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Improved Simulation for the killed Brownian motion in a cone.

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Abstract

In this paper, we first give an error expansion of the weak error associated to a discretely killed Brownian motion in a cone that writes as an intersection of half spaces. We exploit this first result to derive an original correction method to improve the initial convergence rate. This method is based on the sensitivity of the underlying Dirichlet problem with respect to the domain and turns out to be a numerically cheaper and sharper alternative to standard extrapolation techniques.

Key words: Discretely killed Brownian motion, Non smooth domains, Overshoot above the boundary, Domain correction.
1991 MSC: 65C30, 60H35, 60J65.

1 Introduction: statement of the problem

Let $(X_t)_{t\in[0,T]}$ be a d-dimensional Brownian Motion (BM in short) with dynamics

$$X_t = x + \mu t + \sigma W_t \tag{1}$$

with a fixed initial data x and a fixed terminal time T. Here W is a standard d-dimensional BM defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ with the usual assumptions on $(\mathcal{F}_t)_{t\in[0,T]}$. We assume $\sigma\sigma^*$ to be positive definite.

Let D be a domain (i.e. an open connected subset) of $\mathbb{R}^d$. Define $\tau := \inf\{t \geq 0 : X_t \notin D\}$. Consider a regular time mesh of the interval $[0,T]$ with N time steps, $(t_i = ih)_{i\in[\![0,N]\!]}$, $h = T/N$ being the step size. Introduce $\tau^N := \inf\{t_i \geq$

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$0 : X_{t_i} \notin D\}$. For a measurable nonnegative function f and an initial point $x \in D$, denoting $\mathbb{E}_x[.] = \mathbb{E}[.|X_0 = x]$, we refer to the quantity

$$\mathrm{Err}(T, h, f, x) = \mathbb{E}_x[f(X_T)\mathbf{1}_{\tau^N > T}] - \mathbb{E}_x[f(X_T)\mathbf{1}_{\tau > T}]$$

as the weak error associated to the discrete time killing of X w.r.t. the domain D. This is the main quantity studied in this paper. We recall that the numerical approximation of $\mathbb{E}_x[f(X_T)\mathbf{1}_{\tau > T}]$ is a well known issue in finance since it is linked to the price of a barrier option in a Black-Scholes framework, see Andersen and Brotherton-Ratcliffe [ABR96] for some references on the subject. The discrete approximation of the exit time allows to define a simple Monte Carlo procedure to estimate the previous quantity. In this context, $\mathrm{Err}(T, h, f, x)$ can thus be seen as the error associated to the discretization of the exit time.

Let us first recall some controls on $\mathrm{Err}(T, h, f, x)$ given in the literature. In [GM04] and Chapter I of [Men04], we proved, in the wider framework of killed diffusion processes approximated by their discrete Euler scheme, that for smooth domains and functions f satisfying either some support condition w.r.t. D or some smoothness properties and compatibility conditions, one had that $\mathrm{Err}(T, h, f, x)$ was upper and lower bounded at order $1/2$ w.r.t. h. We also showed in [GM05] that, for a large class of Itô processes, the upper bound holds true for an intersection of smooth domains.
Still in [GM04], we stated an expansion and correction result for $\mathrm{Err}(T, h, f, x)$ in the special case of the half-space in a Brownian framework. In this work, we extend these results to the case of an intersection of half-spaces which is of particular interest in mathematical finance since the domain of a multi-asset barrier option is often defined as a product domain.

Let $D \subset \mathbb{R}^d$ be a domain of the form $D = \cap_{i=1}^m D^i,\ m \in [\![1, d]\!]$ where the $(D^i)_{i \in [\![1,m]\!]}$ are d-dimensional half spaces with non-empty intersection. Under suitable smoothness properties up to the boundary for the function $v(t, x) := \mathbb{E}_x[f(X_{T-t})\mathbf{1}_{\tau > T-t}]$, we obtain an error expansion at order $\frac{1}{2}$ w.r.t. h for $\mathrm{Err}(T, h, f, x)$.
As emphasized in [GM04], the leading term in the weak error is still the one associated to the overshoot of the killed process above the boundary (the overshoot being defined as the distance to the boundary of the process when it exits the domain).
In the special case of Brownian Motion, for half spaces or intersections of half spaces forming a cone, we are able to obtain the asymptotic distribution of the overshoot, extending previous results obtained by Siegmund [Sie79]. To derive the error expansion we then use usual techniques based on Taylor's expansions. The smoothness of v is needed for this last step. From a theoretical point of view, the main difficulty is analytical and consists in having good smoothness properties of v up to the boundary of a non-smooth domain.

From a numerical point of view, the error expansion is the preliminary step for a procedure that aims to improve the convergence rate. A standard one in this framework is the Romberg extrapolation, see Talay and Tubaro [TT90] and Section 3 for details.
We propose an alternative correction method based on the recent work of Costantini, El Karoui and Gobet [CKG03] concerning the sensitivity of the Dirichlet problem w.r.t. the domain. Unlike in the Romberg extrapolation, we do not need to refine the time step and thus the procedure is computationally cheaper. Note also that the empirical variance associated to the Monte Carlo estimator is by construction smaller. We simply proceed to the simulation w.r.t. a more constrained domain. Namely, instead of killing the process when it exits from D at one of the discretization times, we kill it when it leaves $D_h := \cap_{i=1}^m D_h^i,\ D_h^i := \{y \in \mathbb{R}^d : y - C_i\sqrt{h}n_i \in D^i\}$ where n_i denotes the inner unit normal associated to the half-space D^i. We will see that, for suitable positive constants C_i, this new choice of discrete time killing allows to remove the leading term in the error development.
We mention that in a one dimensional setting, both the expansion result and the correction procedure could be derived by direct computations from the work of Broadie, Glasserman and Kou [BGK99].

Outline of the paper. We state our main results in Section 2. Numerical results are presented in Section 3. They confirm the correction procedure is rather accurate. The proofs of the main results are developed in Section 4. In Section 5 we give some smoothness properties of v in non-smooth domains. We conclude in Section 6 evoking possible extensions and open problems. Appendices A and B are respectively devoted to the proofs of technical points concerning the asymptotic behavior of the overshoot and the killed heat kernel in a non-smooth domain.

2 Main results

2.1 Current working assumptions.

We suppose our domain satisfies Assumption

(D) $D = \cap_{j=1}^m D^j,\ \forall j \in [\![1, m]\!],\ D^j := \{y \in \mathbb{R}^d : y_j > b_0^j\}$, where $m \in [\![1, d]\!]$.

We introduce

(BM) The d-dimensional process $(X_s)_{s\geq 0}$ has the form $X_s := x + \sigma_0 W_s$, where W is a standard d-dimensional BM and $\sigma_0\sigma_0^* = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbf{I}_{d-m} \end{pmatrix}$ is assumed to be positive definite and Σ is a correlation matrix with coefficients $(\rho_{ij})_{(i,j)\in[\![1,m]\!]^2}$. The integer $m \in [\![1,d]\!]$ is the same as in assumption **(D)**.

Suppose **(BM)**, **(D)** are in force. For a given positive measurable function f we define $\forall (t,y) \in [0,T]\times\mathbb{R}^d,\ v(t,y) := \mathbb{E}_y[f(X_{T-t})\mathbf{1}_{\tau>T-t}]$.

In the following, under **(BM)**, **(D)**, we assume

(S) The function f vanishes on the boundary. The associated function v belongs to $C_b^{1/2+\alpha/2,1+\alpha}([0,T]\times\bar{D})\cap C^{1,2}([0,T)\times D)$, $\alpha>0$, i.e. there exists a constant $C>0$, s.t.

$$|v|_{\infty,[0,T]\times\bar{D}} + |\nabla v|_{\infty,[0,T]\times\bar{D}} + \sup_{\substack{(x,y)\in\bar{D}^2,(s,t)\in[0,T]^2,\\ x\neq y,\, s\neq t}} \frac{|\nabla v(s,x)-\nabla v(t,y)|}{|s-t|^{\alpha/2}+|x-y|^\alpha} \le C.$$

In particular, this means that the function f is at least $C_b^{1+\alpha}(\bar{D})$. We specify in Section 5 sufficient conditions on f to obtain **(S)** in some special cases.

2.2 Statement of the main theorems

Theorem 1 (Error expansion for the correlated Brownian motion in an orthant). *Assume* **(BM)**, **(D)** *and* **(S)**. *The error writes*

$$\mathrm{Err}(T,h,f,x) = C_1\sqrt{h}+o(\sqrt{h})$$

with $C_1 = C_0 \sum_{i=1}^m \mathbb{E}_x[\mathbf{1}_{\tau^i\le T,\wedge_{j\in[\![1,m]\!]\setminus\{i\}}\tau^j>\tau^i}(\partial_{y_i}v(\tau^i,X_{\tau^i}))]$, $\tau^i := \inf\{s\ge 0: X_s^i = b_0^i\}$, $C_0 = \dfrac{\mathbb{E}_0[s_{\tau^+}^2]}{2\mathbb{E}_0[s_{\tau^+}]}$, *where* $s_0 := 0, \forall n\ge 1, s_n := \sum_{i=1}^n G^i$, *the* G^i *being i.i.d. standard centered normal variables and* $\tau^+ := \inf\{n\ge 0: s_n>0\}$.

One knows from [Sie79] and [AGP95] that $\dfrac{\mathbb{E}_0[s_{\tau^+}^2]}{2\mathbb{E}_0[s_{\tau^+}]} = -\dfrac{\zeta(1/2)}{\sqrt{2\pi}} = 0.5823\ldots$, where ζ denotes Riemann's Zeta function.

Under our current assumptions **(BM)**, **(D)**, **(S)**, the next Theorem improves the accuracy of the numerical procedure by removing the term of order $\frac{1}{2}$ in the error. For this, the simulation of $(X_{t_i})_{0\le i\le N}$ is performed in a modified domain, namely $D^h := \{y\in\mathbb{R}^d : \forall i\in[\![1,m]\!],\ y_i > b_0^i + C_0\sqrt{h}\}$ instead of $D := \{y\in\mathbb{R}^d : \forall i\in[\![1,m]\!],\ y_i > b_0^i\}$.

We denote $\tau_{D^h}^N$ (resp. τ_{D^h}) the discrete (resp. continuous) exit time from this domain D^h.

Theorem 2 *Assume* **(BM)**, **(D)**, **(S)**. *We have*

$$\mathrm{Err}'(T,h,f,x) := \mathbb{E}_x[f(X_T)\mathbf{1}_{\tau^N_{D^h}>T}] - \mathbb{E}_x[f(X_T)\mathbf{1}_{\tau>T}] = o(\sqrt{h}).$$

Remark 3 *Consider now the more general case* $X_s = x + \mu s + \sigma W_s, D := \{x \in \mathbb{R}^d : (Ax)_i > b_i,\ \forall i \in [\![1,m]\!]\}$ *where* $A = (a_1\ a_2\ \ldots\ a_m)^*$ *is of rank* m. *Using a clear change of probability measure and a rotation of coordinates using a matrix* Λ *(with the* i^{th} *row equal to* $\frac{\sigma^* a_i}{\|\sigma^* a_i\|}$ *for* $i \in [\![1,m]\!]$ *and the remaining rows forming an orthonormal basis of* $\{\mathrm{Span}((\sigma^* a_j)_{j\in[\![1,m]\!]})\}^\perp$*) preserving the Wiener measure, one easily obtains that for a borelian bounded function* f

$$\mathrm{Err}(T,h,f,x) = \mathbb{E}_0[f_0(\check{W}_T)(\mathbf{1}_{\tau_{D_0^N}>T} - \mathbf{1}_{\tau_{D_0}>T})]$$

where $\check{W}$ *is a centered* d*-dimensional Brownian motion with covariance matrix* $\sigma_0\sigma_0^* = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbf{I}_{d-m}\end{pmatrix}$, *and* $\forall (i,j) \in [\![1,m]\!],\ \Sigma_{ij} = \langle \sigma^* a_i, \sigma^* a_j\rangle/(\|\sigma^* a_i\|\|\sigma^* a_j\|)$.
The domain D_0 *writes* $D_0 := \cap_{j=1}^m D_0^j$, *where for all* $j \in [\![1,m]\!]$, $D_0^j := \{y \in \mathbb{R}^d :\ y_j > b_0^j\}$, $b_0^j = \frac{b_j - a_j\cdot x}{\|\sigma^* a_j\|}$. *Denoting* $\tau_{D_0^j} := \inf\{s \geq 0 : \check{W}_s \notin D_0^j\}$, $\tau^N_{D_0^j} := \inf\{s_i \geq 0 : \check{W}_{s_i} \notin D_0^j\}$, *we have* $\tau_{D_0} := \wedge_{j=1}^m \tau_{D_0^j}, \tau^N_{D_0} := \wedge_{j=1}^m \tau^N_{D_0^j}$. *The function* f_0 *writes* $f_0(y) = \exp(\sigma^{-1}\mu\cdot\Lambda^{-1}y - \frac{\|\sigma^{-1}\mu\|^2}{2}T)f(x+\sigma\Lambda^{-1}y)$.
If assumption **(S)** *is fulfilled by* v_0 *(one could weaken the boundedness condition in* **(S)** *to an exponential growth condition), then the former error expansion remains valid. We can as in the previous theorem remove the leading term of the error by simulating w.r.t.* $D^h := \{x \in \mathbb{R}^d : (Ax)_i > (b + C_0 e\sqrt{h})_i,\ i \in [\![1,m]\!]\}$, $e = (\|\sigma^* a_1\|, ..., \|\sigma^* a_m\|)^*$.
Note also that in the half space case, i.e. for $m = 1$, *this transformation illustrates that the problem is essentially one-dimensional. This last aspect still holds true for a domain delimited by parallel hyperplanes. This is the reason why we did not take this case into consideration in* **(D)**.

3 Numerical Results

In this section we provide some numerical tests and compare the method from Theorem 2 with the usual Romberg correction that we briefly recall.
From Theorem 1, we derive $\frac{1}{\sqrt{2}-1}\mathbb{E}[f(X_T)(\sqrt{2}\mathbf{1}_{\tau^{2N}>T} - \mathbf{1}_{\tau^N>T})] - \mathbb{E}[f(X_T)\mathbf{1}_{\tau>T}] = o(\sqrt{h})$. The Romberg extrapolation technique consists in approximating by a Monte Carlo method the first term in the l.h.s. of the previous equation.
We point out that our correction is numerically less expensive than the Romberg procedure that requires to refine the time step. The Monte Carlo estimator deriving from Theorem 2 also has by construction a smaller empirical variance.

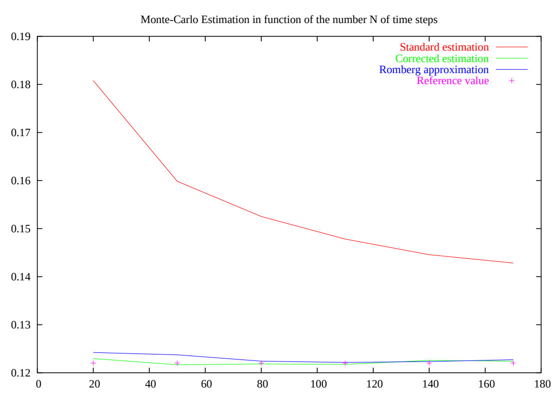
Bidimensional cone. We consider a two-dimensional risky asset following the Black-Scholes-Merton dynamics, $S_t^1 = S_0^1 \exp(\sigma_1 W_t^1 + (r - \frac{\sigma_1^2}{2})t)$, $S_t^2 = S_0^2 \exp(\sigma_2(\rho W_t^1 + (1-\rho^2)^{1/2} W_t^2) + (r - \frac{\sigma_2^2}{2})t)$, where W is a standard bidimensional BM. For a fixed final time T, a given strike K and threshold B, put $D := \{(s_1, s_2) \in \mathbb{R}^2 : s_1 > B, s_2 > B\}$. We are interested in computing $\mathbb{E}[e^{-rT}\mathbf{1}_{\tau > T} h(S_T)]$, where h is a smooth approximation of the indicator function that one expects in the case of a digital barrier option. We take $h(s) := \mathbf{1}^\star_{K,\varepsilon}(s_1)\mathbf{1}^\star_{K,\varepsilon}(s_2)$ with $\mathbf{1}^\star_{K,\varepsilon}(s_1) = 0$, if $s_1 \le K - \varepsilon$, $\mathbf{1}^\star_{K,\varepsilon}(s_1) = 1$ if $s_1 \ge K$, and in between we use the smooth interpolating function $\mathbf{1}^\star_{K,\varepsilon}(s_1) = 10\varepsilon^{-3}(s_1 - (K-\varepsilon))^3 - 15\varepsilon^{-4}(s_1 - (K-\varepsilon))^4 + 6\varepsilon^{-5}(s_1 - (K-\varepsilon))^5$. The previous function h satisfies the compatibility and support conditions from assumption **(F2)**, defined in Section 5.2, that guarantees **(S)** is fulfilled under **(D)**, **(BM)**, cf. Proposition 16, as soon as $K > B + \varepsilon$. One can show that the conditions required on v_0 introduced in Remark 3 are satisfied using a standard Girsanov argument and the explicit controls obtained in Lemma 15. For $r = .04, \sigma_1 = \sigma_2 = .3, \rho = .5, S_0^1 = S_0^2 = 100 = K, B = 90, T = 1, \varepsilon = 5$ we compute the standard Monte-Carlo approximation, the Romberg approximation and the correction proposed in Theorem 2 for 10^6 paths. The reference value has been computed with 10^6 paths and 15000 times steps.

Empirical mean

	MC	MC Shift	Romberg	Reference
$N = 20$	.180807	0.12295	0.124209	0.122017
$N = 50$	.159831	0.121661	0.123736	0.122017
$N = 80$	.15251	0.12184	0.122422	0.122017
$N = 110$	.147821	0.121735	0.122145	0.122017
$N = 140$	.144559	0.122581	0.122338	0.122017
$N = 170$	.142827	0.122393	0.122717	0.122017

Unbiased variance estimator

	MC	MC Shift	Romberg	Reference
$N = 20$	.146365	0.106928	.298985	.10413
$N = 50$	.132417	0.105984	.230244	.10413
$N = 80$	.127643	0.106117	.205762	.10413
$N = 110$	.12445	0.106048	.191531	.10413
$N = 140$	.122845	0.106683	.182411	.10413
$N = 170$	.121412	0.106549	.175806	.10413



The width of the 95%-confidence interval is essentially equal to $1.5\ 10^{-3}$. Furthermore we also observe that the associated empirical variance is lower than the one of the Romberg extrapolation.
From a numerical point of view a natural question concerns the behaviour of the $o(\sqrt{h})$ appearing in Theorem 2. We have not experimentally emphasized

a constant exponent, anyhow it turns out that the numerical rest is smaller than $O(h^{3/2})$.

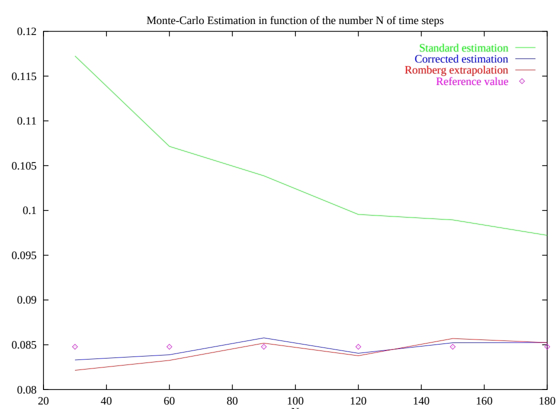
Experiments in higher dimensions. We introduce here a d-dimensional model for which we do not have the required smoothness of assumption **(S)**. Set $d \in \mathbf{N}^*, d > 2$. We consider a d-dimensional BM $X_s = x + \sigma_0 W_s$, where W is a d-dimensional standard BM and $\forall (i,j) \in [\![1,d]\!]^2$, $(\sigma_0\sigma_0^*)_{ij} = \mathbf{1}_{i=j} + \frac{\alpha}{d-1}\mathbf{1}_{i\neq j}, \alpha < 1$, so that $\sigma_0\sigma_0^*$ has dominant diagonal and is thus positive definite. We take $D := \{x \in \mathbb{R}^d : x_i > 0,\ \forall i \in [\![1,d]\!]\}$ and we are interested in approximating the quantity $\mathbb{E}_x[h(X_T)\mathbf{1}_{\tau>T}]$ where $\forall x \in D,\ h(x) = \prod_{i=1}^d \mathbf{1}_{K<S_0^i \exp(sx_i)}$. Here $S_0 \in \mathbb{R}^d$ is a fixed vector and s is a fixed scale factor. Note that h is not as smooth as required in **(S)**.
The results below have been obtained with $d = 5, T = 1, K = 100, \alpha = s = .05, S_0^i = 95, x_i = .85,\ \forall i \in [\![1,d]\!]$. The reference value has been computed with the standard Monte-Carlo procedure for 2×10^6 paths and 2×10^4 steps. It comes

Empirical mean

	MC	MC Shift	Romberg	Reference
$N = 30$	.128265	.0833072	.0821488	.0847798
$N = 60$	.115335	.0838812	.0832571	.0847798
$N = 90$	.111043	.0857746	.0851829	.0847798
$N = 120$	.105783	.0840571	.083779	.0847798
$N = 150$	.104819	.0852239	.0857012	.0847798
$N = 180$	.10276	.0852529	.0852469	.0847798

Unbiased variance estimator

	MC	MC Shift	Romberg	Reference
$N = 30$	.11168	.0763648	.18754	.069879
$N = 60$	.101939	.0768374	.152264	.069879
$N = 90$	.098633	.0784061	.141044	.069879
$N = 120$	.0945224	.0769788	.129368	.069879
$N = 150$	.0937689	.0779463	.124412	.069879
$N = 180$	.0921404	.0779695	.120608	.069879



The size of the 95%-confidence interval is essentially equal to 1.5 10^{-3}.
Even though we are not under the assumptions of the main theorems the correction technique gives good result. Although we can not say any more that it is more accurate than the Romberg extrapolation for the empirical mean, we still observe that it provides a smaller empirical variance.

4 Proof of the main results

4.1 Additional notations and usual controls

For smooth functions $g(t,x)$, the notation $H_g(t,x)$ stands for the Hessian matrix of g w.r.t. x. Time derivatives are denoted by $\partial_t^\beta g(t,x)$, $\beta \in \mathbf{N}^*$.

We will keep the same notation C (or C') for all finite, non-negative constants which will appear in our computations: they may depend on D, T, σ_0, or f, but they will not depend on the number of time steps N and the initial value x. We reserve the notation c and c' for constants also independent of T and f. Other possible dependences for the constants are explicitly indicated.

In the following $O_{pol}(h)$ (resp. $O(h)$) stands for every quantity $R(h)$ such that $\forall n \in \mathbf{N}$, for some $C > 0$, one has $|R(h)| \leq Ch^n$ (resp. $|R(h)| \leq Ch$) (uniformly in x).

Lemma 4 (Bernstein's inequality) . *Assume* **(BM)**. *Consider two stopping times S, S' upper bounded by T with $0 \leq S' - S \leq \Delta \leq T$. Then for any $p \geq 1$, there are some constants $c > 0$ and C, such that for any $\eta \geq 0$, one has a.s:*

$$
\begin{aligned}
\mathbb{P}[\sup_{t\in[S,S']} \|X_t - X_S\| \geq \eta \,\big|\, \mathcal{F}_S] &\leq C \exp\left(-c\frac{\eta^2}{\Delta}\right), \\
\mathbb{E}[\sup_{t\in[S,S']} \|X_t - X_S\|^p \,\big|\, \mathcal{F}_S] &\leq C\Delta^{p/2}.
\end{aligned}
$$

For a proof of the first inequality we refer to Chapter 3, §3 in [RY99]. The other inequality easily follows from the first one or from the BDG inequalities.

4.2 Proof of Theorem 1 (Expansion result)

Let us briefly outline the scheme of the proof. First, we write the error as a sum of increments of the function v. Using Taylor expansions, we then introduce the overshoot terms of the process in the previous development (the overshoot being defined as the distance of the process to the domain when it exits the domain). We finally conclude using the asymptotic independence of the rescaled overshoot, as well as its integrability properties, and the discrete exit time. Recalling that the function v vanishes on D^C, we write

$$
\begin{aligned}
\mathrm{Err}(T,h,f,x) &= \mathbb{E}_x[v(T\wedge\tau^N,\Pi_{\bar D}(X_{T\wedge\tau^N}))]-v(0,x)\\
&=\sum_{i=0}^{N-1}\mathbb{E}_x\Big[\Big(v(t_{i+1}\wedge\tau^N,\Pi_{\bar D}(X_{t_{i+1}\wedge\tau^N}))-v(t_i\wedge\tau^N,\Pi_{\bar D}(X_{t_i\wedge\tau^N}))\Big)\Big]\\
&=\sum_{i=0}^{N-1}\mathbb{E}_x[\mathbf{1}_{\tau^N>t_i}\Big(v(t_{i+1},\Pi_{\bar D}(X_{t_{i+1}}))-v(t_i,X_{t_i})\Big)].
\end{aligned}
$$

Introduce $\forall t\in[0,T),\tau_t:=\inf\{s\ge t: X_s\notin D\}$. Recall also that the function v satisfies the PDE

$$
\begin{cases}
(\partial_t v+\frac{1}{2}\mathrm{tr}(H_v\sigma_0\sigma_0^*))(t,y)=0,\ (t,y)\in[0,T)\times D,\\
v(t,.)|_{\partial D}=0, t\in[0,T],\ v(T,y)=f(y),\ y\in\bar D.
\end{cases}
\tag{2}
$$

Hence, $(v(s\wedge\tau_t,X_{s\wedge\tau_t}))_{s\in[t,T]}$ is a martingale. It comes

$$
\mathrm{Err}(T,h,f,x)=\sum_{i=0}^{N-1}\mathbb{E}_x[\mathbf{1}_{\tau^N>t_i}\mathbf{1}_{\tau_{t_i}<t_{i+1}}(v(t_{i+1},\Pi_{\bar D}(X_{t_{i+1}}))-v(\tau_{t_i},X_{\tau_{t_i}}))].
$$

Remark 5 *This kind of development for the error associated to the discretization of the exit time is still valid in the wider context of Itô processes, see [GM05].*

Define $\forall j\in[\![1,m]\!],\ \tau_t^j:=\inf\{s\ge t: X_s^j=b_0^j\}$. Since $\mathbb{P}[\tau_{t_i}^j=\tau_{t_i}^k|\mathcal{F}_{t_i}]=0,\ j\neq k$, one gets $\mathbb{P}[X_{t_{i+1}}\notin D|\mathcal{F}_{t_i}]=\sum_{j=1}^m\mathbb{E}[\mathbf{1}_{\tau_{t_i}<t_{i+1},\tau_{t_i}=\tau_{t_i}^j}\mathbb{P}[X_{t_{i+1}}\notin D|\mathcal{F}_{\tau_{t_i}^j}]|\mathcal{F}_{t_i}]$ $\ge\frac{1}{2}\mathbb{P}[\tau_{t_i}<t_{i+1}|\mathcal{F}_{t_i}]$.

Remark 6 *From the above control we derive* $\sum_{i=0}^{N-1}\mathbb{P}_x[\tau^N>t_i,\tau_{t_i}\le t_{i+1}]\le 2\sum_{i=0}^{N-1}\mathbb{P}_x[\tau^N=t_{i+1}]\le 2$. *This last identity will be frequently used from now on to isolate the rests, see e.g. last equality below.*

Using **(S)** and Lemma 4 we obtain

$$
\begin{aligned}
&\mathrm{Err}(T,h,f,x)=\\
&\sum_{i=0}^{N-1}\mathbb{E}_x[\mathbf{1}_{\tau^N>t_i}\mathbf{1}_{\tau_{t_i}<t_{i+1}}\{\nabla v(\tau_{t_i},X_{\tau_{t_i}})\cdot(\Pi_{\bar D}(X_{t_{i+1}})-X_{\tau_{t_i}})+o(h^{1/2})\}]\\
&=\sum_{i=0}^{N-1}\sum_{j=1}^{m}\mathbb{E}_x[\mathbf{1}_{\tau^N>t_i}\mathbf{1}_{\tau_{t_i}^j<t_{i+1},\tau_{t_i}=\tau_{t_i}^j}\partial_{x_j}v(\tau_{t_i}^j,X_{\tau_{t_i}^j})(X_{t_{i+1}}^j-b_0^j)^+]+o(h^{1/2}).
\end{aligned}
$$

For the last equality we used the explicit expression of the projection on $\bar D$, namely $\Pi_{\bar D}(y)=(b_0^1+(y_1-b_0^1)^+,...,b_0^m+(y_m-b_0^m)^+,y_{m+1},...,y_d)$ and also that $\forall(j,k)\in[\![1,m]\!]^2,\ j\neq k,\ \forall s\in[0,T],\ \partial_{x_k}v(s,y)|_{y\in\mathbb{R}^d:y_j=b_0^j}=0$. This is a simple consequence of the fact that v vanishes on D^C. By symmetry, assumption **(S)** and the previous arguments we derive

$$\mathrm{Err}(T,h,f,x)=\sum_{i=1}^{m}\mathbb{E}_x[\mathbf{1}_{\tau^N\le T}\partial_{x_j}v(\tau^N,\Pi_{\bar D}(X_{\tau^N}))(X^j_{\tau^N}-b^j_0)^-]+R+o(h^{1/2}) \tag{3}$$

where

$$\begin{aligned}
|R| &:= |\sum_{i=0}^{N-1}\sum_{j=1}^{m}\mathbb{E}_x[\mathbf{1}_{\tau^N>t_i}\mathbf{1}_{\tau^j_{t_i}<t_{i+1},\tau_{t_i}\neq\tau^j_{t_i}}\partial_{x_j}v(\tau^j_{t_i},\Pi_{\bar D}(X_{\tau^j_{t_i}}))(X^j_{t_{i+1}}-b^j_0)^+]|\\
&\le C\sqrt{h}\sum_{i=0}^{N-1}\sum_{j=1}^{m}\sum_{k\in[\![1,m]\!],\ k\neq j}\mathbb{E}_x[\mathbf{1}_{\tau^N>t_i}\mathbf{1}_{\tau^j_{t_i}<t_{i+1},\tau_{t_i}=\tau^k_{t_i}}|\partial_{x_j}v(\tau^j_{t_i},\Pi_{\bar D}(X_{\tau^j_{t_i}}))|].
\end{aligned}$$

Define $\forall\eta>0$, $V^{jk}(h^\eta):=\{y\in\mathbb{R}^d:|y_l-b^l_0|\le h^\eta,\ l\in\{j,k\}\}$. Put also $\mathcal{CO}^{jk}:=\{y\in\mathbb{R}^d:y_l=b^l_0,\ l\in\{j,k\}\}$. Note that $\forall(t,y)\in[0,T]\times\mathcal{CO}^{jk}\cap\bar D,\ \nabla v(t,y)=0$. Thus, from Lemma 4, assumption **(S)** and Remark 6 we derive that for h small enough

$$\begin{aligned}
|R| &\le C\sqrt{h}\sum_{i=0}^{N-1}\sum_{j=1}^{m}\sum_{k\in[\![1,m]\!],\ k\neq j}\mathbb{E}_x[\mathbf{1}_{\tau^N>t_i}\mathbf{1}_{X_{t_i}\in V^{jk}(h^{1/4})}\mathbf{1}_{\tau^j_{t_i}<t_{i+1},\tau_{t_i}=\tau^k_{t_i}}\\
&\times|\partial_{x_j}v(\tau^j_{t_i},\Pi_{\bar D}(X_{\tau^j_{t_i}}))-\partial_{x_j}v(\tau^j_{t_i},\Pi_{\bar D}(\Pi_{\mathcal{CO}^{jk}}(X_{\tau^j_{t_i}})))|]+O_{pol}(h)\\
&\le C\sqrt{h}\sum_{i=0}^{N-1}\sum_{j=1}^{m}\sum_{k\in[\![1,m]\!],\ k\neq j}\mathbb{E}_x[\mathbf{1}_{\tau^N>t_i}\mathbf{1}_{X_{\tau^j_{t_i}}\in V^{jk}(h^{1/8})}\mathbf{1}_{\tau^j_{t_i}<t_{i+1},\tau_{t_i}=\tau^k_{t_i}}\\
&\times|X_{\tau^j_{t_i}}-\Pi_{\mathcal{CO}^{jk}}(X_{\tau^j_{t_i}})|^\alpha]+O_{pol}(h)=O(h^{\frac12+\frac{\alpha}{8}}):=o(h^{1/2}).
\end{aligned}$$

Plugging this last estimate into (3) it comes

$$\begin{aligned}
&\mathrm{Err}(T,h,f,x)=\sum_{j=1}^{m}\mathbb{E}_x[\mathbf{1}_{\tau^N\le T}\partial_{x_j}v(\tau^N,\Pi_{\bar D}(X_{\tau^N}))(X^j_{\tau^N}-b^j_0)^-]+o(h^{1/2})\\
&:=\sum_{j=1}^{m}E_j+o(h^{1/2}).
\end{aligned} \tag{4}$$

Remark 7 *We emphasize that, up to now, we did not have used the specific Brownian dynamics of the process X. The expansion (4) is valid for the error associated to the discretization of a diffusion process approximated by its discretely killed Euler scheme, provided that the process is non adherent to the boundary and that* **(S)** *is fulfilled for the associated function v.*

We now detail the asymptotic behavior of E_1. The other terms could be handled exactly in the same way. The following Lemma, whose proof is postponed to Appendix A and strongly relies on the Brownian setting, is the main tool needed.

Lemma 8 Asymptotic independence of the hitting time and the overshoot
Assume **(BM)**, **(D)**. *Put* $\forall i \in [\![1, m]\!]$, $\tau^i := \inf\{t \geq 0 : X_t^i = b_0^i\}$, $\tau^{N,i} := \inf\{t_j := jh \geq 0 : X_{t_j}^i \leq b_0^i\}$. *One has* $\forall y \in \mathbb{R}^{+,*}$,

$$\mathbb{P}_x[\sqrt{h}^{-1}(X^1_{\tau^N} - b_0^1)^- \geq y, \tau^N \leq t, \tau^{N,1} \leq \wedge_{i=2}^m \tau^{N,i}]$$
$$\underset{N}{\longrightarrow} (1 - H(y))\mathbb{P}_x[\tau^1 \leq t, \tau^1 < \wedge_{i=2}^m \tau^i]$$

where, using the notations of Theorem 1, $H(y) := (\mathbb{E}_0[s_{\tau^+}])^{-1} \int_0^y dz \mathbb{P}_0[s_{\tau^+} > z]$. *The limit is uniform on* $[0, T]$.

In order to isolate the rescaled overshoot $Z_N := \sqrt{h}^{-1}(X^1_{\tau^N} - b_0^1)^-$ in E_1, we rewrite the components $X^2, ..., X^m$, of the correlated part of X in terms of X^1 and an additional correlated $(m-1)$-dimensional BM $\tilde{X}$ independent of X^1 and $(X^i)_{i \in [\![m+1,d]\!]}$. Namely, $\forall i \in [\![2, m]\!]$, $X_s^i = \rho_{1i}X_s^1 + (1-\rho_{1i}^2)^{1/2}\tilde{X}_s^{i-1}$, $\tilde{X}_0^{i-1} = (x_0^i - \rho_{1i}x_0^1)/(1-\rho_{1i}^2)^{1/2}$. Set also $(\rho_{1.}X_s^1 + (1-\rho_{1.}^2)^{1/2}\tilde{X}_s^{.-1})^{2,m} := (\rho_{12}X_s^1 + (1-\rho_{12}^2)^{1/2}\tilde{X}_s^1, ..., \rho_{1m}X_s^1 + (1-\rho_{1m}^2)^{1/2}\tilde{X}_s^{m-1}) = (X_s^2, ..., X_s^m) := X_s^{2,m}$, $X_s^{m+1,d} := (X_s^{m+1}, ..., X_s^d)$.

For notational convenience we introduce $\forall y \in \mathbb{R}^{d-1}$, $\Pi_{\bar{D}^{2,d}}(y) := ((y_1 - b_0^2)^+ + b_0^2, ..., (y_{m-1} - b_0^m)^+ + b_0^m, y_m, ..., y_{d-1})$. The term E_1, defined in (4), writes

$$\begin{aligned}
&E_1 = \sqrt{h}\mathbb{E}_x[Z_N \mathbf{1}_{\tau^N \leq T} \partial_{x_1} v(\tau^N, b_0^1, \Pi_{\bar{D}^{2,d}}((\rho_{1.}(b_0^1 - \sqrt{h}Z_N) \\
&+(1-\rho_{1.}^2)^{1/2}\tilde{X}_{\tau^N}^{.-1})^{2,m}, X_{\tau^N}^{m+1,d}))] \\
&= \sqrt{h}\mathbb{E}_x[Z_N \mathbf{1}_{\tau^N \leq T} \partial_{x_1} v(\tau^N, b_0^1, \Pi_{\bar{D}^{2,d}}((\rho_{1.}b_0^1 + (1-\rho_{1.}^2)^{1/2}\tilde{X}_{\tau^N}^{.-1})^{2,m}, X_{\tau^N}^{m+1,d}))] \\
&+R_1,
\end{aligned}$$

where

$$\begin{aligned}
&R_1 := \sqrt{h}\mathbb{E}_x\Big[Z_N \mathbf{1}_{\tau^N \leq T}\Big(\partial_{x_1} v(\tau^N, b_0^1, \Pi_{\bar{D}^{2,d}}((\rho_{1.}(b_0^1 - \sqrt{h}Z_N) + \\
&(1-\rho_{1.}^2)^{1/2}\tilde{X}_{\tau^N}^{.-1})^{2,m}, X_{\tau^N}^{m+1,d})) \\
&-\partial_{x_1} v(\tau^N, b_0^1, \Pi_{\bar{D}^{2,d}}((\rho_{1.}b_0^1 + (1-\rho_{1.}^2)^{1/2}\tilde{X}_{\tau^N}^{.-1})^{2,m}, X_{\tau^N}^{m+1,d}))\Big)\Big].
\end{aligned}$$

Under **(S)** the function $\partial_{x_1} v$ is continuous and bounded. Proposition 6 from [GM04] gives the uniform integrability of Z_N on the event $\tau^{N,1} \leq T$. We thus derive from Lemma 8 by convergence in law that for h small enough $E_1 = \sqrt{h}\mathbb{E}[Z]\mathbb{E}_x[\mathbf{1}_{\tau^1 \leq T, \wedge_{j=2}^m \tau^j > \tau^1} \partial_{x_1} v(\tau^1, X_{\tau^1})] + o(\sqrt{h}) + R_1$, where the distribution function of Z is given by H defined in Lemma 8. Recalling that $\mathbb{E}[Z] = \frac{\mathbb{E}[s_{\tau^+}^2]}{2\mathbb{E}[s_{\tau^+}]} = C_0$ we finally obtain $E_1 = \sqrt{h}C_0\mathbb{E}[\mathbf{1}_{\tau^1 \leq T, \wedge_{j=2}^m \tau^j > \tau^1} \partial_{x_1} v(\tau^1, X_{\tau^1})] + o(\sqrt{h}) + R_1$. Now, from assumption **(S)** we have $|R_1| \leq Ch^{\frac{1+\alpha}{2}}\mathbb{E}_x[Z_N^{1+\alpha}\mathbf{1}_{\tau^{N,1} \leq T}]$.

Thus, by Proposition 6 in [GM04] it comes $R_1 = O(h^{\frac{1+\alpha}{2}})$ which completes the proof. $\square$

Remark 9 *The controls in the previous proof as well as the residual terms appearing in the computations are locally uniform w.r.t. the domain D.*

Remark 10 *To conclude this section, we would like to emphasize that the main difficulty in order to apply the previous Theorem consists in finding conditions on f that guarantee* **(S)** *is fulfilled. We provide some sufficient conditions in Section 5 but in all generality this is far from being easy. Anyhow, this difficulty is essentially of analytical nature.*

4.3 Proof of Theorem 2 (Correction result)

In this subsection we detail how the arguments from Costantini, El Karoui and Gobet, see [CKG03], can be employed to prove our correction result. We write

$$\begin{aligned}\mathrm{Err}'(T,h,f,x) &= \mathbb{E}_x[f(X_T)\mathbf{1}_{\tau^N_{Dh}>T}] - \mathbb{E}_x[f(X_T)\mathbf{1}_{\tau_{Dh}>T}]\\ &+\mathbb{E}_x[f(X_T)\mathbf{1}_{\tau_{Dh}>T}] - \mathbb{E}_x[f(X_T)\mathbf{1}_{\tau>T}] := E_1 + E_2.\end{aligned}$$

From Remark 9 we derive that one could show just like in Theorem 1 that even though the domain depends on h we have $E_1 = C_1\sqrt{h} + o(\sqrt{h})$, where C_1 denotes the constant introduced in the quoted theorem. For E_2 we adapt some ideas from [CKG03] concerning the sensitivity of the Dirichlet problem w.r.t. the domain.
For a given $c \in \mathbb{R}^d$, let us denote $\forall \eta > 0,\ D_\eta := \{y \in \mathbb{R}^d : y - \eta c \in D\}$. We define $\tau_{D_\eta} := \inf\{s > 0 : X_s \notin D_\eta\}$ and we introduce for all $x \in D$ the mapping $\mathcal{J}^x_c : \eta \longrightarrow \mathbb{E}_x[f(X_T)\mathbf{1}_{\tau_{D_\eta}>T}]$. We show below that under the assumptions of Theorem 2, the mapping $\mathcal{J}^x_c$ is differentiable in $\eta = 0$ and for $c = (\underbrace{1,...,1}_{m},\underbrace{0,...,0}_{d-m})$, one has

$$\partial_\eta \mathcal{J}^x_c(\eta)|_{\eta=0} = -\mathbb{E}_x[\nabla v(\tau, X_\tau).c\mathbf{1}_{\tau<T}] = -\sum_{i=1}^{m}\mathbb{E}_x[\partial_{x_i}v(\tau^i, X_{\tau^i})\mathbf{1}_{\tau^i\leq T,\wedge_{j\neq i}\tau^j>\tau^i}]. \tag{5}$$

From (5) we then derive that $E_2 = \mathcal{J}^x_c(C_0\sqrt{h}) - \mathcal{J}^x_c(0) = \partial_\eta\mathcal{J}^x_c(0)C_0\sqrt{h} + o(\sqrt{h}) = -C_1\sqrt{h} + o(\sqrt{h})$ which proves the Theorem. $\square$

4.3.1 Proof of (5)

Let us define $X^\eta_s := X_s - \eta c,\ \tau^{D,\eta} = \inf\{s > 0 : X^\eta_s \notin D\}$. Note that $\tau_{D_\eta} = \tau^{D,\eta}$. Denoting $\Delta_\eta := \mathbb{E}_x[f(X^\eta_T + \eta c)\mathbf{1}_{\tau^{D,\eta}>T}] - v(0,x)$, we have to identify the limit of Δ_η/η as $\eta \longrightarrow 0$. It comes

$$\begin{aligned}\Delta_\eta =& \mathbb{E}_x[f(X_T^\eta+\eta c)\mathbf{1}_{\tau^{D,\eta}>T}]-\mathbb{E}_x[f(X_T^\eta)\mathbf{1}_{\tau^{D,\eta}>T}]\\ &+\mathbb{E}_x[f(X_T^\eta)\mathbf{1}_{\tau^{D,\eta}>T}-v(T\wedge\tau^{D,\eta},X_{T\wedge\tau^{D,\eta}})]\\ &+\mathbb{E}_x[v(T\wedge\tau^{D,\eta},X_{T\wedge\tau^{D,\eta}})]-v(0,x):=\Delta_{\eta,1}+\Delta_{\eta,2}+\Delta_{\eta,3}.\end{aligned}$$

Since $(M_t)_{t\in[0,T]} := (v(t\wedge\tau, X^{0,x}_{t\wedge\tau}))_{t\in[0,T]}$ is a martingale and $\tau^{D,\eta}<\tau$, we readily get $\Delta_{\eta,3}=0$.
Note that $f(X_T^\eta)\mathbf{1}_{\tau^{D,\eta}>T} = v(T\wedge\tau^{D,\eta}, X^\eta_{T\wedge\tau^{D,\eta}})$. One also has $\tau_{D_\eta} \underset{\eta\longrightarrow 0,\text{ a.s}}{\longrightarrow} \tau$. From Assumption **(S)**, v is continuously differentiable. Thus, one gets $\lim_{\eta\longrightarrow 0}\Delta_{\eta,2}/\eta = -\mathbb{E}_x[\nabla v(T\wedge\tau, X_{T\wedge\tau}).c]$.
On the other hand, since we assumed f to be continuously differentiable we obtain $\lim_{\eta\longrightarrow 0}\Delta_{\eta,1}/\eta = \mathbb{E}_x[\nabla f(X_\tau).c\ \mathbf{1}_{\tau>T}]$. Recalling $\forall x\in\bar{D},\ v(T,x)=f(x)$ we write

$$\begin{aligned}\partial_\eta \mathcal{J}_c^x(\eta)|_{\eta=0} &= -\mathbb{E}_x\Big[\Big(\nabla v(T\wedge\tau, X_{T\wedge\tau})-\nabla f(X_T)\mathbf{1}_{\tau>T}\Big).c\Big]\\ &= -\mathbb{E}_x[\mathbf{1}_{\tau\le T}\nabla v(\tau,X_\tau).c]\end{aligned}$$

which for $c=(\underbrace{1,...,1}_{m},\underbrace{0,...,0}_{d-m})$ proves (5). □

Remark 11 *We mention that in the special case of a half space, an alternative proof based on the explicit expression of the hitting time densities is possible, see Chapter III of [Men04].*

5 Some smoothness properties of v under (D): sufficient conditions to fulfill (S)

In the whole section we assume **(BM)**, **(D)**. We only deal with the case $d\ge 2, m\ge 2$. Indeed, for $m=1$ the domain D is smooth and standard results, based on the explicit expression of the density of the killed BM, can be used to derive the required smoothness in **(S)**.

This section is divided into two parts. In Subsection 5.1 we recall the explicit expression of the killed heat kernel under **(D)** and also state a control of its derivatives in dimension 2. Using these results and some standard PDE techniques we then derive in Subsection 5.2 some sufficient conditions to get **(S)** when $d=m=2$.

Note first that

$$\mathbb{P}_x[\tau > t, X_t \in [y, y+dy)] = |\det(\sigma_0^{-1})|\mathbb{P}_{\tilde{x}}[\tilde{\tau} > t, \tilde{X}_t \in [z, z+dz)] \qquad (6)$$

where $\tilde{x} = \sigma_0^{-1}x,\ z = \sigma_0^{-1}y,\ \tilde{X}_t = \tilde{x} + W_t$ with W standard BM, $\tilde{D} := \{z \in \mathbb{R}^d : (\sigma_0 z)_i > b_0^i\},\ \tilde{\tau} := \inf\{s \geq 0 : \tilde{X}_s \notin \tilde{D}\}$.

Equation (6) gives the expression of the killed density of X under **(D)** in function of the density of the killed standard BM in a convex cone $\tilde{D}$ that writes as an intersection of half spaces. The domain $\tilde{D}$ is piece-wise C^∞ and hence the trace of the cone on the unit sphere $\mathbb{S}^{d-1}$ with center at the vertex of $\tilde{D}$ is a normal domain in the sense of Chavel [Cha84] (see definition page 16 of this reference). Denoting this trace by $\Gamma := \tilde{D} \cap \mathbb{S}^{d-1}$, we derive from p. 169 of the above reference that we have a Sturm-Liouville spectral decomposition of the Laplace-Beltrami operator for the elliptic Dirichlet problem on Γ, i.e. the normalized eigenfunctions $(m_j)_{j\in\mathbf{N}^*}$ of $\Delta_{\mathbb{S}^{d-1}}$ form an orthonormal basis of $L^2(\Gamma)$ and the eigenvalues $(\lambda_j)_{j\in\mathbf{N}^*}$ are s.t. $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \uparrow +\infty$.

As a direct consequence of Equation (2.2) in Bañuelos and Smits [BS97], we derive the following

Proposition 12 *Let $\tilde{D}$ be a cone with origin 0 that writes as a non-empty intersection of half-spaces. One has*

$$\begin{aligned}
&\forall (t,x,y) \in \mathbb{R}^{+*} \times \tilde{D}^2,\ x = r\theta,\ y = \rho\eta,\ (\theta,\eta) \in (\mathbb{S}^{d-1})^2,\ (\rho, r) \in (\mathbb{R}^{+*})^2\\
&\mathbb{P}_x[\tilde{X}_t \in dy, \tilde{\tau} > t] = \frac{e^{-\frac{\rho^2+r^2}{2t}}}{t(\rho r)^{\frac{d}{2}-1}} \sum_{j=1}^{+\infty} I_{\nu_j}\left(\frac{\rho r}{t}\right) m_j(\theta) m_j(\eta) \rho^{d-1} d\rho d\sigma(\eta)\\
&:= q_t(x,y)dy,\ \nu_j = (\lambda_j + (\tfrac{d}{2}-1)^2)^{1/2},
\end{aligned}$$

where λ_j, (resp. m_j) are the eigenvalues (resp. the normalized eigenfunctions) of $\Delta_{\mathbb{S}^{d-1}}$ on Γ for the elliptic Dirichlet problem and I_ν denotes the modified Bessel function of order ν.

Remark 13 *The result of Proposition 12 is standard in the bidimensional case. It is in that case a simple extension of the well known method of images that consists, for special angles, in writing the killed heat kernel as a suitable sum of standard Gaussian kernels alternating heat sources and sinks in order to satisfy the boundary conditions. We refer to Carslaw and Jaeger [CJ59] or to Iyengar [Iye85] for details.*

Remark 14 *In the special case $d = m = 2$ the eigenvalues (resp. the normalized eigenfunctions) write $\lambda_j = (\pi j/\omega)^2$ (resp. $m_j(\theta) = \sqrt{\frac{2}{\omega}}\sin(\frac{\pi j}{\omega}\arg(\theta))$),* *where $\omega \in (0, 2\pi)$ is the angle of the cone. For $m > 2$, we do not have such an explicit expression but analysis techniques, see Weyl's Lemma [Cha84] p. 172, give some controls on the behavior of these eigenvalues, see also Remark 20.*

Lemma 15 Radial control of the derivatives when $d = m = 2$
For a given $\omega \in (0, \pi)$, let $\tilde{D} := \{x = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 : r > 0,\ \theta \in (0, \omega)\}$. Note that $\tilde{D}$ is a convex cone. For all $R > 0, T > 0$, there exist positive constants $C := C(R, T), c, \xi$ s.t. $\forall (t, x, y) \in (0, T] \times (\tilde{D} \cap B(0, R)) \times \tilde{D}$, $x = (r\cos\theta, r\sin(\theta)),\ y = (\rho\cos(\eta), \rho\sin(\eta)),\ (\theta, \eta) \in (0, \omega)^2,\ (\rho, r) \in (\mathbb{R}^{+*})^2$, one has

$$q_t(x, y) + |\partial_t q_t(x, y)| + |\nabla_x q_t(x, y)| \le \frac{C}{t^\xi} \exp\left(-c\frac{|r-\rho|^2}{t}\right)$$

and $\exists \alpha_0 > 0$,

$$\sup_{\substack{(x,x') \in (\tilde{D} \cap B(0,R))^2, \\ x \neq x' = (r'\cos(\theta'), r\sin(\theta'))}} \frac{|\nabla q_t(x, y) - \nabla q_t(x', y)|}{|x - x'|^{\alpha_0}} \le \frac{C}{t^\xi} \exp\left(-c\frac{|r-\rho|^2 \wedge |r'-\rho|^2}{t}\right).$$

The proof of the above Lemma is postponed to Appendix B.

5.2 Derivation of (S) *when* $d = m = 2$

In Remark 13 we mentioned that for special angles of the cone, one could express the killed heat kernel q in terms of a sum of standard Gaussian kernels. To be precise, this can be done when the angle of the cone writes $\omega = \pi/m_0, m_0 \in \mathbf{N}^*$. For our original problem (2), one can establish a connection between the killed heat kernel q and the density of the killed BM in the orthant thanks to (6). One therefore deduces that for some particular correlation coefficients, corresponding to angles that have the previous form, under suitable assumptions on the final condition f one has the "usual" smoothness properties for the solution v of problem (2), and hence **(S)** is satisfied. We now give a smoothness result for the solution v of (2) for general correlation coefficients. We introduce Assumption

(F2) The function $f \in C_b^{2+\alpha}(\bar{D})$, $\alpha > 0$, $f|_{\partial D} = \mathrm{Tr}(H_f \sigma_0 \sigma_0^*)|_{\partial D} = 0$ and $d(\mathrm{supp}(f), b_0) \ge 2\varepsilon > 0$.

Proposition 16 *Assume* **(D)**, **(BM)**, **(F2)**. *For $D := \{x \in \mathbb{R}^2 : x_1 > b_0^1, x_2 > b_0^2\}, \sigma_0\sigma_0^* = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, $\rho \in (-1, 1)$, there exists $\alpha' > 0$ s.t. the unique solution v of (2) belongs to $C_b^{1/2+\alpha'/2, 1+\alpha'}([0, T] \times \bar{D})$. In particular* **(S)** *is* *satisfied.*

Proof. From Proposition 12 we derive that problem (2) has a unique solution $v \in C^{1,2}([0,T) \times D) \cap C^0_b([0,T] \times \bar{D})$.
Let us now note that as a consequence of the support condition in **(F2)** and the radial control of Lemma 15 there exists $\alpha_0 > 0$ s.t. $v \in C^{1/2+\alpha_0/2, 1+\alpha_0}([0,T] \times B(b_0, \varepsilon) \cap \bar{D})$.
Choose now D_1 to be a C^3 domain s.t. $d(\bar{D}_1, b_0) \geq \varepsilon/3 > 0$ and $\{x \in \mathbb{R}^2 : |x - b_0| \geq \varepsilon, x \in \partial D\} = \{x \in \mathbb{R}^2 : |x - b_0| \geq \varepsilon, x \in \partial D_1\}$. From the techniques used in Chapter IV of Friedman [Fri64] to prove the boundary Schauder estimates and Theorem 5.2 Chapter 4 in [LSU68], we derive that $v \in C^{1+\alpha/2, 2+\alpha}([0,T] \times \bar{D}_1)$. Put $\alpha' := \alpha \wedge \alpha_0$. The proof is complete. □

6 Conclusion

In this paper we obtained an expansion result for the weak error in the special case of a discretely killed Brownian Motion in an orthant provided we had smoothness properties of the solution of the underlying Cauchy-Dirichlet problem. We exploited the explicit asymptotic distribution of the overshoot above the boundary that had previously been characterized as the leading term of the weak error, see [GM04]. The correction method proposed to improve the convergence rate also gave promising results. A natural question concerns its possible extension to a wider framework than the Brownian one. We proposed in Chapter III of [Men04] an intuitive algorithm to extend the previous boundary correction to general domains and diffusion processes discretized with the Euler scheme. Its theoretical and numerical analysis will concern further research.

The main motivation that led us to deal with conical cases comes from Mathematical finance. Indeed, with multi assets, one often defines the domain of a barrier option as a product domain. For the moment we are only able to treat in whole generality the case of bidimensional domains in a Black-Scholes framework.

Concerning further extensions in bigger dimensions, let us point out that the remaining efforts to be done concern the smoothness properties of the underlying function $v(t,x) = \mathbb{E}_x[f(X_{T-t})\mathbf{1}_{\tau > T-t}]$. Anyhow, for some special angles, or equivalently for special correlation coefficients, we can extend the method of images to express the transition density as a sum of standard Gaussian kernels. In that case, under suitable assumptions on f, we have the usual smoothness properties on v, and both the expansion and correction results hold true.

Let us mention that the proof of the main results would work if v had a uniform Hölder continuous first spatial derivative with exponential growth only in a neighbourhood of the boundary. This could allow to relax the boundedness assumption on f.

A Asymptotic behavior of the overshoot

This section is dedicated to the proof of Lemma 8 introduced in Section 4.2 concerning the asymptotic behavior of the overshoot. In the following, we freely use the notations introduced in Theorem 1 and Lemma 8.

A.1 Asymptotic independence of the overshoot and the exit time

We first state a one-dimensional result due to Siegmund [Sie79].

Lemma 17 (Asymptotic independence of the overshoot and the discrete exit time) *Let W be a standard linear BM. Put $x > 0$ and consider the domain $D :=]-\infty, x[$. We have for any $y \geq 0$*

$$\lim_{h \longrightarrow 0} \mathbb{P}_0[\tau^N \leq t, (W_{\tau^N} - x) \leq y\sqrt{h}] = \mathbb{P}_0[\tau \leq t] H(y). \tag{A.1}$$

The limits is uniform in $t \in [0, T]$.

Proof. Equation (A.1) is a direct consequence of Lemma 3 in [Sie79] for a fixed t. We derive the uniformity on $[0, T]$ using Dini-like arguments noting that the l.h.s. of (A.1) defines a sequence of (discontinuous) increasing functions and that the simple limit is continuous (see e.g. problem 7.2.3 in [Die71])or Subsection A.2. □

From now on, we assume $m \geq 2$ and proceed to the

Proof of Lemma 18.

Let us first show that $\forall (t, y) \in [0, T] \times \mathbb{R}^{+,*}$, $\zeta_N(t) := \mathbb{P}_x[\sqrt{h}^{-1}(X^1_{\tau^{N,1}} - b^1_0)^- \geq y, \tau^N \leq t, \tau^{N,1} \leq \wedge_{i=2}^m \tau^{N,i}] \underset{N}{\longrightarrow} (1 - H(y))\mathbb{P}_x[\tau^1 \leq t, \wedge_{i=2}^m \tau^i > \tau^1] := \zeta(t)$. We write

$$\begin{aligned} \zeta_N(t) = \ & \mathbb{P}_x[\sqrt{h}^{-1}(X^1_{\tau^{N,1}} - b^1_0)^- \geq y, \tau^{N,1} \leq t] \\ & - \mathbb{P}_x[\sqrt{h}^{-1}(X^1_{\tau^{N,1}} - b^1_0)^- \geq y, \tau^{N,1} \leq t, \tau^{N,1} > \wedge_{i=2}^m \tau^{N,i}] \\ := \ & (\zeta_N^1 - \zeta_N^2)(t). \end{aligned} \tag{A.2}$$

From Lemma 17 one gets

$$\zeta_N^1(t) \underset{N}{\longrightarrow} \zeta^1(t) := (1 - H(y))\mathbb{P}_x[\tau^1 \leq t] \tag{A.3}$$

uniformly on $[0,T]$. Let us turn to the control of ζ_N^2. As a consequence of the strong Markov property of X it comes

$$\begin{aligned}\zeta_N^2(t) = & \ \mathbb{E}_x[\mathbf{1}_{\wedge_{i=2}^m \tau^{N,i}\le t}\mathbf{1}_{\tau^{N,1}>\wedge_{i=2}^m\tau^{N,i}}\mathbb{P}[\sqrt{h}^{-1}(X^1_{\tau^{N,1}}-b_0^1)^- \ge y, \tau^{N,1}\le t|\mathcal{F}_{\wedge_{i=2}^m\tau^{N,i}}]] \\ & := \mathbb{E}_x[\mathbf{1}_{\wedge_{i=2}^m \tau^{N,i}\le t}\mathbf{1}_{\tau^{N,1}>\wedge_{i=2}^m\tau^{N,i}}\xi_N(X^1_{\wedge_{i=2}^m\tau^{N,i}}, \wedge_{i=2}^m\tau^{N,i}, t)]\end{aligned}$$

with $\xi_N(X^1_{\wedge_{i=2}^m\tau^{N,i}}, \wedge_{i=2}^m\tau^{N,i}, t) = \mathbb{P}_{X^1_{\wedge_{i=2}^m\tau^{N,i}}}[\sqrt{h}^{-1}(\tilde{X}^1_{\tilde{\tau}^{N,1}}-b_0^1)^- \ge y, \tilde{\tau}^{N,1}\le t - \wedge_{i=2}^m\tau^{N,i}]$ where $(\tilde{X}^1_t)_{t\ge 0}$ is a standard BM with starting point $X^1_{\wedge_{i=2}^m\tau^{N,i}}$ and $\tilde{\tau}^{N,1} := \inf\{t_i := ih \ge 0 : \tilde{X}^1_{t_i} \le b_0^1\}$.

For a given arbitrary compact interval $\mathcal{K} := [\underline{\mathcal{K}}, \overline{\mathcal{K}}] \subset (b_0^1, +\infty)$ we split $\zeta_N^2(t)$ into two parts.

$$\begin{aligned}\zeta_N^2(t) = & \ \mathbb{E}_x[\mathbf{1}_{\wedge_{i=2}^m \tau^{N,i}\le t}\mathbf{1}_{\tau^{N,1}>\wedge_{i=2}^m\tau^{N,i}}\mathbf{1}_{X^1_{\wedge_{i=2}^m\tau^{N,i}}\in\mathcal{K}}\xi_N(X^1_{\wedge_{i=2}^m\tau^{N,i}}, \wedge_{i=2}^m\tau^{N,i}, t)] \\ & + \ \mathbb{E}_x[\mathbf{1}_{\wedge_{i=2}^m \tau^{N,i}\le t}\mathbf{1}_{\tau^{N,1}>\wedge_{i=2}^m\tau^{N,i}}\mathbf{1}_{X^1_{\wedge_{i=2}^m\tau^{N,i}}\notin\mathcal{K}}\xi_N(X^1_{\wedge_{i=2}^m\tau^{N,i}}, \wedge_{i=2}^m\tau^{N,i}, t)] \\ & := \zeta_N^{21}(t) + \zeta_N^{22}(t).\end{aligned}$$

Fix $\varepsilon > 0$. We now show that one can choose $\mathcal{K}(\varepsilon)$, $N_0 := N_0(\varepsilon, \mathcal{K}(\varepsilon))$ s.t. for $N \ge N_0$,

$$\zeta_N^2(t) = (1-H(y))\mathbb{P}_x[\wedge_{i=2}^m\tau^i < \tau^1, \tau^1 \le t] + O(\varepsilon). \tag{A.4}$$

<u>*Control of* $\zeta_N^{21}(t)$.</u> Write first

$$\begin{aligned}\zeta_N^{21}(t) = & \Big(\zeta_N^{21}(t) - (1-H(y))\mathbb{P}_x[\wedge_{i=2}^m\tau^i < \tau^1, \tau^1 \le t, X^1_{\wedge_{i=2}^m\tau^i}\in\mathcal{K}]\Big) \\ & +(1-H(y))\mathbb{P}_x[\wedge_{i=2}^m\tau^i < \tau^1, \tau^1 \le t] - R(t,\mathcal{K})\end{aligned}$$

where $R(t,\mathcal{K}) = (1-H(y))\mathbb{P}_x[\wedge_{i=2}^m\tau^i < \tau^1, \tau^1 \le t, X^1_{\wedge_{i=2}^m\tau^i}\notin\mathcal{K}]$. Note that

$$\begin{aligned}0 \le R(t,\mathcal{K}) \le & \ \mathbb{P}_x\Big[\wedge_{i=2}^m\tau^i \le T, X^1_{\wedge_{i=2}^m\tau^i} \ge \overline{\mathcal{K}}\Big] \\ & +\mathbb{P}_x\Big[\wedge_{i=2}^m\tau^i \le T, X^1_{\wedge_{i=2}^m\tau^i} \in (b_0^1, \underline{\mathcal{K}}]\Big] := R_1(\overline{\mathcal{K}}) + R_2(\underline{\mathcal{K}}).\end{aligned}$$

Lemma 4 readily gives $R_1(\overline{\mathcal{K}}) \le C\exp(-c\frac{(\overline{\mathcal{K}}-x_1)^2}{T})$. On the other hand, $R_2(\underline{\mathcal{K}}) \underset{\underline{\mathcal{K}}\longrightarrow b_0^1}{\longrightarrow} 0$. Hence, for $\varepsilon > 0$ we can choose $\mathcal{K} = \mathcal{K}(\varepsilon)$ s.t.

$$\begin{aligned}\zeta_N^{21}(t) = & \ \Big(\zeta_N^{21}(t) - (1-H(y))\mathbb{P}_x[\wedge_{i=2}^m\tau^i < \tau^1, \tau^1 \le t, X^1_{\wedge_{i=2}^m\tau^i}\in\mathcal{K}])\Big) \\ & +(1-H(y))\mathbb{P}_x[\wedge_{i=2}^m\tau^i < \tau^1, \tau^1 \le t] + O(\varepsilon) \\ & := \delta_N(t) + (1-H(y))\mathbb{P}_x[\wedge_{i=2}^m\tau^i < \tau^1, \tau^1 \le t] + O(\varepsilon).\end{aligned}$$

For the term $\delta_N(t)$ we introduce the following Lemma whose proof is postponed to the end of the section.

Lemma 18 *Let $\tilde{X}^1$ be a standard BM with starting point $\tilde{x}$ in a given compact interval $\mathcal{K} = [\underline{\mathcal{K}}, \overline{\mathcal{K}}] \subset (b_0^1, +\infty)$. Then*

$$\mathbb{P}_{\tilde{x}}[\sqrt{h}^{-1}(\tilde{X}^1_{\tilde{\tau}^{N,1}} - b_0^1)^- \geq y, \tilde{\tau}^{N,1} \leq u] \underset{N}{\longrightarrow} (1 - H(y))\mathbb{P}_{\tilde{x}}[\tilde{\tau}^1 \leq u]$$

uniformly on $(\tilde{x}, u) \in \mathcal{K} \times [0, T]$.

From Lemma 18, $\forall \varepsilon > 0,\ \exists N_0 := N_0(\mathcal{K}(\varepsilon), \varepsilon)$, s.t. $N \geq N_0$

$$\begin{aligned}
&\delta_N(t) = (1 - H(y)) \Big\{ \mathbb{E}_x[\mathbf{1}_{\wedge_{i=2}^m \tau^{N,i} \leq t} \mathbf{1}_{\wedge_{i=2}^m \tau^{N,i} < \tau^{N,1}} \mathbf{1}_{X^1_{\wedge_{i=2}^m \tau^{N,i}} \in \mathcal{K}} \times \\
&\mathbb{P}_{X^1_{\wedge_{i=2}^m \tau^{N,i}}}[\tilde{\tau}^1 \leq t - \wedge_{i=2}^m \tau^{N,i}]] - \mathbb{P}_x[\wedge_{i=2}^m \tau^i < \tau^1, \tau^1 \leq t, X^1_{\wedge_{i=2}^m \tau^i} \in \mathcal{K}] \Big\} + O(\varepsilon) \\
&:= (1 - H(y)) \Big(\mathbb{E}_x[\mathbf{1}_{\wedge_{i=2}^m \tau^{N,i} \leq t} \mathbf{1}_{\wedge_{i=2}^m \tau^{N,i} < \tau^{N,1}} \mathbf{1}_{X^1_{\wedge_{i=2}^m \tau^{N,i}} \in \mathcal{K}} \xi_t(X^1_{\wedge_{i=2}^m \tau^{N,i}}, \wedge_{i=2}^m \tau^{N,i})] \\
&- \mathbb{E}_x[\mathbf{1}_{\wedge_{i=2}^m \tau^i \leq t} \mathbf{1}_{\wedge_{i=2}^m \tau^i < \tau^1} \mathbf{1}_{X^1_{\wedge_{i=2}^m \tau^i} \in \mathcal{K}} \xi_t(X^1_{\wedge_{i=2}^m \tau^i}, \wedge_{i=2}^m \tau^i)] \Big) + O(\varepsilon).
\end{aligned}$$

Note that $\xi_t(x, u) = \mathbb{P}_x[\tilde{\tau} \leq t - u]$ is continuous in $(x, u) \in (b_0^1, +\infty) \times [0, t]$. Recall that $\tau^{N,i} \xrightarrow[N]{\text{a.s}} \tau^i,\ i \in [\![1, m]\!]$, and by continuity $X^1_{\wedge_{i=2}^m \tau^{N,i}} \xrightarrow[N]{\text{a.s}} X^1_{\wedge_{i=2}^m \tau^i}$. One can check that the law of $(\tau^1, \wedge_{i=2}^m \tau^i, X^1_{\wedge_{i=2}^m \tau^i})$ is absolutely continuous w.r.t. the Lebesgue measure. We thus derive by convergence in law that for N large enough

$$\delta_N(t) = O(\varepsilon),\ \zeta_N^{21}(t) = (1 - H(y))\mathbb{P}_x[\wedge_{i=2}^m \tau^i < \tau^1, \tau^1 \leq t] + O(\varepsilon). \tag{A.5}$$

<u>*Control of $\zeta_N^{22}(t)$.*</u> The arguments we use to control this term are quite similar to those introduced to treat the terms $R_1(\overline{\mathcal{K}}), R_2(\underline{\mathcal{K}})$ above.
Indeed, since $\xi_N \in [0, 1]$ one gets

$$\begin{aligned}
\zeta_N^{22}(t) \leq\ & \mathbb{P}_x[\wedge_{i=2}^m \tau^{N,i} \leq T, X^1_{\wedge_{i=2}^m \tau^{N,i}} \geq \overline{\mathcal{K}}] + \mathbb{P}_x[\wedge_{i=2}^m \tau^{N,i} \leq T, X^1_{\wedge_{i=2}^m \tau^{N,i}} \in (b_0^1, \underline{\mathcal{K}}]] \\
:=\ & R_1^N(\overline{\mathcal{K}}) + R_2^N(\underline{\mathcal{K}}).
\end{aligned}$$

From Lemma 4 we get $R_1^N(\overline{\mathcal{K}}) \leq C \exp(-c\frac{(\overline{\mathcal{K}} - x_1)^2}{T})$. The previous choice of $\overline{\mathcal{K}}$ gives $R_1^N(\overline{\mathcal{K}}) = O(\varepsilon)$. Write now, $R_2^N(\underline{\mathcal{K}}) := (R_2^N(\underline{\mathcal{K}}) - R_2(\underline{\mathcal{K}})) + R_2(\underline{\mathcal{K}})$. On the one hand, the former choice of $\underline{\mathcal{K}}$ yields $R_2(\underline{\mathcal{K}}) = O(\varepsilon)$. On the other hand, for the difference $(R_2^N - R_2)(\underline{\mathcal{K}})$, since $\tau^{N,i} \xrightarrow[N]{\text{a.s}} \tau^i,\ i \in [\![2, m]\!],\ X^1_{\wedge_{i=2}^m \tau^{N,i}} \xrightarrow[N]{\text{a.s}} X^1_{\wedge_{i=2}^m \tau^i}$, with the same arguments we employed to control $\delta_N(t)$, we derive by convergence in law $\exists N_0 := N_0(\underline{\mathcal{K}}, \varepsilon),\ N \geq N_0,\ |(R_2^N - R_2)(\underline{\mathcal{K}})| \leq \varepsilon$. Hence, for $N := N(\mathcal{K}, \varepsilon)$ large enough, we write $\zeta_N^{22}(t) = O(\varepsilon)$ which together with (A.5) gives (A.4). From (A.4), (A.3) and (A.2) we derive the simple convergence of ζ_N to ζ for a fixed $t \in [0, T]$.

The uniformity in $t \in [0, T]$ derives from the fact that $\zeta_N(t)$ is a cumulative distribution function with continuous limit, see also the arguments at the beginning of the proof of Lemma 17. $\square$

A.2 Proof of Lemma 18.

Let us define $\forall (x, u) \in \mathcal{K} = [\underline{\mathcal{K}}, \overline{\mathcal{K}}] \times [0, T]$, $\underline{\mathcal{K}} > b_0^1$, $\Psi_N(x, u) = \mathbb{P}_x[\sqrt{h}^{-1}(\tilde{X}^1_{\tilde{\tau}^{N,1}} - b_0^1)^- \geq y, \tilde{\tau}^{N,1} \leq u]$. For a fixed $x \in \mathcal{K}$, Lemma 17 yields that $\Psi_N(x, u) \underset{N}{\longrightarrow} (1 - H(y))\mathbb{P}_x[\tau^1 \leq u] := \Psi(x, u)$ uniformly on $u \in [0, T]$. Let us now show that for a fixed $u \in [0, T]$ we have the uniform convergence w.r.t. $x \in \mathcal{K}$. Write

$$\Psi_N(x, u) = \mathbb{P}_x[\sqrt{h}^{-1}(\tilde{X}^1_{\tilde{\tau}^{N,1}} - b_0^1)^- \geq y] - \mathbb{P}_x[\sqrt{h}^{-1}(\tilde{X}^1_{\tilde{\tau}^{N,1}} - b_0^1)^- \geq y, \tilde{\tau}^{N,1} > u]$$
$$:= \Psi_N^1(x) - \Psi_N^2(x, u).$$

With the notations of Theorem 1, introducing $\forall a \geq 0, \bar{\tau}_a := \inf\{n \in \mathbf{N} : s_n > a\}$, we write $\Psi_N^1(x) = \mathbb{P}_x[\sqrt{h}^{-1}(\tilde{X}^1_{\tilde{\tau}^{N,1}} - b_0^1)^- \geq y] = \mathbb{P}_0[(s_{\bar{\tau}_{(x-b_0^1)/\sqrt{h}}} - (x - b_0^1)/\sqrt{h}) \geq y]$. Equation (19) from [Sie79] gives $\lim_{b\to\infty} \mathbb{P}_0[s_{\bar{\tau}_b} - b \geq y] = 1 - H(y)$. Hence, $\Psi_N^1(x) \underset{N}{\longrightarrow} (1 - H(y))$ uniformly on $x \in \mathcal{K}$. We develop Ψ_N^2 like in the proof of Lemma 3 from the same reference, controlling that we can isolate uniform rests. We get

$$\begin{aligned}
\Psi_N^2(x, u) &= \mathbb{P}[(s_{\bar{\tau}_{(x-b_0^1)/\sqrt{h}}} - (x - b_0^1)/\sqrt{h}) \geq y, \bar{\tau}_{(x-b_0^1)/\sqrt{h}} > \phi(u)/h] \\
&= \int_0^\infty \mathbb{P}[\bar{\tau}_{(x-b_0^1)/\sqrt{h}} > \phi(u)/h, (x - b_0^1)/\sqrt{h} - s_{\phi(u)/h} \in [z, z + dz)] \\
&\quad \times \mathbb{P}[s_{\bar{\tau}_z} - z \geq y].
\end{aligned}$$

We split the above integral into three terms $\Psi_N^{21}, \Psi_N^{22}, \Psi_N^{23}$ respectively associated to the intervals $(0, \varepsilon(x - b_0^1)/\sqrt{h})$, $(\varepsilon(x - b_0^1)/\sqrt{h}, (x - b_0^1)/(\varepsilon\sqrt{h}))$, $((x - b_0^1)/(\varepsilon\sqrt{h}), \infty)$ for an arbitrary $\varepsilon \in (0, 1)$. One has

$$\begin{aligned}
\Psi_N^{21}(x, u) &\leq \mathbb{P}[\frac{(1-\varepsilon)(x - b_0^1)}{\sqrt{h}\sqrt{\phi(u)/h}} \leq \mathcal{N}(0, 1) \leq \frac{x - b_0^1}{\sqrt{h}\sqrt{\phi(u)/h}}] \\
&\leq \mathbb{P}[\frac{(1-\varepsilon)(x - b_0^1)}{T^{1/2}} \leq \mathcal{N}(0, 1) \leq \frac{x - b_0^1}{T^{1/2}}] \leq \tfrac{C\varepsilon(\overline{\mathcal{K}} - b_0^1)}{T^{1/2}}
\end{aligned}$$

uniformly for $x \in \mathcal{K}$. We also have

$$\Psi_N^{23}(x,u) \le \mathbb{P}[\mathcal{N}(0,1) \le (1-\varepsilon^{-1})\frac{x-b_0^1}{\phi(u)^{1/2}}] \le \mathbb{P}[\mathcal{N}(0,1) \le (1-\varepsilon^{-1})\frac{\underline{\mathcal{K}}-b_0^1}{T^{1/2}}]$$
$$\le \frac{CT^{1/2}}{\underline{\mathcal{K}}-b_0^1}\frac{\varepsilon}{1-\varepsilon}$$

which is still uniform w.r.t. $x \in \mathcal{K}$. From these computations we derive that for N large enough, $\Psi_N^{22}(x,u) = (1-H(y))\mathbb{P}_x[\tilde{\tau}^{N,1} > u] + O(\varepsilon)$, where the rest is uniform w.r.t. $\mathcal{K}$. It therefore remains to show $\mathbb{P}_x[\tilde{\tau}^{N,1} > u] := \gamma_N(u,x) \underset{N}{\longrightarrow} \gamma(u,x) := \mathbb{P}_x[\tilde{\tau}^1 > u]$ uniformly on $\mathcal{K}$. We note that $1 - \gamma_N(u,x) = \mathbb{P}_0[\sup_{i\in[\![0,\phi(u)/h]\!]} \tilde{X}^1_{t_i} \ge (x-b_0^1)]$ is decreasing in x, so that $\gamma_N(u,.)$ is increasing. Since the simple limit is continuous, we derive the uniformity using the same arguments as in the proof of Lemma 17.

Now, we have shown that for a fixed parameter $x \in \mathcal{K}$, $u \in [0,T]$, we have the uniform convergence with respect to the other. Let us now show the joint uniform convergence. The limit Ψ is uniformly continuous on $\mathcal{K} \times [0,T]$. This reads

$$\forall \varepsilon > 0, \exists \eta := \eta(\varepsilon), \forall (x,x') \times (t,t') \in \mathcal{K}^2 \times [0,T]^2,\ |t-t'| + |x-x'| \le \eta,$$
$$|\Psi(x,t) - \Psi(x',t')| \le \varepsilon. \tag{A.6}$$

In particular, $|t-t'| \le \eta \Rightarrow \sup_{x\in\mathcal{K}} |\Psi(x,t) - \Psi(x,t')| \le \varepsilon$. Let us now consider a regular grid $\Lambda := \{s_i\}_{i\in[\![1,a]\!]}$ of $[0,T]$ with step $s = s_{i+1} - s_i \le \eta$. Since for a fixed $t \in [0,T]$ we have uniform convergence in space it comes

$$\forall \varepsilon > 0,\ \exists N_0 = \max_{i\in[\![1,a]\!]} N_0(s_i),\ N \ge N_0,\ \sup_{i\in[\![1,a]\!]} \sup_{x\in\mathcal{K}} |\Psi_N(x,s_i) - \Psi(x,s_i)| \le \varepsilon. \tag{A.7}$$

Noting that both $\Psi_N(x,.), \Psi(x,.)$ are increasing functions we derive from (A.6), (A.7)

$$\forall t \in [s_i, s_{i+1}],\ \Psi(x,s_i) - \Psi(x,s_{i+1}) + \Psi(x,s_{i+1}) - \Psi_N(x,s_{i+1})$$
$$\le \Psi(x,t) - \Psi_N(x,t) \le \Psi(x,s_{i+1}) - \Psi(x,s_i) + \Psi(x,s_i) - \Psi_N(x,s_i),$$
$$\forall \varepsilon > 0, \exists N_0,\ N \ge N_0,\ \sup_{t\in[0,T]} \sup_{x\in\mathcal{K}} |\Psi(x,t) - \Psi_N(x,t)| \le \varepsilon$$

which shows the joint uniformity and completes the proof. □

B Results about the killed heat kernel: proof of Lemma 15

One of the key tools in the proof of the Lemma is the following identity

$$\forall x > 0,\ \forall \mu > \nu \geq 0,\ I_\mu(x) < I_\nu(x). \tag{B.1}$$

Relation (B.1) was proved by Jones in [Jon68]. From identity 9.6.34 in Abramowitz and Stegun we also get that $\exp(z) = I_0(z) + 2\sum_{n=1}^{\infty} I_n(z)$. Hence, from the explicit expression of the killed heat kernel, see Proposition 12 and Remark 14, and recalling that $\nu_n := n\pi/\omega > n$, (B.1) yields

$$q_t(x,y) \leq \tfrac{2\exp(-\frac{r^2+\rho^2}{2t})}{t\omega} \textstyle\sum_{n=1}^{\infty} I_{\nu_n}\left(\frac{r\rho}{t}\right) \leq \tfrac{2\exp(-\frac{r^2+\rho^2}{2t})}{t\omega} \textstyle\sum_{n=1}^{\infty} I_n\left(\frac{r\rho}{t}\right) \leq \tfrac{\exp(-\frac{|r-\rho|^2}{2t})}{t\omega}.$$

Put $A_t := \sum_{n=1}^{\infty} \partial_t(\sin(\nu_n\theta)\sin(\nu_n\eta) I_{\nu_n}\left(\frac{r\rho}{t}\right))$. From the recurrence relations on modified Bessel functions, see formula 9.6.26 in [AS72], one gets $|A_t| \leq \frac{r\rho}{2t^2}\sum_{n=1}^{\infty}(I_{\nu_n+1} + I_{\nu_n-1})\left(\frac{r\rho}{t}\right) \leq C\frac{r\rho}{t^2}\exp\left(\frac{r\rho}{t}\right)$. Thus,

$$\begin{aligned}\exp\left(-\tfrac{r^2+\rho^2}{2t}\right)|A_t| &\leq C\tfrac{r\rho}{t^2}\exp\left(-\tfrac{|r-\rho|^2}{2t}\right) \leq C(\tfrac{R^2}{t^2} + \tfrac{R|r-\rho|}{t^2})\exp\left(-c\tfrac{|r-\rho|^2}{2t}\right)\\ &\leq C(\tfrac{R^2}{t^2} + \tfrac{R}{t^{3/2}})\exp\left(-c\tfrac{|r-\rho|^2}{t}\right) \leq \tfrac{C}{t^\xi}\exp\left(-c\tfrac{|r-\rho|^2}{t}\right)\end{aligned}$$

which gives the result for the time derivative.

The boundedness and Hölder continuity of the gradient is somehow trickier to obtain. Let us show these properties for the partial derivative of the heat kernel w.r.t. the first parameter. They could be obtained for the other ones exactly in the same way. Bare hand calculations yield

$$\begin{aligned}\partial_{x_1} q_t(x,y) = &-\tfrac{x_1}{t}q_t(x,y) + \tfrac{\rho}{t^2\omega}\exp\left(-\tfrac{r^2+\rho^2}{2t}\right)\sum_{n=1}^{\infty}\sin(\nu_n\eta)\times\\ &\left\{\sin((\nu_n-1)\theta)I_{\nu_n-1}\left(\tfrac{r\rho}{t}\right) + \sin((\nu_n+1)\theta)I_{\nu_n+1}\left(\tfrac{r\rho}{t}\right)\right\}.\end{aligned}$$

The previous arguments give the stated control for $|\partial_{x_1} q_t(x,y)|$.

Now, the most "singular" term in the expression of $\partial_{x_1} q_t(x,y)$ is the one involving the modified Bessel functions of lowest order. Thus, we have to prove the Hölder continuity of

$$\begin{aligned}g_t(x,y) &:= \tfrac{\rho}{t^2\omega}\exp\left(-\tfrac{r^2+\rho^2}{2t}\right)\sum_{n=1}^{\infty}\sin(\nu_n\eta)\sin((\nu_n-1)\theta)I_{\nu_n-1}\left(\frac{r\rho}{t}\right)\\ &:= \tfrac{\rho}{t^2\omega}\exp\left(-\tfrac{r^2+\rho^2}{2t}\right)B_t(x,y).\end{aligned}$$

Still by direct computation we get that

$$|\nabla_x B_t(x,y)| \le \frac{C\rho}{t}\left\{\exp\left(\frac{r\rho}{t}\right) + \sum_{n=1}^{\infty} I_{\nu_n - 2}\left(\frac{r\rho}{t}\right)\right\}.$$

Now, since $z \in \mathbb{R}^{+*}$, from equation 9.6.20 in [AS72]

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi \exp(z\cos(\gamma))\cos(\nu\gamma)d\gamma - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty \exp(-z\cosh(t) - \nu t)dt.$$

From this expression it is easily seen that $\forall n \in \mathbf{N},\ z > 0, I_n(z) = I_{-n}(z)$ and $\forall \nu > 0,\ \forall \varepsilon > 0$

$$\begin{aligned}
&|I_\nu - I_{-\nu}|(z) \le C\int_0^\infty \exp(-z\cosh(t))\exp(\nu t)dt \\
&= C\int_0^1 \exp(-\frac{z}{2}(u^{-1}+u))u^{-(1+\nu)}du \le Cz^{-(\nu+\varepsilon)}\int_0^1 u^{-(1-\varepsilon)}du \le \frac{C}{\varepsilon}z^{-(\nu+\varepsilon)}.
\end{aligned}$$

Thus, for all $\varepsilon > 0$

$$|\nabla_x B_t(x,y)| \le \frac{C\rho}{t}\left\{\exp\left(\frac{r\rho}{t}\right) + \varepsilon^{-1}\left(\frac{r\rho}{t}\right)^{-(2-\nu_1+\varepsilon)}\mathbf{1}_{\nu_1<2}\right\}. \tag{B.2}$$

Take $(x,x') \in (B(0,R)\cap\tilde{D})^2$, s.t. $r < r'$. For $\alpha_0 \in (0,1]$ to be specified later on, it comes

$$\begin{aligned}
\tfrac{|g_t(x,y)-g_t(x',y)|}{|x-x'|^{\alpha_0}} &\le \tfrac{\rho}{t^2\omega}\exp\left(-\tfrac{\rho^2}{2t}\right)\Big[\Big|\exp\left(-\tfrac{r'^2}{2t}\right) - \exp\left(-\tfrac{r^2}{2t}\right)\Big|\,|B_t(x,y)| \\
&\quad + \exp\left(-\tfrac{r'^2}{2t}\right)|B_t(x',y) - B_t(x,y)|\Big] \times |x-x'|^{-\alpha_0} := (A_t^1 + A_t^2)(x,x').
\end{aligned}$$

Recalling that $|x-x'| \ge |r-r'|$ and $|B_t(x,y)| \le C\exp\left(\frac{r\rho}{t}\right)$ we derive

$$\begin{aligned}
A_t^1(x,x') &\le \tfrac{C\rho}{t^3}\exp\left(-\tfrac{\rho^2}{2t}\right)\sup_{s\in[r,r']}\exp\left(-\tfrac{s^2}{2t}\right)|r-r'|^{1-\alpha_0} \times \exp\left(\tfrac{r\rho}{t}\right) \\
&\le \tfrac{C}{t^\xi}\exp\left(-c\tfrac{|r-\rho|^2}{t}\right).
\end{aligned} \tag{B.3}$$

We also have $A_t^2(x,x') \le \frac{C\rho}{t^2}\exp\left(-\frac{r'^2+\rho^2}{2t}\right)\sup_{u\in[0,1]}|\nabla_x B_t(ux+(1-u)x',y)||x-x'|^{1-\alpha_0}$. Hence, from (B.2) we get

$$\begin{aligned}
A_t^2(x,x') \le \tfrac{C\rho^2}{t^3}\exp\left(-\tfrac{r'^2+\rho^2}{2t}\right)&\left\{\exp\left(\tfrac{r'\rho}{t}\right) + \varepsilon^{-1}\left(\tfrac{r\rho}{t}\right)^{-(2-\nu_1+\varepsilon)}\mathbf{1}_{\nu_1<2}\right\}\times \\
&|x-x'|^{1-\alpha_0}.
\end{aligned}$$

If $\nu_1 \ge 2$, the above control together with (B.3) give the statement of the proposition with $\alpha_0 = 1$, i.e. the gradient is Lipschitz continuous. So from now on, we consider the case $\nu_1 < 2$.

If $|x-x'|\le r$, we derive from the previous expression that for $\nu_1>\varepsilon$

$$\begin{aligned} A_t^2(x,x') &\le \tfrac{C}{(1\wedge\varepsilon)t^\xi}\exp\left(-c\tfrac{|r'-\rho|^2}{t}\right)\left(1+r^{1-\alpha_0-(2-\nu_1+\varepsilon)}\right) \\ &\le \tfrac{C}{(1\wedge\varepsilon)t^\xi}\exp\left(-c\tfrac{|r'-\rho|^2}{t}\right)\left(1+r^{\nu_1-1-\varepsilon-\alpha_0}\right). \end{aligned} \tag{B.4}$$

On the other hand, if $|x-x'|>r$, we write

$$\begin{aligned} &A_t^2(x,x') \le \tfrac{C\rho}{t^2}\exp\left(-\tfrac{r'^2+\rho^2}{2t}\right)|B_t(x,y)-B_t(x',y)| \times |x-x'|^{-\alpha_0} \\ &\le \tfrac{C\rho}{t^2}\exp\left(-\tfrac{r'^2+\rho^2}{2t}\right)\left(\textstyle\sum_{n=1}^\infty(\nu_n-1)I_{\nu_n-1}\left(\tfrac{r\rho}{t}\right)+\right. \\ &\left.\textstyle\sum_{n=1}^\infty\left|I_{\nu_n-1}\left(\tfrac{r\rho}{t}\right)-I_{\nu_n-1}\left(\tfrac{r'\rho}{t}\right)\right|\right)\times|x-x'|^{-\alpha_0} := \tfrac{C\rho}{t^2}\exp\left(-\tfrac{r'^2+\rho^2}{2t}\right)(D_t^1+D_t^2). \end{aligned} \tag{B.5}$$

From the recurrence relations on modified Bessel functions, see equation 9.6.26 in [AS72], and since $|x-x'|>r$ one gets

$$\begin{aligned} D_t^1 &\le \tfrac{Cr\rho}{t}\textstyle\sum_{n\ge1}(I_{\nu_n-2}+I_{\nu_n})\left(\tfrac{r\rho}{t}\right)r^{-\alpha_0} \le \tfrac{Cr^{1-\alpha_0}\rho}{(1\wedge\varepsilon)t}\exp\left(\tfrac{\rho r}{t}\right)\left\{\left(\tfrac{r\rho}{t}\right)^{-(2-\nu_1+\varepsilon)}+1\right\} \\ &\le \tfrac{C}{1\wedge\varepsilon}\exp\left(\tfrac{r\rho}{t}\right)\left\{\tfrac{\rho}{t}+\left(\tfrac{\rho}{t}\right)^{\nu_1-1-\varepsilon}r^{\nu_1-1-\varepsilon-\alpha_0}\right\}. \end{aligned} \tag{B.6}$$

Recall now formula 9.6.18 from [AS72], i.e.

$$\forall\nu>-1/2,\ I_\nu(z)=\frac{(\frac{1}{2}z)^\nu}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\int_{-1}^1(1-u^2)^{\nu-1/2}\cosh(zu)du.$$

Hence,

$$\begin{aligned} D_t^2 &\le C\left\{\left|I_{\nu_1-1}\left(\tfrac{r\rho}{t}\right)-I_{\nu_1-1}\left(\tfrac{r'\rho}{t}\right)\right|+\tfrac{\rho|r-r'|}{t}\exp\left(\tfrac{r'\rho}{t}\right)\right\}|x-x'|^{-\alpha_0} \\ &\le C\left\{\left(\tfrac{\rho}{t}\right)^{\nu_1-1}|r^{\nu_1-1}-r'^{\nu_1-1}|+\left(\tfrac{\rho}{t}\right)^{\nu_1}|r-r'|+\tfrac{\rho|r-r'|}{t}\right\}\exp\left(\tfrac{r'\rho}{t}\right)|r-r'|^{-\alpha_0} \\ &\le C\left\{\left(\tfrac{\rho}{t}\right)^{\nu_1-1}|r-r'|^{\nu_1-1-\alpha_0}+\left(\tfrac{\rho}{t}\right)^{\nu_1}+\tfrac{\rho}{t}\right\}\exp\left(\tfrac{r'\rho}{t}\right). \end{aligned} \tag{B.7}$$

Plugging (B.6) and (B.7) into (B.5) we derive that for $|x-x'|>r$

$$A_t^2(x,x')\le\frac{C}{(1\wedge\varepsilon)t^\xi}\exp\left(-c\frac{|r'-\rho|^2}{t}\right)\left\{1+|r-r'|^{\nu_1-1-\alpha_0}+r^{\nu_1-1-\varepsilon-\alpha_0}\right\}. \tag{B.8}$$

Take now $\varepsilon>0$ s.t. $\nu_1-1-\varepsilon>0$. Set $\alpha_0=\nu_1-1-\varepsilon$. From (B.4) and (B.8) the proof is complete. □

Remark 19 *The spectral theory suggests our previous Hölder constant for the gradient is somehow optimal. Indeed, if ϕ_1 denotes the first eigenfunction of the elliptic Dirichlet problem for the Laplacian in a bidimensional truncated cone of angle ω and vertex 0, we have from Example 4.6.5 in Davies [Dav89] that $\phi_1(x)=O(r^{\nu_1})$, $\nu_1=\pi/\omega$ when $x\longrightarrow0$ non tangentially, and the heat*

kernel also writes $q_t(x,y) = \sum_{i=1}^{\infty} \exp(-E_i t)\phi_i(x)\phi_i(y)$ *where the* $(E_i)_{i\in\mathbf{N}^*}$ *are the eigenvalues of the Laplacian in the truncated cone* $(0 < E_1 \leq E_2 \leq \ldots \uparrow \infty)$ *and the* $(\phi_i)_{i\in\mathbf{N}^*}$ *the orthonormal eigenfunctions.*
The spectral decomposition of the heat kernel also suggests that we can not expect more spatial smoothness than the one of the elliptic problem. A general study of this kind of problem is far from being easy. A Sobolev approach can be found in Dauge [Dau88] and Kozlov et al. [KMR97]. The possible application of their arguments to the parabolic case will concern further research.

Remark 20 *The control of the time derivative stated in Lemma 15 holds true up to* $d = 4$ *without major changes in the proof. The main tool needed is Weyl's asymptotic Lemma that gives some controls on the behaviour of the eigenfunctions that appear in Proposition 12, see [Cha84] p. 172. The remaining computations are rather similar to the previous ones.*

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