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Pricing functions and risk measures in incomplete markets.

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Abstract

We present a new approach for pricing and making decisions of investment in incomplete markets. This we do without fixing in advance any probability measure. The key concept that we introduce is a notion of pricing function compatible with a family of bid and ask prices observed in the market.

This method links the theory of asset pricing and the theory of risk measuring. Furthermore we prove a first fundamental theorem of asset pricing in this new context.

Keywords: Incomplete market, convex risk measures, risk management, arbitrage, derivative pricing, fundamental theorem of asset pricing.

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1 Introduction

We present a new approach for pricing and making decisions of investment in incomplete markets. This method establishes a link between the theory of asset pricing and the theory of risk measuring.

The problem of pricing is perfectly solved by the arbitrage pricing theory in complete markets. However real markets connot be assumed complete. The question of pricing in incomplete market has therefore become a fundamental question in finance.

To address this question, different kind of approches have been proposed in the litterature.

Most of them assume that a probability P is given and that the processes of the assets used in order to define prices are well known, even their joint stochastic processes. Among these approches are two important families: pricing using maximisation of utility and pricing using Backward Stochastic Differential Equations (BSDE). For pricing using maximisation of utility in a context of no arbitrage we can refer to works by Schachermeyer, among them [Schachermayer(2001)] and [Schachermayer(2004)], and also to [Biagini et al.(2005)]. In the case of pricing via BSDE a probability P is fixed as well as a Brownian motion and a filtration $(\mathcal{F}_t)_{t\leq T}$ adapted to the Brownian motion and one can price square integrable $(\mathcal{F}_T, P)$ measurable variables. For this we refer to [El Karoui et al. (1995)] and [El Karoui et al. (1997)].

Another very interesting approach is that of Avellaneda and Paras [Avellaneda et al. (1995)] and [Avellaneda et al.(1996)]. In these two papers the authors point out the importance of the risk of volatility in pricing theory and the fact that the option prices provide informations about the market's volatility expectations. Within this approach, a pricing theory has to take into account the prices of options available in the market. In addition, in order to modelize the risk of volatility, [Avellaneda et al. (1995)] consider a diffusion model with uncertain volatility varying inside a band, and this leads to probability measures which cannot be all equivalent.

Taking a different point of view for pricing in incomplete markets, Carr, Geman and Madan [Carr et al. (2001)] introduce a new fondamental notion, that of No Strictly Arbitrage Opportunity (N.S.A.O).

Another very important theory in finance is that of risk measuring. The notion of coherent risk measure was first defined by [Artzner et al.(1999)]. This notion was generalized into the notion of convex risk measure and studied in detail by [Föllmer et al. (2002)] and [Föllmer et al. (2002 b)]. In the theory of risk measuring, another notion, that of model uncertainty has been introduced in [Cont (2005)].

The present work takes advantage of both the theory of risk measuring, and the notion of N.S.A.O.

The aim of our paper is the following: introduce in the context of incomplete markets the notion of pricing function which assigns to each financial instrument (stock, bond, derivative, portfolio, basket, index..) a bid price and an ask price in such a way that it is compatible with the observed prices in the market. We do it without fixing in advance any reference probability measure. And then we prove a first fundamental theorem of asset pricing in this new context.

We consider a family $(X_i)_{i\in I}$ of financial instruments for which either a price or a bid price and an ask price are available in the market. Among those with a price that we take into account, there is a non risky asset. We will make use of these financial instruments in order to give a bid price and an ask price for any financial position. We say that these financial instruments are the assets used to calibrate the pricing function, or the reference assets.

In section 2, we introduce the notion of pricing function which assigns to each financial position (stock, derivative, portfolio..) a bid price $\Pi(X)$ and an ask price $-\Pi(-X)$. Taking into account the diversification of risk and the lack of liquidity of some financial instruments, we conclude that the bid price $\Pi(X)$ has to be a concave function. We then define the notion of admissible pricing function i.e. a pricing function compatible with a family of bid and ask prices observed in the market. Rewritting the pricing function in terms of risk measures, we express the condition of admissibility in terms of necessary and sufficiant conditions on the penalty functions. This gives a new comprehension and interpretation of the penalty functions for monetary risk measures. In section 3 we study the perfectly liquid assets. We prove that the set of perfectly liquid assets is a vector space and that the restriction of any pricing function to this set is linear. The set of perfectly liquid assets can be used for superhedging and subhedging. We study also the set of hedgeable financial positions.

As a pricing function is represented by a family of probability measures and penalty functions, a natural question arises. How can we choose the family of probability measures in order to construct a pricing function (subordinated to this family of probability measures)? The second part of this paper is devoted to giving an answer to this question by proving an equivalent, in that context, to the first fundamental theorem of asset pricing [Harrison et al.(1979)].

For this second part, as in [Carr et al.(2001)] the key notions are that of acceptability corresponding to the fact that enough persons would accept a given position, and that of N.S.A.O. In [Carr et al.(2001)], the authors define the new notion of N.S.A.O., assuming that for each asset used for calibration, a price is available in the market. They prove a first fundamental theorem, assuming either that the space state Ω is finite or that a probability measure P is given a priori and that the set $\mathcal{Q}_0$ of valuation probability measures is finite and composed of probability measures absolutely continuous with respect to P.

In our paper there is no restrictive hypothesis on Ω, and the set $\mathcal{Q}_0$ can be any set of probability measures on $(\Omega, \mathcal{G})$, closed for the weak* topology ($\mathcal{G}$ is a fixed σ-algebra and there is no reference probability on it). We define here a notion of N.S.A.O., for that general case, taking into account the fact that for some assets used for calibration only bid and ask prices are available in the market. This definition extends the definition of N.S.A.O. introduced in [Carr et al.(2001)].

The first fundamental theorem of pricing that we prove is the following: If there is no strictly acceptable opportunity for a closed family $\mathcal{Q}_0$ of probabilities, there is an admissible pricing function in the convex hull of $\mathcal{Q}_0$. In section 4 this is done in the case where there is a price available in the market for each one of the reference assets. In section 5 we give the proof in the general case, that is assuming only that there are bid and ask prices available in the market for the reference assets.

The fundamental mathematical tools for the proof are the following: theorems of separation of convexes; compacity of the unit ball of the dual of a Banach space for the weak* topology (Banach Alaoglu theorem); Riesz representation theorem in measure theory.

Sections 6 and 7 discuss applications of the previous sections. In section 6 we study the set of all admissible pricing functions subordinated to a family of probability measures. Section 7 concerns the implications of our work for pricing derivatives and making decisions of investment.

2 Pricing function in an incomplete market

2.1 The economic model

We consider that the set of financial positions is the linear space $\mathcal{X}$ of all bounded measurable functions on a measurable space $(\Omega, \mathcal{G})$. (Ω can be the product of the state space and the time space). We assume that in the market a finite number of financial instruments $(X_i)_{i\in I}$ are priced and among them a non risky asset $X_0 = 1$. We assume that for some of them a price C_i is available in the market and that for some others only bid and ask prices are available; we denote them C_i^{bid} and C_i^{ask} (when C_i is uniquely defined we put $C_i^{bid} = C_i^{ask} = C_i$). Among the $(X_i)_{i\in I}$ there may be some derivatives. We say that the $(X_i)_{i\in I}$ are the financial instruments used for calibration, and we call them the reference financial instruments.

In that context we want to define for each financial position a bid price and an ask price. Selling X can be considered as buying $-X$. Therefore we will model the bid price of X as $\Pi(X)$ and the ask price as $-\Pi(-X)$.

Which properties should satisfy the pricing function Π on the space $\mathcal{X}$?

First of all, we assume that the price of the non risky asset is constant. (This doesn't restrict the generality as we can choose the non risky asset as numeraire). This leads to the translation invariance property. The market is not perfectly liquid so we don't assume linearity of the pricing function. If λ is big the price of λX should be less than the price of X muliplied by λ. On the other end we have to take into account the diversification of risk. Avellaneda and Paras said that therefore the portfolio value has to be super-additive for the buy side [Avellaneda et al. (1996)]. Taking also into account the illiquidity of some products, we conclude that the porfolio value for the buy side i.e. the bid price $\Pi(X)$ has to be a concave function.

So we give the following definition:

Definition 2.1 *A pricing function* Π *on the space* $\mathcal{X}$ *is a map* $\Pi : \mathcal{X} \to \mathbb{R}$ *satisfying the following properties:*

i) monotonicity: $\forall X, Y \in \mathcal{X}$ *if* $X \leq Y$ *then* $\Pi(X) \leq \Pi(Y)$

ii) translation invariance: $\forall m \in \mathbb{R}\ \forall X \in \mathcal{X}\ \Pi(X + m) = \Pi(X) + m$

iii) concavity: $\forall X, Y \in \mathcal{X}\ \ \forall \lambda \in [0, 1]$

$$\Pi(\lambda X + (1 - \lambda)Y) \geq \lambda\Pi(X) + (1 - \lambda)\Pi(Y)$$

iv) normalization: $\Pi(0) = 0$

Remark: Π defined on $\mathcal{X}$ is a pricing function if and only if $-\Pi$ is a normalized monetary convex risk measure in the sense of Föllmer and Schied [Föllmer et al.(2002)].

Definition 2.2 *A pricing function* Π *is admissible if for all* $i \in I$,

$$C_i^{bid} \leq \Pi(X_i) \leq C_i^{ask} \ \ \textit{and} \ \ C_i^{bid} \leq -\Pi(-X_i) \leq C_i^{ask}$$

Definition 2.3 *A pricing function Π is strongly admissible if $\forall i \in I$,*

$$\Pi(X_i) = C_i^{bid} \text{ and } \Pi(-X_i) = -C_i^{ask}$$

Definition 2.4 *A pricing function Π is continuous from below if for every increasing sequence X_n of elements of $\mathcal{X}$ such that $X = lim X_n$, the increasing sequence $\Pi(X_n)$ has the limit $\Pi(X)$.*

2.2 Representation of an admissible pricing function. Interpretation of the penalty functions

Theorem 2.1 *A pricing function Π continuous from below admits a representation of the kind:*

$$\forall X \in \mathcal{X} \ \ \Pi(X) = \min_{Q \in \mathcal{Q}} (E_Q(X) + \alpha(Q)) \tag{1}$$

where $\mathcal{Q}$ is a set of probability measures on $(\Omega, \mathcal{G})$ and for each $Q \in \mathcal{Q}$ $\alpha(Q)$ is a real number called penalty.

The pricing function Π is admissible if and only if any representation of Π of the kind (1) satisfies the two following conditions:

i) $\forall Q \in \mathcal{Q}$, $\alpha(Q) \geq \sup(0, \sup_i (C_i^{bid} - E_Q(X_i), E_Q(X_i) - C_i^{ask}))$

ii) $\mathcal{Q}_0 = \{Q \in \mathcal{Q} \ / \ \alpha(Q) = 0\}$ is non empty and for every $Q_0 \in \mathcal{Q}_0$, $\forall i \in I$, $C_i^{bid} \leq E_{Q_0}(X_i) \leq C_i^{ask}$

Proof:

Let Π be a pricing function continuous from below. Put $\rho(X) = -\Pi(X)$. It is a monetary convex risk measure continuous from below so we can apply the Theorem 4.12 and the Proposition 4.17 of [Föllmer et al.(2002)].

We get a set $\mathcal{Q}$ of probability measures on Ω such that (1) is satisfied. Furthermore $P(0) = 0$ so there is a probability measure Q_0 in $\mathcal{Q}$ such that $P(0) = E_{Q_0}(0) + \alpha(Q_0)$ so $\alpha(Q_0) = 0$ i.e.$\mathcal{Q}_0 \neq \emptyset$ and for all $Q \in \mathcal{Q}$, $\alpha(Q) \geq 0$

Assume now that Π is admissible.

From the inequality $\Pi(X_i) \geq C_i^{bid}$ (resp. $-\Pi(-X_i) \leq C_i^{ask}$) it follows that for all Q, $\alpha(Q) \geq C_i^{bid} - E_Q(X_i)$ (resp. $\alpha(Q) \geq E_Q(X_i) - C_i^{ask}$) so we get i).

Let $Q_0 \in \mathcal{Q}_0$. It follows from i) that $C_i^{bid} \leq E_{Q_0}(X_i) \leq C_i^{ask}$; i.e. ii) is satisfied.

Conversely, assume that a family $\mathcal{Q}$ of probabilities with penalty functions $\alpha(Q)$ satisfy the conditions i) and ii). Define

$$\Pi(X) = \min_{Q \in \mathcal{Q}} (E_Q(X) + \alpha(Q))$$

Then $\Pi(0) = \min_{Q \in \mathcal{Q}}(\alpha(Q)) = 0$ and Π is a pricing function. Let $Q_0 \in \mathcal{Q}_0$

$$\min_{Q \in \mathcal{Q}} (E_Q(X_i) + \alpha(Q)) \leq E_{Q_0}(X_i) \leq C_i^{ask}$$

and for all $Q \in \mathcal{Q}$

$$E_Q(X_i) + \alpha(Q) \geq E_Q(X_i) + C_i^{bid} - E_Q(X_i) = C_i^{bid}$$

So for all i, $C_i^{bid} \leq \Pi(X_i) \leq C_i^{ask}$. In the same way, we prove that $C_i^{bid} \leq -\Pi(-X_i) \leq C_i^{ask}$; so Π is admissible.

q.e.d.

In the case where prices (and not only bid and ask prices) are observed in the market for every financial instrument X_i, the Theorem 2.1 takes the following form:

Corollary 2.2 *:*

Assume that for every financial instrument $i \in I$, a price C_i is available in the market and not only bid and ask prices.

The pricing function Π continuous from below is admissible if and only if any representation of Π of the kind (1) (as in Theorem 2.1) satisfies the two following conditions:

i) $\forall Q \in \mathcal{Q}$, $\alpha(Q) \geq \sup_i(|C_i - E_Q(X_i)|)$

ii) $\mathcal{Q}_0 = \{Q \in \mathcal{Q} \ / \ \alpha(Q) = 0\}$ is non empty and $\forall Q_0 \in \mathcal{Q}_0$, $\forall i \in I$, $C_i = E_{Q_0}(X_i)$

It follows also from Theorem 2.1 that a pricing function gives also the bid ask spread of every financial instrument, indeed:

Corollary 2.3 *:*

Let Π be a pricing function continuous from below (non necessarily admissible). Then $\forall X \in \mathcal{X}$, $-\Pi(-X) \geq \Pi(X)$.

Proof:

As in the proof of Theorem 2.1 , let $Q_0 \in \mathcal{Q}_0$,

$$\begin{aligned} -\Pi(-X) &= -\min_{Q\in\mathcal{Q}}(E_Q(-X)+\alpha(Q)) \\ &= \max_{Q\in\mathcal{Q}}(E_Q(X)-\alpha(Q)) \\ &\geq E_{Q_0}(X) \\ &\geq \min_{Q\in\mathcal{Q}}(E_Q(X)+\alpha(Q)) = \Pi(X) \end{aligned}$$

q.e.d.

This leads to the following definition of bid and ask prices and of uncertainty:

Definition 2.5 *i) $\Pi(X)$ is called the bid price of X and $-\Pi(-X)$ is called the ask price of X.*

ii) The uncertainty function associated to the pricing function Π on $\mathcal{X}$ is the real function μ_Π defined on $\mathcal{X}$ by $\mu_\Pi(X) = -(\Pi(X)+\Pi(-X))$.

This definition of uncertainty generalizes the notion of uncertainty introduced in [Cont (2005)]. The uncertainty satisfies the following properties:

Proposition 2.4 *i) The uncertainty function associated to any pricing function Π is convexe and positive.*

ii) If the pricing function Π is admissible, then for all i,

$$0 \leq \mu_\Pi(X_i) \leq C_i^{ask} - C_i^{bid}$$

Proof: i) As $\mu_\Pi(X) = -(\Pi(X)+\Pi(-X))$, the convexity of μ_Π is an easy consequence of the concavity of Π.

For all $X \in \mathcal{X}$, we know from Corollary (2.3) that $\Pi(X) \leq -\Pi(-X)$, so we get the positivity of μ_Π.

ii) If Π is admissible, for all $i \in I$, $C_i^{bid} \leq P(X_i) \leq C_i^{ask}$ and $-C_i^{ask} \leq \Pi(-X_i) \leq -C_i^{bid}$

And so, $\mu(X_i) = -(\Pi(X_i)+\Pi(-X_i)) \leq C_i^{ask} - C_i^{bid}$.

In all the following, we will always consider pricing functions which admit a representation of the kind (1), as defined in Theorem (2.1).

2.3 Exemples of pricing functions

2.3.1 Linear and homogeneous pricing functions

Consider first a probability measure Q_0 on $(\Omega, \mathcal{G})$. Then $\Pi(X) = E_{Q_0}(X)$ defines a linear pricing function continuous from below.

It follows from Theorem2.1 that Π is an admissible pricing function if and only if

$$\forall i \in I \;\; C_i^{bid} \leq E_{Q_0}(X_i) \leq C_i^{ask} \tag{2}$$

We will see in the next section that any linear admissible pricing function is of this kind.

Consider now a family $\mathcal{Q}$ of probability measures on $(\Omega, \mathcal{G})$.

$\Pi(X) = \max_{Q \in \mathcal{Q}} E_Q(X)$ defines a pricing function such that: $\forall \lambda > 0, \; \Pi(\lambda X) = \; \lambda \Pi(X)$. We say that such a pricing function is a homogeneous pricing function. This pricing function is admissible if and only if every probability measure in $\mathcal{Q}$ satisfies (2).

Remark: A pricing function Π is homogeneous if and only if the corresponding risk measure $-\Pi$ is a coherent risk measure.

2.3.2 Exemples of non homogeneous admissible pricing functions

We will give two more exemples of admissible pricing functions which are not homogeneous.

In both cases we consider a family $\mathcal{Q}$ of probability measures and we assume that in this family there is at least one probability measure Q_0 satisfying the condition (2). (We will prove in sections 4 and 5 that there exists such a probability measure Q_0 in the convex hull of a given family of probability measures as soon as the condition of No Strictly Arbitrage Opportunity is satisfied).

i) In the first case we consider for every probability measure $Q \in \mathcal{Q}$ the penalty function $\alpha_m(Q) = \sup(0, \sup_i(C_i^{bid} - E_Q(X_i), E_Q(X_i) - C_i^{ask}))$.

Then $\Pi(X) = \min_{Q \in \mathcal{Q}}(E_Q(X) + \alpha_m(Q))$ defines an admissible pricing function.

ii) We can also consider the penalty function introduced in [Cont (2005)], in the case where $C_i^{bid} = C_i^{ask}$ for all i in I

$$\alpha(Q) = \sum_{i \in I} |C_i - E_Q(X_i)|$$

Clearly in view of Theorem2.1, this exemple can be extended to the case of bid and ask prices by

$$\alpha(Q) = \sum_{i \in I} \max(0, C_i^{bid} - E_Q(X_i), E_Q(X_i) - C_i^{ask})$$

In both cases the associated pricing function $\Pi(X) = \min_{Q \in \mathcal{Q}}(E_Q(X) + \alpha(Q))$ is admissible.

2.3.3 Pricing functions defined from BSDE

For this exemple, we refer to [El Karoui et al. (1995)], [El Karoui et al. (1997)] and [Coquet et al. (2002)].

This exemple is in the following specific context:

A probability space $(\Omega, \mathcal{F}, P)$ is given.

A multidimensional Brownian motion is given on this probability space. Denote $(\mathcal{F}_t;\ 0 \leq t \leq T)$ the augmented filtration generated by the Brownian motion.

Consider the backward stochastic differential equation

$$-dY_t = f(t, Y_t, Z_t) - Z_t^* dB_t \quad Y_T = -X$$

Assume that X is in $L^\infty(\Omega, \mathcal{F}_T, P)$. Assume that the driver

$g : (\Omega \times I\!R \times I\!R^n) \rightarrow I\!R$ is convex, satisfies the usual assumptions and that $\forall t \in I\!R\ g(t, 0, 0) = 0\ \ P\ a.s.$

Then $\Pi(X) = -Y_0(-X)$ defines a pricing function.

2.3.4 Pricing function associated to a utility function

Consider a loss function $l(x) = -u(-x)$. Consider a probability measure P on $(\Omega, \mathcal{G})$.

Föllmer and Schied [Föllmer et al. (2002)] introduced the convex risk measure defined from the set of acceptable positions

$\mathcal{A}_{x_0} = \{X \in L^\infty(\Omega, \mathcal{G}, P) \mid E_P(l(-X)) \leq x_0\}$.

$\rho(X) = inf\{m \in I\!R;\ X + m \in \mathcal{A}_{x_0}\}$

then $\rho(X) = max_{Q \in \mathcal{M}}((E_Q(-X) - \alpha(Q))$ where $\mathcal{M}$ is the set of all probability measures on $(\Omega, \mathcal{G})$ absolutely continuous with respect to P. They have computed the penalty function:

$$\alpha(Q) = \inf_{\lambda > 0}[\frac{1}{\lambda}(x_0 + E_P(l^*(\lambda \frac{dQ}{dP})))]$$

where l^* is the conjugate function of the convex function l.

Then $\Pi(X) = -\rho(X)$ defines a pricing function considered also in [El Karoui et al.(2005)].

This pricing function is admissible iff for all Q, $\alpha(Q) \geq \alpha_m(Q)$.

For exemple with the exponential utility function $\alpha(x) = -e^{-\alpha x}$; this gives the necessary condition

$$x_0 \leq \inf_{Q \in \mathcal{M}} (\exp(\frac{1}{\alpha} H(Q|P) - \alpha_m(Q)) = x_{\max}$$

So there is a maximal level of expectated utility x_{max} such that there exits an admissible pricing function satisfying: every financial position of positive price is acceptable. More precisely $x_0 \leq x_{\max}$, if and only if there exists an admissible pricing function Π_{x_0} such that $\Pi_{x_0}(X) > 0$ implies that the expectation of the utility of X is greater that x_0.

3 Perfectly liquid assets and hedgeable assets

In all this section, we assume that a pricing function Π is given on the set of financial positions.

Definition 3.1 *An asset $X \in \mathcal{X}$ is perfectly liquid if for all $\lambda \in I\!R$, $\Pi(\lambda X) = \lambda \Pi(X)$. We denote $\mathcal{L}$ the set of perfectly liquid assets.*

We want now to characterize the set of perfectly liquid assets and the restriction of the pricing function to it.

Remark:

None of the financial instruments X_i for which $C_i^{bid} \neq C_i^{ask}$ can be perfectly liquid. Among the X_i for which $C_i^{ask} = C_i^{bid}$, some are perfectly liquid (at least the non risky asset X_0) but it is possible that others are not.

Proposition 3.1 *1) The set $\mathcal{L}$ of perfectly liquid assets is a linear space. The restriction of the pricing function to this linear space $\mathcal{L}$ is a linear form equal to E_Q for all $Q \in \mathcal{Q}$ (the set of probability measures associated to the pricing function Π).*

2) Let $Y \in \mathcal{X}$. $Y \in \mathcal{L}$ if and only if

$$\forall X \in \mathcal{X} \;\; \Pi(X+Y) = \Pi(X) + \Pi(Y)$$

In particular if every financial instrument in the market is perfectly liquid, the pricing function Π is linear and is of the form $\Pi(X) = E_{Q_0}(X)$ for some probability measure Q_0 on $(\Omega, \mathcal{G})$.

Proof:

Let $Y \in \mathcal{L}$

$$\forall \lambda \in \mathbb{R}\,,\ \ \forall Q \in \mathcal{Q}\,,\ \ E_Q(\lambda Y) + \alpha(Q) \geq \lambda \Pi(Y)$$

So $\alpha(Q) \geq \lambda[\Pi(Y) - E_Q(Y)]$. Let λ tends to $+\infty$ or to $-\infty$. The previous inequality is possible only if for all $Q \in \mathcal{Q}$, $\Pi(Y) = E_Q(Y)$. It follows then that for all $X \in \mathcal{X}$, for all $Y \in \mathcal{L}$,

$$\begin{aligned}\Pi(X+Y) = \min_{Q \in \mathcal{Q}}(E_Q(X+Y) + \alpha(Q)) &= \min_{Q \in \mathcal{Q}}(E_Q(X) + \alpha(Q)) + \Pi(Y)\\ &= \Pi(X) + \Pi(Y)\end{aligned}$$

Endly, for all $Y \in \mathcal{L}$, for all $\beta \in \mathbb{R}$, βY is obviously in $\mathcal{L}$ and then

$$\forall (Y,Z) \in \mathcal{L}^2\,, \forall \lambda \in \mathbb{R}\,, \Pi(\lambda(Y+Z)) = \Pi(\lambda Y) + \Pi(\lambda Z) = \lambda(\Pi(Y) + \Pi(Z))$$

So $\mathcal{L}$ is a linear subspace of $\mathcal{X}$ and the restriction of Π to $\mathcal{L}$ is linear and equal to E_Q for every $Q \in \mathcal{Q}$.

Conversely Let $Y \in \mathcal{X}$ such that for all $X \in \mathcal{X}$ $\Pi(X+Y) = \Pi(X) + \Pi(Y)$.

First by an obvious recursion, we prove that for all $n \in \mathbb{N}$, $\Pi(nY) = n\Pi(Y)$.

Then for all $Q \in \mathcal{Q}$, $E_Q(nY) + \alpha(Q) \geq n\Pi(Y)$.

It follows then that $\Pi(Y) - E_Q(Y) \leq 0$

Moreover, $0 = \Pi(0) = \Pi(-Y) + \Pi(Y)$. And then by an obvious recursion we get that for all $n \in \mathbb{N}$, $\Pi(-nY) = -n\Pi(Y)$.

And it follows that $\Pi(Y) - E_Q(Y) \geq 0$ for all $Q \in \mathcal{Q}$

We have thus proved that $\forall Q \in \mathcal{Q}$ $\Pi(Y) = E_Q(Y)$.

And so for all $\lambda \in \mathbb{R}$,

$$\Pi(\lambda Y) = \min_{Q \in \mathcal{Q}}(E_Q(\lambda Y) + \alpha(Q)) = \lambda \Pi(Y) + \min_{Q \in \mathcal{Q}}(\alpha(Q)) = \lambda \Pi(Y)$$

q.e.d.

We define now the set of acceptable financial positions associated to a price function Π.

Definition 3.2 *The set $\mathcal{A}_\Pi$ of acceptable financial positions associated to the price function Π is:*

$$\mathcal{A}_\Pi = \{X \in \mathcal{X};\ \Pi(X) \geq 0\} \tag{3}$$

Remark: This set is also the acceptance set associated to the risk measure $\rho(X) = -\Pi(X)$ [Föllmer and Schied (2002)].

We prove now that the pricing function can be recovered from $\mathcal{A}_\Pi$ and $\mathcal{L}$.

Proposition 3.2 *For all X in $\mathcal{X}$,*

$$1)\ \ \Pi(X) = \max_{\{Y \in \mathcal{L}\ /\ X-Y \in \mathcal{A}_\Pi\}} \Pi(Y)$$

$$\forall Q \in \mathcal{Q}\quad \Pi(X) = \max_{\{Y \in \mathcal{L}\ /\ X-Y \in \mathcal{A}_\Pi\}} E_Q(Y)$$

Any $Y_b \in \mathcal{L}$ such that $\Pi(X) = \Pi(Y)$ and $X - Y \in \mathcal{A}_\Pi$ can be considered as a subhedge for X.

$$2)\quad -\Pi(-X) = \min_{\{Y \in \mathcal{L}\ /\ Y-X \in \mathcal{A}_\Pi\}} \Pi(Y)$$

$$\forall Q \in \mathcal{Q}\quad -\Pi(-X) = \min_{\{Y \in \mathcal{L}\ /\ Y-X \in \mathcal{A}_\Pi\}} E_Q(Y)$$

Any $Y_a \in \mathcal{L}$ such that $\Pi(X) = \Pi(Y)$ and $Y - X \in \mathcal{A}_\Pi$ can be considered as a superhedge for X.

Proof:

Let $X \in \mathcal{X}$. Let $Y \in \mathcal{L}$ such that $X - Y \in \mathcal{A}_\Pi$.

From Proposition 3.1, $\Pi(X) - \Pi(Y) = \Pi(X - Y)$ so $\Pi(X) \geq \Pi(Y)$ Moreover the restriction of Π to $\mathcal{L}$ is onto so there is $Y \in \mathcal{L}$ such that $\Pi(X) = \Pi(Y)$. Then $X - Y \in \mathcal{A}_\Pi$ and we get 1), as the restriction of Π to $\mathcal{L}$ is equal to E_Q for every Q. We apply now 1) to $-X$ and use the fact that $\mathcal{L}$ is a linear space to get 2).

q.e.d.

Definition 3.3 *A financial position $X \in \mathcal{X}$ is called hedgeable if there is a perfectly liquid asset $Y \in \mathcal{L}$ such that $X-Y$ and $-X+Y$ are both acceptable. Such a Y is then called a hedge fo X and satisfies $\Pi(Y) = \Pi(X)$.*

Proposition 3.3 *Let $X \in \mathcal{X}$. The bid price $\Pi(X)$ of X is equal to the ask price of X (i.e. there is no uncertainty on X) if and only if X is hedgeable.*

Proof:

i) Assume that X is hedgeable. There is $Y \in \mathcal{L}$ such that $X - Y$ and $-X + Y$ are acceptable. $\Pi(X - Y) \geq 0$ and $\Pi(-X + Y) \geq 0$ It follows that $-\Pi(-X) \leq \Pi(Y) \leq \Pi(X)$ and so $-\Pi(-X) = \Pi(Y)$.

ii) Conversely assume that $\Pi(X) = -\Pi(-X)$.

Let $Y \in \mathcal{L}$ such that $\Pi(Y) = \Pi(X)$

From Proposition 3.1, $\Pi(X - Y) = \Pi(X) - \Pi(Y) = 0$, so $X - Y \in \mathcal{A}_{\Pi}$.

As $\Pi(-X) = -\Pi(X)$. It follows that $\Pi(-X + Y) = \Pi(-X) + \Pi(Y) = 0$ and $-X + Y \in \mathcal{A}_{\Pi}$

q.e.d.

Remark: In general, when Π is an admissible pricing function not homogeneous, the set of perfectly liquid assets is strictly contained in the set of hedgeable positions. In the case where Π is homogeneous, $\mathcal{A}_{\Pi}$ is a cone, and in such case every hedgeable position is perfectly liquid.

4 First fundamental theorem when a price is available for each reference financial instrument

As we have seen in section 2, every admissible pricing function is expressed in terms of a family of probability measures on $(\Omega, \mathcal{G})$ and of penalties. So a natural question arises: How can we choose a family of probability measures in order to be able to construct an admissible pricing function from this family of probabilities?

Assume in all this section as in [Carr et al. (2001)], that two families of probability measures, $\mathcal{Q}_0$ and $\mathcal{Q}_1$, are given on $(\Omega, \mathcal{G})$. For each of them a penalty function δ is defined. We assume that it is equal to 0 on $\mathcal{Q}_0$ and that it is strictly positive on $\mathcal{Q}_1$. Following [Carr et al. (2001), we call valuation test measures the measures in the first set and stress tests measures the mesures of the second set.

We denote $\mathcal{Q} = \mathcal{Q}_0 U \mathcal{Q}_1$.

In all this section we assume that the prices of the financial instruments $(X_i)_{i \in I}$ (I is finite) are available in the market and that among them is the non risky asset $X_0 = 1$. We denote C_i the price of X_i ($C_0 = 1$).

The aim of this section is to prove a first fundamental therem of pricing in that context.

Carr Geman and Madan [Carr et al. (2001)] have proved it in the case where the space Ω is finite, and also in the case where Ω is infinite but where a reference probability is fixed, and under the assumption that the set $\mathcal{Q}$ of valuation measures is finite and that all these measures are absolutely continuous with respect to this reference probability.

In the present paper no reference probability is fixed, and there is no restriction on the set Ω.

We assume in a first step (Theorem 4.3) that the set of valuation measures $\mathcal{Q}_0$ is finite.

After that we prove the first fundamental theorem in the general case (Theorem 4.4). This last proof involves more topological arguments.

Definition 4.1 *i) The set of acceptable positions associated to the family $\mathcal{Q}_0$ is*

$$\mathcal{A}_{\mathcal{Q}_0} = \{X \in \mathcal{X} \;/\; \forall Q \in \mathcal{Q}_0\;,\; E_Q(X) \geq 0\}$$

ii) The set of strictly acceptable positions associated to the family $\mathcal{Q}_0$ is

$$\mathcal{A}^+_{\mathcal{Q}_0} = \{X \in \mathcal{A}_{\mathcal{Q}_0} \;/\; \exists Q_0 \in \mathcal{Q}_0 \; E_{Q_0}(X) > 0\}$$

iii) Denote also:

$$\mathcal{A}_{(\mathcal{Q},\delta)} = \{X \in \mathcal{X} \;/\; \forall Q \in \mathcal{Q} \; E_Q(X) + \delta(Q) \geq 0\}$$

iv) and:

$$\mathcal{A}^+_{(\mathcal{Q},\delta)} = \{X \in \mathcal{A}_{(\mathcal{Q},\delta)} \;/\; \exists Q_0 \in \mathcal{Q}_0 \; E_{Q_0}(X) > 0\}$$

We want to give a meaning to the notion of admissible pricing function subordinated to the families of measures $\mathcal{Q}_0$ and $\mathcal{Q}_1$.

The first idea is to consider the function $\Pi(X) = -\rho_{\mathcal{A}}(X)$ where $\mathcal{A} = \mathcal{A}_{(\mathcal{Q},\delta)}$ as in Definition 4.1 and $\rho_{\mathcal{A}}$ is the risk measure defined in [Föllmer et al.(2002)] associated to the acceptance set $\mathcal{A}$. We can remark that Π is a pricing function as soon as Q_0 is non empty.

This pricing function is admissible if and only if $\mathcal{Q}_0 \; U \; \mathcal{Q}_1$ satisfies the conditions i) and ii) of Theorem 2.1. The condition i) implies conditions on the penalty function and the condition ii) implies that there is a probability measure Q_0 in the set $\mathcal{Q}_0$ such that $E_{Q_0}(X_i) = C_i$ for all $i \in I$. This is a very restrictive condition which will be often not satisfied.

We will soften this condition replacing it by the existence of a probability measure in the closed convex hull of $\mathcal{Q}_0$ (for the weak* topology) such that $E_{Q_0}(X_i) = C_i$ for all $i \in I$.

Recall that the weak* topology on the dual space E^* of a Banach space E is the $\sigma(E^*, E)$ topology and that from Banach Alaoglu theorem the unit ball of the dual space is compact for the weak* topology [Dunford et al. (1958)].

We give then the following definition:

Definition 4.2 *We say that a pricing function* Π *is subordinated to* $(\mathcal{Q}_0, \mathcal{Q}_1)$ *if there is a family* $\tilde{\mathcal{Q}}$ *of probability measures contained in the closed convex hull of* $\mathcal{Q}_0 \; U \; \mathcal{Q}_1$ *(for the weak* topology) such that the intersection of* $\tilde{\mathcal{Q}}$ *with the closed convex hull of* $\mathcal{Q}_0$ *is non empty; and a penalty function* α *defined on* $\tilde{\mathcal{Q}}$ *with values in* $I\!R+$ *such that* $\alpha(Q) = 0$ *for all* Q *in the closed convex hull of* $\mathcal{Q}_0$ *such that*

$$\forall X \in \mathcal{X} \;\; \Pi(X) = \min_{Q \in \tilde{\mathcal{Q}}} (E_Q(X) + \alpha(Q))$$

In case where $\mathcal{Q}_1 = \emptyset$, a pricing function subordinated to $(\mathcal{Q}_0, \emptyset)$ will be called a pricing function subordinated to $\mathcal{Q}_0$.

Remark: We impose that the penalty function is equal to 0 on the elements of $\tilde{\mathcal{Q}}$ belonging to the closed convex hull of $\mathcal{Q}_0$ because we want that these measures are valuation measures. Now we want to study the existence of an admissible pricing function subordinated to $(\mathcal{Q}_0, \mathcal{Q}_1)$.

Following [Carr et al. (2001)], we define the notion of No Strictly Acceptable Opportunity (N.S.A.O.). Our definition is not exactly the same - it refers only to $\mathcal{Q}_0$ and not to $\mathcal{Q}_1$. In the case considered in [Carr et al. (2001)], the set $\mathcal{Q}_1$ is finite, and as we will see with the next lemma, the two definitions are equivalent in that particular case.

Definition 4.3 *We say that there is No Strictly Acceptable Opportunity (N.S.A.O.) with respect to the family of probability measures* $\mathcal{Q}_0$ *if there is no family* $(\alpha_i)_{i \in I}$ *such that* $\sum_{i \in I} \alpha_i C_i = 0$ *and* $\sum_{i \in I} \alpha_i X_i \in \mathcal{A}^+_{\mathcal{Q}_0}$.

The next lemma will prove that our definition coincides with that of [Carr et al. (2001)] in many cases and in particular in the context of their study.

Lemma 4.1 *Assume that the penalty function δ is bounded from below on $\mathcal{Q}_1$ by a strictly positive constant a (this is for exemple the case when $\mathcal{Q}_1$ is finite or when the function δ is strictly positive and continuous on the weak* closure of $\mathcal{Q}_1$).*

There is N.S.A.O. with respect to $\mathcal{Q}_0$ if and only if there is no family $(\alpha_i)_{i\in I}$ such that $\sum_{i\in I}\alpha_i C_i = 0$ and $\sum_{i\in I}\alpha_i X_i \in \mathcal{A}^+_{(\mathcal{Q},\delta)}$.

Proof:

i) As $\mathcal{A}^+_{(\mathcal{Q},\delta)}$ is a subset of $\mathcal{A}^+_{\mathcal{Q}_0}$, it is obvious that if there is N.S.A.O. with respect to $\mathcal{Q}_0$, there is no family $(\alpha_i)_{i\in I}$ such that $\sum_{i\in I}\alpha_i C_i = 0$ and $\sum_{i\in I}\alpha_i X_i \in \mathcal{A}^+_{(\mathcal{Q},\delta)}$.

ii) Conversely assume that there is no family $(\beta_i)_{i\in I}$ such that $\sum_{i\in I}\beta_i C_i = 0$ and $\sum_{i\in I}\beta_i X_i \in \mathcal{A}^+_{(\mathcal{Q},\delta)}$.

Assume that there is a family $(\alpha_i)_{i\in I}$ such that $\sum_{i\in I}\alpha_i C_i = 0$ and $\sum_{i\in I}\alpha_i X_i \in \mathcal{A}^+_{\mathcal{Q}_0}$.

Denote $\overline{\mathcal{Q}_1}$ the weak* closure of $\mathcal{Q}_1$. $\{E_Q(\sum_{i\in I}\alpha_i X_i);\quad Q \in \overline{\mathcal{Q}_1}\}$ is compact. So it has a minimum m.

Either $m \geq 0$; then $E_Q(\sum_{i\in I}\alpha_i X_i) \geq -\delta(Q)$ for all $Q \in \mathcal{Q}_1$.

Or $m < 0$; then there is $\lambda > 0$, such that $\lambda m > -a$.
Then $E_Q(\lambda(\sum_{i\in I}\alpha_i X_i)) > -\delta(Q)$ for all $Q \in \mathcal{Q}_1$.

Hence $\sum_{i\in I}\lambda\alpha_i C_i = 0$ and $\sum_{i\in I}\lambda\alpha_i X_i \in \mathcal{A}^+_{(\mathcal{Q},\delta)}$.

Contradiction.

q.e.d.

We want now to prove an equivalent of the first fundamental theorem of pricing theory [Harrison et al. (1979)]. More precisely we want to prove the equivalence between N.S.A.O. and the existence of an admissible pricing function subordinated to the closed convex hull of $\mathcal{Q}_0$.

First of all notice that if there exists an admissible pricing function, then there is N.S.A.O. with respect to the family of probability measures whose penalty function is equal to zero. More precisely:

Lemma 4.2 *Let Π an admissible pricing function. Consider its representation: $\Pi(X) = \min_{Q\in\mathcal{Q}}(E_Q(X) + \alpha(Q))$.*

Denote $\mathcal{Q}_0 = \{Q \in \mathcal{Q} \ / \ \alpha(Q) = 0\}$. There is N.S.A.O. with respect to $\mathcal{Q}_0$.

Proof:

Consider a family $(\alpha_i)_{i\in I}$ such that $\sum_{i\in I}\alpha_i X_i \in \mathcal{A}^+_{\mathcal{Q}_0}$. There is $Q_0 \in \mathcal{Q}_0$ such that $\sum_{i\in I}\alpha_i E_{Q_0}(X_i) > 0$. On the other end $E_{Q_0}(X_i) = C_i$. So $\sum_{i\in I}\alpha_i C_i > 0$.

q.e.d.

4.1 First fundamental theorem in case of a finite number of valuation measures

In this subsection, we prove the first fundamental theorem under the hypothesis that the set $\mathcal{Q}_0$ of valuation measures is finite and without restriction on Ω. It generalizes the first fundamental theorem proved in [Carr et al.(2001)].

Theorem 4.3 *Assume that $\mathcal{Q}_0$ is a finite set. The economy satisfies N.S.A.O. if and only if there is a probability measure Q_0 which is a strictly convex combination of the elements of $\mathcal{Q}_0$ such that for all i in I, $E_{Q_0}(X_i) = C_i$.*

Proof:

i) Assume first that such a probability measure Q_0 exists. $Q_0 = \sum_{Q\in\mathcal{Q}_0}\lambda_Q Q$ for some $\lambda_Q > 0$ such that $\sum_{Q\in\mathcal{Q}_0}\lambda_Q = 1$. Let $(\alpha_i)_{i\in I}$ be such that $\sum_{i\in I}\alpha_i X_i \in \mathcal{A}^+_{\mathcal{Q}_0}$

$$\sum_{i\in I}\alpha_i C_i = E_{Q_0}(\sum_{i\in I}\alpha_i X_i) = \sum_{Q\in\mathcal{Q}_0}(\lambda_Q E_Q(\sum_{i\in I}\alpha_i X_i))$$

Each term of this sum is non negative and at least one of them is strictly positive it follows that $\sum_{i\in I}\alpha_i C_i > 0$.

And N.S.A.O. is satisfied.

ii) Conversely:

Denote

$$\mathcal{C}_1 = \{(\sum_{i\in I}\alpha_i E_Q(X_i))_{Q\in\mathcal{Q}_0} \;/\; \sum_{i\in I}\alpha_i C_i = 0\}$$

and

$$\mathcal{C}_2 = \{(\gamma_Q)_{Q\in\mathcal{Q}_0} \;/\; \gamma_Q \geq 0 \;\forall Q\in\mathcal{Q}_0 \text{ and } \sum_{Q\in\mathcal{Q}_0}\gamma_Q = 1\}$$

As $\mathcal{Q}_0$ is finite, $\mathcal{C}_2$ is a convex compact subset of $\mathbb{R}^{\mathcal{Q}_0}$. Let $L : \mathbb{R}^I \to \mathbb{R}^{\mathcal{Q}_0}$ be the linear map defined by $L((\alpha_i)_{i\in I}) = (\sum_{i\in I}\alpha_i E_Q(X_i)))_{Q\in\mathcal{Q}_0}$.

$\mathcal{C}_1$ is the image by L of the hyperplane $H = \{(\alpha_i)_{i\in I} \ / \ \sum_{i\in I} \alpha_i C_i = 0\}$ and therefore $\mathcal{C}_1$ is a linear subspace of $I\!R^{\mathcal{Q}_0}$ and so it is closed (because it is of finite dimension).

By hypothesis $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. Hence from Hahn Banach theorem, there is a linear form F defined on $I\!R^{\mathcal{Q}_0}$ such that $\mathcal{C}_1$ is contained into $KerF$ and such that $\forall u \in \mathcal{C}_2 \ \ F(u) > 0$.

F is a linear form so there are $(\beta_Q)_{Q\in\mathcal{Q}_0}$ such that

$$F((Y_Q)_{Q\in\mathcal{Q}_0}) = \sum_{Q\in\mathcal{Q}_0} \beta_Q Y_Q$$

$$\sum_{i\in I} \alpha_i C_i = 0 \ \ (i.e.(\alpha_i)_{i\in I} \in H) \ \ implies \sum_{Q\in\mathcal{Q}_0} \beta_Q \sum_{i\in I} \alpha_i E_Q(X_i) = 0$$

$$i.e. \sum_{i\in I} \alpha_i (\sum_{Q\in\mathcal{Q}_0} \beta_Q E_Q(X_i)) = 0$$

But H is the Kernel of a linear form unique up to a scalar. It follows that there is a real number γ such that

$$\forall i \in I, \ \gamma C_i = \sum_{Q\in\mathcal{Q}_0} \beta_Q E_Q(X_i) \tag{4}$$

Furthermore for every $u \in \mathcal{C}_2$, $F(u) > 0$. So for every $Q \in \mathcal{Q}_0$ with $\gamma_Q = 1$ and $\gamma_{Q'} = 0$ for $Q \neq Q'$, we get: $\beta_Q > 0 \ \forall Q \in \mathcal{Q}_0$.

Applying now the equality (4) for the non risky asset X_{i_0}, we get $\gamma = \sum_{Q\in\mathcal{Q}_0} \beta_Q$ It follows that $\gamma > 0$ and that if we put $\forall Q \ \beta'_Q = \frac{\beta_Q}{\gamma}$, then

$\beta'_Q > 0 \ \ \sum_{Q\in\mathcal{Q}_0} \beta'_Q = 1$ and the probability measure $Q_0 = \sum_{Q\in\mathcal{Q}_0} \beta'_Q Q$ satisfies all the required conditions.

q.e.d.

Remark: When $\mathcal{Q}_0$ is not finite but when the closed convex hull of $\mathcal{Q}_0$ has a finite number of extremal points, we can do the same replacing $\mathcal{Q}_0$ by the set of these extremal points.

In the next subsection, we study the general case.

4.2 First fundamental theorem for any closed set of valuation measures

In all this subsection, the set of valuation measures is any set of probability measures on $(\Omega, \mathcal{G})$ closed for the weak* topology.

Theorem 4.4 *Assume that $\mathcal{Q}_0$ is a set of probability measures on $(\Omega, \mathcal{G})$ closed for the weak* topology. Assume that there is N.S.A.O. with respect to $\mathcal{Q}_0$.*

There is a probability measure Q_0 in the convex hull of $\mathcal{Q}_0$ such that for all $i \in I$, $E_{Q_0}(X_i) = C_i$. E_{Q_0} is thus a linear admissible pricing function.

Proof:

In all this proof we consider the weak * topology on the dual of the Banach space E of all bounded $(\Omega, \mathcal{G})$ measurable functions.

Denote C_0 the convex hull of $\mathcal{Q}_0$ for the weak* topology. C_0 is compact and is a subset of the set of probability measures on $(\Omega, \mathcal{G})$

We adapt the proof of Theorem 4.3 to the fact that $\mathcal{Q}_0$ is no more finite. For this we replace $I\!R_0^{\mathcal{Q}}$ by the set $\mathcal{C}(C_0, I\!R)$ of continuous maps from C_0 to $I\!R$. We replace also the linear map L of the previous proof by $L : I\!R^I \to \mathcal{C}(C_0, I\!R)$ defined by $L((\alpha_i)_{i\in I})(Q) = E_Q(\sum_{i\in I} \alpha_i X_i)$.

As in the previous proof $\mathcal{C}_1$ is the image by L of the hyperplane $H = \{(\alpha_i)_{i\in I} \;/\; \sum_{i\in I} \alpha_i C_i = 0\}$. Therefore $\mathcal{C}_1$ is a linear subspace of $\mathcal{C}(C_0, I\!R)$.

Denote now $\mathcal{C}_2$ the set of continuous maps from C_0 into $I\!R_+$ which are non equal to 0. By N.S.A.O. $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. The interior of $\mathcal{C}_2$ is non empty. It follows that the linear space $\mathcal{C}_1$ is not dense in $\mathcal{C}(C_0, I\!R)$ and therefore from Hahn Banach theorem, it is contained in a closed hyperplane= $KerF$ for a continuous linear form F on $\mathcal{C}(C_0, I\!R)$ such that $\forall u \in \mathcal{C}_2$, $F(u) \geq 0$.

Now from the Riesz representation theorem [Rudin W. (1987)], there is a bounded measure μ on C_0 such that $F(X) = \int_{C_0} X(Q) d\mu(Q)$ $\forall X \in \mathcal{C}(C_0, I\!R)$.

$\forall X \in \mathcal{C}_2$, $F(X) \geq 0$. So μ is positive. Furthermore FoL is a linear form null on the hyperplane H. As there is a unique non zero linear form on $I\!R^I$ up to a scalar which is equal to zero on H, we get the existence of $\gamma \in I\!R$ such that

$$\int_{C_0} E_Q(X_i) d\mu(Q) = \gamma C_i \quad \forall i \in I$$

$\mu \neq 0$ so $\gamma \neq 0$ (considering the non risky asset $X_0 = 1$).

Consider now Q_0 defined on $\mathcal{G}$ by: $Q_0(A) = \int\limits_{C_0} Q(A)\frac{d\mu(Q)}{\gamma}$. Q_0 belongs to the convex hull of C_0 i.e. to C_0. Q_0 is then a probability measure on $(\Omega, \mathcal{G})$ which satisfies the required properties.

q.e.d.

Remark: As we will see in the following lemma, the condition that $\mathcal{Q}_0$ is a set of probability measures on $(\Omega, \mathcal{G})$ closed for the weak* topology is not restrictive if we want to define a price function on a separable closed subalgebra A of the algebra of bounded $(\Omega, \mathcal{G})$ measurable functions. Furthermore A is separable as soon as it is generated by a numerable family of functions, and this is always the case if A is the algebra of continuous functions in the $(X_i)_{i\in I}$.

Lemma 4.5 *Assume that A is a closed subalgebra of the Banach algebra E of bounded $(\Omega, \mathcal{G})$ measurable functions. Assume that A is separable. The closure of $\{(E_Q)|_A;\ Q \in \mathcal{Q}_0\}$ for the weak* topology in the dual of A is always contained in $\{(E_P)|_A;\ P$ probability measure on $(\Omega, \mathcal{G})\}$.*

Proof:

From [Dunford et al.(1958)],, the unit sphere S^* of the dual A^* of A is metrisable (as A is separable) and compact.

Each element ϕ in the closure of $\{(E_Q)|_A;\ Q \in \mathcal{Q}_0\}$ is then limit of a sequence $\{(E_{Q_n})|_A;\ Q_n \in \mathcal{Q}_0\}$. Q_n is a sequence in the unit ball of the dual space E^*. This unit ball is a compact set for the weak* topology. Therefore we can extract from the sequence Q_n a converging subsequence $Q_{\psi(n)}$. From Vitali Hahn Saks Theorem [Dellacherie et al.(1975)] its limit is a probability measure $\tilde{Q}$ on $(\Omega, \mathcal{G})$ and from unicity of the limit it follows that $\phi = (E_{\tilde{Q}})|A$.

q.e.d.

We have proved, when N.S.A.O. is satisfied with respect to $\mathcal{Q}_0$, the existence of linear admissible pricing functions subordinated to $\mathcal{Q}_0$(and then also the existence of admissible pricing functions subordinated to $(\mathcal{Q}_0, \mathcal{Q}_1)$ for every $\mathcal{Q}_1$). We will study in the section 6 the general admissible pricing functions subordinated to $(\mathcal{Q}_0, \mathcal{Q}_1)$.

5 First fundamental theorem in the general case of incomplete markets

As in the preceding section we consider two families of probability measures $\mathcal{Q}_0$ and $\mathcal{Q}_1$ with a penalty function δ equal to 0 on $\mathcal{Q}_0$.

The difference with the preceding section is that we assume now that the market gives for some financial instruments $(X_i)_{i\in I}$ bid and ask prices C_i^{bid} and C_i^{ask} (if for some of the X_i a price C_i is given we simply put $C_i^{bid} = C_i^{ask} = C_i$). We assume also that in the market there is a non risky asset X_0.

We denote $I_0 = \{i \in I \ / \ C_i^{bid} = C_i^{ask}\}$.

We keep the same definitions of acceptable and strictly acceptable positions as in section 4.

We extend to that context the notion of no strictly acceptable opportunity:

Definition 5.1 *We say that there is No Strictly Acceptable Opportunity (N.S.A.O.) with respect to the family of probability measures $\mathcal{Q}_0$ if there is no family $(\alpha_i, \beta_i)_{i\in I} \in (I\!R^+)^{2I}$ such that $\sum_{i\in I} \alpha_i C_i^{bid} - \sum_{i\in I} \beta_i C_i^{ask} \geq 0$ and $\sum_{i\in I}(\beta_i - \alpha_i)X_i \in \mathcal{A}^+_{\mathcal{Q}_0}$.*

Remark: this notion of N.S.A.O. generalizes the notion of N.S.A.O. given in the previous section. That is, if N.S.A.O. is satisfied (in the new sense) then the economy restricted to the financial instruments $(X_i)_{i\in I_0}$ satisfies N.S.A.O. in the sense of the previous definition.

As in the preceding section we want now to prove an equivalent of the first fundamental theorem of Harrison and Kreps [Harrison et al.(1979)]. More precisely we want to prove that N.S.A.O. implies the existence of an admissible pricing function which belongs to the closed convex hull of $(E_Q)_{Q\in\mathcal{Q}_0}$.

As in the previous case we begin with a lemma:

Lemma 5.1 *Assume that the penalty function δ is bounded from below on $\mathcal{Q}_1$ by a strictly positive constant a.*

There is N.S.A.O. with respect to $\mathcal{Q}_0$ if and only if there is no family $(\alpha_i, \beta_i)_{i\in I} \in (I\!R^+)^{2I}$ such that $\sum_{i\in I} \alpha_i C_i^{bid} - \sum_{i\in I} \beta_i C_i^{ask} \geq 0$ and $\sum_{i\in I}(\beta_i - \alpha_i)X_i \in \mathcal{A}^+_{(\mathcal{Q},\delta)}$.

The proof of this lemma is exactly the same as the proof of Lemma 4.1.

As in the preceding section we prove that if Π is an admissible pricing function, there is N.S.A.O. with respect to the family of probability measures whose penalty function is equal to zero.

Lemma 5.2 *Let Π an admissible pricing function. Consider the representation: $\Pi(X) = \min_{Q\in\mathcal{Q}}(E_Q(X) + \alpha(Q))$.*

Denote $\mathcal{Q}_0 = \{Q \in \mathcal{Q} \ / \ \alpha(Q) = 0\}$. There is N.S.A.O. with respect to $\mathcal{Q}_0$.

Proof:

Consider a family $((\alpha_i, \beta_i)_{i\in I}$ such that $\sum_{i\in I}(\beta_i - \alpha_i)X_i \in \mathcal{A}^+_{\mathcal{Q}_0}$. There is $Q_0 \in \mathcal{Q}_0$ such that $\sum_{i\in I}(\beta_i - \alpha_i)E_{Q_0}(X_i) > 0$. On the other end $C_i^{bid} \leq E_{Q_0}(X_i) \leq C_i^{ask}$. So $\sum_{i\in I}\beta_i C_i^{ask} > \sum_{i\in I}\alpha_i C_i^{bid}$.

q.e.d.

Now we will prove the equivalent of the first fundamental theorem:

As in the previous section we prove it first in the case where the set of valuation measures is finite.

5.1 Case of a finite number of valuation measures

Theorem 5.3 *Assume that $\mathcal{Q}_0$ is a finite set. Assume that there is N.S.A.O. with respect to $\mathcal{Q}_0$. There is a probability measure Q_0 which is a convex combination of elements of $\mathcal{Q}_0$ such that E_{Q_0} is an admissible pricing function (i.e. for all $i \in I$, $C_i^{bid} \leq E_{Q_0}(X_i) \leq C_i^{ask}$).*

Proof:

We have to adapt the proof of theorem4.3 to that new context.

There is no family $(\alpha_i, \beta_i)_{i\in I} \in (I\!R^+)^{2I}$ such that $\sum_{i\in I}\alpha_i C_i^{bid} - \sum_{i\in I}\beta_i C_i^{ask} \geq 0$ and $\sum_{i\in I}(\beta_i - \alpha_i)X_i \in \mathcal{A}^+_{\mathcal{Q}_0}$.

Denote

$$\mathcal{C} = \{(\alpha_i, \beta_i)_{i\in I} \in (I\!R^+)^{2I} \ / \ \sum_{i\in I}\alpha_i C_i^{bid} - \sum_{i\in I}\beta_i C_i^{ask} \geq 0\}$$

$\mathcal{C}$ is convex.

Consider the linear map $L : I\!R^{2I} \to I\!R^{\mathcal{Q}_0}$ defined by

$$L((\alpha_i, \beta_i)_{i\in I}) = (\sum_{i\in I}\beta_i E_Q(X_i) - \sum_{i\in I}\alpha_i E_Q(X_i))_{Q\in\mathcal{Q}_0}$$

L is linear and $\mathcal{C}$ is convex, it follows that $L(\mathcal{C})$ is convex. Consider

$$\mathcal{C}_2 = \{(\gamma_Q)_{Q\in\mathcal{Q}_0} \; / \; \gamma_Q \geq 0 \; \forall Q \in \mathcal{Q}_0 \; and \sum_{Q\in\mathcal{Q}_0} \gamma_Q > 0\}$$

$\mathcal{C}_2$ is a convex subset of $I\!R^{\mathcal{Q}_0}$.

$\mathcal{C}_2$ has an interior point because it contains the ball $\mathcal{B}(1, \frac{1}{2}) = \{(\alpha_Q)_{Q\in\mathcal{Q}_0} \; / \; |\alpha_Q - 1| < \frac{1}{2} \; \forall Q \in \mathcal{Q}_0\}$; and $\mathcal{C}_2$ is disjoint from $L(\mathcal{C})$ as N.S.A.O. is assumed.

So from the theorem of separation of convexes, there is an hyperplane separating $L(\mathcal{C})$ and $\mathcal{C}_2$; i.e. there is a non zero continuous linear form F on $I\!R^{\mathcal{Q}_0}$ and $\alpha \in I\!R$ such that for all $X \in \mathcal{C}_2$, $F(X) \geq \alpha$ and for all $X \in L(\mathcal{C})$, $F(X) \leq \alpha)$.

As $\mathcal{C}_2$ and $L(\mathcal{C})$ are cones, it follows that $\alpha = 0$ There are $(\beta_Q)_{Q\in\mathcal{Q}_0}$ such that $F((Y_Q)_{Q\in\mathcal{Q}_0}) = \sum_{Q\in\mathcal{Q}_0} \beta_Q Y_Q$.

Consider now $I_0 = \{i \in I \; / \; C_i^{bid} = C_i^{ask}\}$. We already know that I_0 is non empty because we have assumed that there is a non risky asset.

We continue the proof as in theorem4.3, restricting first our attention to the set of indices I_0.

Let $(\tilde{\alpha}_i)_{i\in I_0}$ a family such that $\sum_{i\in I_0} \tilde{\alpha}_i C_i = 0$ Let $I_1 = \{i \in I_0 \; / \; \tilde{\alpha}_i \geq 0\}$ and $I_2 = \{i \in I_0 \; / \; \tilde{\alpha}_i < 0\}$. Now put for all $i \in I$ $\alpha_i = \tilde{\alpha}_i$ if $i \in I_1$ and $\alpha_i = 0$ otherwise; and $\beta_i = -\tilde{\alpha}_i$ if $i \in I_2$ and $\beta_i = 0$ otherwise. So for all i, $(\alpha_i, \beta_i) \in I\!R+^2$ and furthermore $(\alpha_i, \beta_i)_{i\in I} \in \mathcal{C}$ and $(\beta_i, \alpha_i)_{i\in I} \in \mathcal{C}$. It follows that

$$FoL((\alpha_i, \beta_i)_{i\in I}) \leq 0$$

In the same way $FoL((\beta_i, \alpha_i)_{i\in I}) \leq 0$. So it is equal to 0 (indeed, $FoL((\beta_i, \alpha_i)_{i\in I}) = -FoL((\alpha_i, \beta_i)_{i\in I})$.

We have proved that $\sum_{i\in I_0} \tilde{\alpha}_i C_i = 0$ implies $\sum_{i\in I_0} \tilde{\alpha}_i \; (\sum_{Q\in\mathcal{Q}_0} \beta_Q E_Q(X_i)) = 0$. This means that the kernel of the first linear form (on $I\!R^{I_0}$) is contained in the kernel of the second linear form and then the second linear form is proportional to the first one; i.e. there is $\lambda \in I\!R$ such that for all $i \in I_0$, $\lambda C_i = \sum_{Q\in\mathcal{Q}_0} \beta_Q E_Q(X_i)$.

The interior of $\mathcal{C}_2$ is non empty and F is a non zero linear form positive on $\mathcal{C}_2$ so there is an element X in $\mathcal{C}_2$ such that $F(X) > 0$. Considering for each $P \in \mathcal{Q}_0$ the element $(\delta_{(P,Q)})_{Q\in\mathcal{Q}_0}$, we get that for all $P \in \mathcal{Q}_0$ $\beta_P \geq 0$ and that there is at least one $P \in \mathcal{Q}_0$ such that $\beta_P > 0$. Applying the equality $\lambda C_i = \sum_{Q\in\mathcal{Q}_0} \beta_Q E_Q(X_i)$ to the non risky asset, we get $\lambda > 0$.

So if we put $Q_0 = \frac{1}{\lambda} \sum_{Q\in\mathcal{Q}_0} \beta_Q Q$ then for all $i \in I_0$, $E_{Q_0}(X_i) = C_i$ and Q_0 is a convex combination of the elements of $\mathcal{Q}_0$.

Now it only remains to prove that for all $i \in I$, $C_i^{bid} \leq E_{Q_0}(X_i) \leq C_i^{ask}$. Choose $i_0 \in I_0$. Let $\tilde{i} \in I$. Let $\mu_{\tilde{i}} = \frac{C_{\tilde{i}}^{bid}}{C_{i_0}}$.

Define $\alpha_i = 1$ if $i = \tilde{i}$ and $\alpha_i = 0$ otherwise. Define $\beta_i = \mu_{\tilde{i}}$ if $i = i_0$ and $\beta_i = 0$ otherwise.

$(\alpha_i, \beta_i)_{i \in I} \in \mathcal{C}$ So $FoL((\alpha_i, \beta_i)_{i \in I}) \leq 0$ i.e.

$$\mu_{\tilde{i}} E_{Q_0}(X_{i_0}) - E_{Q_0}(X_{\tilde{i}}) \leq 0$$

Now $E_{Q_0}(X_{i_0}) = C_{i_0}$ and from definition of $\mu_{\tilde{i}}$ we get

$$E_{Q_0}(X_{\tilde{i}}) \;\geq\; C_{\tilde{i}}^{bid} \quad \forall \tilde{i} \in I$$

The other inequality

$$E_{Q_0}(X_{\tilde{i}}) \;\leq\; C_{\tilde{i}}^{ask} \quad \forall \tilde{i} \in I$$

is obtained in the same way.

q.e.d.

As in the previous section we can generalize this theorem without assuming that Q_0 is a finite set. This is the object of the next subsection.

5.2 First fundamental theorem in incomplete markets

In this subsection, as in section 4.2, $\mathcal{Q}_0$ is any set of probability measures closed for the weak* topology.

The proof of the first fundamental theorem of pricing in this general case involves topological arguments.

Theorem 5.4 *Assume that $\mathcal{Q}_0$ is a set of probability measures on $(\Omega, \mathcal{G})$ closed for the weak* topology. Assume that there is N.S.A.O. with respect to $\mathcal{Q}_0$. There is a probability measure Q_0 belonging to the convex hull of $\mathcal{Q}_0$ such that E_{Q_0} is an admissible pricing function (i.e. for all $i \in I$, $C_i^{bid} \leq E_{Q_0}(X_i) \leq C_i^{ask}$).*

Proof:

Denote C_0 the convex hull of $\mathcal{Q}_0$ for the weak* topology as in the proof of Theorem 4.4. The proof of this theorem follows the proof of the Theorem 5.3 and uses also parts of the proof of Theorem 4.4.

The changes are the following ones:

The linear map L in the proof of Theorem 5.3 is now defined from $\mathbb{R}^{2I}$ to $\mathcal{C}(C_0, \mathbb{R})$ (the set of continuous maps from C_0 to $\mathbb{R}$) by

$$L((\alpha_i, \beta_i)_{i\in I})(Q) = E_Q(\sum_{i\in I}(\beta_i - \alpha_i)X_i)$$

and now $\mathcal{C} = (\{(\alpha_i, \beta_i) \in \mathbb{R}_+^{2I} \ / \ \sum_{i\in I}(\alpha_i C_i^{bid} - \beta_i C_i^{ask}) \geq 0\}$. $L(\mathcal{C})$ is a convex cone (as the image of a convex cone by a linear map).

$\mathcal{C}_2$ is the set of continuous maps from C_0 into $\mathbb{R}_+$ which are non equal to zero as in the proof of Theorem 4.4. $\mathcal{C}_2$ is convex ; its interior is non empty and $L(\mathcal{C}) \cap \mathcal{C}_2 = \emptyset$. So from the theorem of separation of convexes, there is a continuous linear form F on $\mathcal{C}(C_0, \mathbb{R})$ and $\alpha \in \mathbb{R}$ such that $\forall X \in L(\mathcal{C})$ $F(X) \leq \alpha$ and $\forall X \in \mathcal{C}_2$ $F(X) \geq \alpha$.

$L(\mathcal{C})$ and $\mathcal{C}_2$ are cones so $\alpha = 0$.

Consider now as in the proof of Theorem 5.3. $I_0 = \{i \in I \ / \ C_i^{bid} = C_i^{ask}\}$. Exactly as in that proof, we get that $\sum_{i\in I_0} \tilde{\alpha}_i C_i = 0$ implies $F(f_{\tilde{\alpha}_i}) = 0$ where $f_{\tilde{\alpha}_i}$ is the continuous map on C_0 defined by $f_{\tilde{\alpha}_i}(Q) = E_Q(\sum_{i\in I_0} \tilde{\alpha}_i X_i)$.

We come back to the proof of Theorem 4.4. and using the Riesz representation theorem, we get a probability measure Q_0 belonging to C_0 such that for all $i \in I_0$ $E_{Q_0}(X_i) = C_i$.

Endly we prove that for all $i \in I$, $C_i^{bid} \leq E_{Q_0}(X_i) \leq C_i^{ask}$, exactly as in the proof of Theorem 5.3.

q.e.d.

The preceding theorem can also be expressed in the following way:

Corollary 5.5 *Assume that C_0 is a convex set of probability measures on $(\Omega, \mathcal{G})$ closed for the weak* topology. There is N.S.A.O. with respect to a closed subset of C_0 if and only if there is a probability measure Q_0 belonging C_0 such that E_{Q_0} is an admissible pricing function (i.e. for all $i \in I$, $C_i^{bid} \leq E_{Q_0}(X_i) \leq C_i^{ask}$).*

Proof:

This is a consequence of Theorem 5.4 and of the fact that if E_{Q_0} is an admissible pricing function, there is N.S.A.O. with respect to $\{Q_0\}$ (lemma 5.2).

5.3 Extension of the first fundamental theorem to the case of calibration on infinitely many financial instruments

In this subsection we extend the first fundamental theorem to the case where we assume that the market gives bid and ask prices for infinitely many financial instruments i.e. I is infinite numerable and even I_0 can be an infinite set.

We denote

$$I\!R_+^{2(I)} = \{(\alpha_i, \beta_i)_{i \in I} \;/\; (\alpha_i, \beta_i) \neq (0,0) \; only \; for \; a \; finite \; number \; of \; indices \; i \in I\}$$

The definition of No Strictly Acceptable Opportunity becomes the following one:

Definition 5.2 *There is No Strictly Acceptable Opportunity (N.S.A.O.) with respect to $\mathcal{Q}_0$ if there is no family $(\alpha_i, \beta_i)_{i\in I} \in (I\!R+)^{2(I)}$ such that $\sum_{i\in I} \alpha_i C_i^{bid} - \sum_{i\in I} \beta_i C_i^{ask} \geq 0$ and $\sum_{i\in I}(\beta_i - \alpha_i)X_i \in \mathcal{A}^+_{\mathcal{Q}_0}$.*

We prove also in that case the first fundamental theorem of pricing.

Theorem 5.6 *Assume that $\mathcal{Q}_0$ is a set of probability measures closed for the weak* topology. Assume that there is N.S.A.O. with respect to $\mathcal{Q}_0$. There is a probability measure Q_0 belonging to the convex hull of $\mathcal{Q}_0$ such that E_{Q_0} is an admissible price function (i.e. for all $i \in I$, $C_i^{bid} \leq E_{Q_0}(X_i) \leq C_i^{ask}$).*

The proof of Theorem 5.4 can be extended to the case where I is an infinite set replacing everywhere $I\!R^{2I}$ by $I\!R^{2(I)}$.

6 Maximal bid ask spread for the pricing functions subordinated to the family $(\mathcal{Q}_0, \mathcal{Q}_1)$ of probability measures

As in the two previous sections we assume that for a family of financial instruments $(X_i)_{i\in I}$ either a price or bid and ask prices are available in the market. Assume that I is finite. We assume that two families of probabilities $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are given.

Assuming that there is N.S.A.O. with respect to $\mathcal{Q}_0$, we have proved in sections 4 and 5 the existence of an admissible pricing function subordinated to $(\mathcal{Q}_0, \mathcal{Q}_1)$ which is linear.

Now we want to study all the admissible pricing functions subordinated to $(\mathcal{Q}_0, \mathcal{Q}_1)$.

We denote here I_0 a subset of I corresponding to assets considered as liquid. Necessarily I_0 is a subset of $\{i \in I \ / \ C_i^{bid} = C_i^{ask}\}$ (But the inclusion can be strict).

We then obtain the following characterization:

Denote $\mathcal{C}$ the closed convex hull of $\mathcal{Q}_0 \ U \ \mathcal{Q}_1$

Proposition 6.1 *Every pricing function* Π *subordinated to* $(\mathcal{Q}_0, \mathcal{Q}_1)$ *has a representation of the kind*

$$\forall X \in \mathcal{X} \ \ \Pi(X) = \min_{Q \in \mathcal{Q}}(E_Q(X) + \alpha(Q))$$

where $\mathcal{Q}$ *is a subset of* $\mathcal{C}$. *It is admissible if and only if it satisfies the conditions i) and ii) of Theorem 2.1. Furthermore the pricing function* Π *is perfectly liquid on* $(X_i)_{i \in I_0}$ *if and only if* $E_Q(X_i) = C_i$ *for all* $Q \in \mathcal{Q}$ *and all* $i \in I_0$.

Denote now $\tilde{\mathcal{Q}} = \{Q \in \mathcal{C} \ / \ E_Q(X_i) = C_i \ \forall i \in I_0\}$ and $\tilde{\mathcal{Q}}_0 = \{Q \in \tilde{\mathcal{Q}} \ / \ C_i^{bid} \leq E_Q(X_i) \leq C_i^{ask} \ \forall i \in I\}$

Proposition 6.2 *Assume that N.S.A.O. is satisfied with respect to* $\mathcal{Q}_0$. *There exist admissible pricing functions subordinated to* $(\mathcal{Q}_0, \mathcal{Q}_1)$ *perfectly liquid on* $(X_i)_{i \in I_0}$. *Among these pricing functions there is a minimal one denoted* Π_m.

$$\Pi_m = \min_{Q \in \tilde{\mathcal{Q}}}(E_Q(X) + \alpha_m(Q))$$

where $\alpha_m(Q) = \max(0, \max_i(C_i^{bid} - E_Q(X_i), E_Q(X_i) - C_i^{ask}))$

Every pricing function is less than

$$P_M = \max_{Q \in \tilde{\mathcal{Q}}_0}(E_Q(X))$$

The bid ask spread $[\Pi(X); -\Pi(-X)]$ *is always contained in* $[\Pi_m(X); -\Pi_m(-X)]$

Proof:

This is an easy consequence of Theorem 5.4 and of the preceding proposition.

We define now the notion of admissible completeness of the market:

Definition 6.1 *We say that the market is admissibly complete if there is only one element in $\tilde{Q}$ i.e. if the map $\Phi : \mathcal{C} \to I\!R^{I_0}$ defined by $\Phi(Q) = (E_Q(X_i))_{i \in I_0}$ is injective.*

Proposition 6.3 *If the market is admissibly complete there is at most one admissible pricing function Π subordinated to $(\mathcal{Q}_0, \mathcal{Q}_1)$ perfectly liquid on $(X_i)_{i \in I_0}$*

If the market is admissibly complete and satisfies N.S.A.O.there is exactly one admissible pricing function Π subordinated to $(\mathcal{Q}_0, \mathcal{Q}_1)$ perfectly liquid on $(X_i)_{i \in I_0}$.The unique pricing function is linear. For all $X \in \mathcal{X}$, $\Pi(X) = -\Pi(-X)$

Remark: In many cases, in an incomplete market, when there is no arbitrage, it is possible to find a family of probabilities $\mathcal{Q}_0$ such that there is N.S.A.O. with respect to $\mathcal{Q}_0$ and such that the market is admissibly complete.

7 Choice of investments

This section concerns some implications of our work for a market-maker or an investor who has to price derivatives and to make decisions of investment.

7.1 Constructing a pricing function

As in the previous sections, we assume that the bid and ask prices of some financial instruments $(X_i)_{i \in I}$ are available in the market. These financial instruments are considered as the reference financial instruments or the instruments used for calibration. An investor wants to price other financial instruments and has to make decisions of investment.

Assume now that the investor has his own preferred reference family of probabilities.

- This family may have been obtained by choosing models for the $(X_i)_{i \in I}$. For exemple in a case of a diffusion model where the uncertain volatility is allowed to vary inside a band as in [Avellaneda et al. (1995)], the corresponding set of probability measures is infinite and the probabilities are not all equivalent. This requires to consider the case of an infinite set of probability measures, as we have done in sections 4.2 and 5.2.

- In the case where the financial instruments considered are partitionned into two subsets $(X_i)_{i \in I_0}$ and $(X_i)_{i \in I_1}$, and the $(X_i)_{i \in I_1}$ are derivatives on the

$(X_i)_{i\in I_0}$, a natural family is the set of probability measures for which the $(X_i)_{i\in I_0}$ are martingales.

We denote C_0 a reference family of probability measures. We assume that C_0 is convex and closed for the weak* topology. Assume that there is N.S.A.O. with respect to a subset of C_0. We know from the theorems of sections 4 and 5 the existence of probability measures $Q \in C_0$ such that E_Q is a pricing function (i.e. such that $\forall i \in I \quad C_i^{bid} \leq E_Q(X_i) \leq C_i^{ask}$). Denote $\mathcal{M}_0$ the set of such probabilities. From Theorem 2.1, the choice of an admissible pricing function Π is then the choice of:

- a subset $\tilde{\mathcal{Q}}_0$ of $\mathcal{M}_0$

- another set of probabilities $\mathcal{Q}_1$, and for each $Q \in \mathcal{Q}_1$ of a penalty $\alpha(Q)$ such that $\alpha(Q) \geq \sup(0, \sup_i(C_i^{bid} - E_Q(X_i), E_Q(X_i) - C_i^{ask}))$.

Then

$$\Pi(X) = \min_{Q\in\tilde{\mathcal{Q}}_0 U \mathcal{Q}_1} (E_Q(X) + \alpha(Q))$$

Denote $I_0 = \{i \in I \; / \; \Pi(\lambda X) = \lambda\Pi(X) \; \forall\lambda \in I\!R\}$.

Recall that from Proposition 3.2,

$$\Pi(X) = \max_{\{(X-\sum_{i\in I_0}\alpha_i X_i)\in\mathcal{A}_\Pi\}} \sum_{i\in I_0} \alpha_i C_i$$

$$-\Pi(-X) = \min_{\{(\sum_{i\in I_0}\beta_i X_i - X)\in\mathcal{A}_\Pi\}} \sum_{i\in I_0} \beta_i C_i$$

where $\mathcal{A}_\Pi = \{Y \; / \; \forall Q \in \tilde{\mathcal{Q}}_0 \; E_Q(Y) \geq 0 \text{ and } \forall Q \in \mathcal{Q}_1 \; E_Q(Y) \geq \alpha(Q)\}$.

7.2 What to do when the preferred reference family doesn't satisfy N.S.A.O.?

In some cases, the investor has his preferred reference family of probability measures, which may not satisfy the property N.S.A.O. In such case, the investor has two main choices in order to construct a pricing function.

Either the investor considers a bigger family satisfying N.S.A.O. This is always possible if there is no arbitrage in the following sense: There is no family $(\alpha_i, \beta_i)_{i\in I} \in (I\!R^+)^{2I}$ such that $\sum_{i\in I}\alpha_i C_i^{bid} - \sum_{i\in I}\beta_i C_i^{ask} \geq 0$ and $\sum_{i\in I}(\beta_i - \alpha_i)X_i$ is a non negative function non equal to zero.

Or the investor prefers to keep his reference family of probability measures, but considers that the prices (or the bid and ask prices) of some of the reference financial instruments are not enough significant. Then he can remove some assets from the set of assets used for calibration. He can also consider that the bid ask spread observed in the market for some financial instruments i are too small and he can replace C_i^{bid} (resp.C_i^{ask}) by $C_i^{bid} - \epsilon_i$ (resp.$C_i^{ask} + \epsilon_i$) (this is even possible for some i for which a price is observed in the market). These changes have to be done in such a way that for the restricted set of calibration and the new bid and ask prices N.S.A.O. is satisfied with respect to the reference family.

When the investor has constructed and choosen his admissible pricing function Π, he can assign a bid price and an ask price to every financial instrument. He will accept to sell (resp. buy) a position X only at a price greater or equal to $\Pi(X)$ (resp less or equal to $-\Pi(-X)$.

8 Conclusion

In this paper we have two main parts. In the first one, we have defined in the context of incomplete markets a general notion of pricing function Π which assigns to every financial position X a bid price $\Pi(X)$ and an ask price $-\Pi(-X)$. We say that a pricing function is admissible if it is compatible with the prices (or bid and ask prices) observed in the market for some family of financial instruments $(X_i)_{i\in I}$. Taking into account conditions of illiquidity and of diversification of risk in incomplete market we have derived the condition that Π is concave.

Using then the theory of convex risk measures [Föllmer et al.(2002)(b)], we have proved the main result of this first part: each admissible pricing function has a representation of the kind

$$\Pi(X) = \min_{Q\in\mathcal{Q}}(E_Q(X) + \alpha(Q))$$

where $\mathcal{Q}$ is a set of probability measures, and the set of probability measures and of penalty functions have to satisfy the two fundamental following properties:

$\mathcal{Q}_0 = \{Q \in \mathcal{Q} \;/\; \alpha(Q) = 0\}$ is non empty;

$\forall Q \in \mathcal{Q}$, $\alpha(Q) \geq \sup(0, \sup_i(C_i^{bid} - E_Q(X_i), E_Q(X_i) - C_i^{ask}))$.

In the second part of the paper we have addressed the question of characterizing the families of probability measures for which it is possible to construct an admissible pricing function.

Our main result of this second part is the proof, in that context of incomplete markets, of a fundamental theorem of asset pricing. Generalizing the notion of N.S.A.O. introduced in [Carr et al (2001)] to the case where for some financial instruments used for calibration only bid and ask prices are available in the market, we have proved the following result:

Assume that C_0 is a closed (for the weak* topology) convex set of probabilities. There is N.S.A.O. with respect to a subset of C_0 if and only if there exists an admissible pricing function (on the algebra generated by the X_i) defined from the set of probability measures C_0.

Hence there are constraints on the set of valuation probability measures that one can use in order to construct an admissible pricing function defined on the algebra generated by the $(X_i)_{i \in I}$.

The extension of the second part of the paper (condition of N.S.A.O. and first fundamental theorem of pricing) to a dynamic setting will be the subject of future work, using the notion of conditional risk measure [Bion-Nadal (2004)].

References

[Artzner et al.(1999)] Artzner P., Delbaen F., Eber J.M., Heath D. (1999), Coherent measures of risk, Mathematical Finance, 9 (3), pp. 203-228

[Avellaneda et al. (1995)] Avellaneda M., Levy A., Paras A. (1995), Pricing and hedging derivative securities in markets with uncertain volatilities, Applied Mathematical Finance, 2, pp. 73-88.

[Avellaneda et al.(1996)] Avellaneda M., Paras A. (1996), Managing the Volatility Risk of Portfolios of Derivative Securities: The Lagrangian Uncertain Volatility Model, Applied Mathematical Finance, 3, pp. 31-52.

[Biagini et al. (2005)] Biagini S., Frittelli M. (2005), Utility Maximization in incomplete markets for unbounded processes, to appear in Finance and Stochastics.

[Bion-Nadal (2004)] Bion-Nadal J. (2004), Conditional risk measure and robust representation of convex conditional risk measures, preprint CMAP 557.

[Carr et al (2001)] Carr P., Geman H. and Madan D. (2001), Pricing and hedging in incomplete markets, Journal of financial economics, 62 pp. 131-167

[Cont (2005)] Cont R. (2005), Model uncertainty and its impact on the pricing of derivative instruments, to appear in Mathematical Finance

[Coquet et al. (2002)] Coquet F., Hu Y.,Memin J., Peng S. (2002): Filtration-consistent nonlinear expectations and related g-expectations, Probability Theory and Related Fields, 123, 1-27.

[Dunford et al.(1958)] Dunford N., Schwartz J. (1958), Linear Operators. Part I: General Theory. Interscience Publishers, New York.

[El Karoui N., Quenez M. C.(1995)] El Karoui N., Quenez M. C. (1995), Dynamic programming and the pricing of contingent claims in incomplete markets, SIAM Journal of Control and Optimization, 33 (1), pp. 29-66.

[El Karoui et al.(1997)] El Karoui N., Peng S., Quenez M. C. (1997), Backward stochastic differential equations in finance, Mathematical Finance, 7, pp 1-71.

[El Karoui N., Barrieu P.(2005)] El Karoui N., Barrieu P. (2005), Inf-convolution of risk measures and optimal risk transfer, Finance and Stochastics, 9, pp 269-298.

[Föllmer et al.(2002)] Föllmer H. and Schied A. (2002),Convex measures of risk and trading constraints, Finance and Stochastics, 6, pp 429-447.

[Föllmer et al.(2002)(b)] Föllmer H. and Schied A. (2002), Stochastic Finance:An introduction in discrete time , de Gruyter studies in mathematics 27.

[Harrison et al.(1979)] Harrison J. and Kreps D. (1979), Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory 20, pp 381-408.

[Rudin W.(19)] Rudin W., Real and complex analysis, New York: McGraw-hill, third edition.

[Schachermeyer W.(2001)] Schachermeyer W. (2001), Optimal Investment in Incomplete Financial Markets, Mathematical Finance:Bachelier Congress 2000, Springer, pp. 427-462

[Schachermeyer W.(2004)] Schachermeyer W. (2004), Utility Maximisation in Incomplete Markets, lectures given at the CIME-EMS Summer School in Bressanone/Buxen Italy July 6-12 2003, Springer Lecture Notes in Mathematics, 1856, pp. 225-288