F})$. Let $X \in \mathcal{X}$. From lemma 2, there is a finitely additive set function Q_X such that the equality (I) is satisfied for all $B \in \mathcal{F}$ and such that the restriction of Q_X to $\mathcal{F}$ is equal to P.

It remains to prove that Q_X is a probability measure on $(\Omega, \mathcal{G})$ i.e. that Q_X is σ-additive.

Consider an increasing sequence $(A_n)_{n\in I\!N}$ of elements of $\mathcal{G}$ whose union is equal to Ω we have to prove that $Q_X(A_n)$ converges to 1. Apply equality (I) to $B = \Omega$. We get

$$E_P(\rho_{\mathcal{F}}(X)) = E_{Q_X}(-X) - \alpha(Q_X)$$

with $\alpha(Q_X) = sup_{Y\in\mathcal{A}_{\rho_{\mathcal{F}}}} E_{Q_X}(-Y)$; X and $\rho_{\mathcal{F}}(X)$ are bounded so $\alpha(Q_X)$ is finite.

Let $\lambda > 0$. Apply now the inequality (II) to $\lambda 1_{A_n}$ We get

$$E_{Q_X}(\lambda 1_{A_n}) \geq -E_P(\rho_{\mathcal{F}}(\lambda 1_{A_n})) - \alpha(Q_X)$$

As n tends to infinity, $\rho_{\mathcal{F}}(\lambda 1_{A_n})$ tends to $\rho_{\mathcal{F}}(\lambda) = -\lambda$ so

$$\liminf_{n\to\infty} \ E_{Q_X}(A_n) \geq 1 - \frac{\alpha(Q_X)}{\lambda}$$

As λ tends to ∞ we get $\liminf_{n\to\infty} \ E_{Q_X}(A_n) \geq 1$.

This ends the proof of 2).

3) From 1) for every probability measure Q on $(\Omega, \mathcal{G})$,

$$\rho_{\mathcal{F}}(X) \geq (E_Q(-X|\mathcal{F}) - ess\ sup_{\{Y\in\mathcal{A}_{\rho_{\mathcal{F}}}\}}(E_Q(-Y|\mathcal{F}))\ \ Q\ a.s.$$

And if g is-$\mathcal{F}$-measurable and satisfies the relation
$g \geq E_Q(-X|\mathcal{F}) - ess\ sup_{Y\in\mathcal{A}_{\rho_{\mathcal{F}}}} E_Q(-Y|\mathcal{F})\ \ Q\ a.s.$
for every probability measure P on $(\Omega, \mathcal{G})$,

Let $B = \{\omega | g(\omega) < \rho_{\mathcal{F}}(X)(\omega)\}$. If B is non empty, there is a probability measure P on $(\Omega, \mathcal{F})$ such that $P(B) > 0$.

Now from 2) there is $Q_X \in M_1(\mathcal{G}, \mathcal{F}, P)$ such that
$\rho_{\mathcal{F}}(X) = (E_{Q_X}(-X|\mathcal{F}) - ess\ sup_{Y\in\mathcal{A}_{\rho_{\mathcal{F}}}} E_{Q_X}(-Y|\mathcal{F})\ \ Q_X\ a.s.$ And so we get a contradiction, and 3) is proved.

Corollary 1

Let $\rho_{\mathcal{F}}$ be a convex coherent risk measure conditional to $\mathcal{F}$

1)There is a set $\mathcal{M}_f$ of finitely additive set functions such that:

For all $X \in \mathcal{X}$, for all $Q \in \mathcal{M}_f$, for all $B \in \mathcal{F}$,

$$E_Q(\rho_{\mathcal{F}}(X) 1_B) \geq E_Q(-X 1_B)$$

For all $X \in \mathcal{X}$, there is $Q_X \in \mathcal{M}_f$, such that for all $B \in \mathcal{F}$,

$$E_{Q_X}(\rho_{\mathcal{F}}(X) 1_B) = E_{Q_X}(-X 1_B)$$

2)If $\rho_{\mathcal{F}}$ is furthermore continuous from below, it can be expressed in terms of conditional expextations for a family Q of probability measures such that the penality function of each Q $\alpha(Q)$is equal to 0.

Proof: This corollary follows from the theorems 1 and 2, from the fact that $\mathcal{A}_{\rho_{\mathcal{F}}}$ is a cone and from the observation that for each Q_X and for all $B \in \mathcal{F}$, $\sup_{Y \in \mathcal{A}_{\rho_{\mathcal{F}}}} E_{Q_X}(-Y 1_B)$ is finite and $\alpha(Q_X)$ is bounded.

4.2 Representation of risk measures conditional to a probability space

In the case of a risk measure conditional to a probability space $(\Omega, \mathcal{F}, \mathcal{P})$,the restriction to $\mathcal{F}$ of the measures Q and Q_X of the representation theorem must be absolutely continuous with respect to P and they can all be choosen equal to P. For example in the case of continuity from below we get the theorem:

Theorem 3: Let $\rho_{\mathcal{F}}$ be a convex risk measure conditional to the probability space $(\Omega, \mathcal{F}, P)$.

Assume that $\rho_{\mathcal{F}}$ is continuous from below then for all $X \in \mathcal{X}$

$$\rho_{\mathcal{F}}(X) = ess\ max_{Q \in \mathcal{M}}((E_Q(-X|\mathcal{F}) - \alpha(Q))$$

where $\alpha(Q) = ess\ sup_{\{Y \in \mathcal{A}_{\rho_{\mathcal{F}}}\}}(E_Q(-Y|\mathcal{F})$

Where $\mathcal{M}$ is a set of probability measures on $(\Omega, \mathcal{G})$ whose restriction to $\mathcal{F}$ is equal to P. If in addition $\rho_{\mathcal{F}}$ is coherent, then $\rho_{\mathcal{F}}(X) = ess\ max_{Q \in \mathcal{M}}(E_Q(-X|\mathcal{F})$ Remark:$\rho_{\mathcal{F}}(X) = ess\ max_{Q \in \mathcal{M}}((E_Q(-X|\mathcal{F}) - \alpha(Q))$ means that $\rho_{\mathcal{F}}(X)$ is the *ess sup* and that this *ess sup* is attained for one $Q \in \mathcal{M}$

Proof:

First, as in theorem 1, it is easy to verify that for all X in $\mathcal{X}$, for all B in $\mathcal{F}$ and for all Q absolutely continuous with respect to P,

$$E_Q(\rho_{\mathcal{F}}(X) 1_B) \geq E_Q(-X 1_B) - \sup_{Y \in \mathcal{A}_{\rho_{\mathcal{F}}}} E_Q(-Y 1_B) \quad (II')$$

We prove then the existence of a finitely additive set function Q_X satisfying the equality:

$$E_{Q_X}(\rho_{\mathcal{F}}(X)1_B) = E_{Q_X}(-X1_B) - \sup_{Y\in\mathcal{A}_{\rho_{\mathcal{F}}}} E_{Q_X}(-Y1_B) \quad (I')$$

The proof is exactly the same as that of theorem 1 replacing $\mathcal{C}$ by the convex hull $\tilde{\mathcal{C}}$ of

$$\{(Y-X)1_B; \rho_{\mathcal{F}}(Y) < 0 \; P \; a.s.;\; B \in \mathcal{F} \; and \; P(B) \neq 0\}$$

Let A in $\mathcal{F}$ such that $P(A) = 0$. For all $Z \in \tilde{\mathcal{C}}$,for all $\beta \in I\!R$, $Z + \beta 1_A = Z \quad P \quad a.s.$.It follows from the inequality $E_{Q_X}(Z) \geq 0$ applied to $Z + \beta 1_A$ for all $\beta \in I\!R$ that $Q_X(A) = 0$

So Q_X must be absolutely continuous with respect to P.

The end of the proof follows easily from the proofs of lemma 2 and theorem 2.

In the particular case of conditional risk measures defined on a probability space the representation theorem takes the following form:

Proposition 5:

Let $\rho_{\mathcal{F}}$ be a convex risk measure on $L^\infty(\Omega, \mathcal{G}, P)$ conditional to the probability space $(\Omega, \mathcal{F}, P)$.

Assume that $\rho_{\mathcal{F}}$ is continuous from below then for all $X \in \mathcal{X}$

$$\rho_{\mathcal{F}}(X) = ess\; max_{Q\in\mathcal{M}}(E_Q(-X|\mathcal{F})) - \alpha(Q)$$

where $\alpha(Q) = ess\; sup_{\{Y\in\mathcal{A}_{\rho_{\mathcal{F}}}\}}(E_Q(-Y|\mathcal{F}))$ Here $\mathcal{M}$ is a set of probability measures on $\mathcal{G}$ absolutely continuous with respect to P whose restriction to $\mathcal{F}$ is equal to P.

Remark : K.Detlefsen [7] has also obtained independently a theorem of representation, however expressed differently and only in this specific case of a conditional risk measure defined on a probability space.

We now apply this representation result to the case of backward stochastic differential equations.

Proposition 6: Consider a probability space $(\Omega, \mathcal{G}, P)$ and an $I\!R^n$-valued Brownian motion. Consider the backward stochastic differential equation

$$-dY_t = g(t, Z_t) - Z_t^* dB_t$$
$$Y_T = -X$$

Assume that X is in $L^\infty(\Omega, \mathcal{F}_T, P)$. Assume that the driver g : $(\Omega \times I\!R \times I\!R^n) \to I\!R)$ satisfies the usual assumptions and that $\forall t \in I\!R$ $g(t, 0) = 0$ P $a.s.$ then

$$\rho_{\mathcal{F}_t}(X) = Y_t$$

defines a risk measure conditional to $\mathcal{F}_t$ (with values in $L^\infty(\Omega, \mathcal{F}_t, P)$).

Furthermore if g is convex in z, then $\rho_{\mathcal{F}_t}$ is a convex conditional risk measure continuous from below and so

$$\rho_{\mathcal{F}_t}(X) = ess\ max_{Q\in\mathcal{M}}[(E_Q(-X|\mathcal{F}_t) - \alpha(Q)]$$

with $\alpha(Q) = ess\ sup_{\{Y \in A_{\rho_\mathcal{F}}\}}(E_Q(-Y|\mathcal{F}_t))]$

Moreover if g is sub-linear in z, then $\rho_{\mathcal{F}_t}$ is coherent and can be written

$$\rho_{\mathcal{F}_t}(X) = ess\ max_{Q\in\mathcal{M}}[(E_Q(-X|\mathcal{F}_t)]$$

$\mathcal{M}$ is a set of probability measures on $(\Omega, \mathcal{G})$ absolutely continuous with respect to P whose restriction to $\mathcal{F}$ is equal to P.

Remark : The above representation result in the case where g is sub-linear was already obtained in the particular case where the process Z is one dimensional and $g(z) = |z|$ by Chen and Peng [2].

Proof of the proposition:

The existence of a unique solution (Y, Z) in $H^2(I\!R) \times H^2(I\!R^n)$ is proved in Pardoux and Peng [13] and also in El Karoui, Peng, Quenez [9]. From the comparaison theorem [9] as X is bounded it follows that $(Y_t)_{0\le t\le T}$ is uniformly bounded and in particular $Y_t \in L^\infty(\Omega, F_t, P)$. The property of monotonicity follows also from the comparaison theorem. The properties of translation and of multiplication invariance are proved in [5] (see also [13] and [15]). So $\rho_{\mathcal{F}_t}$ is a risk measure conditional to $\mathcal{F}_t$. For the properties of convexity (resp coherence) of $\rho_{\mathcal{F}_t}$ when g is convex (resp sublinear) we refer to [5] [9] [13] and [15].

Consider now an increasing sequence X_n such that $X = lim\ X_n$ P $a.s.$. It follows from proposition 3.2 of [9] (as $-X = inf(-X_n)$) that for each t, $\rho_{\mathcal{F}_t}(X) = ess\ inf(\rho_{\mathcal{F}_t}(X_n))$ and as the sequence $(\rho_{\mathcal{F}_t}(X_n))$ is decreasing from the comparaison theorem,

$$\rho_{\mathcal{F}_t}(X) = lim\ \rho_{\mathcal{F}_t}(X_n) \ \ P\ a.s.$$

The conditional risk measure is then continuous from below, and if g is convex (resp sublinear) we can apply the proposition 5 and it gives the announced representation.

5 Conditional risk measures in financial markets

In this section we assume that an investor wants to choose financial products, that this investor has his own convex loss function and that he has only access to partial information (represented by the σ-algebra $\mathcal{F}$). We assume that a probability measure P is given on $(\Omega, \mathcal{G})$.

Recall that a loss function l is an increasing non constant function $l : I\!R \rightarrow I\!R$. We will only consider convex loss functions.

In this context it is natural to define the set of acceptable positions by

$$\mathcal{A} = \{X \in L^\infty(\Omega, \mathcal{G}, P) \mid E(l(-X)|\mathcal{F}) \leq g \ \ P\ a.s.\}.$$

where g is a bounded $\mathcal{F}$-measurable map.

The purpose of this section is to study the $\mathcal{F}$ conditional risk measure associated to $\mathcal{A}$ and to find an explicit formula for the penality function. We prove that the result obtained by H. Föllmer and A. Schied [10] and [11]. For monetary risk measures associated to a loss function can be extended to the case of conditional risk measures in a more technical proof using the uniform continuity of l on each compact and the Lebesgue dominated convergence theorem. The penality function can be expressed in terms of the conjugate function of the loss function l.

Theorem 4:

Let l a convex loss function strictly increasing on$I\!R^+$. Let g a bounded $\mathcal{F}$-measurable map. Assume that there is a stricly positive constant c such that $g \geq c \ \ P\ a.s.$ Define

$$\tilde{\mathcal{A}} = \{X \in L^\infty(\Omega, \mathcal{G}, P) \ / \ E(l(-X)|\mathcal{F}) \leq l(g) \ \ P\ a.s.\}$$

and

$$\mathcal{A} = \tilde{\mathcal{A}} + g = \{X + g; X \in \tilde{\mathcal{A}}\}$$

Then $\mathcal{A}$ is a $\mathcal{F}$-acceptance set.
For all $X \in \mathcal{X}$ define

$$\rho(X) = ess\ inf\{Y \in \mathcal{E}_{\mathcal{F}};\ X + Y \in \tilde{\mathcal{A}}\}$$

Then $\rho_{\mathcal{F}}(X) = \rho(X) + g$ is the risk measure $\rho_{\mathcal{F}}$ conditional to $\mathcal{F}$ associated to $\mathcal{A}$. It is convex, continuous from below. $\rho(X)$ admits the following representation:

$$\rho(X) = ess\ max_{Q \in \mathcal{M}}((E_Q(-X|\mathcal{F}) - \alpha(Q))$$

where $\mathcal{M}$ is a set of probability measures Q on $(\Omega, \mathcal{G})$ absolutely continuous with respect to P and such that the restriction of Q to $\mathcal{F}$ is equal to P.

Furthermore

$$\alpha(Q) = ess\ inf\{f \in \mathcal{E}_{\mathcal{F}}; f > 0\ a.s.\}[\frac{1}{f}(l(g) + E_P(l^*(f\frac{dQ}{dP})|\mathcal{F}))]$$

l^* is the conjugate function of the convex function l, i.e. $l^*(z) = sup_{x \in I\!R}(zx - l(x))$.

This theorem is a generalization to the case of conditional risk measures of the theorem 4.61 of [12].

In order to prove this theorem, we prove the following results:

Lemma 3:
Let l and $\mathcal{A}$ as in theorem 4.
Then $\mathcal{A}$ is a $\mathcal{F}$-acceptance set.

Proof:
We verify that $\mathcal{A}$ satisfies the properties of the definition 4.

- $0 \in \mathcal{A}$. The continuity of l and the Lebesgue dominated convergence theorem imply that $\mathcal{A}$ is norm closed. - Let $X \in \mathcal{A}$. Let $Y \in L^\infty(\Omega, G, P)$ such that $Y \geq X \ \ P\ a.s.$

As l is increasing $l(-Y) \leq l(-X) \quad P\ a.s.$ so $E_P(l(-Y)|\mathcal{F})) \leq E_P(l(-X)|\mathcal{F}))$ a. s. and $Y \in \mathcal{A}$. So property 1) is true.

- Let $X_1, X_2 \in \mathcal{A}$. $X_1 - g$ and $X_2 - g$ are in $\tilde{\mathcal{A}}$ Let B_1, B_2 disjoint sets in $\mathcal{F}$.

$$E_P(l(-X_1 1_{B_1} - X_2 1_{B_2} + g)|\mathcal{F}) = E_P(l(-X_1+g)|\mathcal{F})1_{B_1} + E_P(l(-X_2+g)|\mathcal{F})1_{B_2} + l(g)1_{(\Omega - B_1 - B_2)}$$

as $X_1 - g$ and$X_2 - g$ are in $\tilde{\mathcal{A}}$ it follows that $X_1 1_{B_1} + X_2 1_{B_2}$ is in $\mathcal{A}$ and property 2) is satisfied.

- Let $X \in A \cap \mathcal{E}_\mathcal{F}$. Let $B = \{\omega \in \Omega | X(\omega) < 0\}$.

As l is strictly increasing on $I\!R^+$, $E_P(l(-X+g)|\mathcal{F})1_{|B} > l(g)1_{|B} \ \ P\ a.s.$ so $P(B) = 0$. So property 3) is true.

And so $\mathcal{A}$ is a $\mathcal{F}$-acceptance set.

Lemma 4:

Let l and $\mathcal{A}$ be as in theorem 4. Let $\rho_\mathcal{F}$ the risk measure conditional to $\mathcal{F}$ associated to the acceptance set $\mathcal{A}$.

Then $\rho(X) = \rho_F(X) - g = ess\ inf\{Y \in \mathcal{E}_\mathcal{F} | X + Y \in \tilde{\mathcal{A}}\}$ is the unique $Y \in L^\infty(\Omega, \mathcal{F}, P)$ such that $E(l(-X - Y)|\mathcal{F})) = l(g) \ \ P\ a.s.$

Proof

By definition of he risk measure associated to $\mathcal{A}$, $\rho_\mathcal{F}(X) = ess\ inf\{Y \in \mathcal{E}_\mathcal{F}; X + Y \in \mathcal{A}\}$. Let $\rho(X) = \rho_\mathcal{F}(X) - g$ Then $\rho(X) = ess\ inf B_X$ where $B_X = \{Y \in \mathcal{E}_\mathcal{F}; X + Y \in \tilde{\mathcal{A}}\}$ If $Y_1, Y_2 \in B_X$ let $C = \{\omega \in \Omega; Y_1(\omega) > Y_2(\omega)\}$ then from the bifurcation property of $\mathcal{A}$ it follows that $inf(Y_1, Y_2) = Y_1 1_{\Omega - C} + Y_2 1_C$ is also in B_X.

So from Theorem A.18 of [12]. a decreasing family Y_n of elements of B_X such that $\rho_F(X) = lim_{n\to\infty}(Y_n)P\ a.s.$ Notice that this family is uniformly bounded as $\rho_{calF}(X)$, g and Y_1 are bounded.

l is continuous and increasing. So the sequence $l(-X - Y_n)$ is increasing uniformly bounded and tends to $l(-X - \rho(X)) \ \ P\ a.s.$.

From the properties of the conditional expectation, it follows that $E(l(-X - Y_n)|\mathcal{F})$ is an increasing sequence and that $E(l(-X - \rho(X))|\mathcal{F}) = \lim_{n\to\infty}(E(l(-X - Y_n))|\mathcal{F}) \ \ P\ a.s.$ So

$$E(l(-X-\rho(X))|\mathcal{F}) \leq l(g) \;\; P\; a.s.$$

Denote now $B_n = \{\omega \in \Omega / E(l(-X-\rho(X))|\mathcal{F})(\omega) < l(g) - \frac{1}{n}\}$.

Then $B_n \in \mathcal{F}$ and $E(l(-X-\rho_(X))1_{B_n}|\mathcal{F}) < (l(g) - \frac{1}{n})1_{B_n}$.

$X+\rho(X)$ is bounded and l is uniformly continuous on each compact so there is $\epsilon_n > 0$ such that for all $\omega \in \Omega$, $l(-X(\omega) - \rho(X)(\omega) + \epsilon_n) < l(-X(\omega) - \rho(X)(\omega)) + \frac{1}{n}$.

Then $E(l(-X - [\rho(X) - \epsilon_n])1_{B_n}|\mathcal{F}) < l(g)1_{B_n}$.

$\rho(X) - \epsilon_n 1_{B_n}$ is an element of B_X and this is possible only if $P(B_n) = 0$.

We have then proved that

$$E(l(-X-\rho_{\mathcal{F}}(X))|\mathcal{F}) = l(g) \;\; P\; a.s.$$

Assume now that Y is another $\mathcal{F}$ measurable map such that

$E(l(-X-Y)|\mathcal{F})) = l(g) \;\; P\; a.s.$

Let $C = \{\omega \in \Omega / \rho(X)(\omega) > Y(\omega)\}$.

C is $\mathcal{F}$-measurable, l is strictly increasing on $I\!R+$ so

$l([-X-\rho(X)]1_C) < max(l([-X-Y]1_C), l(g)) \;\; P\; a.s.$

So $E(l([-X-\rho(X)]1_C)|\mathcal{F}) < l(g)1_C \quad P\; a.s.$and this is possible only if $P(C) = 0$; i.e. $\rho(X) \leq Y \quad P\; a.s.$ The other inequality is obtained in the same way so $Y = \rho(X)$ in $L^\infty(\Omega, \mathcal{F}, P)$.

Q.E.D.

Lemma 5:

Under the same hypothesis, $\rho_{\mathcal{F}}$ is convex and continuous from below.

Proof:

-Note first that the convexity of l and the linearity of the conditional expectation implies the convexity of $\mathcal{A}$ and so also the convexity of $\rho_{\mathcal{F}}$ from proposition 2.

-We prove now that $\rho_{\mathcal{F}}$ is continuous from below. Let $X \in \mathcal{X}$ and $X_n \in \mathcal{X}$ an increasing sequence such that $X = \lim X_n \;\; P\; a.s.$ The sequence $\rho(X) = \rho_{\mathcal{F}}(X_n) - g$ is decreasing uniformly bounded (as X_1 and X are bounded); it has a limit $Y = \lim \rho(X_n) \;\; P\; a.s.$ From the continuity of l and the Lebesgue dominated convergence theorem, it follows that, $E(l(-X-Y)|\mathcal{F}) = l(g) \;\; P\; a.s.$ which proves, using the result of the preceding lemma that $Y = \rho(X) = \rho_{\mathcal{F}}(X) - g$. So $\rho_{\mathcal{F}}$ is continuous from below.

Proposition 7:
Assume the same hypothesis as in theorem 4 then

$$\rho(X) = ess\ sup_{\mathcal{M}}(E_Q(-X|\mathcal{F}) - \alpha(Q))$$

where

$$\alpha(Q) = ess\ inf\{f \in \mathcal{E}_{\mathcal{F}}; f > 0\ a.s.\}[\frac{1}{f}(l(g) + E_P(l^*(f\frac{dQ}{dP})|\mathcal{F}))]$$

where l^* is the conjugate function of the convex function l.

Proof: We already know from the preceding lemmas that $\rho_{\mathcal{F}}$ is a conditional convex risk measure continuous from above. So from proposition 5 of section 4.2. $\rho_{\mathcal{F}}$ has a representation:

$$\rho_{\mathcal{F}}(X) = ess\ max_{Q\in\mathcal{M}}((E_Q(-X|\mathcal{F}) - \beta(Q))$$

with $\beta(Q) = ess\ sup_{\{Y\in\mathcal{A}\}}(E_Q(-Y|\mathcal{F}))$),so

$$\rho(X) = \rho_{\mathcal{F}}(X) - g = ess\ max_{Q\in\mathcal{M}}((E_Q(-X|\mathcal{F}) - \alpha(Q))$$

with $\alpha(Q) = ess\ sup_{\{Y\in\tilde{\mathcal{A}}\}}(E_Q(-Y|\mathcal{F}))$).

It remains to express $\alpha(Q)$ in terms of the loss function l.

l is convex $l : I\!R \to I\!R$.

Consider its conjugate function l^* defined by $l^*(y) = sup_{\{x\in I\!R\}}(yx - l(x)$ defined on $I\!R$ with values in $I\!R\ U\ \{+\infty\}$. We denote D the domain of l^*. Recall that for all $x, y \in I\!R$, $xy \leq l(x) + l^*(y)$ and that we have equality for all $x \in I\!R$ and $y \in [l'_-(x), l'_+(x)]$, and also for all $x \in I\!R$, $l(x) = sup_{y\in I\!R}(xy - l^*(y)$. Recall also that the right derivative of l^* exists in every point of D and that it is right continuous. We will denote J this right derivative. We have then the following equality:

for all $z \in I\!R$ $zJ(z) = l(J(z)) + l^*(z)$

Let $Q \in M_1((P,G)a.c.,(P,F))$ Denote $\phi_Q = \frac{dQ}{dP}$ the Radon Nikodym derivative of Q with respect to P. Let $Y \in \tilde{\mathcal{A}}$. For all $f \in \mathcal{E}_{\mathcal{F}}$ such that $f > 0$ a.s.,

$$-Y\phi_Q = \frac{1}{f}(-Y)(f\phi_Q) \leq \frac{1}{f}[l(-Y) + l^*(f\phi_Q)]$$

Notice that for all h $\mathcal{G}$-measurable,for all $B \in \mathcal{F}$, $E_Q(h1_B) = E_P(h1_B\phi_Q) = E_P(E_P(h\phi_Q|\mathcal{F})1_B) = E_Q(E_P(h\phi_Q|\mathcal{F}1_B)$.(as the restriction of Q to $\mathcal{F}$ is equal to P). So

$$E_Q(h|\mathcal{F}) = E_P(h\phi_Q|calF)$$

It follows that for all $Y \in \tilde{\mathcal{A}}$ for all $f \in \mathcal{E}_{\mathcal{F}}$ such that $f > 0$ a.s.,

$$E_Q(-Y|\mathcal{F}) = E_P(-Y\phi_Q|\mathcal{F}) \leq \frac{1}{f}[l(g) + E_P[l^*(f\phi_Q)|\mathcal{F}]$$

(because $E_P(l(-Y)|calF) \leq l(g)$)

Hence

$$\alpha(Q) \leq ess\ inf_{\{f\in\mathcal{E}_{\mathcal{F}}; f>0 a.s.\}}[l(g) + E_P(l^*(f\phi)|\mathcal{F}))]$$

It remains to prove the converse inequality.

We give a complete proof of this inequality under the following simplified assumptions (as in theorem 4.61 of [12].

H: Assume that there is $k \in I\!R$ such that $l(x) = infl$ for all $x \leq k$, that l^* is finite on $I\!R+$ and that J is continuous on $I\!R^*+$.

As for all $z \in I\!R+$,$l^*(z) \geq -l(0)$,

$$\lim_{z\to 0^+} l(J(z)) \leq l(0) < l(g)\ P\ a.s. \quad (1)$$

For all $n \in I\!N$ denote $A_n = \{\omega \in \Omega; \phi_Q(\omega) \leq n\}$ and $\phi_n = \phi_Q 1_{A_n} + n1_{\Omega-A_n}$ Denote

$$E_n = \{f \in \mathcal{E}_{\mathcal{F}}; f > 0\ and\ E_P(l(J(f\phi_n))|\mathcal{F} \leq l(g)$$

As ϕ_n is bounded, it follows from (1) that E_n always contains a strictly positive constant; and also as $lim_{z\to\infty} J(z) = \infty$ it follows that E_n is bounded. Let $f_n = ess\ supE_n f_n \in \mathcal{E}_{\mathcal{F}}$. f_n is the increasing limit of a sequence of elements of E_n so using the continuity of l and J and the dominated convergence theorem, it follows that

$$E_P(l(J(f_n\phi_n))|\mathcal{F} \leq l(g)$$

We want to prove that the preceding inequality is in fact an equality. Otherwise there is $\epsilon > 0$ such that $P(B_\epsilon) > 0$ where $B_\epsilon = \{\omega \in \Omega; E_P(l(J(f_n\phi_n))|\mathcal{F}(\omega) < l(g)(\omega) - \epsilon\}$. As f_n and ϕ_n are bounded it follows from the uniform continuity of loJ on every compact that there is $\eta > 0$ such that $E_P(l(J(f_n + \eta 1_{B_\epsilon})\phi_n))|\mathcal{F} \leq l(g)1_{B_\epsilon}$ And this contradicts the definition of f_n. So for all n, there is $f_n > 0$ in $\mathcal{E}_\mathcal{F}$ such that $E_P(l(J(f_n\phi_n))|\mathcal{F} = l(g)$

Denote now $Y_n = -J(f_n\phi_n)$

$$E_P(l(-Y_n)|\mathcal{F}) = E_P(l(J(f_n\phi_n))|\mathcal{F}) = l(g)$$

so $Y_n \in \tilde{\mathcal{A}}$

and

$$E_Q(-Y_n|\mathcal{F}) = E_P(J(f_n\phi_n)\phi_Q)|\mathcal{F})$$

$$\geq E_P(J(f_n\phi_n)\phi_n)|\mathcal{F}) = \frac{1}{f_n}[E_P((J(f_n\phi_n))\phi_n f_n|\mathcal{F})]$$

$$= \frac{1}{f_n}[l(g) + E_P(l^*(f_n\phi_n)|\mathcal{F})] \geq \frac{1}{f_n}(l(g) - l(0))$$

As ϕ_n is an increasing sequence, E_n is decreasing so f_n is a decreasing sequence of strictly positive bounded $\mathcal{F}$-measurable maps. It follows that it has a limit $f \in \mathcal{E}_\mathcal{F}$. Q is such that $\alpha(Q)$ is in $\mathcal{E}_\mathcal{F}$ For all n, $\alpha(Q) \geq \frac{1}{f_n}(l(g) - l(0))$. It follows that $f > 0$ and that f and $\frac{1}{f}$ are bounded. Furthermore we have proved that for all $n \in I\!N$,

$$\alpha(Q) \geq \frac{1}{f_n}[l(g) + E_P(l^*(f_n\phi_n)|\mathcal{F})]$$

We can apply Fatou's lemma and we get

$$\alpha(Q) \geq lim\ inf \frac{1}{f_n}[l(g) + E_P(l^*(f_n\phi_n)|\mathcal{F})]$$

$$\geq \frac{1}{f}[l(g) + E_P(l^*(f\phi)|\mathcal{F})]$$

This ends the proof of proposition 7 and also of theorem 4 under the hypothesis H.

In the case where we don't assume the hypothesis H, we end the proof as in the proof of theorem 4.61 of [12].

Corollary 2:

Let l be a convex loss function. Let g a bounded $\mathcal{F}$-measurable function such that for all $\omega \in \Omega$,$g(\omega)$ is in the interior of the image of l. Let

$$\tilde{\mathcal{A}} = \{X \in L^\infty(\Omega, \mathcal{G}, P)\ /\ E(l(-X)|\mathcal{F}) \leq g\ \ P\ a.s.\}$$

For all $X \in L^\infty(\Omega, \mathcal{G}, P)$ define

$$\rho(X) = ess \inf\{Y \in \mathcal{E}_{\mathcal{F}} \ / \ X + Y \in \tilde{\mathcal{A}}\}$$

Then

$$\rho(X) = ess \max_{Q \in \mathcal{M}}((E_Q(-X|\mathcal{F}) - \alpha(Q))$$

where $\mathcal{M}$ is a set of probability measures Q on $(\Omega, \mathcal{G})$ absolutely continuous with respect to P and such that the restriction of Q to $\mathcal{F}$ is equal to P.

Furthermore

$$\alpha(Q) = ess\ inf\{f \in \mathcal{E}_{\mathcal{F}}; f > 0 \ \ P\ a.s.\}[\frac{1}{f}(g + E_P(l^*(f\frac{dQ}{dP})|\mathcal{F}))]$$

(l^* is the conjugate function of the convex function l).

Proof:

For all ω, $g(\omega)$ is in the interior of the image of l and l is continuous strictly increasing from I onto the interior of the image of l. so $f = l^{-1}(g)$ is a bounded $\mathcal{F}$-measurable map,there is a real number a such that $a + f > 1$ and the loss function $\tilde{l}$ defined by $\tilde{(l)}(x) = l(x - a)$ satisfies the hypothesis of theorem 4

We apply the result of theorem 4 to $\tilde{l}$. So we get the result.

Remark Assume that there is no probability measure given a priori on $(\Omega, \mathcal{G})$ and that for the investor there is a whole family $\mathcal{Q}$ of possible probability measures on $(\Omega, \mathcal{G})$ whose restriction to $\mathcal{F}$ is equal to P. We replace then the $\mathcal{F}$ acceptance set of the theorem 4 by

$$\mathcal{A} = \{X \in \mathcal{X} \ / \forall Q \in \mathcal{Q}; E_Q(l(-X)|\mathcal{F}) \leq g \ \ P\ a.s.\}$$

where $\mathcal{X}$ is the set of all bounded $\mathcal{G}$-measurable maps. This gives a risk measure conditional to the probability space $(\Omega, \mathcal{F}, P)$. The penality function is now

$$\alpha(Q) = ess\ inf\{(f, R) \ / \ \in \mathcal{E}_{\mathcal{F}}; f > 0\ a.s.; R \in \mathcal{Q}\}[\frac{1}{f}(g+E_P(l^*(f\frac{dQ}{dR})|\mathcal{F}))]$$

Applications:

1) Conditional entropic risk measure:
Consider the exponential loss function
$l(x) = e^{\alpha x}$ with $\alpha > 0$
Let $g > 0$ a bounded $\mathcal{F}$-measurable map.

$$\rho(X) = ess\ inf\{Y \in \mathcal{E}_{\mathcal{F}}\ /\ E(e^{[-\alpha(X+Y)]}|\mathcal{F}) \leq g\} = \frac{1}{\alpha}[lnE(e^{-\alpha X})|\mathcal{F}) - ln(g)\}$$

Then $\alpha(Q) = \frac{1}{\alpha}(E_P(ln(\frac{dQ}{dP})\frac{dQ}{dP}|\mathcal{F}) - ln(g))$.

2)Let $p > 1$ Consider the loss function
$l(x) = \frac{x^p}{p}\ if x \geq 0$
$l(x) = 0$ else
The conjugate function is
$l^*(x) = \frac{x^q}{q}\ if x \leq 0$
$l^*(x) = \infty$ else where q is the conjugate exponent of p.
Let $g > 0$.
Then

$$\alpha(Q) = (pg)^{\frac{1}{p}} E_P[(\frac{dQ}{dP})^q|\mathcal{F}]^{\frac{1}{q}}.$$

Indeed

$$\alpha(Q) = ess\ inf\{f \in \mathcal{E}_{\mathcal{F}}; f > 0\ \ P\ a.s.\}[\frac{1}{f}(g + E_P(\frac{1}{q}(f\frac{dQ}{dP})^q|\mathcal{F}))].$$

For all $\omega \in \Omega$, the maximum of

$$\frac{1}{f(\omega)}(g(\omega) + (\frac{1}{q}(f(\omega)^q E_P[(\frac{dQ}{dP})^q|\mathcal{F})](\omega)))$$

is obtained for $-g(\omega) + \frac{q-1}{q}(f(\omega)^q E_P[(\frac{dQ}{dP})^q|\mathcal{F}](\omega) = 0$ i.e.

$$f = (\frac{pg}{E_P[(\frac{dQ}{dP})^q|\mathcal{F}]})^{\frac{1}{q}}\ \ P\ a.s.$$

The results obtained for a convex loss function can also be extended to the case of a random convex loss function $l(\omega, x)$ under conditions such as the uniform equicontinuity on each compact of the family $l(\omega, .)$.

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