

ECOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES APPLIQUÉES
UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 41 50. Fax: 01 69 33 30 11

<http://www.cmap.polytechnique.fr/>

**Optimized Schwarz methods for unsymmetric
layered problems with strongly discontinuous and
anisotropic coefficients**

Luca Gerardo-Giorda and Frédéric Nataf

R.I. N° 561

December 2004

Optimized Schwarz methods for unsymmetric layered problems with strongly discontinuous and anisotropic coefficients

Luca GERARDO-GIORDA ^{*} and Frédéric NATAF [†]

December 8, 2004

Abstract

In this paper we consider unsymmetric elliptic problems of advection-diffusion-reaction type, with strongly heterogeneous and anisotropic diffusion coefficients. We use non-overlapping Optimized Schwarz Methods (OSM) and we study new interface conditions where only one or two real parameters have to be chosen along the entire interface. Using one real parameter it is possible to design interface conditions of Robin type, whereas the use of two real parameters and of more general interface conditions allows to better take into account the heterogeneities of the medium. The analysis is made at the semi-discrete level, where the equation is discretized in the direction parallel to the interface, and kept continuous in the normal direction. Numerical results are given to validate the proposed interface conditions.

1 Introduction

High fluid pressures within the rock layers of the subsurface are among the biggest problems an oil company has to deal with when drilling. A mathematical model for the prediction of fluid pressures on a geological time scale is based on conservation of mass and Darcy's law (see for instance [5]). This can be generalized to a time-dependent advection-diffusion equation, where the region also changes in time as rocks are deposited or eroded. An Euler backward method is used for the time integration, and a numerical method such as finite volumes or finite differences is applied at any time step in order to solve the advection-diffusion equation, yielding a linear system of equations.

A further complication of the physical problem is given by the heterogeneities of the underground: the presence of layers with very large differences in permeability yields contrasts up to seven orders of magnitude in the different regions of the computational domain. The widespread availability of parallel computers makes domain decomposition methods a natural candidate to take into account such problems. Such methods are based on the subdivision of the computational domain into several subdomains (which may or may not overlap) and the parallel solution of the local problems. This procedure leads to an iterative method that converges to the solution of the original problem if the solutions in the subdomains are related by means of suitable boundary conditions at the interface. The performance of the method depends drastically on the design of interface conditions, which has been the subject of several works (see e.g. [30, 32, 39] and references therein).

We consider here unsymmetric elliptic problems of advection-diffusion-reaction type, with strongly heterogeneous and anisotropic viscosity coefficients. Such problems arise naturally in several applications of practical interest, when dealing with the modeling of transport and diffusion of a species through porous media, but also in other engineering applications as electrical power networks, groundwater flows, semiconductors, and electromagnetics modeling. The generalized Robin/Robin preconditioner introduced in [19] is especially

^{*}Dép. Math. App., Institut Français du Pétrole, Rueil Malmaison, France. This author's work is supported by a Marie Curie Industry Fellowship, Contract nr. HPMI-GH-99-00012-05

[†]CMAP, CNRS, UMR 7641, École Polytechnique, 91128 Palaiseau Cedex - France

suites to treat strong discontinuities in the viscosity coefficients when the surfaces of discontinuity coincide with the subdomain interfaces, but its efficiency decays when the subdomains are not homogeneous. To override this drawback, we propose a different approach based on a semi-continuous factorization of the advection-diffusion operator. The original Schwarz Algorithm uses Dirichlet interface conditions, and overlapping is necessary to ensure convergence. In [29] Robin interface conditions are introduced, ensuring convergence without resorting to overlap.

Usually, general domain decomposition procedures are analyzed at continuous level (see for instance [1, 3, 6, 18, 19, 13], and references therein) and the resulting optimized interface conditions are then discretized (see [16, 26, 34, 10]). Recently, the problem of optimization has been analyzed also at the discrete level (see [38, 15, 30]). We choose herein an halfway between the two approaches, where the model problem is discretized in the direction parallel to the interface, and kept continuous in the normal direction.

The paper is organized as follows. In Section 2 the setting of the semi-discrete problem is given. In Section 3 an Optimized Schwarz Method is defined at the semi-discrete level. In Section 4 we outline the substructuring procedure to reduce the global problem and we study the spectrum of the resulting interface problem. In Section 5 are designed Robin interface conditions depending on a real parameter which is optimized. In Section 6 we design more general interface conditions based on two real parameters: the choice of these parameters is also addressed. Finally, in Section 7 we validate the proposed interface conditions by means of several numerical tests.

2 Statement of the problem

Let $\Omega = \mathbf{R} \times Q$ be an infinite tube, where Q is a bounded domain of \mathbf{R}^p , for some $p \geq 1$. A point in Ω is denoted with (x, \mathbf{y}) . We consider the problem

$$\begin{aligned} Lu &= f && \text{in } \Omega \\ \mathcal{B}u &= g && \text{on } \mathbf{R} \times \partial Q, \end{aligned} \tag{2.1}$$

with u bounded at infinity, where $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ are given functions, \mathcal{B} is a suitable boundary operator, and where L is an elliptic partial differential operator of advection-diffusion-reaction type, with coefficients independent of x , namely

$$L := -\frac{\partial}{\partial x} c_x(\mathbf{y}) \frac{\partial}{\partial x} + b_x(\mathbf{y}) \frac{\partial}{\partial x} + \mathcal{A}(\mathbf{y}).$$

As an example, for $p = 1$, $\mathbf{y} = y$ and a possible form of the operator $\mathcal{A}(y)$ is

$$\mathcal{A}(y) = \eta(y) - \frac{\partial}{\partial y} c_y(y) \frac{\partial}{\partial y} + \frac{\partial}{\partial y} b_y(y) + b_y(y) \frac{\partial}{\partial y}, \tag{2.2}$$

with boundary conditions depending on \mathcal{B} , with $\eta \geq 0$, $c_x, c_y > 0$, $\mathbf{b} = (b_x, b_y) \in [W^{1,\infty}(Q)]^2$ given, which corresponds to the advection-diffusion-reaction problem

$$\begin{aligned} -\operatorname{div}(c\nabla u) + \operatorname{div}(\mathbf{b}u) + \eta u &= f && \text{in } \Omega \\ \mathcal{B}u &= g && \text{on } \mathbf{R} \times \partial Q, \end{aligned}$$

with coefficients depending only on y .

We solve problem (2.1) by a non overlapping domain decomposition method. We decompose Ω into the half tubes $\Omega_1 = (-\infty, 0) \times Q$ and $\Omega_2 = (0, +\infty) \times Q$. We denote in the sequel with u_i and g_i ($i = 1, 2$) the restrictions of the solution u and the boundary term g to Ω_i and $\partial\Omega_i \cap \partial\Omega$, respectively.

Following what is done in [15] for symmetric problems, we consider a semi-continuous version of problem (2.1) where only the directions tangential to the interface $\Gamma = \{0\} \times Q$ are discretized, whereas the normal direction x is kept continuous. We denote with \mathcal{L} the semi-discrete version of L , where the original operator is discretized in the \mathbf{y} direction, namely

$$\mathcal{L} = -\frac{\partial}{\partial x} C \frac{\partial}{\partial x} + B \frac{\partial}{\partial x} + A, \tag{2.3}$$

where A , B , and C are $N_y \times N_y$ matrices, where N_y is the number of discretization points of the open set $Q \subset \mathbb{R}$, and can be obtained via a finite volume or finite difference discretization of (2.2) on a given mesh or triangulation of $Q \subset \mathbb{R}$. In such framework, we may assume

Assumption 2.1 *Matrices B and C are diagonal and matrices A and C are positive definite .*

3 An Optimized Schwarz Method without overlap

We consider an Optimized Schwarz Method (OSM) based on arbitrary interface conditions \mathcal{Q}_1 and \mathcal{Q}_2 , which reads as follows.

Given arbitrary initial guesses u_1^0 and u_2^0 , solve for $n \geq 1$ until convergence

$$\begin{aligned} \mathcal{L}u_1^{n+1} &= f && \text{in } \Omega_1 && \mathcal{L}u_2^{n+1} &= f && \text{in } \Omega_2 \\ \mathcal{B}(u_1^{n+1}) &= g && \text{on } \partial\Omega_1 \cap (\mathbf{R} \times \partial Q) && \mathcal{B}(u_2^{n+1}) &= g && \text{on } \partial\Omega_2 \cap (\mathbf{R} \times \partial Q) \\ \mathcal{Q}_1(u_1^{n+1}) &= \mathcal{Q}_1(u_2^n) && \text{on } \Gamma, && \mathcal{Q}_2(u_2^{n+1}) &= \mathcal{Q}_2(u_1^n) && \text{on } \Gamma, \end{aligned} \quad (3.1)$$

where, with a little abuse of notation, \mathcal{B} denotes the semi-discrete version of the boundary condition in (2.1). Such methods, relying on the solution of a fixed point iteration, are intrinsically slow; however, it is possible to both increase the robustness of the method and speed up the algorithm convergence by replacing the fixed point iterative solver by a Krylov type method. This can be accomplished by substructuring the algorithm in terms of the interface unknowns

$$H_1 = \mathcal{Q}_1(u_2)(0, \cdot) \quad H_2 = \mathcal{Q}_2(u_1)(0, \cdot),$$

and introducing the operator

$$\mathbf{T} : (H_1, H_2, f, g) \mapsto (\mathcal{Q}_2(v_1), \mathcal{Q}_1(v_2))^T,$$

where, for $i = 1, 2$, v_i is the solution of

$$\begin{aligned} \mathcal{L}(v_i) &= f && \text{in } \Omega_i \\ \mathcal{B}(v_i) &= g && \text{on } \partial\Omega_i \cap (\mathbf{R} \times \partial Q) \\ \mathcal{Q}_i(v_i) &= H_i && \text{on } \Gamma. \end{aligned} \quad (3.2)$$

So far, the substructured problem is then obtained by matching the auxiliary variables on the interface and reads

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} - \mathbf{\Pi}\mathbf{T}(H_1, H_2, 0, 0) = \mathbf{\Pi}\mathbf{T}(0, 0, f, g) \quad (3.3)$$

where $\mathbf{\Pi}$ is the swap operator on the interface, $\mathbf{\Pi}((H_1, H_2)^T) = (H_2, H_1)^T$, having the block form

$$\mathbf{\Pi} = \begin{pmatrix} 0 & \mathbf{Id} \\ \mathbf{Id} & 0 \end{pmatrix}.$$

As it is well known in domain decomposition literature, the OSM can be interpreted as a fixed point iterative procedure to solve (3.3).

The convergence rate of the OSM depends clearly on the choice of the interface conditions \mathcal{Q}_i ($i = 1, 2$). An essential role is played by the Steklov-Poincaré operator for each subdomain, which is also known in literature as Dirichlet to Neumann (DtN). Such operator is defined in the following way. Let $u_\Gamma : \Gamma \rightarrow \mathbf{R}$; the Dirichlet to Neumann operator for Ω_i ($i = 1, 2$) is defined as

$$\Phi_i : u_\Gamma \mapsto C \frac{\partial u_i}{\partial n_i},$$

where u_i is the solution of

$$\begin{aligned} \mathcal{L}u_i &= 0 && \text{in } \Omega_i \\ \mathcal{B}u_i &= 0 && \text{on } \partial\Omega_i \cap (\mathbf{R} \times \partial Q) \\ u_i &= u_\Gamma && \text{on } \Gamma, \end{aligned} \quad (3.4)$$

namely, it associates to u_Γ the normal derivative on the interface of its \mathcal{L} -harmonic extension in Ω_i . Let us consider the Hilbert spaces V^+ and V^- defined as

$$V^\pm := (H^1(\mathbb{R}^\pm))^{N_y}.$$

An element $u \in V^\pm$ is defined as $u = (u_1, \dots, u_{N_y})^T$, the spaces V^\pm are endowed with the norms

$$\|u\|_{V^\pm} := \left(\sum_{j=1}^{N_y} \|u_j\|_{H^1(\mathbb{R}^\pm)}^2 \right)^{1/2},$$

and it is easy to see, by a standard Lax-Milgram argument and using Assumption 2.1, that problem (3.4) admits a unique solution in V^\pm for any u_0 given.

The use of the interface conditions

$$\mathcal{Q}_i = C \frac{\partial}{\partial n_i} + \Phi_j \quad 1 \leq i \neq j \leq 2,$$

would entail an optimal convergence rate, ensuring convergence in a number of iteration equal to the number of subdomains (see [35]). In general, such operators are unfortunately nonlocal and cannot be used in practical computations. An approximation is thus mandatory and is usually based in literature on the symbol of the Steklov-Poincaré operator which is obtained via a Fourier transform of the operator in the direction parallel to the interface and on a factorization of the transformed operator $\widehat{\mathcal{L}}$ (see for instance [4, 8, 17, 33] for various problems).

In order to design suitable approximations of the Steklov-Poincaré operators, we introduce

$$\Lambda_1 = \Phi_1 - \frac{B}{2}, \quad \Lambda_2 = \Phi_2 + \frac{B}{2}, \quad (3.5)$$

which are square matrices of order N_y . We can prove the following

Lemma 3.1 *The operators Λ_1 and Λ_2 defined in (3.5) are positive definite.*

Proof. We prove the result for Λ_1 , the result for Λ_2 being analogous. Multiplying equation (3.4) in Ω_1 by u_1 and integrating by parts, we get

$$\int_{\Omega_1} C \left(\frac{\partial u_1}{\partial x} \right)^2 + (A u_1, u_1) = \int_{\Gamma} \left(C \frac{\partial u_1}{\partial x} + \frac{B}{2} u_\Gamma \right) u_\Gamma = \int_{\Gamma} (\Lambda_1 u_0, u_0) \geq 0.$$

Moreover, it is easy to see that if $\int_{\Gamma} (\Lambda_1 u_0, u_0) = 0$, we have $u_1 \equiv 0$ in Ω_1 , and this concludes the proof. \square From the definition of the Steklov-Poincaré operator, we have, for $x = 0$,

$$0 = -C \frac{\partial u_2}{\partial x} - \Phi_2 u_2 = -C \frac{\partial u_2}{\partial x} + \left(\frac{B}{2} - \Lambda_2 \right) u_2. \quad (3.6)$$

The x -invariance of the problem entails this equality for all $x > 0$, thus u_2 in (3.4) is given by

$$u_2(x) = e^{C^{-1} \left(\frac{B}{2} - \Lambda_2 \right) x} u_\Gamma.$$

The fact that u_2 must satisfy equation (3.4) yields a compatibility condition for Λ_2 . Applying the operator \mathcal{L} to u_2 we get

$$\mathcal{L}(u_2) = e^{C^{-1} \left(\frac{B}{2} - \Lambda_2 \right) x} \left(- \left(\frac{B}{2} - \Lambda_2 \right) C^{-1} \left(\frac{B}{2} - \Lambda_2 \right) + B C^{-1} \left(\frac{B}{2} - \Lambda_2 \right) + A \right) u_\Gamma = 0,$$

and a simple algebra provides

$$\Lambda_2 C^{-1} \Lambda_2 + \frac{1}{2} (B C^{-1} \Lambda_2 - \Lambda_2 C^{-1} B) - \left(A + \frac{1}{4} B C^{-1} B \right) = 0, \quad (3.7)$$

In a similar way, we can easily see that the matrix Λ_1 must satisfy the compatibility condition

$$\Lambda_1 C^{-1} \Lambda_1 + \frac{1}{2} (\Lambda_1 C^{-1} B - B C^{-1} \Lambda_1) - \left(A + \frac{1}{4} B C^{-1} B \right) = 0. \quad (3.8)$$

Multiplying equation (3.8) by $C^{-1/2}$ (which is always defined as C is a diagonal positive definite matrix) on the left and on the right, we have

$$\tilde{\Lambda}_1^2 + \frac{1}{2} [\tilde{\Lambda}_1, \tilde{B}] = \left(\tilde{A} + \frac{1}{4} \tilde{B}^2 \right) \quad (3.9)$$

where we have set, for any square matrix M of order N_y , $\tilde{M} := C^{-1/2} M C^{-1/2}$, and where $[\cdot, \cdot]$ is the Lie bracket, defined, for any pair of square matrices of the same order (P, Q) , as $[P, Q] := PQ - QP$.

In a similar way we easily see that, with the same notations as in equation (3.9), equation (3.7) reduces to

$$\tilde{\Lambda}_2^2 + \frac{1}{2} [\tilde{B}, \tilde{\Lambda}_2] = \left(\tilde{A} + \frac{1}{4} \tilde{B}^2 \right), \quad (3.10)$$

Equations (3.9) and (3.10) are a special case of the well-known Algebraic Riccati Equation (ARE), which is widely studied in Optimal Control Theory (see for instance [7, 24, 31] and references therein), and whose most general shape is

$$XPX + QX + XR - S = 0,$$

where P, Q, R, S are real matrices of order $m \times m$, $m \times m$, $n \times n$, and $n \times m$, respectively ($m, n \in \mathbf{N}$). It is immediate to see that equations (3.12) and (3.13) belong to such a framework by taking $m = n = N_y$, and $P = Id$, $Q = -R = \tilde{B}/2$, and $S = \tilde{A} + \tilde{B}^2/4$.

The bracket products in (3.9) and (3.10) do not vanish in general, unless the matrices \tilde{B} and Λ_j commute (this, for instance, the case of constant coefficients, where $\tilde{B} = \gamma Id$ for some constant γ). Thus, an explicit formula for the solutions $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$, and, consequently, for Λ_1 and Λ_2 , is not available, differently from the purely elliptic case (see [15]). However, under some mild assumptions, we can prove the following proposition, which will be useful in the sequel.

Proposition 3.1 *Let $p = 1$. Assume that the matrix $\tilde{A} + \frac{1}{4} \tilde{B}^2$ is symmetrizable, and let Λ_1 be a positive solution of equation (3.8). Then there exists a diagonal matrix V such that*

$$\Lambda_2 = (V^{-1})^2 \Lambda_1^T (V)^2 \quad (3.11)$$

is a positive solution of equation (3.7).

The result is based on the following proposition (a standard result in linear algebra).

Proposition 3.2 *Let $D \in \mathbb{R}^{n \times n}$ be a tridiagonal matrix such that*

$$D = \begin{pmatrix} d_1 & -p_1 a_{1,2} & 0 & \dots & 0 \\ -\frac{a_{2,1}}{p_1} & d_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & d_{n-1} & -p_{n-1} a_{n-1,n} \\ 0 & \dots & 0 & -\frac{a_{n,n-1}}{p_{n-1}} & d_n \end{pmatrix}$$

where $a_{i,i-1} = a_{i-1,i}$ and $p_i \neq 0, \forall i$. The transformation

$$\begin{aligned} V D V^{-1} &= \bar{D} \\ V &= \text{diag} (1, p_1, p_1 p_2, \dots, \prod_{i=1}^{n-1} p_i) \end{aligned}$$

yields a symmetric matrix \bar{D} :

$$\bar{D} = \begin{pmatrix} d_1 & -a_{1,2} & 0 & \dots & 0 \\ -a_{2,1} & d_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & d_{n-1} & -a_{n-1,n} \\ 0 & \dots & 0 & -a_{n,n-1} & d_n \end{pmatrix}.$$

Remark 3.1 *The matrix A in (3.7) and (3.8) comes from the discretization of an elliptic differential operator of advection-diffusion-reaction type: either in a finite volume or a finite difference setting, with an upwind scheme for the velocity field, the resulting matrix is tridiagonal and it can be easily seen to satisfy the hypotheses of Proposition 3.2. The same can be easily seen to hold true also for \tilde{A} , as C is diagonal positive definite.*

Proof (of Proposition 3.1). We apply Proposition 3.2 to matrix \tilde{A} , and let V be the symmetrizing matrix: we multiply equations (3.9) and (3.10) on the left by V and on the right by V^{-1} . Since the matrix B comes from the discretization of the transport coefficients in the x direction, it is diagonal, and the same holds true also for \tilde{B} : thus, the multiplication by V and V^{-1} does not affect the matrix \tilde{B} , and we obtain

$$\bar{\Lambda}_1^2 + \frac{1}{2}[\tilde{B}, \bar{\Lambda}_1] = \bar{A} + \frac{1}{4}\tilde{B}^2, \quad (3.12)$$

where we have set $\bar{\Lambda}_1 = V\tilde{\Lambda}_1V^{-1}$ and $\bar{A} = \tilde{A}^T = V\tilde{A}V^{-1}$. Similarly, equation (3.10) for Λ_2 becomes

$$\bar{\Lambda}_2^2 + \frac{1}{2}[\bar{\Lambda}_2, \tilde{B}] = \bar{A} + \frac{1}{4}\tilde{B}^2, \quad (3.13)$$

where $\bar{\Lambda}_2 = V\tilde{\Lambda}_2V^{-1}$. It is now easy to see that if $\bar{\Lambda}_1$ is a solution of equation (3.12), then $\bar{\Lambda}_2 = \bar{\Lambda}_1^T$ is a solution of (3.13). In fact, taking the transpose of equation (3.12) we get

$$(\bar{\Lambda}_1^T)^2 + \frac{1}{2}\left((\tilde{B}\bar{\Lambda}_1)^T - (\bar{\Lambda}_1\tilde{B})^T\right) = \bar{A}^T + \frac{1}{4}(\tilde{B}^T)^2,$$

namely

$$(\bar{\Lambda}_1^T)^2 + \frac{1}{2}\left(\bar{\Lambda}_1^T\tilde{B} - \tilde{B}\bar{\Lambda}_1^T\right) = \bar{A} + \frac{1}{4}\tilde{B}^2,$$

which is nothing but equation (3.13).

So far, we get

$$\begin{cases} \Lambda_1 = C^{1/2}V^{-1}\bar{\Lambda}_1VC^{1/2} \\ \Lambda_2 = C^{1/2}V^{-1}\bar{\Lambda}_1^TVC^{1/2} = C^{1/2}V^{-1}\left(C^{-1/2}V\Lambda_1V^{-1}C^{-1/2}\right)^TVC^{1/2}, \end{cases} \quad (3.14)$$

and the thesis follows since the matrices C and V commute. \square

Remark 3.2 *Under the hypotheses of Proposition 3.1, since V is diagonal, an immediate consequence of (3.11) is that Λ_1 and Λ_2 share the same diagonal, i.e.*

$$\text{diag}(\Lambda_1) = \text{diag}(\Lambda_2).$$

Notice moreover that in general $\Lambda_2 \neq \Lambda_1^T$, unless \tilde{A} is already a symmetric matrix (i.e. $V = Id$): this is the case, for instance, of an advective field normal to the interface, yielding a symmetric operator in the y direction.

4 Substructuring

If we could use the interface conditions

$$\mathcal{Q}_1 = C \frac{\partial}{\partial x} + \Phi_2 \quad \mathcal{Q}_2 = -\frac{\partial}{\partial x} C + \Phi_1 \quad (4.1)$$

the Schwarz algorithm would converge in two steps, and the result would be optimal in the sense of iteration counts. Unfortunately, as the matrices Φ_1 and Φ_2 are not sparse, they cannot be used in practice due to their high computational cost. We therefore consider the approximated matrices

$$\Phi_1^{\text{app}} := \frac{B}{2} + \Lambda_1^{\text{app}} \quad \Phi_2^{\text{app}} := -\frac{B}{2} + \Lambda_2^{\text{app}}, \quad (4.2)$$

where Λ_1^{app} and Λ_2^{app} are sparse and suitably approximate Λ_1 and Λ_2 (recall that B is diagonal), and we consider the Schwarz algorithm (3.1) with the interface conditions (4.1), where the exact interface operators Φ_j ($j = 1, 2$) are replaced by the approximated ones Φ_j^{app} .

We introduce the auxiliary variables

$$H_1 = \left(C \frac{\partial}{\partial x} + \Phi_2^{\text{app}} \right) u_2 \quad H_2 = \left(-C \frac{\partial}{\partial x} + \Phi_1^{\text{app}} \right) u_1, \quad (4.3)$$

where u_1 and u_2 denote the restrictions of the solution u to Ω_1 and Ω_2 , respectively, and we substructure the original problem in terms of these latter variables.

Lemma 4.1 *Let \mathcal{L} be defined in (2.3). The Schwarz algorithm (3.1) with interface conditions $\mathcal{Q}_1 = C \frac{\partial}{\partial x} + \Phi_2^{\text{app}}$ and $\mathcal{Q}_2 = -\frac{\partial}{\partial x} C + \Phi_1^{\text{app}}$ can be substructured in form (3.3), where the matrix \mathbf{T} is given by*

$$\mathbf{T} = \begin{pmatrix} [-\Lambda_2 + \Lambda_2^{\text{app}}][\Lambda_2 + \Lambda_1^{\text{app}}]^{-1} & 0 \\ 0 & [-\Lambda_1 + \Lambda_1^{\text{app}}][\Lambda_1 + \Lambda_2^{\text{app}}]^{-1} \end{pmatrix}. \quad (4.4)$$

Proof. We have to compute $\mathbf{T}(H_1, H_2, 0, 0)$. The general solution in Ω_1 is given by

$$v_1(x) = \exp \left\{ \frac{1}{C} \left(\frac{B}{2} + \Lambda_1 \right) x \right\} \alpha_1,$$

whereas the general solution within Ω_2 is given by

$$v_2(x) = \exp \left\{ \frac{1}{C} \left(\frac{B}{2} - \Lambda_2 \right) x \right\} \alpha_2,$$

where the vectors α_1 and α_2 are completely determined by the interface conditions on Γ .

We have, within Ω_1 ,

$$H_1 = \left(C \frac{\partial}{\partial x} + \Phi_2^{\text{app}} \right) v_1(x) \Big|_{x=0} = \left(C \frac{\partial}{\partial x} + \Phi_2^{\text{app}} \right) \exp \left\{ \frac{1}{C} \Phi_1 x \right\} \alpha_1 \Big|_{x=0} = (\Phi_1 + \Phi_2^{\text{app}}) \alpha_1,$$

and, within Ω_2 ,

$$H_2 = \left(-C \frac{\partial}{\partial x} + \Phi_1^{\text{app}} \right) \exp \left\{ \frac{1}{C} \Phi_2 x \right\} \beta_2 \Big|_{x=0} = [-\Phi_2 + \Phi_1^{\text{app}}] \beta_2.$$

Thus,

$$v_1(x) = \exp \left\{ \frac{1}{C} \Phi_1 x \right\} [\Phi_1 - \Phi_2^{\text{app}}]^{-1} H_1 \quad v_2(x) = \exp \left\{ \frac{1}{C} \Phi_2 x \right\} [-\Phi_2 + \Phi_1^{\text{app}}]^{-1} H_2.$$

We then have:

$$\mathcal{Q}_1(v_2) = \left(C \frac{\partial}{\partial x} - \Phi_2^{\text{app}} \right) v_2(x) = [\Phi_2 - \Phi_2^{\text{app}}] [-\Phi_2 + \Phi_1^{\text{app}}]^{-1} H_2,$$

$$\mathcal{Q}_2(v_1) = \left(-C \frac{\partial}{\partial x} + \Phi_1^{\text{app}} \right) v_1(x) = [-\Phi_1 + \Phi_1^{\text{app}}] [\Phi_1 - \Phi_2^{\text{app}}]^{-1} H_1.$$

Owing to (3.5) and (4.2), the thesis follows. \square

4.1 Spectral analysis of the substructured problem

In this section we focus on the spatial distribution of the spectrum of the substructured matrix $\mathbf{Id} - \mathbf{\Pi T}$. In the following, we denote, for any square matrix M , with $\sigma(M)$ the spectrum of M . For sake of simplicity, we set

$$\Sigma_1 := [-\Lambda_2 + \Lambda_2^{\text{app}}][\Lambda_2 + \Lambda_1^{\text{app}}]^{-1} \quad \Sigma_2 := [-\Lambda_1 + \Lambda_1^{\text{app}}][\Lambda_1 + \Lambda_2^{\text{app}}]^{-1},$$

and the substructured matrix can be rewritten as

$$\mathbf{Id} - \mathbf{\Pi T} = \begin{pmatrix} \mathbf{Id} & -\Sigma_2 \\ -\Sigma_1 & \mathbf{Id} \end{pmatrix}, \quad (4.5)$$

and we can prove the following result, giving a characterization of the eigenvalues of the substructured problem.

Lemma 4.2 *Let the matrix $\mathbf{Id} - \mathbf{\Pi T}$ be defined in (4.5). We have*

$$\begin{aligned} \gamma \in \sigma(\mathbf{Id} - \mathbf{\Pi T}) &\implies (1 - \gamma)^2 \in \sigma(\Sigma_1 \Sigma_2) \\ \mu \in \sigma(\Sigma_1 \Sigma_2) &\implies (1 \pm \sqrt{\mu}) \in \sigma(\mathbf{Id} - \mathbf{\Pi T}) \end{aligned}$$

Proof. For the first part of the proof, let γ be an eigenvalue of $(\mathbf{Id} - \mathbf{\Pi T})$, associated with the eigenvector (\mathbf{x}, \mathbf{y}) , *i.e.*

$$\begin{pmatrix} \mathbf{Id} & -\Sigma_2 \\ -\Sigma_1 & \mathbf{Id} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \gamma \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \quad (4.6)$$

Equation (4.6) can be equivalently rewritten as

$$\begin{cases} (1 - \gamma) \mathbf{x} = \Sigma_2 \mathbf{y} \\ (1 - \gamma) \mathbf{y} = \Sigma_1 \mathbf{x} \end{cases} \quad (4.7)$$

So far, applying Σ_1 to the first equation in (4.7) and using the second one we get

$$\Sigma_1 \Sigma_2 \mathbf{y} = (1 - \gamma) \Sigma_1 \mathbf{x} = (1 - \gamma)^2 \mathbf{y}. \quad (4.8)$$

For the second part of the proof, we recall that if μ is an eigenvalue of $\Sigma_1 \Sigma_2$, with associated eigenvector \mathbf{w} , *i.e.*

$$\Sigma_1 \Sigma_2 \mathbf{w} = \mu \mathbf{w},$$

then, μ is an eigenvalue also for $\Sigma_2 \Sigma_1$ with associated eigenvector $\Sigma_2 \mathbf{w}$. So far, let then

$$\mathbf{x} = \Sigma_2 \mathbf{w} \quad \mathbf{y} = \pm \sqrt{\mu} \mathbf{w}, \quad (4.9)$$

where (μ, \mathbf{w}) is an eigenpair of $\Sigma_1 \Sigma_2$. Computing $\Sigma_1 \mathbf{x}$ and $\Sigma_2 \mathbf{y}$, we get

$$\begin{cases} \Sigma_1 \mathbf{x} = \Sigma_1 \Sigma_2 \mathbf{w} = \mu \mathbf{w} = \pm \sqrt{\mu} \mathbf{y} \\ \Sigma_2 \mathbf{y} = \pm \sqrt{\mu} \Sigma_2 \mathbf{w} = \pm \sqrt{\mu} \mathbf{x}, \end{cases}$$

which, setting $\gamma = 1 \pm \sqrt{\mu}$, are equivalent to equations (4.7), and this concludes the proof. \square

The previous lemma states that the eigenvalues of the substructured matrix $(\mathbf{Id} - \mathbf{\Pi T})$ lie in the interior of a disc of the complex plane centered in 1 with radius equal to square root of the spectral radius of the matrix $\Sigma_1 \Sigma_2$, which corresponds to the reduction factor of the Schwarz algorithm with interface conditions $\mathcal{Q}_1 = (C \frac{\partial}{\partial x} - \Phi_2^{\text{app}})$ and $\mathcal{Q}_2 = (-\frac{\partial}{\partial x} C + \Phi_1^{\text{app}})$. To ensure convergence, the matrices Λ_1^{app} and Λ_2^{app} in the interface conditions have thus to be chosen in order to have $\rho(\Sigma_1 \Sigma_2) < 1$. A partial result in this direction is given by the following lemma.

Lemma 4.3 Let $\Lambda_1^{\text{app}} = \Lambda_2^{\text{app}} = \Lambda$, where Λ is a symmetric positive definite matrix. Then we have

$$\rho(\Sigma_1) < 1, \quad \rho(\Sigma_2) < 1.$$

Proof. Since Λ is symmetric positive definite, we can write

$$\Sigma_j = [-\Lambda_j + \Lambda][\Lambda_j + \Lambda]^{-1} = \Lambda^{1/2} \left[-\Lambda^{-1/2} \Lambda_j \Lambda^{-1/2} + \text{Id} \right] \left[\Lambda^{-1/2} \Lambda_j \Lambda^{-1/2} + \text{Id} \right]^{-1} \Lambda^{-1/2},$$

and the spectrum of Σ_j coincides with the spectrum of $\left[-\Lambda^{-1/2} \Lambda_j \Lambda^{-1/2} + \text{Id} \right] \left[\Lambda^{-1/2} \Lambda_j \Lambda^{-1/2} + \text{Id} \right]^{-1}$. Let then (γ, \mathbf{w}) be an eigenpair of this latter matrix, *i.e.*

$$\left[-\Lambda^{-1/2} \Lambda_j \Lambda^{-1/2} + \text{Id} \right] \left[\Lambda^{-1/2} \Lambda_j \Lambda^{-1/2} + \text{Id} \right]^{-1} \mathbf{w} = \gamma \mathbf{w}.$$

Multiplying on the right by $\left[\Lambda^{-1/2} \Lambda_j \Lambda^{-1/2} + \text{Id} \right]$, a simple algebra leads

$$\Lambda^{-1/2} \Lambda_j \Lambda^{-1/2} \mathbf{w} = \frac{1-\gamma}{1+\gamma} \mathbf{w}.$$

Owing to Lemma 3.1, $\Lambda^{-1/2} \Lambda_j \Lambda^{-1/2}$ is a real positive matrix: we then have, for any $\mathbf{z} \in \mathbb{C}^{N_y}$,

$$\text{Re}(\Lambda^{-1/2} \Lambda_j \Lambda^{-1/2} \mathbf{z}, \bar{\mathbf{z}}) > 0.$$

Thus,

$$\text{Re} \left(\frac{\gamma+1}{\gamma-1} \right) = - \left| \frac{1-\gamma}{1+\gamma} \right|^2 \text{Re} \left(\frac{1-\gamma}{1+\gamma} \right) < 0,$$

and owing to the Cayley transformation we can conclude that $|\gamma| < 1$. \square

Remark 4.1 From Lemma 4.2 it is immediate to see that if one could take $\Phi_1^{\text{app}} = \Phi_1$ or $\Phi_2^{\text{app}} = \Phi_2$ in (4.2), the preconditioning would be exact, as $\Sigma_1 = \Sigma_2 = \mathbf{0}$.

The following sections are dedicated to the choice of the approximate matrices Λ_1^{app} and Λ_2^{app} .

5 Robin interface conditions

For sake of simplicity at the computational level, the most natural choice consists in approximating Λ_1 and Λ_2 by suitable positive definite diagonal matrices D_1 and D_2 . Owing to Proposition 3.1 (which states that under some not so restrictive assumptions the matrices Λ_1 and Λ_2 have the same diagonal) and to Lemma 4.3, we choose $D_1 = D_2 = \alpha D$, where D is an approximation of the diagonal of $\Lambda_{1,2}$. The Robin interface conditions are thus given by

$$\mathcal{Q}_1 = C \frac{\partial}{\partial x} - \frac{B}{2} + \alpha D \quad \mathcal{Q}_2 = -C \frac{\partial}{\partial x} + \frac{B}{2} + \alpha D$$

where $\alpha \in \mathbb{R}$ is a parameter to be tuned for sake of optimization of the convergence rate of the OSM. The problem of optimization is widely studied for the symmetric positive case, but no theoretical result exists in this sense for the unsymmetric case. The optimal parameter α_{opt} is sought here in order to minimize the maximum between the spectral radii $\rho^2(\Sigma_1^\alpha)$ and $\rho^2(\Sigma_2^\alpha)$, where we have set $\Sigma_j^\alpha = [-\Lambda_j + \alpha D][\Lambda_j + \alpha D]^{-1}$ ($j = 1, 2$). Since, for any $\alpha \in \mathbb{R}^+$, $\rho(\Sigma_j^\alpha) = \max_{\lambda \in \sigma(\Lambda_j D^{-1})} \left| \frac{\lambda - \alpha}{\lambda + \alpha} \right|$, we set

$$\zeta(\alpha) := \max_{\lambda \in \sigma(\Lambda_2 D^{-1}) \cup \sigma(\Lambda_1 D^{-1})} \left| \frac{\lambda - \alpha}{\lambda + \alpha} \right|,$$

and we would like to solve the optimization problem

$$\zeta(\alpha_{\text{opt}}) = \min_{\alpha \in \mathbb{R}^+} \zeta(\alpha). \quad (5.1)$$

A simple manipulation leads to the equivalent problem

$$\begin{aligned}
\zeta(\alpha_{\text{opt}}) &= \min_{\alpha \in \mathbf{R}^+} \max_{\lambda \in \sigma(\Lambda_1 D^{-1}) \cup \sigma(\Lambda_2 D^{-1})} \left| \frac{\lambda - \alpha}{\lambda + \alpha} \right|^2 = \\
&= \min_{\alpha \in \mathbf{R}^+} \max_{\lambda \in \sigma(\Lambda_1 D^{-1}) \cup \sigma(\Lambda_2 D^{-1})} \left| \frac{(\text{Re } \lambda - \alpha) + i \text{Im } \lambda}{(\text{Re } \lambda + \alpha) + i \text{Im } \lambda} \right|^2 = \\
&= 1 - 4 \max_{\alpha \in \mathbf{R}^+} \left(\min_{\lambda \in \sigma(\Lambda_1 D^{-1}) \cup \sigma(\Lambda_2 D^{-1})} \frac{\alpha \text{Re } \lambda}{(\alpha + \text{Re } \lambda)^2 + (\text{Im } \lambda)^2} \right).
\end{aligned} \tag{5.2}$$

Differently from what is done in [12], we do not have an explicit formula linking the real part and the imaginary part of the eigenvalues of the matrices $\Lambda_j D^{-1}$ ($j = 1, 2$), and we cannot solve the optimization problem (5.1) analytically. Let

$$r := \min_{\lambda \in \bigcup_j \sigma(\Lambda_j D^{-1})} \text{Re } \lambda \quad R := \max_{\lambda \in \bigcup_j \sigma(\Lambda_j D^{-1})} \text{Re } \lambda \quad I := \max_{\lambda \in \bigcup_j \sigma(\Lambda_j D^{-1})} \text{Im } \lambda. \tag{5.3}$$

Owing to (5.2), we reduce ourselves to solve the optimization problem

$$\max_{\alpha \in \mathbf{R}^+} \min_{(x, y) \in [r, R] \times [0, I]} \frac{\alpha x}{(\alpha + x)^2 + y^2}, \tag{5.4}$$

the latter interval being justified by symmetry with respect to 0 in the y variable, whose solution is given by the following lemma.

Lemma 5.1 *The solution of the optimization problem (5.4) is given by*

$$\alpha_{\text{opt}} = \max \left\{ \sqrt{r^2 + I^2}, \sqrt{rR - I^2} \right\} \tag{5.5}$$

and the optimal reduction factor is bounded from above by

$$\zeta_{\text{opt}} \leq \begin{cases} \frac{2r(r - \sqrt{r^2 + I^2}) + I^2}{2r(r + \sqrt{r^2 + I^2}) + I^2} & \text{if } rR - I^2 \leq r^2 + I^2 \\ \frac{r + R - 2\sqrt{rR - I^2}}{r + R + 2\sqrt{rR - I^2}} & \text{if } rR - I^2 > r^2 + I^2 \end{cases} \tag{5.6}$$

Proof. Let $\varphi(\alpha, x, y) := \frac{\alpha x}{(\alpha + x)^2 + y^2}$. As

$$\partial_y \varphi(\alpha, x, y) = -2 \frac{\alpha x y}{[(\alpha + x)^2 + y^2]^2},$$

and $\alpha \in \mathbf{R}^+$, it is not difficult to see that $\varphi(\alpha, x, y)$ is decreasing with respect to y in $[0, I]$, thus the minimum has to be sought for $y = I$.

Since

$$\partial_x \varphi(\alpha, x, y) = \frac{\alpha}{[(\alpha + x)^2 + y^2]^2} [\alpha^2 + y^2 - x^2],$$

we immediately have that, if $\alpha^2 < r^2 - I^2$, $\varphi(\alpha, x, y)$ is decreasing in x and thus the minimum is attained at $x = R$. If $r^2 - I^2 < \alpha^2 < R^2 - I^2$, there exists $x_0 \in (r, R)$, $x_0 = \alpha^2 + y^2$, that maximizes $\varphi(\alpha, x, y)$, and the minimum is attained at one endpoint of the interval. Finally, if $\alpha^2 > R^2 - I^2$, $\varphi(\alpha, x, y)$ is increasing in x and the minimum is attained at $x = r$. Notice that, if $R^2 - I^2 < 0$, this is always the situation that occurs. Thus, for any $\alpha \in \mathbf{R}^+$, and whatever the sign of $r^2 - I^2$ and $R^2 - I^2$, the minimum of $\varphi(\alpha, x, y)$ is attained either at (r, I) or at (R, I) , and we have

$$\frac{\alpha R}{(\alpha + R)^2 + I^2} - \frac{\alpha r}{(\alpha + r)^2 + I^2} = \frac{\alpha(R - r)(\alpha^2 - rR + I^2)}{[(\alpha + r)^2 + I^2][(\alpha + R)^2 + I^2]}.$$

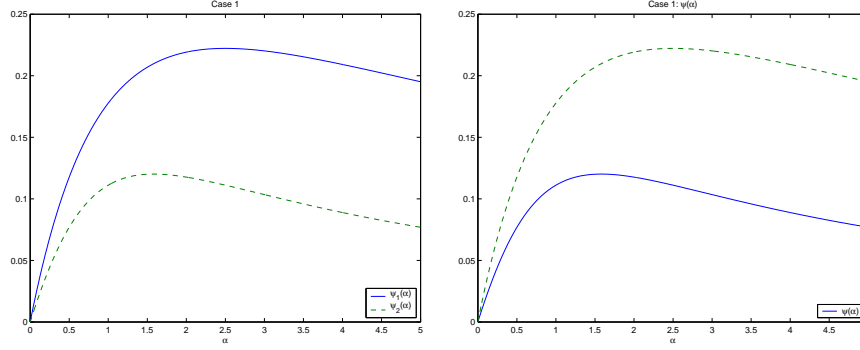


Figure 1: $[rR - I^2 \leq 0]$ Left: $\psi_1(\alpha)$ (solid) and $\psi_2(\alpha)$ (dashed). Right: $\psi(\alpha)$ (solid)

If $\alpha^2 < rR - I^2$ the minimum is attained in (R, I) , whereas if $\alpha^2 > rR - I^2$ the minimum is attained in (r, I) . The optimization problem reduces thus to seek the maximum of the function $\psi(\alpha)$ defined as follows:

$$\psi(\alpha) := \begin{cases} \psi_1(\alpha) = \frac{\alpha R}{(\alpha + R)^2 + I^2} & \text{if } \alpha^2 < rR - I^2 \\ \psi_2(\alpha) = \frac{\alpha r}{(\alpha + r)^2 + I^2} & \text{if } \alpha^2 > rR - I^2. \end{cases}$$

For any fixed (x, y) , we have

$$\partial_\alpha \varphi(\alpha, x, y) = \frac{x}{[(\alpha + x)^2 + y^2]^2} [x^2 + y^2 - \alpha^2],$$

and it is immediate to see that the maximum of $\varphi(\alpha, x, y)$ is attained in

$$\alpha_{\max}^2 = x^2 + y^2.$$

We have thus to consider three different occurrences.

i) $rR - I^2 \leq 0$ (Figure 1): in this case $\psi(\alpha) = \psi_2(\alpha)$ for any $\alpha \in \mathbf{R}^+$, and the optimal parameter is given by

$$\alpha_{\text{opt}} = \sqrt{r^2 + I^2}.$$

ii) $0 < rR - I^2 < r^2 + I^2$ (Figure 2): in this case the maxima of both $\psi_1(\alpha)$ and $\psi_2(\alpha)$ occur for $\alpha^2 > rR - I^2$, and the optimal parameter is again given by

$$\alpha_{\text{opt}} = \sqrt{r^2 + I^2}.$$

iii) $r^2 + I^2 < rR - I^2$ (Figure 3): in this latter case $\alpha^2 = rR - I^2$ separates the maxima of $\psi_1(\alpha)$ and $\psi_2(\alpha)$, and $\psi(\alpha)$ can be easily seen to be increasing for $\alpha^2 < rR - I^2$ and decreasing afterwards. The optimal parameter is thus given by

$$\alpha_{\text{opt}} = \sqrt{rR - I^2}.$$

So far, a simple algebra provides estimate (5.6), and concludes the proof. \square

Remark 5.1 A very simple situation occurs when the advection field vanishes. In this case $B = 0$, the operator is symmetric, and we have an explicit expression for $\Lambda = \Lambda_1 = \Lambda_2$, given by

$$\Lambda = C^{1/2} \tilde{A}^{1/2} C^{1/2} = C^{1/2} (C^{-1/2} A C^{-1/2})^{1/2} C^{1/2}.$$

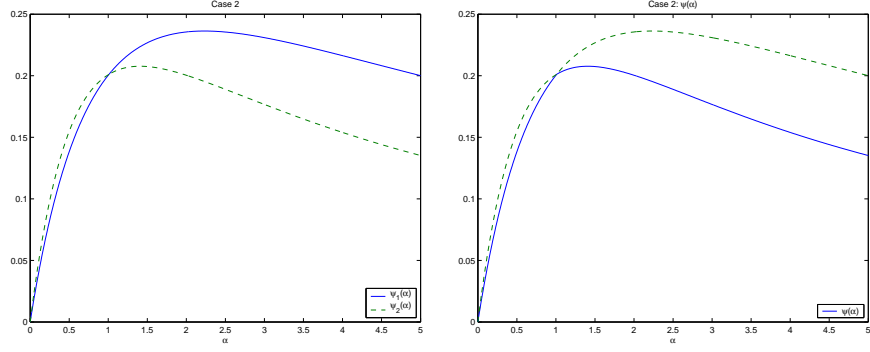


Figure 2: $[rR - I^2 < r^2 + I^2]$ Left: $\psi_1(\alpha)$ (solid) and $\psi_2(\alpha)$ (dashed). Right: $\psi(\alpha)$ (solid)

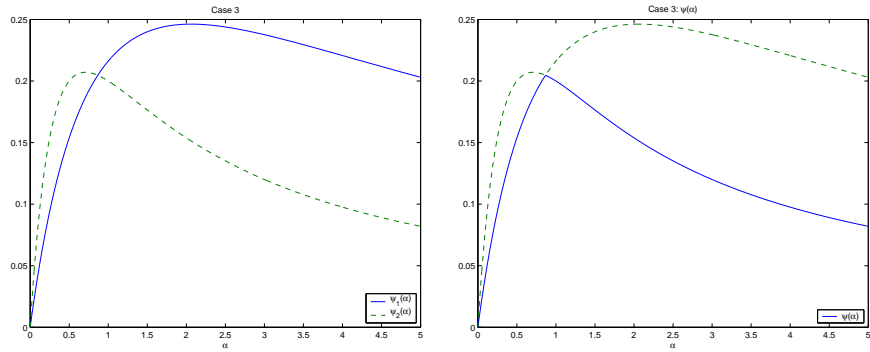


Figure 3: $[r^2 + I^2 < rR - I^2]$ Left: $\psi_1(\alpha)$ (solid) and $\psi_2(\alpha)$ (dashed). Right: $\psi(\alpha)$ (solid)

The Robin interface condition can thus be written as

$$C \frac{\partial}{\partial x} + \alpha C^{1/2} \text{diag}(\tilde{A})^{1/2} C^{1/2}$$

and the tuning of the parameter $\alpha \in \mathbb{R}$ coincides with the optimization done in [15] in the case of purely elliptic problems.

6 Second order interface conditions

In the previous section, we used an interface condition of Robin type which reads, for domain Ω_1 :

$$C \frac{\partial}{\partial x} - \frac{B}{2} + \alpha D$$

where D is a suitable diagonal matrix. We are interested here in considering more general interface conditions, whose performance is better than the Robin ones.

Taking inspiration from Higdon's trick (see [25]) for absorbing boundary conditions, we consider interface conditions of the form

$$\tilde{Q} := \left(C \frac{\partial}{\partial x} - \frac{B}{2} + \alpha D \right) \left(C \frac{\partial}{\partial x} - \frac{B}{2} + \beta D \right)$$

where D is a positive diagonal matrix, and α and β are positive parameters. The product yields a second order derivative with respect to x , the direction normal to the interface,

$$\tilde{\mathcal{Q}} := C\left(\frac{\partial}{\partial x}C\frac{\partial}{\partial x} - B\frac{\partial}{\partial x}\right) + (\alpha + \beta)CD\frac{\partial}{\partial x} - \frac{\alpha + \beta}{2}BD + \alpha\beta D^2 + \frac{B^2}{4}.$$

Owing to (2.3), we can replace the terms in the parenthesis by CA , and we get the interface condition

$$\tilde{\mathcal{Q}} := CA + (\alpha + \beta)CD\frac{\partial}{\partial x} - \frac{\alpha + \beta}{2}BD + \alpha\beta D^2 + \frac{B^2}{4}.$$

In order to write the interface condition in the form $C\partial/\partial x - B/2 + \Lambda^{\text{app}}$, and, owing on one hand to the fact that interface conditions are equivalent up to an invertible transformation on the interface, and on the other hand to the fact that B , C , and D are diagonal and commute, we can multiply on the left $\tilde{\mathcal{Q}}$ by the inverse of $(\alpha + \beta)D$, obtaining finally the interface condition

$$\mathcal{Q} := C\frac{\partial}{\partial x} - \frac{B}{2} + \frac{\alpha\beta D + D^{-1}CA + (1/4)D^{-1}B^2}{\alpha + \beta}. \quad (6.1)$$

This amount to choose an approximation of the matrices Λ_1 and Λ_2 by means of

$$\Lambda^{\text{app}} := \frac{\alpha\beta D + D^{-1}CA + (1/4)D^{-1}B^2}{\alpha + \beta},$$

where the parameters α and β have to be suitably tuned: the optimization of the parameters in the unsymmetric case is still an open problem. Following what is done in [15] for purely elliptic problems, in the numerical tests of the following section, we choose α and β such that:

$$\alpha\beta = rR, \quad \alpha + \beta = \sqrt{2(r+R)\sqrt{rR}}, \quad (6.2)$$

where r and R are defined in (5.3).

7 Numerical results

In this section we test several different interface conditions and algorithms in the semi-continuous framework of the previous sections. More precisely, we deal with an infinite tube in 2D, $\Omega = \mathbb{R} \times (0, 1)$, and consider the operator

$$L := -\left(\frac{\partial}{\partial x}c(y)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}d(y)\frac{\partial}{\partial y}\right) + p(y)\frac{\partial}{\partial x} + q(y)\frac{\partial}{\partial y} + \eta(y)$$

with a Dirichlet boundary condition at the bottom and a Neumann boundary condition on the top. We use a finite volume discretization of the operator in the y direction yielding a tridiagonal matrix A of order ny . We build the matrices of the substructured problem for various interface conditions and we study their spectra. We give in the tables the ratio of the largest modulus of the eigenvalues over the smallest real part. We also give the number of subdomain solves corresponding to the solution of the substructured problem (3.3) by a GMRES algorithm with a random right hand side G . The numerical tests are performed with MATLAB[®] 6.1, and the stopping criterion is a reduction of the residual by a factor 10^{-10} . Although we do not consider a discretization in the x direction, the results are a good indication of what would happen in the corresponding fully discrete case.

We use interface conditions of the form

$$\mathcal{Q}_1 = C\frac{\partial}{\partial x} - \frac{B}{2} + \Lambda_2^{\text{app}}, \quad \mathcal{Q}_2 = -\frac{\partial}{\partial x}C + \frac{B}{2} + \Lambda_1^{\text{app}},$$

and we list hereafter the different approximations of Λ_i ($i = 1, 2$) we used in the numerical tests.

- **Diagonal:** The interface condition uses the approximated matrix $\Lambda_i^{\text{app}} = D_\Lambda = \text{diag}(\Lambda_1) = \text{diag}(\Lambda_2)$.
- Λ^{opt} : The interface condition uses the diagonal matrix D_Λ weighted by the optimal parameter α_{opt} defined in (5.5).

- **Robin:** Let D be the square root of the diagonal matrix $Q = \text{diag}(\tilde{A} + \frac{\tilde{B}^2}{4})$, where \tilde{A} and \tilde{B} are defined in Section 3. The approximate matrix in the interface condition is given by the diagonal matrix D weighted by the optimal parameter α_D given by formula (5.5) where we have replaced (r, R, I) with

$$\tilde{r} := \min_{\lambda \in \sigma((\tilde{A} + \frac{\tilde{B}^2}{4})D^{-2})} \text{Re } \lambda^{1/2}, \quad \tilde{R} := \max_{\lambda \in \sigma((\tilde{A} + \frac{\tilde{B}^2}{4})D^{-2})} \text{Re } \lambda^{1/2}, \quad \tilde{I} := \max_{\lambda \in \sigma((\tilde{A} + \frac{\tilde{B}^2}{4})D^{-2})} \text{Im } \lambda^{1/2}. \quad (7.1)$$

- **Order 2:** The interface condition is given by formula (6.1) where D is defined as for the **Robin** interface condition, and the optimal parameters $\alpha\beta$ and $\alpha + \beta$ are given by formula (6.2), where we have replaced r and R by \tilde{r} and \tilde{R} as defined in (7.1).

For the first two interface conditions (**Diagonal** and $\mathbf{\Lambda}^{\text{opt}}$) it is necessary to compute explicitly the matrices Λ_i ($i = 1, 2$). Since no explicit formula is available in this direction, we use an iterative procedure to solve equations (3.9) and (3.10), and at each step we have to compute the square root of a full matrix. Such procedure turns out to be too costly to be conceivable in practice. To avoid such a drawback, we use in the **Robin** and **Order 2** interface conditions a simpler approach that can be used in practice. The approximations of Λ_1 and Λ_2 are built neglecting the Lie bracket in equations (3.9) and (3.10). Moreover, to avoid the computation of the square root of a matrix, we simply compute the square root of its diagonal, which is extremely cheap.

We have also implemented the Generalized Robin/Robin algorithm (see [19]): we will refer to it as **GRR**. This choice corresponds to apply the GMRES algorithm to the interface problem

$$(\Phi_1 + \Phi_2) \mathbf{u} = \mathbf{g}$$

preconditioned by

$$P = C_1(C_1 + C_2)^{-1} \left(\Phi_1 - \frac{B}{2} \right)^{-1} C_1(C_1 + C_2)^{-1} + C_2(C_1 + C_2)^{-1} \left(\Phi_2 + \frac{B}{2} \right)^{-1} C_2(C_1 + C_2)^{-1},$$

where C_j ($j = 1, 2$) is the matrix of the viscosity coefficients in Ω_j . Notice that in this case the size of the interface problem to be solved is half the size of other cases, but, on the other hand, each iteration consists in solving both a Dirichlet and a Robin boundary value problem in each subdomain. We report in the tables the total number of subdomains solves.

7.1 Layered coefficients with strong discontinuities

In this first series of tests, we consider the case where the viscosity coefficients show strong discontinuities in the y direction. The domain $\Omega = \mathbb{R} \times (0, 1)$ is divided into ten slabs of height $hy = .1$, where the viscosity coefficients are constant. The difference between the viscosities in two neighboring slabs can be of order 10^4 . The viscosity coefficients in the i -th slab is given by $c = d = \nu(i)$, the latter being the i -th component of the vector $\nu = [\alpha, \beta, \gamma, \beta, \beta, \beta, \alpha, \alpha, \gamma, \alpha]$, where $\alpha = 1.e0$, $\beta = 1.e4$, and $\gamma = 1.e2$. We take a constant reaction term $\eta = 10$, and the following advective fields

- i)* $p = 100, q = 0$: the velocity is normal with respect to the interface.
- ii)* $p = 0, q = 100$: the velocity is parallel with respect to the interface.
- iii)* $p = q = 100$: the velocity field is diagonal with respect to the interface.
- iv)* $p(y) = 100y^2, q(y) = 100 \cos(4\pi y)$: the velocity is no longer constant.

In Table 1 we report the results for different mesh parameters $hy = 1/ny$. The proposed interface conditions show on one hand an increase in both subdomain solves and condition number with respect to the meshsize, although they remain reasonable (apart for the condition number of the diagonal preconditioning). On the other hand all the proposed interface conditions show a fair insensitivity with respect to the velocity field. For an advection field parallel to the interface the Generalized Robin/Robin algorithm shows the expected exact preconditioning due to the symmetry of the problem. The best performances among the proposed interface conditions are obtained with the second order ones.

Velocity	ny		10	20	40	80	160
$p = 100$ $q = 0$	subd. solves	Diagonal Λ^{opt} Robin Order 2 GRR	13	21	24	35	50
			19	30	36	47	61
			18	26	35	47	61
			16	18	21	23	25
			10	12	12	12	12
	cond	Diagonal Λ^{opt} Robin Order 2 GRR	108.17	299.14	675.8	1.42e+03	2.92e+03
			14.27	24.97	34.8	49.2	69.7
			16.71	31.53	49.5	73.2	105.3
			2.79	5.46	7.41	9.32	11.25
			1.14	1.18	1.19	1.18	1.20
$p = 0$ $q = 100$	subd. solves	Diagonal Λ^{opt} Robin Order 2 GRR	12	20	25	36	52
			18	30	37	49	62
			18	28	37	49	62
			14	17	20	23	25
			2	2	2	2	2
	cond	Diagonal Λ^{opt} Robin Order 2 GRR	45.65	124.14	280.66	592.77	1.21e+03
			8.83	15.48	21.66	30.92	44.1
			9.19	14.54	21.36	30.77	43.9
			1.98	2.70	3.26	3.92	4.7
			1.00	1.00	1.00	1.00	1.00
$p = 100$ $q = 100$	subd. solves	Diagonal Λ^{opt} Robin Order 2 GRR	12	20	25	35	51
			19	30	36	48	61
			18	26	36	48	61
			14	17	20	23	26
			8	8	8	8	8
	cond	Diagonal Λ^{opt} Robin Order 2 GRR	45.62	124.05	280.43	592.29	1.21e+03
			8.86	15.51	21.70	30.94	44.1
			9.05	15.54	23.30	33.78	48.2
			2.00	2.71	3.56	4.43	5.3
			1.01	1.01	1.02	1.02	1.02
$p = 100y^2$ $q = 100 \cos(4\pi y)$	subd. solves	Diagonal Λ^{opt} Robin Order 2 GRR	13	21	25	37	54
			19	30	37	48	61
			18	26	36	49	62
			16	19	22	24	26
			6	8	8	8	8
	cond	Diagonal Λ^{opt} Robin Order 2 GRR	81.02	205.20	394.87	796.31	1.61e+03
			11.79	19.94	25.72	35.85	50.74
			12.06	19.34	25.92	36.49	51.70
			2.27	3.02	3.50	4.20	5.03
			1.02	1.06	1.09	1.09	1.09

Table 1: Results for layered subdomains, strongly heterogeneous coefficients

7.2 Different subdomains

Up to now we have considered discontinuities in the y direction, which yield two subdomains symmetric with respect to the interface. In this section we consider discontinuities in the viscosity coefficients also in the x direction, yielding two different subdomains. The operators in Ω_1 and Ω_2 are thus not the same, and the model problem reads

$$\begin{aligned} \mathcal{L}_{1,h}u_1 &= f & \text{in } \Omega_1 & & \mathcal{L}_{2,h}u_2 &= f & \text{in } \Omega_2 \\ C_1 \frac{\partial u_1}{\partial x} &= C_2 \frac{\partial u_2}{\partial x} & \text{on } \Gamma & & u_1 &= u_2 & \text{on } \Gamma \end{aligned} \quad (7.2)$$

where $\mathcal{L}_{i,h}$ ($i = 1, 2$) is a finite volume discretization of

$$\mathcal{L}_i = \left(\frac{\partial}{\partial x} c_i(y) \frac{\partial}{\partial x} + \frac{\partial}{\partial y} d_i(y) \frac{\partial}{\partial y} \right) + p_i(y) \frac{\partial}{\partial x} + q_i(y) \frac{\partial}{\partial y} + \eta_i(y)$$

We solve problem (7.2) by an additive Schwarz method which reads

$$\begin{aligned} \mathcal{L}_{1,h}u_1^{n+1} &= f & \text{in } \Omega_1 \\ \left(C_1 \frac{\partial}{\partial x} - \frac{B_2}{2} + \Lambda_{2,ap} \right) u_1^{n+1} &= \left(C_2 \frac{\partial}{\partial x} - \frac{B_2}{2} + \Lambda_{2,ap} \right) u_2^n & \text{on } \Gamma \end{aligned} \quad (7.3)$$

$$\begin{aligned} \mathcal{L}_{2,h}u_2^{n+1} &= f & \text{in } \Omega_2 \\ \left(-C_2 \frac{\partial}{\partial x} + \frac{B_1}{2} + \Lambda_{1,ap} \right) u_2^{n+1} &= \left(-C_1 \frac{\partial}{\partial x} + \frac{B_1}{2} + \Lambda_{1,ap} \right) u_1^n & \text{on } \Gamma \end{aligned} \quad (7.4)$$

where the matrices $\Lambda_{i,ap}$ are suitable approximation of the matrices Λ_i for the domain Ω_i , and where B_i is the matrix of the coefficients p_i . The matrices $\Lambda_{i,ap}$ are built separately as in Sections 5 and 6: these approximations don't take into account the fact that they are used in a domain decomposition where the operators vary from one subdomain to the other.

We focus here on three different situations. Firstly we consider the subdomains Ω_1 and Ω_2 to be homogeneous, namely, the viscosity coefficients do not change in the interior of each subdomain, and the discontinuities are located along the interfaces. Then, we allow the viscosity coefficients to have discontinuities also inside the subdomains. Finally, we consider anisotropic coefficients, where the permeability in the x direction differs from the permeability in the y direction. Such situations typically arise in the simulation of reservoirs and sedimentary basins, due the compaction of rocks. In all cases we compare the performances of the proposed interface conditions with the Generalized Robin/Robin algorithm, which is especially suited to treat the first occurrence.

In all cases, we take a uniform reaction term $\eta_1 = \eta_2 = 10$, and the same advection fields described in Section 7.1. In Tables 2 through 4 we report the results for different mesh parameters $hy = 1/ny$. In the homogeneous subdomains case the viscosity coefficients are $c_1 = d_1 = 1.e4$, and $c_2 = d_2 = 1$ respectively. In the heterogeneous subdomains case, the subdomains Ω_1 and Ω_2 are again divided into ten slabs of height $hy = .1$, where the viscosity coefficients are constant. Let $\alpha = 1.e4$, $\beta = 1.e2$, and $\gamma = 1.e0$. The viscosity coefficients in the i -th slab of Ω_1 is given by $c_1 = d_1 = \nu_1(i)$, the latter being the i -th component of the vector $\nu_1 = [\alpha, \alpha, \beta, \alpha, \alpha, \alpha, \beta, \gamma, \alpha]$, whereas the viscosity coefficients in the i -th slab of Ω_2 is given by $c_2 = d_2 = \nu_2(i)$, the latter being the i -th component of the vector $\nu_2 = [\gamma, \alpha, \alpha, \alpha, \beta, \gamma, \alpha, \alpha, \alpha]$. In the anisotropic case, finally, the viscosity coefficients in the i -th slab are given, in the x direction, by $c_1 = \nu_1(i)$ in Ω_1 and by $c_2 = \nu_2(i)$ within Ω_2 , while in the y direction are given by $d_1 = \mu_1(i)$ in Ω_1 and by $d_2 = \mu_2(i)$ within Ω_2 , the latter being the i -th components of the vectors $\mu_1 = [\gamma, \alpha, \gamma, \alpha, \beta, \alpha, \gamma, \gamma, \beta, \gamma]$ and $\mu_2 = [\beta, \alpha, \beta, \alpha, \gamma, \alpha, \beta, \beta, \gamma, \beta]$, respectively.

In the case of homogeneous subdomains (Table 2), the Generalized Robin/Robin algorithm shows the expected optimal condition number, whereas in terms of subdomain solves the other interface conditions are slightly better (recall that one iteration of GRR amounts to two subdomain solves instead of one for the optimized Schwarz methods), and there is no appreciable difference among them. In the case of heterogeneous

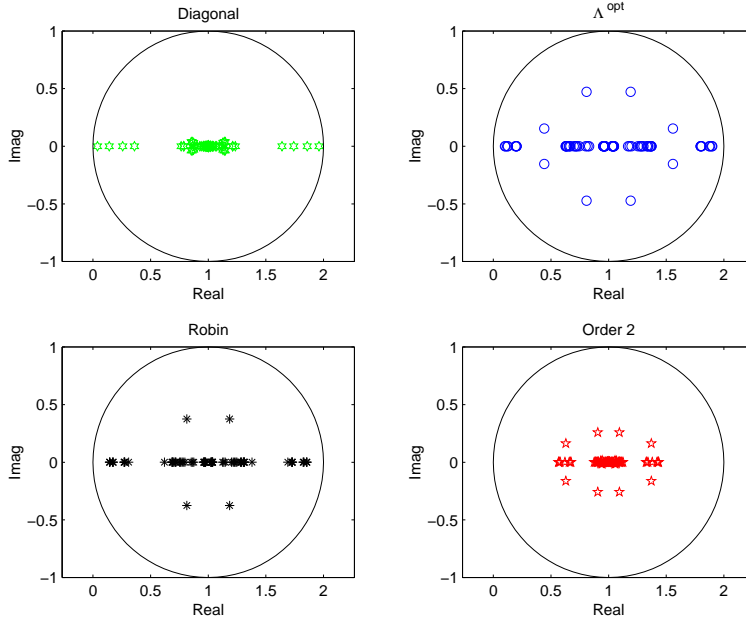


Figure 4: Heterogeneous subdomains: spectra of the substructured problems.

subdomains (Table 3), the condition number for the Generalized Robin/Robin algorithm grows significantly, although not so remarkably as for the diagonal preconditioner. This is particularly evident in the case of an advection field normal to the interface. The best performances in terms of both iteration counts and condition number are obtained for the second order interface conditions. In fact, in terms of subdomain solves, the second order conditions and the Generalized Robin/Robin algorithm are comparable, but the condition number provided by this latter is significantly higher. In the case of anisotropic coefficients (Table 4) the Generalized Robin/Robin algorithm shows a quick growth in terms of subdomain solves, although insensitive to the advection field, and it is comparable to the first order interface conditions. In this latter case the diagonal preconditioner, although giving a bad condition number for the interface problem, shows a good performance in terms of number of subdomains solves. Once again, the second order interface conditions provide the best results in terms of both condition number and number of subdomain solves. In Figure 4 we give the spectra of the substructured problems for the various interface conditions in the heterogeneous case with $ny = 40$, in Figure 5 the spectrum of the substructured problem for the Generalized Robin/Robin Algorithm in the same situation, in Figure 6 we give the convergence history for the proposed interface conditions, and in Figure 7 the convergence history of the Generalized Robin/Robin algorithm.

8 Conclusions

We proposed here a way to build optimized interface conditions for unsymmetric layered elliptic problems, based on a semi-continuous formulation of the differential problem, in order to handle strong discontinuities and anisotropies in the coefficients. The numerical tests performed in a semi-discrete setting enlighten the robustness of such interface conditions with respect to the meshsize, the advective field, the heterogeneities and the anisotropies in the viscosity coefficients. The comparison with the Generalized Robin/Robin algorithm, especially suited to handle discontinuous viscosity coefficients provided the discontinuities are located across the subdomains interfaces, is quite promising towards a fully discrete approach, in terms of both iteration counts and, most of all, of condition number of the problem. Further work needs to be done in this direction, in order to extend such optimization of interface conditions to a fully discrete problem, and

Velocity	ny		10	20	40	80	160
$p = 100$ $q = 0$	subd. solves	Diagonal Λ^{opt} , Robin Order 2 GRR	3	5	5	5	5
			5	5	5	5	5
			4	5	5	5	5
			6	6	6	8	8
	cond	Diagonal Λ^{opt} , Robin Order 2 GRR	1.02	1.05	1.08	1.14	1.23
			1.02	1.04	1.07	1.11	1.15
			1.01	1.02	1.01	1.02	1.03
			1.01	1.01	1.01	1.01	1.01
$p = 0$ $q = 100$	subd. solves	Diagonal Λ^{opt} , Robin Order 2 GRR	5	5	5	5	5
			5	5	5	5	5
			5	5	5	5	5
			6	6	6	6	6
	cond	Diagonal Λ^{opt} , Robin Order 2 GRR	1.06	1.09	1.13	1.18	1.26
			1.03	1.04	1.05	1.06	1.07
			1.01	1.01	1.02	1.02	1.03
			1.01	1.01	1.01	1.01	1.01
$p = 100$ $q = 100$	subd. solves	Diagonal Λ^{opt} , Robin Order 2 GRR	5	5	5	5	5
			5	5	5	5	5
			5	5	5	5	5
			8	8	8	8	8
	cond	Diagonal Λ^{opt} , Robin Order 2 GRR	1.04	1.07	1.11	1.17	1.24
			1.05	1.05	1.06	1.07	1.08
			1.01	1.01	1.02	1.02	1.03
			1.01	1.01	1.01	1.01	1.01
$p = 100y^2$ $q = 100 \cos(4\pi y)$	subd. solves	Diagonal Λ^{opt} Robin Order 2 GRR	5	5	5	5	7
			5	5	5	5	5
			5	5	5	5	5
			5	5	5	5	5
	cond	Diagonal Λ^{opt} Robin Order 2 GRR	1.06	1.08	1.12	1.17	1.25
			1.04	1.05	1.06	1.07	1.08
			1.04	1.05	1.06	1.07	1.08
			1.01	1.01	1.02	1.02	1.03
		GRR	1.01	1.01	1.01	1.01	1.01

Table 2: Results for homogeneous subdomains

Velocity	ny		10	20	40	80	160
$p = 100$ $q = 0$	subd. solves	Diagonal Λ^{opt} Robin Order 2 GRR	10	16	20	27	37
			18	32	48	64	85
			15	27	37	50	62
			9	15	19	23	26
			14	18	20	22	24
	cond	Diagonal Λ^{opt} Robin Order 2 GRR	6.54	27.15	64.27	137.94	285.19
			5.89	25.26	35.75	50.70	71.94
			2.95	11.72	17.86	25.86	37.05
			2.03	2.45	2.99	3.59	4.30
			32.99	42.31	51.84	61.46	71.11
$p = 0$ $q = 100$	subd. solves	Diagonal Λ^{opt} Robin Order 2 GRR	9	16	20	27	36
			17	28	40	52	66
			15	25	35	45	57
			9	15	19	22	25
			14	18	20	22	24
	cond	Diagonal Λ^{opt} Robin Order 2 GRR	6.02	22.56	53.27	114.35	236.47
			3.60	14.23	20.43	29.15	41.46
			2.69	8.79	14.45	21.55	31.14
			1.84	2.11	2.68	3.27	3.94
			10.11	12.70	15.56	18.52	21.45
$p = 100$ $q = 100$	subd. solves	Diagonal Λ^{opt} Robin Order 2 GRR	10	16	20	26	36
			17	28	39	52	66
			15	25	34	44	57
			9	15	19	22	25
			14	18	20	22	24
	cond	Diagonal Λ^{opt} Robin Order 2 GRR	6.02	22.56	53.27	114.35	236.47
			3.59	14.21	20.41	29.12	41.42
			2.69	8.42	13.72	20.40	29.53
			1.85	2.06	2.61	3.18	3.84
			10.42	12.56	15.58	18.40	21.35
$p = 100y^2$ $q = 100 \cos(4\pi y)$	subd. solves	Diagonal Λ^{opt} Robin Order 2 GRR	9	16	20	26	36
			17	27	37	51	62
			15	27	33	43	55
			9	15	19	22	25
			14	18	20	22	24
	cond	Diagonal Λ^{opt} Robin Order 2 GRR	5.71	21.59	50.77	108.86	225.05
			3.55	13.42	18.89	26.74	37.92
			2.61	8.01	12.98	19.43	28.17
			1.96	2.01	2.53	3.11	3.75
			11.81	13.96	16.97	20.10	23.28

Table 3: Results for different subdomains

Velocity	ny		10	20	40	80	160
$p = 100$ $q = 0$	subd. solves	Diagonal Λ^{opt} Robin Order 2 GRR	12	15	20	26	36
			18	33	46	60	72
			18	32	41	51	60
			12	16	19	20	25
			20	34	46	54	62
	cond	Diagonal Λ^{opt} Robin Order 2 GRR	6.40	15.02	32.04	65.47	131.51
			3.03	11.58	20.64	31.35	45.58
			5.26	9.97	15.63	22.77	32.55
			3.71	3.36	3.98	4.71	5.74
			8.82	24.57	25.49	25.52	25.52
$p = 0$ $q = 100$	subd. solves	Diagonal Λ^{opt} Robin Order 2 GRR	11	15	19	26	34
			15	28	39	49	59
			19	33	45	56	66
			14	17	19	22	24
			18	32	44	56	74
	cond	Diagonal Λ^{opt} Robin Order 2 GRR	2.49	7.64	16.25	33.16	66.70
			2.03	7.03	12.59	19.17	27.90
			5.50	10.65	16.70	24.35	34.80
			5.10	5.03	5.10	5.38	6.14
			3.89	13.83	17.02	20.16	22.43
$p = 100$ $q = 100$	subd. solves	Diagonal Λ^{opt} Robin Order 2 GRR	11	15	19	26	34
			15	27	38	49	59
			17	31	40	49	59
			12	16	18	20	24
			20	32	44	56	74
	cond	Diagonal Λ^{opt} Robin Order 2 GRR	2.51	7.67	16.29	33.25	66.87
			2.03	7.02	12.59	19.17	27.89
			4.54	8.72	13.65	19.89	28.44
			4.48	4.33	4.35	4.71	5.58
			3.89	13.85	17.04	20.17	22.44
$p = 100y^2$ $q = 100 \cos(4\pi y)$	subd. solves	Diagonal Λ^{opt} Robin Order 2 GRR	11	15	19	26	36
			18	35	54	72	88
			19	32	42	54	64
			13	16	19	21	24
			18	32	44	56	74
	cond	Diagonal Λ^{opt} Robin Order 2 GRR	3.15	9.87	21.44	44.35	90.01
			3.35	14.26	27.06	43.11	64.68
			5.56	10.28	16.08	23.43	33.51
			9.48	6.12	5.35	5.18	5.45
			6.93	17.46	19.52	21.52	23.22

Table 4: Results for anisotropic coefficients

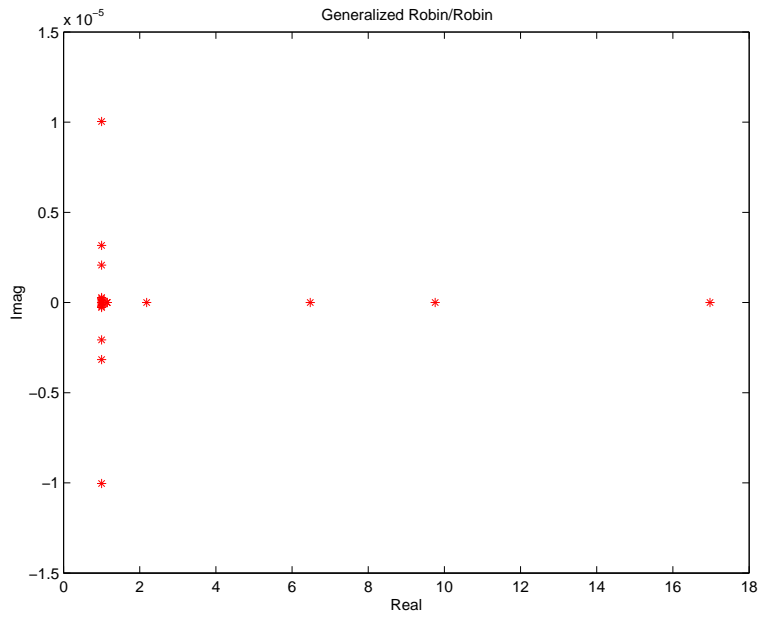


Figure 5: Heterogeneous subdomains: spectra of the substructured problem for the GRR algorithm.

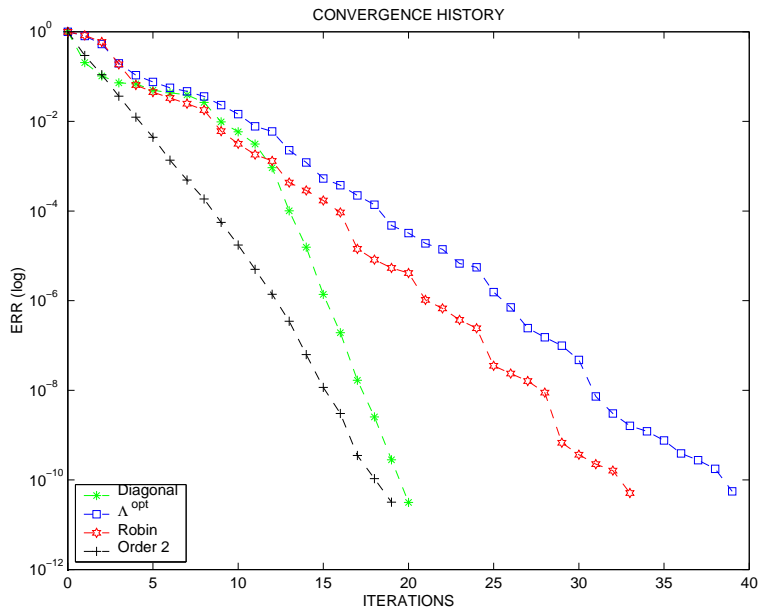


Figure 6: Heterogeneous subdomains: convergence history for different interface conditions.

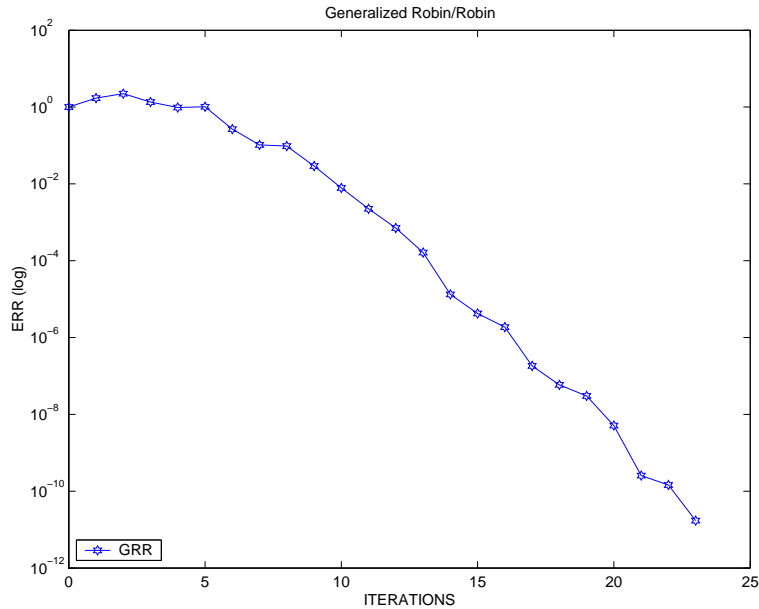


Figure 7: Anisotropic coefficients: convergence history for the GRR algorithm.

it will be the subject of a forthcoming paper ([20]).

References

- [1] Y. Achdou, P. Le Tallec, F. Nataf, and M. Vidrascu. A domain decomposition preconditioner for an advection-diffusion problem. *Comp. Meth. Appl. Mech. Engrg.*, 184:145–170, 2000.
- [2] Y. Achdou and F. Nataf. A Robin-Robin preconditioner for an advection-diffusion problem. *C. R. Acad. Sci. Paris*, 325, Série I:1211–1216, 1997.
- [3] A. Alonso, R. L. Trotta, and A. Valli. Coercive domain decomposition algorithms for advection-diffusion equations and systems. *J. Comput. Appl. Math.*, 96 (1):51–76, 1998.
- [4] Ana Alonso-Rodríguez and Luca Gerardo-Giorda. New non-overlapping domain decomposition methods for the harmonic maxwell system. Technical Report 667, Dep. of Mathematics - University of Trento, 2004. submitted.
- [5] J. Bear. *Dynamics of fluids in Porous media*. Elsevier, New York, 1972.
- [6] Luigi C. Berselli and Fausto Saleri. New substructuring domain decomposition methods for advection-diffusion equations. *J. Comput. Appl. Math.*, 116:201–220, 2000.
- [7] R. Byers. Solving the algebraic Riccati equation with the matrix sign. *Linear Algebra Appl.*, 85:267–279, 1987.
- [8] P. Collino, G. Delbue, P. Joly, and A. Piacentini. A new interface condition in the non-overlapping domain decomposition for the Maxwell equations. *Comput. Methods Appl. Mech. Engrg.*, 148:195–207, 1997.
- [9] Lawrence C. Cowsar, Jan Mandel, and Mary F. Wheeler. Balancing domain decomposition for mixed finite elements. *Math. Comp.*, 64(211):989–1015, July 1995.
- [10] Bruno Després. Domain decomposition method and the Helmholtz problem.II. In *Second International Conference on Mathematical and Numerical Aspects of Wave Propagation (Newark, DE, 1993)*, pages 197–206, Philadelphia, PA, 1993. SIAM.

- [11] Maksymilian Dryja, Marcus V. Sarkis, and Olof B. Widlund. Multilevel Schwarz methods for elliptic problems with discontinuous coefficients in three dimensions. *Numer. Math.*, 72(3):313–348, 1996.
- [12] O. Dubois. Optimized Schwarz methods for the advection-diffusion equation. Technical report, Master Thesis Mc Gill University, 2003.
- [13] Bjorn Engquist and Hong-Kai Zhao. Absorbing boundary conditions for domain decomposition. *Appl. Numer. Math.*, 27(4):341–365, 1998.
- [14] Charbel Farhat, Jan Mandel, and Francois-Xavier Roux. Optimal convergence properties of the FETI domain decomposition method. *Comput. Methods Appl. Mech. Engrg.*, 115:367–388, 1994.
- [15] Eric Flauraud and Frédéric Nataf. Optimized interface conditions in domain decomposition methods. Application at the semi-discrete and at the algebraic level to problems with extreme contrast in the coefficients. Technical Report 524, CMAP (Ecole Polytechnique), 2004.
- [16] Martin J. Gander, Laurence Halpern, and Frédéric Nataf. Optimized Schwarz methods. In Tony Chan, Takashi Kako, Hideo Kawarada, and Olivier Pironneau, editors, *Twelfth International Conference on Domain Decomposition Methods, Chiba, Japan*, pages 15–28, Bergen, 2001. Domain Decomposition Press.
- [17] Martin J. Gander, Frédéric Magoulès, and Frédéric Nataf. Optimized schwarz methods without overlap for the helmholtz equation. *SIAM J. Numer. Anal.*, 24(1):38–60, 2002.
- [18] Fabio Gastaldi, Lucia Gastaldi, and Alfio Quarteroni. Adaptive domain decomposition methods for advection dominated equations. *East-West J. Numer. Math.*, 4:165–206, 1996.
- [19] Luca Gerardo-Giorda, Patrick Le Tallec, and Frederic Nataf. A Robin-Robin preconditioner for advection-diffusion equations with discontinuous coefficients. *Comp. Meth. Appl. Mech. Engng*, 193(9-11):745–764, 2004.
- [20] Luca Gerardo Giorda and Frédéric Nataf. Optimized interface conditions in domain decomposition methods for strongly heterogeneous unsymmetric problems. Technical report, CMAP (Ecole Polytechnique) - in preparation, 2004.
- [21] M. Giles. *Diagenesis: a quantitative perspective; implication for basin modeling and rock property prediction*. Kluwer, Dordrecht, 1997.
- [22] Paulo Goldfeld, Luca F. Pavarino, and Olof B. Widlund. Balancing Neumann-Neumann preconditioners for mixed approximations of heterogeneous problems in linear elasticity. Technical Report 825, Courant Institute of Mathematical Sciences, Computer Science Department, April 2002.
- [23] C.-H. Guo. A note on the minimal nonnegative solution of a nonsymmetric algebraic Riccati equation. *Linear Algebra Appl.*, 357:299–302, 2002.
- [24] C.-H. Guo. On a quadratic matrix equation associated with an M -matrix. *IMA J. Num. Anal.*, 23:11–27, 2003.
- [25] R.L. Higdon. Absorbing boundary conditions for difference approximations to the multi-dimensional wave equations. *Mathematics of Computation*, 47(176):437–459, 1986.
- [26] Caroline Japhet, Frederic Nataf, and Francois Rogier. The optimized order 2 method. application to convection-diffusion problems. *Future Generation Computer Systems FUTURE*, 18, 2001.
- [27] V. Kucera. The matrix equation $AX + XB = C$. *SIAM J. Appl. Math.*, 20(1):15–25, 1974.
- [28] P. Le Tallec and M. Vidrascu. Generalized Neumann-Neumann preconditioners for iterative substructuring. In Petter E. Bjørstad, Magne Espedal, and David Keyes, editors, *Domain Decomposition Methods in Sciences and Engineering*. John Wiley & Sons, 1997. Proceedings from the Ninth International Conference, June 1996, Bergen, Norway.
- [29] Pierre-Louis Lions. On the Schwarz alternating method. III: a variant for nonoverlapping subdomains. In Tony F. Chan, Roland Glowinski, Jacques Périaux, and Olof Widlund, editors, *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, held in Houston, Texas, March 20-22, 1989*, Philadelphia, PA, 1990. SIAM.

- [30] F. Magoulès, F.-X. Roux, and S. Salmon. Optimal discrete transmission conditions for a non-overlapping domain decomposition method for the helmholtz equation. *SIAM J. Sci. Comp.*, 25(5):1497–1515, 2004.
- [31] H.-B. Meyer. The matrix equation $AZ + B - ZCZ - ZD = 0$. *SIAM J. Appl. Math.*, 30(1):136–142, 1976.
- [32] F. Nataf. Interface connections in domain decomposition methods. In *Modern Methods in Scientific Computing and Applications*, volume 75 of *NATO Advanced Study Institute, Université de Montréal, in the NATO Science Ser.II.*, 2001.
- [33] Frédéric Nataf and Francis Nier. Convergence rate of some domain decomposition methods for overlapping and nonoverlapping subdomains. *Numerische Mathematik*, 75(3):357–77, 1997.
- [34] Frédéric Nataf and Francois Rogier. Factorization of the convection-diffusion operator and the Schwarz algorithm. *M³AS*, 5(1):67–93, 1995.
- [35] Frédéric Nataf, Francois Rogier, and Eric de Sturler. Optimal interface conditions for domain decomposition methods. Technical Report 301, CMAP (Ecole Polytechnique), 1994.
- [36] Luca F. Pavarino and Olof B. Widlund. Balancing Neumann-Neumann methods for incompressible Stokes equations. *Comm. Pure Appl. Math.*, 55(3):302–335, 2002.
- [37] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations*. Oxford University Press, 1999.
- [38] F.-X. Roux, F. Magloulès, S. Salmon, and L. Series. Optimization of interface operator based on algebraic approach. In Ismael Herrera, David Keyes, Olof B. Widlund, and Robert Yates, editors, *Domain Decomposition Methods in Sciences and Engineering*, pages 297–304. UNAM, 2003. Proceedings from the Fourteenth International Conference, January 2002, Cocoyoc, Mexico.
- [39] F.X. Roux, F. Magoulès, L. Series, and Y. Boubendir. Approximation of optimal interface boundary conditions for two-lagrange multiplier feti method. In Ismael Herrera, David E. Keyes, Olof B. Widlund, and Robert Yates, editors, *15th International Conference on Domain Decomposition Methods, Berlin, Germany*, 2004.
- [40] Y. Saad and H. Schultz. GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM J. Sci. Stat. Comput.*, 7(3):856–869, 1986.
- [41] C. Vuik, A. Segal, and J. A. Meijerink. An efficient preconditioned CG method for the solution of a class of layered problems with extreme contrasts in the coefficients. *J. Comput. Phys.*, 152:385–403, 1999.