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The modeling of deformable bodies with frictionless (self-)contacts

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Abstract

We propose a mathematical model for m-dimensional nonlinear hyperelastic bodies moving in $\mathbb{R}^n$ for all $m \leq n$, which enables frictionless contacts or self-contacts but not (self-)intersection. To this end, we define a set of admissible deformations, and prove the existence of at least one minimizer of the energy, under suitable assumptions on the stored energy. Moreover, we give a partial equivalence result between the minimization problem and the Euler-Lagrange equations.

1. Introduction

Although contacts and self-contacts arise in many practical situations in nonlinear problems involving deformable bodies, surprisingly they have not been extensively studied, at least from a theoretical point of view. Ciarlet and Nečas [4] proposed a model for n-dimensional hyperelastic bodies moving in $\mathbb{R}^n$ (see also Tang Qi [12], Giaguinta & al [5][6]). Their modeling allows frictionless contacts or self-contacts and ensures almost everywhere injectivity of the deformations. We address the case of m-dimensional deformable bodies moving in $\mathbb{R}^n$ for all $m \leq n$.

For brevity and clarity, we will only consider the case of hyperelastic bodies. Nevertheless, most of our work could be applied to other types of materials. One way of describing a system of elastics bodies is to use a variational approach. Minimizers of the energy over the set of admissible deformations are stable equilibrium states. The challenge is to define a good set of admissible deformations which allows for (self-)contacts while forbidding (self-)intersection. First of all, this set must be chosen in such a way that problems that are well-posed without the non self-intersection constraint remain well posed with the constraint added. Secondly, we must be able (maybe under reasonable assumptions on the equilibrium state) to show that solutions of the minimization problem fulfill the expected Euler-Lagrange equations. The importance of the first condition is obvious. The second one is no less important: It ensures that we actually solve the right problem.

Let M be a submanifold of $\mathbb{R}^n$. The reference injection of M into $\mathbb{R}^n$ is denoted $j_{_M}$. We define the set of admissible deformations simply as the closure of the embeddings isotopic to the reference injection $j_{_M}$ for an appropriate topology. With such a definition, it is straightforward to prove the existence of minimizers of the energy (under suitable assumptions on the stored energy). Nevertheless, the implicit definition of the admissible set of deformation has at least one drawback: it is not obvious to recover the Euler-Lagrange equations. In order to achieve such a goal, it is usual to perform small variations around the minimizer in the admissible set. Then, the differentiability of the energy functional leads to the Euler-Lagrange equations. Unfortunately, the definition of the admissible set does not give us an explicit description of the neighborhood of a minimizer, hence of the allowed variations. To overcome this problem, we prove that admissible deformations fulfill an explicit topological constraint. More precisely, we show that the self-intersections of a deformation φ are, at least partially, described by a topological invariant $\phi(\varphi)$. A deformation such that $\phi(\varphi)$ is equal to $\phi(j_{_M})$ is called ϕ-admissible. We prove that any admissible deformation is ϕ-admissible. Conversely, in the cases $dim(M) = n$ or $dim(M) = 1$ and $n = 2$, every immersion which is ϕ-admissible belongs to the admissible set. This allows us to prove that any immersion which is a minimizer of the energy fulfills the expected Euler-Lagrange equations.

The plan of the paper is as follows. We first recall some basic definitions of differential geometry. Then, we give a description of the self-intersections of a deformation and define the ϕ-admissible set. The main properties of the ϕ-admissible set are studied in 3.3. In section 4, we set up the minimization problem for nonlinear hyperelastic bodies and prove the existence of at least one solution. Section 5 is devoted to the study of the equivalence between the minimization problem and the Euler-Lagrange equations. We consider the case of n-dimensional bodies moving in $\mathbb{R}^n$ and compare our model with the one introduced by Ciarlet and Nečas [4]. Then, we examine the case of one-dimensional structures moving in a two-dimensional Euclidean space (proofs for this part have been postponed to the end of the article). Finally, the case of shells, i.e, surfaces moving in $\mathbb{R}^3$ is discussed.

Let us specify the various notations that we shall us:

j_A^B : injection of A into B.
$\triangle(A) = \{(x, y) \in A \times A \ : \ x = y\}$: diagonal of $A \times A$.
A^c : complement of the set A.
$A \setminus B = A \cap B^c$.
$B^n(x, r)$: open ball centered in $x \in \mathbb{R}^n$ and of radius r.
$S^{n-1}(x, r)$: sphere centered in $x \in \mathbb{R}^n$ and of radius r.

$\dot{\varphi}$: derivative of φ (If φ is a regular one variable function).
$D_x\varphi$: differential of φ at x (If φ is a multi-variable function).
TM : tangent bundle of the manifold M.
T_xM : fiber at x of the tangent bundle TM.
$\Omega^k(M)$: set of differential forms of degree k on the manifold M.
$H^k(M)$: real cohomology group of M of degree k. $f^*(\alpha)$: pull back by f of the differential form α.

2. Preliminaries

We recall in this section basic definitions and notions of differential geometry and topology. For a comprehensive treatment of the topic, we refere, for instance, to C. Godbillon [7] , V. Arnold [1] , R. Bott and L. Tu [3].

2.1. Differential geometry

Let M and N be differentiable manifolds of dimension m and n respectively. Let f be a regular map from M into N. The map f is an immersion if and only if D_xf is of rank m for every x. An injective immersion is an embedding. Two embeddings f and g are said isotopic if there exists a regular map F from $M \times [0,1]$ into N such that $F(0) = f$, $F(1) = g$, and $F(t)$ is an embedding for every t in $[0,1]$ ($F(t)$ is the map from M into N defined by $F(t)(x) = F(x,t)$).

2.2. Differential forms

Let us recall the definition of differential forms for open subsets of $\mathbb{R}^n$. Let Λ^k be the set of k-linear antisymmetric forms on $\mathbb{R}^n$. The exterior product between a k form α and a l form ω is the $k+l$ form $\alpha \wedge \omega$ defined by

$$\alpha \wedge \omega(X_1, \ldots, X_{k+l}) = \sum_{\sigma} (-1)^{|\sigma|} \alpha(X_{\sigma_1}, \ldots, X_{\sigma_k}) \omega(X_{\sigma_{k+1}}, \ldots, X_{\sigma_{k+l}}),$$

where the sum is taken over the permutations σ of $\{1, \ldots, k+l\}$. Let $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{R}^n$ and $(dx_1, \ldots, dx_n)$ the canonical basis of $(\mathbb{R}^n)^*$, that is $dx_k(e_i) = \delta_{i,k}$. Let $I = (i_1, \ldots, i_k)$, with $1 \leq i_1 < \ldots < i_k \leq n$, we denote dx_I the k-linear alternate form defined by

$$dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

Let U be an open subset of $\mathbb{R}^n$. A differential form of degree k on U is a $\mathcal{C}^\infty$-mapping from U to Λ^k. The set of k-differential from on U is denoted $\Omega^k(U)$.

Let $f : V \to U$ be a $\mathcal{C}^\infty$ mapping from an open subset V of $\mathbb{R}^p$ into an open subset U of $\mathbb{R}^n$. Let $\alpha \in \Omega^k(U)$ be a differential form. We define the pullback $f^*\alpha$ of α by f as the k-differential form on V defined by

$$f^*(\alpha)(x)(X_1, \dots, X_k) = \alpha(f(x))(D_x f(X_1), \dots, D_x f(X_k)).$$

The operator of differentiation d defined for any $\mathbb{R}$ valued function g by

$$dg = \sum_k \frac{\partial g}{\partial x_k} dx_k,$$

can be extended to an operator $d : \Omega^k(V) \to \Omega^{k+1}(V)$ defined by

$$d\alpha = \sum_I df_I \wedge dx_I,$$

where $\alpha = \sum_I f_I dx_I$, and df_I is the differential of the real valued function f_I. By the Schwarz equality, we have $d \circ d = 0$. The cohomology group $H^k(V)$ of degree k of V is the quotient space of the kernel of d (as a mapping from $\Omega^k(V)$ into $\Omega^{k+1}(V)$ by the image of d (as a mapping from $\Omega^{k-1}(V)$ into $\Omega^k(V)$). If f is a $\mathcal{C}^\infty$ mapping from $V \to U$, the mapping f^* from $\Omega^k(U) \to \Omega^k(V)$ induced a mapping from $H^k(U) \to H^k(V)$. Moreover, if two mappings f and g are homotopic, then $f^* = g^*$ as maps from $H^*(U)$ into $H^*(V)$.

All those notions can be extended to differentiable manifolds.

3. Description of the self-intersections of a deformation

Let M be a m-dimensional submanifold (with or without boundary) of $\mathbb{R}^n$ and j_M be the injection of M in $\mathbb{R}^n$. We say that a deformation $\psi : M \to \mathbb{R}^n$ is admissible if it belongs to the $\mathcal{C}^0$-closure of the embeddings isotopic to the reference injection j_M. The set of admissible deformations is denoted $\mathcal{A}(j_M)$. In this section we address the problem of describing the self-intersection of a deformation. This leads us to associate to any deformation φ a topological invariant $\phi(\varphi)$ which describe, at least partially, the self-intersections of the deformation φ. The set $\mathcal{A}_\phi(j_M)$ of deformations which have the same topological invariant than the reference injection j_M is called the ϕ-admissible set. Every admissible deformation is ϕ-admissible. It follows, for instance, that no deformation with transversal self-intersection belongs to the set $\mathcal{A}(j_M)$. Before broaching the general case, we focus our attention on the case of thin structure moving in $\mathbb{R}^2$.

3.1. Case of thin structures in $\mathbb{R}^2$

The aim of this section is to give a pedestrian description of the self-intersections of a continuous deformation. The approach in this section is heuristic and is not meant to give complete proofs of the statements made.

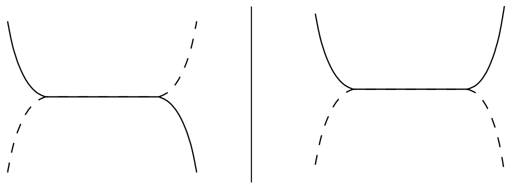

Fig. 1. Non-transverse intersections

We begin with the simplest case, that is the study of the intersection between two deformations from one-dimensional manifolds M_1 and M_2 into $\mathbb{R}^2$. If the intersection is transversal (see below), the intersection is completely describe by a set of oriented points in $M_1 \times M_2$. The definition of this oriented set could be extend to the non transversal case. Nevertheless, this generalization failed to detect some self-intersections of deformations of the circle S^1 into $\mathbb{R}^2$. A later definition is introduced in order to solve this problem.

3.1.1. The transversal case

Let us consider two one-dimensional bodies M_1 and M_2 moving in $\mathbb{R}^2$ (M_1 and M_2 are assumed to be diffeomorphic either to $[0,1]$ or S^1). For two given deformations φ and ψ of M_1 and M_2 respectively, i.e, mappings from M_1 or M_2 into $\mathbb{R}^2$, we want to define the intersection between φ and ψ. We define the set of common points between φ and ψ as

$$K(\varphi, \psi) := \{(x, y) \in M_1 \times M_2 \;:\; \varphi(x) = \psi(y)\}. \tag{1}$$

The intersection is said transverse, if for every $(x, y) \in K(\varphi, \psi)$, the family $(\dot{\varphi}(x), \dot{\psi}(y))$ is free. In this case, the set $K(\varphi, \psi)$ is a finite set of points and is stable under small $\mathcal{C}^1$-perturbations of φ and ψ. Furthermore, each of them could be endowed with a sign $s_{\varphi,\psi}(x, y)$, depending on the orientation of the basis $(\dot{\varphi}(x), \dot{\psi}(y))$.

$$s_{\varphi,\psi}(x, y) := \operatorname{sign}(\det(\dot{\varphi}(x), \dot{\psi}(y))).$$

As transversal intersections are stable under small perturbation, a deformation without self-intersection could only have non transversal intersections.

3.1.2. The non transversal case

It remains to consider the non-transverse case. Figure 1 represents two cases of non-transverse intersections. The image of M_1 under φ and of M_2 under ψ are represented with a dashed line and a continuous line respectively. In the first configuration the beams are intersecting themselves whereas they are just in contact in the second one. Let us first notice, that it is clear, from this example, that only a global criterion will enable us to

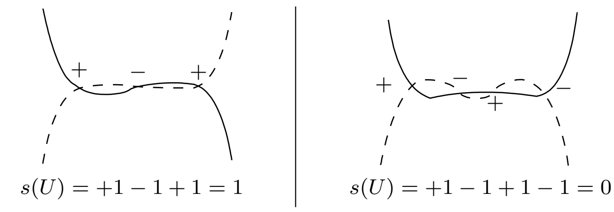

Fig. 2. Perturbations of non-transverse configurations

distinguish deformations with intersection from deformations without intersection. Moreover, the set $K(\varphi,\psi)$ is the same in both cases, and thus does not fully describe the intersection between φ and ψ.

Let V be a small neighborhood of $K(\varphi,\psi)$. There exists two deformations, $\widetilde{\varphi}$ and $\widetilde{\psi}$, close to φ and ψ respectively, such that $K(\widetilde{\varphi},\widetilde{\psi}) \subset V$, and such that the intersection between $\widetilde{\varphi}$ and $\widetilde{\psi}$ is transverse. To each connected component U of V, one can associate an integer $s_{\varphi,\psi}(U)$, equal to the sum of the sign of the points $(x,y) \in K(\widetilde{\varphi},\widetilde{\psi}) \cap U$.

$$s_{\varphi,\psi}(U) := \sum_{(x,y)\in K(\widetilde{\varphi},\widetilde{\psi})\cap U} s_{\widetilde{\varphi},\widetilde{\psi}}(x,y).$$

This integer does not depend on the choice of $\widetilde{\varphi}$ and $\widetilde{\psi}$ made, as soon as $\widetilde{\varphi}$ and $\widetilde{\psi}$ are close enough from φ and ψ. If φ and ψ belongs to the admissible set, one can choose $\widetilde{\varphi}$ and $\widetilde{\psi}$ such that $K(\widetilde{\varphi},\widetilde{\psi}) = \emptyset$, thus,

$$s_{\varphi,\psi}(U) = 0 \text{ for every connected component } U \text{ of } V \text{ and for any neighborhood } V \text{ of } K(\varphi,\psi). \quad (2)$$

Let us compute $s_{\varphi,\psi}(U)$ in the two configurations represented in the figure 1. We obtain $s_{\varphi,\psi}(U) = +1$ in the first case. Thus, this configuration is not admissible. In the second case, $s_{\varphi,\psi}(U) = 0$ (see figure 2).

3.1.3. The case of self-intersections

We investigate now the case of self-intersections. If φ belongs to the admissible set $\mathcal{A}(j_M)$, the condition (2) is satisfied with $M_2 = M_1$ and $\psi = \varphi$. Nevertheless, the converse is not true. Assume that M_1 is homeomorphic to S^1. For any integer k, the deformation φ_k defined by

$$\begin{aligned} \varphi_k : S^1 &\to \mathbb{R}^2 \\ \theta &\mapsto (\cos(k\theta), \sin(k\theta)) \end{aligned} \quad (3)$$

where the circle S^1 is parametrized by the angle θ fulfills the criterion (2). Indeed, let $\widetilde{\varphi}_k = (1+\varepsilon)\varphi_k$, where ε is a small positive real, then $K(\widetilde{\varphi}_k,\varphi_k) = \emptyset$ and

$$s_{\varphi,\varphi}(U) = \sum_{(x,y)\in K(\widetilde{\varphi}_k,\varphi_k)} s_{\widetilde{\varphi}_k,\varphi_k}(x,y) = 0.$$

Even so the deformations φ_k are not always intersection-free: There is no embedding close to φ_k, as soon as $k \neq \pm 1$. Thus, the mapping $s_{\varphi,\varphi}$ does not give us a complete description of the self-intersections of φ.

In order to solve this particular problem, let us go back to the study of the intersections between two deformations. As before, φ and ψ denote deformations from one-dimensional manifolds M_1 and M_2 into $\mathbb{R}^2$. We define the mapping $d_{\varphi,\psi}$ from $M_1 \times M_2 \setminus K(\varphi,\psi)$ into $\mathbb{R}^2_*$ by

$$d_{\varphi,\psi}(x,y) := \varphi(x) - \psi(y).$$

Let $\phi_{\mathbb{R}^2_*}$ be the closed 1-form defined on $\mathbb{R}^2_*$ by

$$\phi_{\mathbb{R}^2_*} := \frac{1}{2\pi}\left(\frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx\right).$$

We denoted $\phi(\varphi,\psi)$ the pullback of $\phi_{\mathbb{R}^2_*}$ by $d_{\varphi,\psi}$

$$\phi(\varphi,\psi) := d^*_{\varphi,\psi}(\phi_{\mathbb{R}^2_*}).$$

Let U be an oriented open set of $M_1 \times M_2$, such that $\partial U \subset M_1 \times M_2 \setminus K(\varphi,\psi)$. We assert that (see Proposition 5)

$$\int_{\partial U} \phi(\varphi,\psi) = s_{\varphi,\psi}(U). \tag{4}$$

Hence, the mapping $s_{\varphi,\psi}$ is completely described by the integration of the 1-form $\phi(\varphi,\psi)$ on loops in $M \times M \setminus K(\varphi,\psi)$. It remains to apply this analysis to the study of self-intersections.

Let φ be a deformation, we define $\phi(\varphi)$ as the 1-form on $M \times M \setminus K(\varphi,\varphi)$

$$\phi(\varphi) = \phi(\varphi,\varphi).$$

If φ belongs to the set of admissible deformations, and if U is an open set such that

$$|d_{\varphi,\varphi}(U)| > \delta > 0,$$

then the restriction of $d_{\varphi,\varphi}$ to U is homotopic to d_{j_M,j_M}. Thus, there exists a mapping $u : U \to \mathbb{R}$ such that

$$\phi_U(\varphi) - \phi_U(j_M) = du, \tag{5}$$

where $\phi_U(\varphi)$ and $\phi_U(j_M)$ are the restriction of $\phi(\varphi)$ and $\phi(j_M)$ to the open set U. In other words, $\phi_U(\varphi)$ and $\phi_U(j_M)$ are equal up to an exact form. The set of ϕ-admissible deformations will be defined in 3.2 as the deformations which fulfill this criteria. The condition (5) implies that the integration of the 1-form $\phi(\varphi)$ and $\phi(j_M)$ on any loop of $M \times M \setminus K(\varphi,\varphi)$ are equal. Thus, by (4), this condition is at least as strong as the condition (2). Actually, the 1-form $\phi(\varphi)$ contains more informations about the intersection than $s_{\varphi,\varphi}$, and the condition (5) is strictly stronger than (2). Indeed, let γ be the loop

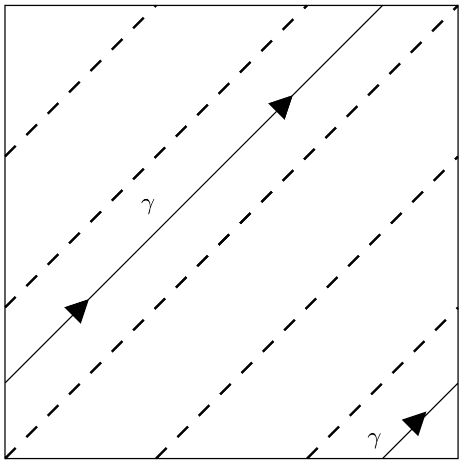

Fig. 3. The test loop γ in the tore $S^1 \times S^1$, with $\varphi = \varphi_3$

in $S^1 \times S^1$ defined by

$$\begin{aligned}\gamma : S^1 &\to S^1 \times S^1\\ \theta &\mapsto (\theta, \theta + h),\end{aligned}$$

where h is a small positive real. A simple computation shows that

$$\int_\gamma \phi(\varphi_k) ds = k,$$

The integer k is called the turning number of the deformation. The Figure 3.1.3 represents the loop γ in the tore $S^1 \times S^1$ (a square which opposite edges are identified) for $\varphi = \varphi_3$. The set $K(\varphi_3, \varphi_3)$ is drawn with dashed lines.
If j_{S^1} is the canonical injection of S^1 into $\mathbb{R}^2$ (that is $j_{S^1} = \varphi_1$), we have

$$\int_\gamma \phi(j_{S^1}) ds = 1.$$

As soon as $k \neq 1$,

$$\int_\gamma \phi(\varphi_k) ds \neq \int_\gamma \phi(j_{S^1}) ds,$$

$\phi(\varphi_k) - \phi(j_{S^1})$ is not an exact form and the condition (5) is not fulfilled. Hence, the deformation φ_k is not an admissible deformation as soon as $k \neq 1$.

Remark 1. Using the de Rahm duality Theorem, one can prove that the condition (5) is equivalent to

$$\langle \mathcal{I}(\varphi), \omega \rangle = \langle \mathcal{I}(j_M), \omega \rangle \tag{6}$$

for any closed form ω on $M \times M$ with compact support included in the set $(K(\varphi,\varphi) \cup \partial(M \times M))^c$, where

$$\langle \mathcal{I}(\varphi), \omega \rangle := \int_{M \times M} \phi(\varphi) \wedge \omega.$$

3.2. Definition of the ϕ-admissible deformations

Let n be an integer and M be a submanifold of $\mathbb{R}^n$. Let $j_{_M}$ be the injection of M into $\mathbb{R}^n$.
For all mappings φ from M into $\mathbb{R}^n$, we denote by d_φ the mapping

$$\begin{aligned} d_\varphi : M \times M &\to \mathbb{R}^n \\ (x, y) &\mapsto \varphi(x) - \varphi(y) \end{aligned}$$

and by $K(\varphi)$ the non injective set, that is

$$K(\varphi) := \{(x, y) \in M \times M \ : \ \varphi(x) = \varphi(y)\}.$$

For any open subset U of $M \times M$ such that there exists a positive real δ for which

$$|d_\varphi(U)| > \delta > 0, \tag{7}$$

we denote $\phi_U(\varphi)$ the element of $H^{n-1}(U)$ defined as the pullback of $\phi_{\mathbb{R}^n_*}$ by $d_{\varphi,U}$, the restriction of d_φ to the open U,

$$\phi_U(\varphi) := d^*_{\varphi,U}(\phi_{\mathbb{R}^n_*}) \in H^{n-1}(U).$$

where $\phi_{\mathbb{R}^n_*}$ is the canonical $n-1$ non exact closed form on $\mathbb{R}^n_*$ defined by

$$\phi_{\mathbb{R}^n_*}(x)(X_1, \ldots, X_{n-1}) := \det(x/|x|, X_1, \ldots, X_{n-1})/|S^{n-1}|$$

and $|S^{n-1}|$ is the $n-1$ Hausdorff measure of the unit sphere S^{n-1} of $\mathbb{R}^n$.

Remark 2. We recall that $H^{n-1}(U)$ is the quotient space of $n-1$-closed forms by the $n-1$ exact forms on U.

The mapping which maps any open subset U of $M \times M$ for which (7) holds to $\phi_U(\varphi)$ is denoted $\phi(\varphi)$.

We say that a deformation is ϕ-admissible if and only if for any open subset U

$$\phi_U(\varphi) = \phi_U(j_{_M}), \tag{8}$$

as element of $H^{n-1}(U)$, that is, if $\phi_U(\varphi) - \phi_U(j_{_M})$ is an exact form on U.

We denote by $\mathcal{A}_\phi(j_{_M})$ the set of ϕ-admissible deformations, that is

$$\begin{aligned} \mathcal{A}_\phi(j_{_M}) := \Big\{ &\varphi \in \mathcal{C}^0(M; \mathbb{R}^n) \ : \ \phi_U(\varphi) = \phi_U(j_{_M}) \text{ in } H^{n-1}(U), \\ &\text{for any open set } U \text{ such that } |d_\varphi(U)| > \delta > 0 \Big\}. \end{aligned} \tag{9}$$

Remark 3. The element $\phi_U(\varphi)$ of $H^{n-1}(U)$ is well defined, even if φ is only continuous. Indeed, if $\widetilde{\varphi}$ is a regular approximation of φ, then $\phi_U(\widetilde{\varphi}) \in H^{n-1}(U)$ is independent of $\widetilde{\varphi}$ as soon as $\|\varphi - \widetilde{\varphi}\|_{\mathcal{C}^0}$ is small enough.

Remark 4. One can define $\phi(\varphi)$ as an element of the inverse limit of the groups $H^1(U)$, where U is any open subset of $M \times M$ which fulfills the condition (7). Then $\phi(\varphi)$ is a mapping, which maps every open subset U which fulfills (7) to an element $\phi_U(\varphi)$ of $H^1(U)$. Moreover, if $U \subset V$, then

$$\phi_U(\varphi) = j_U^{V*}(\phi_V(\varphi)),$$

where j_U^V is the injection of U in V.

3.3. Elementary properties of the ϕ-admissible set

3.3.1. $\mathcal{C}^0$-Closure

The set of ϕ-admissible deformations is closed for the $\mathcal{C}^0$ topology. Furthermore, the set of admissible deformations is included in the set of ϕ-admissible deformations.

Proposition 1. *$\mathcal{A}_\phi(j_M)$ is closed for the $\mathcal{C}^0$ topology.*

Proof. Let φ_n be a sequence of ϕ-admissible deformations and φ be q deformation of M such that φ_n converges toward φ for the $\mathcal{C}^0(M;\mathbb{R}^n)$ topology. Let U be an open subset of $M \times M$ and δ be a positive real such that

$$|d_\varphi(U)| > \delta > 0.$$

Let $\widetilde{\varphi}$ be a $\mathcal{C}^\infty$ regularization of φ such that

$$\|\varphi - \widetilde{\varphi}\|_{\mathcal{C}^0} < \delta/3.$$

Let $\widetilde{\varphi}_n$ be $\mathcal{C}^\infty$ regularization of φ_n such that

$$\|\varphi_n - \widetilde{\varphi}_n\|_{\mathcal{C}^0} < \delta/3.$$

There exists N such that $\|\varphi - \varphi_N\|_{\mathcal{C}^0} < \delta/3$, so that if $\widetilde{\varphi}_t = t\widetilde{\varphi} + (1-t)\widetilde{\varphi}_N$, then for all $(x,y) \in U$,

$$|\widetilde{\varphi}_t(x) - \widetilde{\varphi}_t(y)| > 0.$$

Thus, the restriction $d_{\widetilde{\varphi}_t,U}$ of $d_{\widetilde{\varphi}_t}$ to U defines a homotopy from $d_{\widetilde{\varphi},U} : U \to \mathbb{R}^n_*$ to $d_{\widetilde{\varphi}_N,U} : U \to \mathbb{R}^n_*$. As $\widetilde{\varphi}_N$ belongs to the set of ϕ-admissible deformations,

$$\begin{aligned}\phi_U(\varphi) = \phi_U(\widetilde{\varphi}) &= d^*_{\widetilde{\varphi},U}(\phi_{\mathbb{R}^n_*}) = d^*_{\widetilde{\varphi}_N,K}(\phi_{\mathbb{R}^n_*}) \\ &= \phi_U(\widetilde{\varphi}_N) = \phi_U(\varphi_N) = \phi_U(j_M).\end{aligned}$$

Hence, we have proved that $\phi_U(\varphi) = \phi_U(j_M)$ for every open subset U of $M \times M$, such that (7) holds. In other words that φ belongs to the ϕ-admissible set $\mathcal{A}_\phi(j_M)$.

Remark 5. The proof of the Proposition 1 shows that the element $\phi_U(\varphi)$ is, as stated in the previous section, correctly defined for any continuous deformation and any open subset U for which (7) holds.

Proposition 2. *The admissible set $\mathcal{A}(j_M)$ is included in the ϕ-admissible set $\mathcal{A}_\phi(j_M)$.*

Remark 6. Under some conditions on the dimension of M and the dimension n of the space, we proved that $\mathcal{A}(j_M) cap \operatorname{Imm}(M;\mathbb{R}^n) = \mathcal{A}_\phi(j_M) cap \operatorname{Imm}(M;\mathbb{R}^n)$ (see sections 5.1, 5.2 and 5.3).

Proof. Let φ be an embedding isotopic to j_M. There exists an isotopy φ_t such that $\varphi_0 = \varphi$ and $\varphi_1 = j_M$. Let U be an open subset of $\triangle(M)^c$. Then $d_{\varphi_t,U} : [0,1] \times U \to \mathbb{R}^n_*$ is a regular homotopy from $d_{\varphi,U} : U \to \mathbb{R}^n_*$ to $d_{j_M,U} : U \to \mathbb{R}^n_*$, and

$$d^*_{\varphi,U}(\phi_{\mathbb{R}^n_*}) = d^*_{j_M,U}(\phi_{\mathbb{R}^n_*}),$$

as element of $H^1(U)$. Hence, φ belongs to the set of ϕ-admissible deformations. The conclusion follows from the previous proposition.

Remark 7. As soon as $n \geq 3$ and $\dim(M) \geq 2$, there exists deformations $\varphi : M \to \mathbb{R}^n$ which belong to the ϕ-admissible but not to $\mathcal{A}(j_M)$ (see section 5.3). Moreover, we do not know whether or not $\mathcal{A}_\phi(j_M) = \mathcal{A}(j_M)$ when $n = 2$.

3.3.2. Right and left invariance

Proposition 3. *Let $g : M \to M$ be a homeomorphism isotopic to the identity, then*

$$(\varphi \in \mathcal{A}_\phi(j_M)) \Rightarrow (\varphi \circ g \in \mathcal{A}_\phi(j_M)).$$

Proof. Let U be an open set of $M \times M$ such that

$$|d_\varphi(U)| > \delta > 0,$$

for a real δ. Let $V = (g,g)^{-1}(U)$. There exists regularization $\widetilde{g}$ and $\widetilde{\varphi}$ of g and φ such that

$$\phi_U(\varphi) = d^*_{\widetilde{\varphi}}(\phi_{\mathbb{R}^n_*})$$
$$\phi_V(\varphi \circ g) = d^*_{\widetilde{\varphi}\circ\widetilde{g},V}(\phi_{\mathbb{R}^n_*}).$$

Moreover, $\widetilde{g}$ can be chosen such that it is diffeomorphic to the identity. In the following, $(\widetilde{g},\widetilde{g})$ will be understood as its restriction to v with values in U.

$$d_{\widetilde{\varphi},U} \circ (\widetilde{g},\widetilde{g}) = d_{\widetilde{\varphi}\circ\widetilde{g},V}$$

$$\begin{aligned} d^*_{\widetilde{\varphi}\circ\widetilde{g},V}(\phi_{\mathbb{R}^n_*}) &= (\widetilde{g},\widetilde{g})^* \circ d^*_{\widetilde{\varphi},U}(\phi_{\mathbb{R}^n_*}) = (\widetilde{g},\widetilde{g})^* \circ d^*_{j_M,U}(\phi_{\mathbb{R}^n_*}) \\ &= d^*_{j_M\circ\widetilde{g},V}(\phi_{\mathbb{R}^n_*}) = d^*_{j_M,V}(\phi_{\mathbb{R}^n_*}). \end{aligned}$$

Proposition 4. *Let g be a diffeomorphism from $\mathbb{R}^n$ into itself which preserves the orientation, then*

$$(\varphi \in \mathcal{A}_\phi(j_{_M})) \Rightarrow (g \circ \varphi \in \mathcal{A}(j_{_M})).$$

Proof. As g is a diffeomorphism which preserves the orientation, (g, g) defines a diffeomorphism from $\mathbb{R}^n \times \mathbb{R}^n \setminus \triangle(\mathbb{R}^n)$ into itself and $(g, g)^*$ from $H^{n-1}(\mathbb{R}^n \times \mathbb{R}^n \setminus \triangle(\mathbb{R}^n))$ into itself is nothing else but the identity. Let U be a subset of $M \times M$ such that

$$|d_\varphi(U)| > \delta > 0$$

for a real δ. There exists a positive real δ' such that

$$|d_{g\circ\varphi}(U)| > \delta' > 0.$$

Let p be the mapping

$$\begin{aligned} p : \mathbb{R}^n \times \mathbb{R}^n \setminus \triangle(\mathbb{R}^n) &\to \mathbb{R}^n_* \\ (x, y) &\mapsto x - y. \end{aligned}$$

Up to replace φ by as $\mathcal{C}^\infty$ regularization, on can assume that φ is regular and that

$$\phi_U(\varphi) = d^*_{\varphi,U}(\phi_{\mathbb{R}^n_*}) \text{ and } \phi_U(g \circ \varphi) = d^*_{g\circ\varphi,U}(\phi_{\mathbb{R}^n_*}).$$

We have $d_{g\circ\varphi,U} = p \circ (g, g) \circ (\varphi, \varphi)_{|U}$ and

$$\begin{aligned} d^*_{g\circ\varphi,U}(\phi_{\mathbb{R}^n_*}) &= (p \circ (g, g) \circ (\varphi, \varphi)_{|U})^*(\phi_{\mathbb{R}^n_*}) = (\varphi, \varphi)^*_{|U} \circ (g, g)^* \circ p^*(\phi_{\mathbb{R}^n_*}) \\ &= (\varphi, \varphi)^*_{|U} \circ p^*(\phi_{\mathbb{R}^n_*}) = d^*_{\varphi,U}(\phi_{\mathbb{R}^n_*}) = d^*_{j_{_M},U}(\phi_{\mathbb{R}^n_*}). \end{aligned}$$

3.3.3. Transversal self-intersections

In this section, we prove that any deformation with transverse self-intersection does not belong to the set of ϕ-admissible deformations $\mathcal{A}_\phi(j_{_M})$ and thus to the admissible set $\mathcal{A}(j_{_M})$. More precisely,

Proposition 5. *Let $\varphi : M \to \mathbb{R}^n$ be a continuous mapping. Assume that φ has a transverse self-intersection at one couple of points $(x, y) \in M \times M$ such that $x \neq y$, that is*

- *$\varphi(x) = \varphi(y)$*
- *The mapping φ is of class $\mathcal{C}^1$ in neighborhoods of x and y*
- *$D_x\varphi(T_xM) + D_y\varphi(T_yM) = \mathbb{R}^n$.*

Then φ does not belong to the set of ϕ-admissible deformations $\mathcal{A}_\phi(j_{_M})$.

Proof. We will construct two functions γ_φ and γ_{j_M} from S^{n-1} into S^{n-1}. Their degree will depend respectively on $\phi(\varphi)$ and $\phi(j_M)$. Moreover, we will show that $\deg(\gamma_\varphi) = 1$ as $\deg(\gamma_{j_M}) = 0$. It will imply that $\phi(\varphi) \neq \phi(j_M)$. In other words, we obtain that φ does not belong to the set $\mathcal{A}_\phi(j_M)$ of ϕ-admissible deformations.

As $D_x\varphi(T_xM) + D_y\varphi(T_yM) = \mathbb{R}^n$, the mapping

$$\begin{aligned} d_\varphi : M \times M &\to \mathbb{R}^n \\ (a,b) &\mapsto \varphi(a) - \varphi(b) \end{aligned}$$

is a submersion at (x,y). From the Implicit Function Theorem, we deduce that there exists a neighborhood V of 0 in $\mathbb{R}^{2m}$, a neighborhood W of (x,y) in $M \times M$ and a local diffeomorphism $g : V \to W$ such that $g(0) = (x,y)$ and

$$d_\varphi \circ g(x_1, \ldots, x_{2m}) = (x_1, \ldots, x_n).$$

Furthermore, one can choose g such that the unit disk D^n is included in V, and such that $W \cap \triangle(M) = \emptyset$ where

$$\triangle(M) = \{(a,b) \in M \times M : a = b\}.$$

Let $\delta = 1/2$ and

$$V_\delta = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^{2m-n} \cap V \ : \ |x| > \delta\}.$$

Let $h : V_\delta \to U = g(V_\delta) \subset W \setminus \{(x,y)\}$ be the restriction of g to V_δ. There exists a real δ' such that

$$|d_\varphi(U)| > \delta' > 0.$$

We set

$$\begin{aligned} \gamma_\varphi &= \theta \circ d_{\varphi,U} \circ h \circ j^{V_\delta}_{S^{n-1}} \\ \gamma_{j_M} &= \theta \circ d_{j_M,U} \circ h \circ j^{V_\delta}_{S^{n-1}}, \end{aligned}$$

where $\theta(z) = z/|z|$. The degree of the mapping γ_φ and γ_{j_M} depends only on $\phi_U(\varphi)$ and $\phi_U(j_M)$. Indeed, denoting by $\phi_{S^{n-1}}$ the generator of $H^{n-1}(S^{n-1})$, that is the $n-1$ differential form defined by

$$\phi_{S^{n-1}}(x)(X_1, \ldots, X_n) = \det(n, X_1, \ldots, X_n)/|S^{n-1}|,$$

we have

$$\begin{aligned} \deg(\gamma_\varphi) &= \int_{S^{n-1}} \gamma_\varphi^*(\phi_{S^{n-1}}) \\ &= \int_{S^{n-1}} (h \circ j^{V_\delta}_{S^{n-1}})^* \circ d^*_{\varphi,U}(\theta^*(\phi_{S^{n-1}})) \\ &= \int_{S^{n-1}} (h \circ j^{V_\delta}_{S^{n-1}})^* \circ d^*_{\varphi,U}(\phi_{\mathbb{R}^n_*}). \end{aligned}$$

Thus, the degree of γ_φ is

$$\deg(\gamma_\varphi) = \int_{S^{n-1}} (h \circ j^{V_\delta}_{S^{n-1}})^*(\phi_U(\varphi)), \tag{10}$$

whereas the degree of γ_{j_M} is

$$\deg(\gamma_{j_M}) = \int_{S^{n-1}} (h \circ j^{V_\delta}_{S^{n-1}})^*(\phi_U(j_M)). \tag{11}$$

As $\gamma_\varphi : S^{n-1} \to S^{n-1}$, is nothing else but the identity,

$$\deg(\gamma_\varphi) = 1.$$

On the other hand,

$$\begin{aligned}
\deg(\gamma_{j_M}) &= \int_{S^{n-1}} (h \circ j^{V_\delta}_{S^{n-1}})^*(\phi_U(j_M)) \\
&= \int_{S^{n-1}} (h \circ j^{V}_{D^n})^*(\phi_W(j_M)) \\
&= \int_{\partial D^n} (h \circ j^{V}_{D^n})^*(\phi_W(j_M)) \\
&= \int_{D^n} d\left((h \circ j^{V}_{D^n})^*(\phi_W(j_M))\right) = 0.
\end{aligned}$$

As claimed, $\deg(\gamma_{j_M}) \neq \deg(\gamma_\varphi)$ and from (10) and (11), we deduce that $\phi_U(\varphi) \neq \phi_U(j_M)$ and that φ does not belong to the set of ϕ-admissible deformations.

4. The minimization problem

In this section, we consider nonlinear hyperelastic bodies. With suitable assumptions on the stored energy W of the material, we show that there exists at least one minimizer to the energy over the set of admissible deformation. Moreover, this existence results remains true if one consider ϕ-admissible deformations instead.

4.1. Setting of the problem

Let M be a differentiable submanifold of $\mathbb{R}^n$ (with or without border), m the dimension of M and j_M the injection of M into $\mathbb{R}^n$. The manifold M is implicitly endowed with the differential structure and the Riemann metric induces by j_M. Furthermore, the $m-$dimensional Hausdorff measure in $\mathbb{R}^n$ induced a measure on M noted dx. We define $\mathcal{F}(M;\mathbb{R}^n)$ as the vector bundle of base M whose fiber at $x \in M$ is the set of linear mappings from $(T^*_x M)^m$ into $\mathbb{R}^n$. Let π be the projection of this vector bundle on its base. The stored energy W is a mapping from $\mathcal{F}(M;\mathbb{R}^n)$ into $\mathbb{R}^+$. We assume that W

is a Carathéodory function: The restriction of W to a fiber $\pi^{-1}(x)$ is $\mathcal{C}^0$ for almost every $x \in M$, and the restriction of W to any section is measurable (for any regular mapping $G : M \to \mathcal{F}(M;\mathbb{R}^n)$ such that $\pi \circ G = \mathrm{Id}_M$, the mapping $W \circ g$ is mesurable). We assume W to be quasi-convex, that is for every $F \in \mathcal{F}(M;\mathbb{R}^n)$,

$$\int_U W(F)dx \leq \int_U W(F + D\varphi)dx,$$

where U is the unit square of $T_{\pi(F)}M$, $\varphi \in \mathcal{C}_0^\infty(U;\mathbb{R})$. Moreover, we assume that there exists $p > m$ such that the following growth and coercivity conditions are fulfilled

$$\forall F \in \mathcal{F}(M;\mathbb{R}^n),\ |W(F)| \leq C(1 + |F|^p)$$

$$\forall F \in \mathcal{F}(M;\mathbb{R}^n),\ W(F) \geq \alpha|F|^p + \beta.$$

where C, α, β are constants and $\alpha > 0$. We consider the case where M is submitted to dead body forces $f \in L^2(M;\mathbb{R}^n)$ and clamped on a subset N of M such that every connected component of M intersects N. Let $I : W^{1,p}(M;\mathbb{R}^n) \to \mathbb{R}$ be the energy functional

$$I(\psi) = \int_M W(D\psi)dx - \int_M f(x).\psi(x)dx. \tag{12}$$

The set of admissible deformations of finite energy is

$$\mathcal{A}^p(j_{_M}) = \left\{\varphi \in W^{1,p}(M;\mathbb{R}^n) \cap \mathcal{A}(j_{_M})\ :\ \varphi(x) = j_{_M}(x) \text{ for all } x \in N\right\}, \tag{13}$$

whereas the set of ϕ-admissible deformations with finite energy is

$$\mathcal{A}^p_\phi(j_{_M}) = \left\{\varphi \in W^{1,p}(M;\mathbb{R}^n) \cap \mathcal{A}_\phi(j_{_M})\ :\ \varphi(x) = j_{_M}(x) \text{ for all } x \in N\right\}. \tag{14}$$

We consider the two following minimization problems

$$\text{Find } \varphi \in \mathcal{A}^p(j_{_M}) \ \text{ such that } \ I(\varphi) = \inf_{\psi \in \mathcal{A}^p(j_M)} I(\psi) \tag{$\mathcal{P}$}$$

and

$$\text{Find } \varphi \in \mathcal{A}^p_\phi(j_{_M}) \ \text{ such that } \ I(\varphi) = \inf_{\psi \in \mathcal{A}^p_\phi(j_M)} I(\psi). \tag{$\mathcal{P}_\phi$}$$

Remark 8. We would like to emphasize that our formulation is not limited to the study of a single body. If one considers two bodies whose reference configurations are the submanifolds A and B respectively, we set $M = A \cup B$. Note that we assume for simplicity that all the connected components of M have the same dimension m. This assumption could be removed at no extra cost.

Remark 9. The reference configuration used in our formulation is a differentiable manifold M. Usually, an open subset of $\mathbb{R}^m$ is used instead. Our choice allows us to treat more kinds of topology.

Remark 10. As we will see in the section 5, not only the minimization problem $(\mathcal{P})$ has a physical meaning but also the minimization problem $(\mathcal{P}_\phi)$, at least if $n = 2$ or $\dim(M) = n$. Indeed, in such cases, those minimization problems are partially equivalent to the Euler-Lagrange equations, describing an elastic body with frictionless contacts.

4.2. Existence

Proposition 6. *Both minimization problems $(\mathcal{P})$ and $(\mathcal{P}_\phi)$ have at least one solution.*

Proof. The quasi-convexity of W, coercivity and growing conditions imply that the functional I is sequentially lower semi-continuous (see Morrey [9], [10] and Ball [2]) for the weak topology of $W^{1,p}(M;\mathbb{R}^n)$. Let φ_n be a minimization sequence of I over $\mathcal{A}^p(j_M)$. The clamping conditions combined with the coercivity ensure us that the sequence φ_n in bounded in $W^{1,p}(M;\mathbb{R}^n)$. One can extract a subsequence φ_{n_k} weakly converging toward an element $\varphi \in W^{1,p}(M;\mathbb{R}^n)$. As I is sequentially lower semi-continuous, we have

$$I(\varphi) \leq \inf_{\psi \in \mathcal{A}^p(M)} I(\psi).$$

Since $p > m$, the injection of $W^{1,p}(M;\mathbb{R}^n)$ into $\mathcal{C}^0(M;\mathbb{R}^n)$ is compact. Hence, φ_{n_k} converges in $\mathcal{C}^0(M;\mathbb{R}^n)$ and as $\mathcal{A}^p(j_M)$ is closed for the $\mathcal{C}^0$ topology, φ belongs to $\mathcal{A}^p(j_M)$. Thus, the minimization problem $(\mathcal{P})$ has at least one solution. The existence of a solution to the problem $(\mathcal{P}_\phi)$ ensues from the closure of the set $\mathcal{A}^p_\phi(j_M)$ for the $\mathcal{C}^0$ topology given by Proposition 1.

Remark 11. Existence results could also be obtained if N does not intersect every connected components of M. Yet, the loads f have to fulfill compatibility conditions to ensure the existence of a minimizer. For instance, if M has only one connected component and if $\int_M f(x)dx = 0$, then there exists at least one solution to the minimization problem.

Remark 12. The existence could also be obtained under other assumptions on the stored energy W. In particular, it remains true under the hypothesis made by Ball [2], which allows for the case $W(F) \to +\infty$, if $\det(F) \to 0$ in the case $m = n$. Furthermore, the definition of the admissible set given here only holds for continuous functions and thus requires $p > m$. This condition could probably be weakened using methods similar to the one used by Tang [12] and Šverák [11].

5. On the equivalence with the Euler-Lagrange equations

In this section, we examine whether or not solutions of the minimization problems $(\mathcal{P})$ or $(\mathcal{P}_\phi)$ are solutions of the Euler-Lagrange equations describing the behavior of elastic bodies with frictionless contacts. All results and proofs are expressed with respect to the problem $(\mathcal{P}_\phi)$. They can easily be adapted to the study of the problem $(\mathcal{P})$. To this end, it suffices to use that the admissible set $\mathcal{A}(j_M)$ is included in the ϕ-admissible set $\mathcal{A}_\phi(j_M)$ (Proposition 2).

5.1. The case of n-dimensional bodies in $\mathbb{R}^n$

As recalled in the introduction, the case of n-dimensional bodies moving in $\mathbb{R}^n$ has been studied by Ciarlet and Nečas in [4]. Our modeling differs from theirs in the definition of the admissible set of deformations. Furthermore, their assumptions on the stored energy W are stronger than ours. They choose W in a way that forces local injectivity almost everywhere for deformations with finite elastic energy. To this end, they assume that W takes its values in $\overline{\mathbb{R}}^+$ and verifies

$$\begin{cases} W(F) = +\infty \text{ if } \det(F) \leq 0 \\ W(F) \to +\infty \text{ as } \det(F) \to 0. \end{cases} \tag{15}$$

Their admissible deformations are defined as those that satisfy

$$\int_M \det(\nabla\varphi) dx \leq \mathrm{Vol}(\mathrm{Im}(\varphi)). \tag{16}$$

It is easy to show that if a deformation fulfills this constraint and has a finite energy, it is almost everywhere injective not only locally but globally. This follows from the equation (16) and the identity

$$\int_{\varphi(M)} \mathrm{Card}(\varphi^{-1}(y)) dy = \int_M \det(\nabla\varphi(x)) dx$$

fulfilled by every $W^{1,p}$ function such that $\det(\nabla\varphi) > 0$ almost everywhere. Using the notation introduced in the section 4.1, Ciarlet and Nečas define their set of admissible deformations as

$$\mathcal{A}^p_{CN}(j_M) = \Big\{ \varphi \in W^{1,p}(M;\mathbb{R}^n) \ : \ \varphi(x) = j_M(x) \text{ for all } x \in N,$$
$$\det(D\varphi) > 0 \text{ a.e.}, \int_{\varphi(M)} \mathrm{Card}(\varphi^{-1}(y)) dy = \int_M \det(\nabla\varphi(x)) dx \Big\}.$$

Under other assumptions on the stored energy W, Ciarlet and Nečas prove that there exists a minimizer of the energy on their set of admissible deformations whenever the infimum is finite.

The condition $\phi(\varphi) = \phi(j_M)$ is at least as strong as the condition (16) introduce by Ciarlet and Nečas (see Proposition 8). In particular, every element $\varphi \in \mathcal{A}_\phi(j_M)$ is injective almost everywhere. Consequently, our model inherits all the properties of theirs. This feature enables us to apply their result of partial equivalence between the minimization problem and the

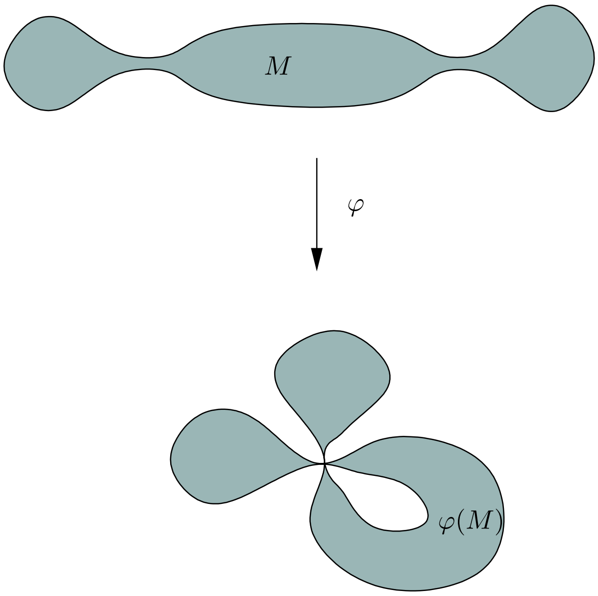

Fig. 4. Admissible deformation according to the Ciarlet and Nečas model with transversal self-intersection

Euler-Lagrange equations (see Proposition 7 and its proof). In our case, the proof can be run word by word as they do. Actually, our criterion is stronger: some deformations which belong to the set $\mathcal{A}^p_{CN}(j_M)$ do not belong to $\mathcal{A}_\phi(j_M)$. For instance, the deformation represented in Figure 4 fulfills criterion (16) but does not belong to $\mathcal{A}_\phi(j_M)$ as it has a transversal self-intersection. The existence of such deformations prevent us from hoping that an asymptotic analysis, performed on the Ciarlet and Nečas model, could lead to a reasonable model of thin structure with frictionless self-contacts, without self-intersection. Another unpleasant consequence of this feature is that it will probably be difficult to implement a robust numerical scheme using their modeling: the thinner the structure is, the more trouble is expected.

Finally, let us remark that every deformation φ belonging to $\mathcal{A}^p_\phi(j_M)$ preserves the orientation, without any assumption on the stored energy W (see Proposition 9). This is another important difference between our model and the Ciarlet and Nečas formulation. Nevertheless, a deformation $\varphi \in \mathcal{A}^p_\phi(j_M)$ only satisfies

$$\det(\nabla\varphi) \geq 0,$$

and not

$$\det(\nabla\varphi) > 0 \text{ almost everywhere.}$$

In the next proposition, we will assume that $N \subset \partial M$, that is the body is only clamped on its boundary.

Proposition 7. *Let M be a n-dimensional submanifold of $\mathbb{R}^n$ such that ∂M is smooth. Let φ be a smooth enough solution of the minimization problem*

$$\varphi \in \mathcal{A}^p_\phi(j_M), I(\varphi) = \inf_{\psi \in \mathcal{A}^p_\phi(j_M)} I(\psi), \tag{$\mathcal{P}_\phi$}$$

where the set $\mathcal{A}^p_\phi(j_M)$ and the functional I are defined as in (12) and (13). Assume that $\det(\nabla\varphi(x)) > 0$ for every $x \in M$, then the minimizer φ is a solution of the following boundary-value problem:

$$-\operatorname{div} DW(\nabla\varphi) = f \text{ in } M \setminus \partial M, \tag{17}$$

$$\varphi = j_M \text{ on } N, \tag{18}$$

$$DW(\nabla\varphi(x)).n'(x) = \lambda(x)n(x) \text{ with } \lambda(x) \leq 0, \text{ for all } x \in \partial M \setminus N; \tag{19}$$

the last equations correspond to one of the following situations (for all $x \in \partial M \setminus N$):

$$\varphi^{-1}(\varphi(x)) = \{x\} \text{ whence } \lambda(x) = 0, \tag{20}$$

$$\varphi^{-1}(\varphi(x)) = \{x, y\}, \text{ with } y \in \partial M \setminus N \text{ whence}$$
$$n(x) + n(y) = 0 \text{ and } \lambda(x)da(x) = \lambda(y)da(y), \tag{21}$$

where $n'(z)$ and $n(z)$ denote the unit outer normal vectors along ∂M and $\varphi(\partial M)$ at the point z and $\varphi(z)$, respectively, and $da(z)$ denotes the differential area along ∂M at the point z.

Proof. Let φ be a smooth ϕ-admissible deformation such that $\det(\nabla\varphi) > 0$. By Proposition 8, φ belongs to the Ciarlet and Nečase admissible set $\mathcal{A}^p_{CN}(j_M)$. In [4], Ciarlet and Nečas prove there is engough variations $F : [0,1] \times M \to \mathbb{R}^3$ such that, $F(0,x) = \varphi(x)$ and $F(t,.)$ is an embedding for every $t > 0$ to recover the given Euler-Lagrange equations, using the differentiability of the energy $I(.)$.

Proposition 8. *Let M be an n-dimensional differentiable submanifold of $\mathbb{R}^n$ (with border). Every deformation $\varphi \in \mathcal{A}_\phi(j_M) \cap W^{1,p}(M;\mathbb{R}^n)$ is such that*

$$\int_M |\det(\nabla\varphi)| dx = \operatorname{Vol}(\operatorname{Im}(\varphi)),$$

and φ is injective almost everywhere.

Proof. Let $\varphi \in \mathcal{A}_\phi(j_M) \cap W^{1,p}(M;\mathbb{R}^n)$. Let x and y be two distinct elements of the interior of M such that

$$\varphi(x) = \varphi(y).$$

We consider a chart $g : B^n(0,2) \to M$ such that $y \notin \operatorname{Im} g$ (We recall that $B^n(0,2)$ is the open ball of radius 2 centered at the origin). For all $r > 1$, we set

$$S_r^{n-1} = g(S^{n-1}(0,r)),$$

and

$$K_r = \{x\} \times S_r^{n-1} \cup \{y\} \times S_r^{n-1}.$$

Assume that

$$\varphi \circ g(S_r^{n-1}) \cap \varphi(x) = \emptyset, \tag{22}$$

then, K_r is a compact subset of ${d_\varphi}^{-1}(\mathbb{R}^n_*)$ and as φ belongs to the set of admissible deformations $\mathcal{A}_\phi(j_M)$, we have

$$\int_{\{x\}\times g(S_r^{n-1})} d_\varphi^*(\phi_{\mathbb{R}^n_*}) = \int_{\{x\}\times g(S_r^{n-1})} d_{j_M}^*(\phi_{\mathbb{R}^n_*}) = 1,$$

and

$$\int_{\{y\}\times g(S_r^{n-1})} d_\varphi^*(\phi_{\mathbb{R}^n_*}) = \int_{\{y\}\times g(S_r^{n-1})} d_{j_M}^*(\phi_{\mathbb{R}^n_*}) = 0,$$

On the other hand, as $\varphi(x) = \varphi(y)$, we have

$$\int_{\{x\}\times g(S_r^{n-1})} d_\varphi^*(\phi_{\mathbb{R}^n_*}) = \int_{\{y\}\times g(S_r^{n-1})} d_\varphi^*(\phi_{\mathbb{R}^n_*}),$$

which can not hold together with the previous equations. Hence, our assumption (22) is false. For all positive real $r < 1$, we have

$$\varphi \circ g(S_r^{n-1}) \cap \varphi(x) \neq \emptyset.$$

We just have shown that for all $x \in M \setminus \partial M$,

$$\operatorname{Card}(\varphi^{-1}(x)) > 1 \Rightarrow \operatorname{Card}(\varphi^{-1}(x)) = +\infty \tag{23}$$

(Here and in the following of the proof, φ is understood as a mapping from $M \setminus \partial M$ into $\mathbb{R}^n$). Let P be the set of non injective points, that is

$$P = \{z \in \mathbb{R}^n : \ \operatorname{Card}(\varphi^{-1}(z)) > 1\}.$$

For all $z \in P$, from (23) , we deduce that $\operatorname{Card}(\varphi^{-1}(z)) = +\infty$. Moreover, a theorem from Marcus & Mizel [8] shows that as $\varphi \in W^{1,p}(\Omega, \mathbb{R}^n)$ with $p > n$,

$$\int_M |\det(\nabla\varphi)| dx = \int_{\varphi(M)} \operatorname{Card}(\varphi^{-1}(z)) dz.$$

It brings up

$$\begin{aligned}\int_M |\det(\nabla\varphi)| dx &= \int_P \operatorname{Card}(\varphi^{-1}(z)) dz + \int_{\varphi(M)\setminus P} \operatorname{Card}(\varphi^{-1}(z)) dz \\ &= +\infty |P| + \operatorname{Vol}(\operatorname{Im}(\varphi)) - |P|.\end{aligned}$$

It implies that the measure $|P|$ of the set of non injective points is zero and that

$$\int_M |\det(\nabla\varphi)| dx = \operatorname{Vol}(\operatorname{Im}(\varphi)).$$

Proposition 9. *Let M be a n-dimensional differentiable submanifold of $\mathbb{R}^n$ (with boundary). Every deformation $\varphi \in \mathcal{A}_\phi(j_{_M}) \cap W^{1,p}(M; \mathbb{R}^n)$ is such that* $\det(\nabla\varphi) \geq 0$.

Proof. Let $z \in \mathbb{R}^n \setminus \varphi(\partial M)$, the degree $\deg(\varphi, M, z)$ is defined as

$$\deg(\varphi, M, z) = \int_{\partial M} d(\varphi, z, \partial M)^*(\phi_{\mathbb{R}^n_*})$$

where

$$\begin{aligned} d(\varphi, z, \partial M) : \partial M &\to \mathbb{R}^n_* \\ x &\mapsto \varphi(x) - z. \end{aligned}$$

The following formula holds (see [11], Corollary 1)

$$\int_\Omega \det(\nabla\varphi) dx = \int_{\mathbb{R}^n} \deg(\varphi, M, z) dz. \tag{24}$$

Furthermore, if $z \in \mathrm{Im}(\varphi)$, there exists $y \in M$ such that $\varphi(y) = z$, and

$$\begin{aligned} \deg(\varphi, \partial M, z) &= \int_{\partial M} d(\varphi, z, \partial M)^*(\phi_{\mathbb{R}^n_*}) \\ &= \int_{\partial M} (p_y \circ d_{\varphi,U})^*(\phi_{\mathbb{R}^n_*}) \end{aligned}$$

where $p_y : \partial M \to M \times M$ is defined by $p_y(t) = (t, y)$ and U is a neighborhood of $\partial M \times \{y\}$. Thus,

$$\deg(\varphi, \partial M, z) = \int_{\partial M} d^*_{\varphi,U} \circ p^*_y(\phi_{\mathbb{R}^n_*})$$

As φ belongs to $\mathcal{A}_\phi(j_{_M})$, we have

$$\begin{aligned} \deg(\varphi, \partial M, z) &= \int_{\partial M} d^*_{j_M,U} \circ p^*_y(\phi_{\mathbb{R}^n_*}) \\ &= \int_{\partial M} d(j_{_M}, j_{_M}(y), \partial M)^*(\phi_{\mathbb{R}^n_*}) \\ &= \deg(j_{_M}, \partial M, j_{_M}(y)) \\ &= 1. \end{aligned}$$

Moreover, if $z \notin \mathrm{Im}(\varphi)$, $\deg(\varphi, \partial M, z) = 0$. From (24), it follows that

$$\int_M \det(\nabla\varphi) dx = \int_{\varphi(M)} 1 dx = \mathrm{Vol}(\mathrm{Im}(\varphi)),$$

and from Proposition 8, we deduce that

$$\int_M \det(\nabla\varphi) dx = \int_M |\det(\nabla\varphi)| dx.$$

Hence, $\det(\nabla\varphi) \geq 0$ almost everywhere.

5.2. Case of thin structures in $\mathbb{R}^2$

In this section, we state a partial equivalence result between the minimization problem and the Euler for thin structures moving in $\mathbb{R}^2$. To avoid unnecessary technicalities, we will only consider the case of one not clamped circle ($M = S^1$ and $N = \emptyset$). The results given could be extended to other cases, that is $M = [-1, 1]$ or $N \neq \emptyset$ or even to a problem involving several bodies. Let us first set the main result of this section

Proposition 10. *Let $M = S^1$ and $N = \emptyset$. Let $\varphi : S^1 \to \mathbb{R}^2$ be an immersion, solution of the minimization problem:*

$$\varphi \in \mathcal{A}^p_\phi(j_{_M}),\ I(\varphi) = \inf_{\psi \in \mathcal{A}^p_\phi(j_{_M})} I(\psi), \qquad (\mathcal{P}_\phi)$$

where the functional I and the set $\mathcal{A}^p_\phi(j_{_M})$ are defined as in (12) and (13). Then, for all $z \in \mathrm{Im}(\varphi)$, there exists a family $(x_0, \dots, x_N)$ of elements of S^1 given by Proposition 14, and a family $(\lambda_{-1}, \dots, \lambda_N)$ of nonnegative reals such that

$$\varphi^{-1}(z) = \{x_0, \dots, x_N\},$$

$$\lambda_{-1} = \lambda_N = 0$$

and for all $k \in \{0, \dots, N\}$,

$$-\frac{dDW(\dot{\varphi})}{dx}(x_k) = f(x_k) + (\lambda_{k-1} - \lambda_k)|\dot{\varphi}(x_k)|n,$$

where n is a unitary normal to $\mathrm{Im}(\varphi)$ *independent of k.*

This result relies on Proposition 14, which will be stated later. This proposition just gives an effective (and unique) way to order $\varphi^{-1}(z)$.

The proof of Proposition 10 is rather long and technical, even if it is not difficult. The main idea is to give a geometrical definition of the ϕ-admissible set of deformations. Next, we prove that this definition is equivalent with the algebraic one (used before to prove the existence) for the immersion. Then, using the geometric definition, we prove that the set of regular embeddings isotopic to $j_{_{S^1}}$ are dense in the set of ϕ-admissible immersions for the $\mathcal{C}^1$ topology. This allows us to build enough "variations" around the minimizer in the set of ϕ-admissible deformations $\mathcal{A}_\phi(j_{_{S^1}})$ to obtain the Euler-Lagrange equations. Moreover, we prove that (see Corollary 1), that the ϕ-admissible immersions are the admissible immersions, that is

$$\mathcal{A}_\phi(j_{_M}) \cap \mathrm{Imm}(S^1; \mathbb{R}^2) = \mathcal{A}(j_{_M}) \cap \mathrm{Imm}(S^1; \mathbb{R}^2).$$

The complete proof is postponed to the next section.

Remark 13. Actually, we conjecture that $\mathcal{A}_\phi(j_{_M}) = \mathcal{A}(j_{_M})$ in the case $n = 2$.

5.3. The case of shells

In the case of shells, that is $n = 3$ and $\dim(M) = 2$, an equivalence result between the minimization problem $(\mathcal{P}_\phi)$ and the Euler-Lagrange equations associated to elastic shells, for which frictionless contacts are allowed is not expected. Indeed, even if a solution of the minimization problem $(\mathcal{P}_\phi)$ has no transversal self-intersection, it might have non-transversal self-intersections. In such a case, not enough test functions can be build to recover the Euler-Lagrange equations. Let us give an example of a deformation which belongs to the set of admissible deformations $\mathcal{A}_\phi(j_{_M})$, but which has self-intersections and does not belong to the admissible set $\mathcal{A}(j_{_M})$. Let $M = S^1 \times [0,1]$. We define the reference deformation of M as the mapping

$$\begin{aligned} j_{_M} : M &\to \mathbb{R}^3 \\ (\theta, h) &\mapsto (\cos(\theta), \sin(\theta), h). \end{aligned}$$

Let k be an integer and φ_k the deformation of M defined by

$$\varphi_k(\theta, h) = (\cos(k\theta), \sin(k\theta), h).$$

One can easily show that for every k, the deformation φ_k belongs to $\mathcal{A}_\phi(j_{_M})$. Nevertheless, if $|k| > 1$, it does not belong to the $\mathcal{C}^0$-closure of the embeddings. This counterexample is very similar to the one that we have met (and solved) in the study of self-contacts of a body homeomorphic to S^1, moving in $\mathbb{R}^2$ (see section 3.1).

6. Proof of Proposition 10

6.1. A geometric criterion of non self-intersection

In all this section φ is an immersion from S^1 into $\mathbb{R}^2$.

6.1.1. Contact set

We define the contact set of φ as

$$K^*(\varphi) = \left\{(x,y) \in S^1 \times S^1 \ : \ \varphi(x) = \varphi(y) \text{ and } x \neq y\right\}.$$

We denote $\mathcal{E}(\varphi)$ the set of connected components of $K^*(\varphi)$. Let Λ_φ be the function which maps every element of $K^*(\varphi)$ to the connected component it belongs.

6.1.2. Tubular neighborhood

One can define a tubular neighborhood $g_\varphi : S^1\times]-T,T[\to \mathbb{R}^2$ of φ with the same regularity by setting

$$g_\varphi(x,t) = (\varphi_1(x) - \varphi_2(x+t) + \varphi_2(x), \varphi_2(x) + \varphi_1(x+t) - \varphi_1(x)). \quad (25)$$

In this definition, we have implicitly identified S^1 to $\mathbb{R}/2\pi\mathbb{Z}$ to give a meaning to the addition between an element x of S^1 and an element t of $\mathbb{R}$. For T small enough, g_φ is a local diffeomorphism at any point and

$$D_{(x,0)}g_\varphi = |\dot{\varphi}(x)|(\tau_x, n_x), \quad (26)$$

where τ_x is the tangent unitary vector $\dot{\varphi}(x)/|\dot{\varphi}(x)|$ and n_x is the normal vector to τ_x such that (τ_x, n_x) is a direct base of $\mathbb{R}^2$. As S^1 is compact, there exists δ_x, δ_h and r_0 nonnegative reals such that for every $x \in S^1$, if

$$V_x(\varphi) =]x - \delta_x, x + \delta_x[\times] - \delta_h, \delta_h[,$$

then $g_{\varphi|V_x(\varphi)}$ is a diffeomorphism on its image such that the ball $B(\varphi(x), r_0)$ of radius r_0 centered at $\varphi(x)$ is included in $g_\varphi(V_x(\varphi))$. For simplicity, we have dropped the dependence of δ_x, δ_h and r_0 on φ. In the following, this dependence will often be understood.

For all $x \in S^1$, we define the mappings

$$\begin{array}{rcl} H_x(\varphi) : B(\varphi(x), r_0) & \to &]-T;T[\\ z & \mapsto & P_{\mathbb{R}} \circ (g_{\varphi|V_x})^{-1}(z) \end{array}$$

and

$$\begin{array}{rcl} \Pi_x(\varphi) : B(\varphi(x), r_0) & \to & S^1 \\ z & \mapsto & P_{S^1} \circ (g_{\varphi|V_x})^{-1}(z). \end{array} \quad (27)$$

where P_{S^1} and $P_{\mathbb{R}}$ are the projections of $S^1 \times \mathbb{R}$ on S^1 and $\mathbb{R}$ respectively. We define a neighborhood of $K^*(\varphi)$

$$V_{K^*(\varphi)} = \{(x,y) \in S^1 \times S^1; |\varphi(x) - \varphi(y)| < r_0\}. \quad (28)$$

Finally, we set

$$\begin{array}{rcl} h_\varphi : V_{K^*(\varphi)} & \to &]-T;T[\\ (x,y) & \mapsto & H_x(\varphi(y)). \end{array}$$

For all element (x,y) of $V_{K^*(\varphi)}$, $h_\varphi(x,y)$ is the ordinate of the point $\varphi(y)$ in the local coordinate system induced by the tubular neighborhood g_φ at the point x.

Remark 14. Whenever x and x' are close enough, $\Pi_x(\varphi) = \Pi_{x'}(\varphi)$ and $H_x(\varphi) = H_{x'}(\varphi)$ on their common set of definition. In particular, $h_\varphi(x,y) = h_\varphi(x',y)$. In the following, we will make an extensive use of this simple remark.

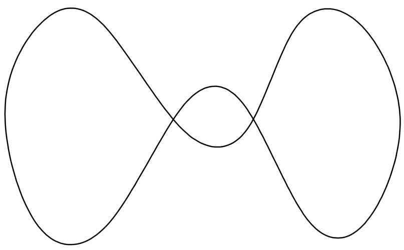

Fig. 5. Deformation with transversal self-intersection

6.1.3. The geometric definition of smooth admissible deformations

Let $\mathcal{P}_1(\varphi)$ and $\mathcal{P}_2(\varphi)$ be the following propositions:

$\mathcal{P}_1(\varphi)$: For all $\Lambda \in \mathcal{E}(\varphi)$ connected component of $K^*(\varphi)$, there exists $(x,y) \in \Lambda$ such that for every neighborhood $V \subset V_{K^*(\varphi)}$ of (x,y), $h_{\varphi|V} \neq 0$.

$\mathcal{P}_2(\varphi)$: For all $\Lambda \in \mathcal{E}(\varphi)$ connected component of $K^*(\varphi)$, there exists a neighborhood $V_\varphi(\Lambda)$ of Λ such that either $h_\varphi(x,y) \geq 0$ for all $(x,y) \in V_\varphi(\Lambda)$, or $h_\varphi(x,y) \leq 0$ for all $(x,y) \in V_\varphi(\Lambda)$.

If $\mathcal{P}_2(\varphi)$ is true, then the ordinate $h_\varphi(x,y)$ of the point $\varphi(y)$ in the local coordinate system induced by g_φ has a constant sign in the neighborhood of Λ. Proposition $\mathcal{P}_1$ make sure that h_φ can not be equal to zero on a neighborhood of Λ. In other words, if two parts of S^1 are in contact under φ, they must separate somewhere. We define the set of immersions without self-intersection as

$$\mathcal{A}_G = \{\varphi \in \mathrm{Imm}(S^1;\mathbb{R}^2); \mathcal{P}_1(\varphi) \text{ and } \mathcal{P}_2(\varphi) \text{ are true } \}.$$

6.1.4. Examples

It is not completely obvious that both $\mathcal{P}_1(\varphi)$ and $\mathcal{P}_2(\varphi)$ have to be true in order for the immersion φ to be without self-intersection. If the deformation φ has a transversal self-intersection, proposition $\mathcal{P}_2(\varphi)$ is obviously false. On the other hand, even if proposition $\mathcal{P}_2(\varphi)$ is true, the deformation φ could have a degenerate self-intersection. Typically, the deformation

$$\begin{aligned} \varphi : S^1 &\to \mathbb{R}^2 \\ \theta &\mapsto (\cos(2\theta), \sin(2\theta)) \end{aligned}$$

has a self-intersection whereas $\mathcal{P}_2(\varphi)$ is true. However, in this case, $\mathcal{P}_1(\varphi)$ is false, hence φ does not belong to $\mathcal{A}_G$.

6.1.5. Description of the contacts

Let σ be the mapping from $S^1 \times S^1$ into itself which maps every couple (x,y) to (y,x). For every immersion $\varphi \in \mathcal{A}_G$, we define two mappings P_φ

and O_φ from $\mathcal{E}(\varphi)$ into $\{-1;+1\}$ by

$$\begin{cases} P_\varphi(\Lambda) = \mathrm{sign}(h_\varphi(x,y)) & \text{where} \quad (x,y) \in V_\varphi(\Lambda) \text{ and } h_\varphi(x,y) \neq 0 \\ O_\varphi(\Lambda) = n_x.n_y & \text{where} \quad (x,y) \in \Lambda. \end{cases}$$

The existence of a couple $(x,y) \in V_\varphi(\Lambda)$ such that $h_\varphi(x,y) \neq 0$ is ensured by proposition $\mathcal{P}_1(\varphi)$. The proposition $\mathcal{P}_2(\varphi)$ makes sure us that the sign of $h_\varphi(x,y)$ does not depend on the choice of the couple (x,y) made to define $P_\varphi(\Lambda)$. Moreover, for all couples $(x,y) \in \Lambda$, $n_x.n_y = \pm 1$ (see Lemma 1). As Λ is a connected and $n_x.n_y$ is continuous, the scalar product $n_x.n_y$ is constant on Λ. Hence, the mapping O_φ is also well defined.

Lemma 1. *Let $\varphi \in \mathcal{A}_G$, for all $(x,y) \in K^*(\varphi)$,*

$$n_x.\tau_y = 0.$$

Proof. The canonical projection of $\mathbb{R}^2$ on the second coordinate is noted p_2. Let φ be an element of $\mathcal{A}_G$. Let $(x,y) \in S^1 \times S^1$ be such that $(x,y) \in K^*(\varphi)$, that is $\varphi(x) = \varphi(y)$ and $x \neq y$. We recall that $V_{K^*(\varphi)}$ is a neighborhood of $K^*(\varphi)$. Therefore, there exists $\varepsilon > 0$ such that $\{x\} \times]y-\varepsilon, y+\varepsilon[\subset V_{K^*(\varphi)}$. For all $t \in]-\varepsilon, \varepsilon[$, we have

$$\begin{aligned} h_\varphi(x, y+t) &= P_\mathbb{R} \circ g_{\varphi|V_x}^{-1}(\varphi(y+t)) \\ &= P_\mathbb{R} \circ g_{\varphi|V_x}^{-1}(\varphi(y)) + D_{\varphi(y)}(P_\mathbb{R} \circ g_{\varphi|V_x}^{-1})\dot{\varphi}(y)t + o(t) \\ &= D_{\varphi(y)}(P_\mathbb{R} \circ g_{\varphi|V_x}^{-1})\dot{\varphi}(y)t + o(t) \\ &= p_2((D_{(x,0)}g_\varphi)^{-1}\dot{\varphi}(y))t + o(t) \end{aligned}$$

From (26), $D_{(x,0)}g = |\dot{\varphi}(x)|(\tau_x, n_x)$. Thus,

$$(D_{(x,0)}g_\varphi)^{-1} = |\dot{\varphi}(x)|^{-1} \, (\tau_x, \, n_x)$$

and

$$h_\varphi(x, y+t) = n_x . \frac{\dot{\varphi}(y)}{|\dot{\varphi}(x)|} t + o(t) = \frac{|\dot{\varphi}(y)|}{|\dot{\varphi}(x)|}(n_x.\tau_y)t + o(t).$$

As $\mathcal{P}_2(\varphi)$ is true, the sign of h is constant on a neighborhood of (x,y), hence $n_x.\tau_y = 0$.

At this stage, it could be useful to give a little example in order to illustrate the different objects introduced.

In the case represented in Figure 6, the deformation has a self-contact and without self-intersecting. The set $\mathcal{E}(\varphi)$ has two elements: the two connected components Λ and $\sigma(\Lambda)$ of the contact set $K^*(\varphi)$ (see figure 7). The value $O_\varphi(\Lambda)$ indicates whether or not the two subsets of S^1 put in contact under the deformation φ are oriented in the same direction. In this particular case, their orientations are opposite, thus

$$O_\varphi(\Lambda) = O_\varphi(\sigma(\Lambda)) = -1.$$

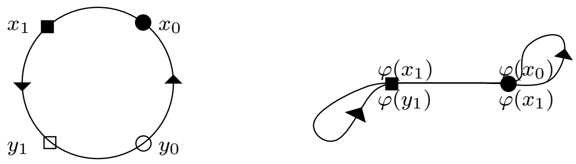

Fig. 6. A deformation with contact

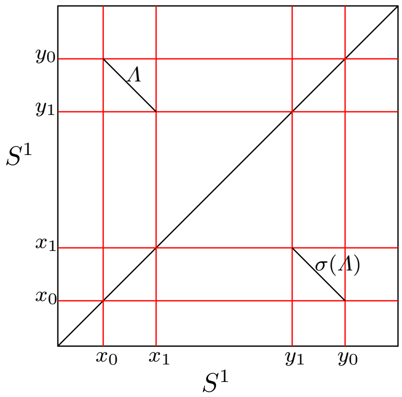

Fig. 7. The contact set $K^*(\varphi)$

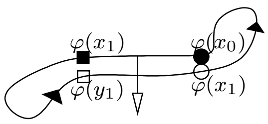

Fig. 8. How to remove the contact ?

As for the value of $P_\varphi(\Lambda)$, it indicates in which direction along the normal one has to pull away the segment $\varphi([y_1, y_0])$ from $\varphi([x_0, x_1])$ in order to remove the contact. In the case represented, $P_\varphi(\Lambda) = +1$. The segment $\varphi([y_1, y_0])$ has to be pushed in the direction of the normal to the segment $\varphi([x_0, x_1])$ as shown in the figure 8.

It remains now to study some basic properties fulfilled by the functions O_φ and P_φ.

Proposition 11. *For all deformations $\varphi \in \mathcal{A}_G$, for all $\Lambda \in \mathcal{E}(\varphi)$,*

$$O_\varphi(\Lambda) = O_\varphi(\sigma(\Lambda)).$$

Proof. Let $\Lambda \in \mathcal{E}(\varphi)$ and $(x, y) \in \Lambda$,

$$O_\varphi(\Lambda) = n_x.n_y = n_y.n_x = O_\varphi(\sigma(\Lambda)).$$

Proposition 12. *For all deformations $\varphi \in \mathcal{A}_G$, for all $\Lambda \in \mathcal{E}(\varphi)$,*

$$P_\varphi(\sigma(\Lambda)) = -O_\varphi(\Lambda) P_\varphi(\Lambda).$$

Proof. Let $\varphi \in \mathcal{A}_G$. From Proposition $\mathcal{P}_1(\varphi)$, there exists a couple $(x, y) \in \Lambda$ such that the restriction of h_φ to any neighborhood of (y, x) is not equal to zero. There exists a neighborhood W of $\varphi(x) = \varphi(y)$, U_x and U_y small enough neighborhoods of $x \times 0$ and $y \times 0$ respectively in $S^1 \times \mathbb{R}$ such that $g_{\varphi|U_x}$ and $g_{\varphi|U_y}$ are diffeomorphisms into W.

For all $\overline{y}$ and $\overline{x}$ belonging respectively to small enough neighborhoods of y and x,

$$\operatorname{sign}(h(\overline{x}, \overline{y})) = P_\varphi(\Lambda) \text{ or } \operatorname{sign}(h(\overline{x}, \overline{y})) = 0.$$

Thence,

$$g_\varphi(U_y \cap S^1 \times \{0\}) \cap g_\varphi(U_x \cap S^1 \times \mathbb{R}_*^{-P_\varphi(\Lambda)}) = \emptyset, \tag{29}$$

where g_φ is the tubular neighborhood defined at (25). One can choose U_x such that $g_\varphi(U_x \cap S^1 \times \mathbb{R}_*^{-P_\varphi(\Lambda)})$ is connected. It follows from (29) that one of those two situations occurs:

$$g_\varphi(U_x \cap S^1 \times \mathbb{R}_*^{-P_\varphi(\Lambda)}) \subset \begin{cases} g_\varphi(U_y \cap S^1 \times \mathbb{R}_*^+) \\ \text{or} \\ g_\varphi(U_y \cap S^1 \times \mathbb{R}_*^-). \end{cases}$$

A straightforward computation leads to

$$(g_{\varphi|U_y})^{-1} \circ g_\varphi(x, t) = (y, 0) + (0, t|\dot{\varphi}(x)||\dot{\varphi}(y)|^{-1} n_x(\varphi).n_y(\varphi)) + o(t).$$

Thus, for t small enough, such that $\operatorname{sign}(t) = -P_\varphi(\Lambda)$, we have

$$(g_{\varphi|U_y})^{-1} \circ g_\varphi(x, t) \in U_y \cap S^1 \times \mathbb{R}_*^{-(n_x.n_y)P_\varphi(\Lambda)}.$$

In other words,

$$g_\varphi\left(U_x \cap S^1 \times \mathbb{R}_*^{-P_\varphi(\Lambda)}\right) \subset g_\varphi\left(U_y \cap S^1 \times \mathbb{R}_*^{-(n_x.n_y)P_\varphi(\Lambda)}\right).$$

As $g_{\varphi|U_x}$ and $g_{\varphi|U_y}$ are diffeomorphisms, we deduce that

$$g_\varphi\left(U_x \cap S^1 \times \mathbb{R}^{-P_\varphi(\Lambda)}\right) \subset g_\varphi(U_y \cap S^1 \times \mathbb{R}^{-(n_x.n_y)P_\varphi(\Lambda)}). \tag{30}$$

There exists $(\widetilde{y}, \widetilde{x})$ in the neighborhood of (y, x) such that

$$h_\varphi(\widetilde{y}, \widetilde{x}) = h_\varphi(y, \widetilde{x}) \neq 0.$$

We have

$$\varphi(\widetilde{x}) \in g_\varphi(U_x \cap S^1 \times \{0\}) \subset g_\varphi(U_x \cap S^1 \times \mathbb{R}^{-P_\varphi(\Lambda)}).$$

Finally, (30) implies that

$$\varphi(\widetilde{x}) \in g_\varphi(U_y \cap S^1 \times \mathbb{R}^{-(n_x.n_y)P_\varphi(\Lambda)}),$$

and

$$\begin{aligned} P_\varphi(\sigma(\Lambda)) &= h_\varphi(\widetilde{y}, \widetilde{x}) = h_\varphi(y, \widetilde{x}), \\ &= -(n_x.n_y)P_\varphi(\Lambda) = -O_\varphi(\Lambda)P_\varphi(\Lambda). \end{aligned}$$

Let us introduce another notation. For every $\varphi \in \mathcal{A}_G$, for every couple $(x, y) \in K^*(\varphi)$, we will denote $\Lambda_\varphi(x, y)$ the connected component of $K^*(\varphi)$ to which (x, y) belongs. We now state a technical proposition

Proposition 13. *For all deformations $\varphi \in \mathcal{A}_G$, for all elements x, x_- and x_+ of S^1 such that*

$$P_\varphi(\Lambda_\varphi(x, x_-)) = -P_\varphi(\Lambda_\varphi(x, x_+)),$$

we have

$$P_\varphi(\Lambda_\varphi(x_-, x_+)) = P_\varphi(\Lambda_\varphi(x_-, x)).$$

Proof. Up to swapping x_- and x_+, one can assume that $P_\varphi(\Lambda_\varphi(x, x_+)) = +1$. Furthermore, by Lemma 2 below, we can also assume that for all neighborhoods of (x_-, x_+), $h_\varphi \neq 0$. Following the same procedure as in the proof of proposition 12, one can find V_x, V_{x_-}, V_{x_+} neighborhoods of $(x, 0)$, $(x_-, 0)$ and $(x_+, 0)$ in $S^1 \times \mathbb{R}$ respectively such that

$$g_\varphi(V_{x_+} \cap S^1 \times \{0\}) \subset g_\varphi(V_x \cap S^1 \times \mathbb{R}^+) \tag{31}$$

and

$$g_\varphi(V_x \cap S^1 \times \mathbb{R}^+) \subset g_\varphi(V_{x_-} \cap S^1 \times \mathbb{R}^{P_\varphi(\Lambda_\varphi(x_-, x))}). \tag{32}$$

There exists $(\overline{x}_-, \overline{x}_+)$, in a small neighborhood of (x_-, x_+) such that

$$h_\varphi(\overline{x}_-, \overline{x}_+) \neq 0.$$

From (31) and (32),

$$\varphi(\overline{x}_+) \in g_\varphi(V_{x_-} \cap S^1 \times \mathbb{R}^{P_\varphi(\Lambda_\varphi(x_-, x))})$$

thus,

$$\operatorname{sign}(h_\varphi(x_-, \overline{x}_+)) = \begin{cases} P_\varphi(\Lambda_\varphi(x_-, x)) \\ \text{or} \\ 0 \end{cases}$$

Moreover, $h_\varphi(x_-, \overline{x}_+) = h_\varphi(\overline{x}_-, \overline{x}_+) \neq 0$. Hence,

$$\begin{aligned} P_\varphi(\Lambda_\varphi(x_-, x_+)) &= \operatorname{sign}(h_\varphi(\overline{x}_-, \overline{x}_+)) \\ &= \operatorname{sign}(h_\varphi(x_-, \overline{x}_+)) = P_\varphi(\Lambda_\varphi(x_-, x)). \end{aligned}$$

Lemma 2. *For all deformations $\varphi \in \mathcal{A}_G$, for all elements x, x_- and x_+ of S^1 such that $\varphi(x) = \varphi(x_-) = \varphi(x_+)$, and*

$$P_\varphi(\Lambda_\varphi(x, x_-)) = -P_\varphi(\Lambda_\varphi(x, x_+))$$

there exists $\widetilde{x}$, $\widetilde{x}_+$ and $\widetilde{x}_-$ in S^1 such that

$$\varphi(\widetilde{x}) = \varphi(\widetilde{x}_-) = \varphi(\widetilde{x}_+),$$

$$\Lambda_\varphi(\widetilde{x}, \widetilde{x}_-) = \Lambda_\varphi(x, x_-)\,;\, \Lambda_\varphi(\widetilde{x}, \widetilde{x}_+) = \Lambda_\varphi(x, x_+)$$
$$\Lambda_\varphi(\widetilde{x}_-, \widetilde{x}_+) = \Lambda_\varphi(x_-, x_+),$$

and for all neighborhood of $(\widetilde{x}_-, \widetilde{x}_+)$, $h_\varphi \neq 0$.

Proof. Let φ, x, x_- and x_+ such be as in Lemma. Without loss of generality, one can assume that

$$P_\varphi(\Lambda_\varphi(x, x_-)) = -1 \text{ and } P_\varphi(\Lambda_\varphi(x, x_+)) = +1.$$

We set

$$\Lambda = \Big\{(\bar{x}_-, \bar{x}_+) \in \Lambda_\varphi(x_-, x_+) : \exists \bar{x} \in S^1 \text{ such that } (\bar{x}, \bar{x}_-) \in \Lambda_\varphi(x, x_-),$$
$$(\bar{x}, \bar{x}_+) \in \Lambda_\varphi(x, x_+), \text{ and } h_\varphi = 0 \text{ on a neighborhood of } (\bar{x}_-, \bar{x}_+)\Big\}.$$

Assume that the lemma is false, then Λ is a closed non-empty subset of $\Lambda_\varphi(x_-, x_+)$. Furthermore, we will show that Λ is also an open subset of $\Lambda_\varphi(x_-, x_+)$. As $\Lambda_\varphi(x_-, x_+)$ is connected, this would imply that

$$\Lambda = \Lambda_\varphi(x_-, x_+)$$

and that for all $(\overline{x}_-, \overline{x}_+) \in \Lambda_\varphi(x_-, x_+)$,

$$h_\varphi = 0 \text{ on a neighborhood of } (\overline{x}_-, \overline{x}_+).$$

As $\mathcal{P}_1(\varphi)$ is true, this cannot hold.

It remains to prove that Λ is actually an open subset of $\Lambda_\varphi(x_-, x_+)$. Let $(\overline{x}_-, \overline{x}_+) \in \Lambda$. Let W be a small enough connected neighborhood of $(\overline{x}_-, \overline{x}_+)$ in $\Lambda_\varphi(x_-, x_+)$. Let $(\widetilde{x}_-, \widetilde{x}_+) \in W$. We set

$$\widetilde{x} = \Pi_{\overline{x}}(\varphi)(\varphi(\widetilde{x}_+)) = \Pi_x(\varphi)(\varphi(\widetilde{x}_+))$$

(see (27) for the definition of $\Pi_x(\varphi)$). First of all, as $(\widetilde{x}_-, \widetilde{x}_+)$ belongs to $\Lambda_\varphi(x_-, x_+)$, we have $\varphi(\widetilde{x}_+) = \varphi(\widetilde{x}_-)$ and

$$h_\varphi(\widetilde{x}, \widetilde{x}_+) = h_\varphi(\widetilde{x}, \widetilde{x}_-). \tag{33}$$

As $(\widetilde{x}, \widetilde{x}_+)$ belongs to a small neighborhood of $(\overline{x}, \overline{x}_+) \in \Lambda_\varphi(x, x_+)$ and as $P_\varphi(\Lambda_\varphi(x, x_+)) = +1$, we have

$$h_\varphi(\widetilde{x}, \widetilde{x}_+) \geq 0. \tag{34}$$

On the other hand, we get

$$h_\varphi(\widetilde{x}, \widetilde{x}_-) \leq 0. \tag{35}$$

From (33), (34) and (35), we deduce that

$$h_\varphi(\widetilde{x}, \widetilde{x}_+) = h_\varphi(\widetilde{x}, \widetilde{x}_-) = 0.$$

As $\widetilde{x} = \Pi_{\widetilde{x}}(\varphi)(\varphi(\widetilde{x}_+)) = \Pi_{\widetilde{x}}(\varphi)(\varphi(\widetilde{x}_-))$, we get

$$\varphi(\widetilde{x}) = \varphi(\widetilde{x}_-) = \varphi(\widetilde{x}_+).$$

Let π_1 and π_2 be the projections of $S^1 \times S^1$ onto S^1,

$$\pi_1(a, b) = a,$$
$$\pi_2(a, b) = b.$$

We have just shown that

$$(\Pi_{\bar{x}}(\varphi) \circ \varphi \times \pi_1)(W) \subset K^*(\varphi)$$
$$\text{and } (\Pi_{\bar{x}}(\varphi) \circ \varphi \times \pi_2)(W) \subset K^*(\varphi).$$

Furthermore, as W is connected, both $(\Pi_{\bar{x}}(\varphi) \circ \varphi \times \pi_1)(W)$ and $(\Pi_{\bar{x}}(\varphi) \circ \varphi \times \pi_2)(W)$ are connected. Thus,

$$(\Pi_{\bar{x}}(\varphi) \circ \varphi \times \pi_1)(W) \subset \Lambda_\varphi(\Pi_{\bar{x}}(\varphi) \circ \varphi \times \pi_1(\bar{x}_-, \bar{x}_+)) = \Lambda_\varphi(\bar{x}, \bar{x}_-).$$

We have as well,

$$(\Pi_{\bar{x}}(\varphi) \circ \varphi \times \pi_2)(W) \subset \Lambda_\varphi(\bar{x}, \bar{x}_+).$$

It follows from these latter relations that every element of W belongs to Λ. Hence, Λ is an open subset of $\Lambda_\varphi(x_-, x_+)$ and the proof is complete.

We are now in a position to state and prove the proposition called in the main result of this section (Proposition 10).

Proposition 14. *For all $\varphi \in \mathcal{A}_G$, for all $z \in \mathrm{Im}(z)$, and for all unitary vector n orthogonal to $\mathrm{Im}(\varphi)$ at z, there exists a family $(x_i)_{i=0,\ldots,N}$ of elements of S^1 such that*

$$\varphi^{-1}(z) = \{x_0, \ldots, x_N\};$$

then for all l, $k \in \{0, \ldots, N\}$ such that $l > k$, we have

$$P_\varphi(\Lambda_\varphi(x_k, x_l)) = n.n_{x_k}(\varphi).$$

Proof. Let $\varphi \in \mathcal{A}_G$, $z \in \mathbb{R}^2$ and n a unitary vector orthogonal to $\mathrm{Im}(z)$ at z. The relation

$$x \preceq y \Leftrightarrow ((x = y) \text{ or } (x \neq y \text{ and } P_\varphi(\Lambda_\varphi(x, y)) = n.n_x(\varphi)))$$

is a total ordering of the set $\varphi^{-1}(z)$. Indeed,

1. Let $x,\ y \in \varphi^{-1}(z)$ be such that we do not have $x \preceq y$. Then, $x \neq y$. Thus,
$$P_\varphi(\Lambda_\varphi(x,y)) = -n.n_x(\varphi).$$
From Proposition 12, we get
$$\begin{aligned}P_\varphi(\Lambda_\varphi(y,x)) &= -O_\varphi(\Lambda_\varphi(y,x))(-n.n_x)\\ &= (n_x.n_y)(n.n_x) = n.n_y,\end{aligned}$$
and $y \preceq x$. We have shown that
$$\boxed{x \preceq y \text{ or } y \preceq x}.$$

2. Let $x, y \in \varphi^{-1}(z)$ be such that $x \preceq y$ and $x \neq y$. Proceeding as above, we show that $P_\varphi(\Lambda_\varphi(y,x)) = n.n_x$ implies that $P_\varphi(\Lambda_\varphi(x,y)) = -n.n_y$. Therefore, we do not have $y \preceq x$. We have show that
$$\boxed{(x \preceq y \text{ and } y \preceq x) \Rightarrow (x = y)}.$$

3. Let x_-, x and $x_+ \in \varphi^{-1}(z)$ be such that $x_- \preceq x$, $x \preceq x_+$, $x_- \neq x$ and $x \neq x_+$. We have
$$\begin{aligned}P_\varphi(\Lambda_\varphi(x,x_+)) &= n.n_x = (n_x.n_{x_-})(n.n_{x_-})\\ &= O_\varphi(\Lambda_\varphi(x,x_-))P_\varphi(\Lambda_\varphi(x_-,x)));\end{aligned}$$
Proposition 12 and 13, brings up
$$P_\varphi(\Lambda_\varphi(x,x_+)) = -P_\varphi(\Lambda_\varphi(x,x_-)).$$
and
$$P_\varphi(\Lambda_\varphi(x_-,x_+)) = P_\varphi(\Lambda_\varphi(x_-,x)) = n.n_{x_-},$$
that is $x_- \preceq x_+$. We have shown that
$$\boxed{(x_- \preceq x \text{ and } x \preceq x_+) \Rightarrow (x_- \preceq x_+)}.$$

It remains to sort $\varphi^{-1}(z)$ with this ordering relationship to obtain the family expected.

6.1.6. Inclusion in the closure of the embeddings

We are now in a position to prove the following Proposition

Proposition 15. *The set $\mathcal{A}_G$ is included in the $\mathcal{C}^1$-closure of the embeddings.*

We split the proof of this proposition in several steps. For a given deformation $\varphi \in \mathcal{A}_G(\varphi)$, we will construct explicitly an embedding arbitrarily close to φ for the $\mathcal{C}^1$ topology. To this end, we define a set $\mathcal{B}_G(\varphi)$ which is more or less the set of admissible deformations which have at most as many contact points as φ. Then we construct a finite sequence ψ_k of deformations belonging to $\mathcal{B}_G(\varphi)$, close to φ, removing at each step another contact zone. Finally, we prove that the last deformation obtained has no contact point. In other words, it is an embedding.

For all deformations $\varphi \in \mathcal{A}_G$, we define the set

$$\mathcal{B}_G(\varphi) = \Big\{ \psi : S^1 \to \mathbb{R}^2 \,;\, \psi \text{ immersion} \,;\, K^*(\psi) \subset K^*(\varphi)\,; \\ \text{For all } (x,y) \in K^*(\psi),\ n_x(\psi).n_y(\psi) = O_\varphi(\Lambda_\varphi(x,y)) \\ \text{and } \operatorname{sign}(h_\psi) \neq -P_\varphi(\Lambda_\varphi(x,y)) \\ \text{on a neighborhood } W_\psi(\Lambda_\varphi(x,y)) \text{ of } \Lambda_\varphi(x,y) \cap K^*(\psi) \Big\}. \tag{36}$$

Let $\alpha \in \mathcal{C}^\infty(\mathbb{R};[0,1])$ with support included in $]-1/2,1/2[$ such that $\alpha([-1/4,1/4]) = 1$. For all $\phi \in \mathcal{C}^1(S^1;[0,1])$ such that $\|\phi\|_{\mathcal{C}^0}\|\alpha\|_{\mathcal{C}^1} < 1$ and for all $\varepsilon > 0$, we define the functions $T_\varepsilon^+(\phi)$ and $T_\varepsilon^-(\phi)$ from the normal bundle $\nu(S^1) = S^1 \times \mathbb{R}$ of S^1 into itself by

$$T_\varepsilon^\pm(\phi)(x,t) = (x, t \pm \varepsilon^2 \phi(x)\alpha(t/\varepsilon)).$$

The mappings $T_\varepsilon^\pm(\phi)$ are diffeomorphisms.

Let $\psi \in \mathcal{B}_G(\varphi)$ and $\phi \in \mathcal{C}^1(S^1;[0,1])$ be such that ψ is injective on the support K of ϕ. The function g_ψ is injective on a neighborhood of $K \times \{0\}$. Thus, there exists a neighborhood U of K in S^1 and a real $r > 0$ such that the restriction of g_ψ to $U\times]-r,r[$ defines a diffeomorphism onto its image. We set $V = g_\psi(U\times]-r,r[)$. For all $\varepsilon < r$, we define $F_\varepsilon^+(\phi,\psi)$ and $F_\varepsilon^-(\phi,\psi) : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_\varepsilon^\pm(\phi,\psi)(x) = \begin{cases} g_\psi \circ T_\varepsilon^\pm(\phi) \circ g_\psi{}_{|U\times]-r,r[}^{-1} & \text{if } x \in g_\psi(U\times]-r,r[) = V \\ x & \text{otherwise.} \end{cases}$$

For ε small enough, $F_\varepsilon^+(\phi,\psi)$ and $F_\varepsilon^-(\phi,\psi)$ are diffeomorphisms of $\mathbb{R}^2$ such that

$$F_\varepsilon^\pm(\phi,\psi)(V) = V.$$

Moreover, $F_\varepsilon^\pm$ does not depend on r. The figure 9, illustrates the action of $F_\varepsilon^+(\phi,\psi)$ and $F_\varepsilon^-(\phi,\psi)$ respectively. The tubular neighborhood defines a coordinate system on V. We set

$$\begin{aligned} \Pi_U : V &\to U \\ x &\mapsto P_{S^1} \circ g_\psi{}_{|U\times]-r;r[}^{-1} \end{aligned}$$

and

$$\begin{aligned} H_U : V &\to U \\ x &\mapsto P_{\mathbb{R}} \circ g_\psi{}_{|U\times]-r;r[}^{-1}. \end{aligned}$$

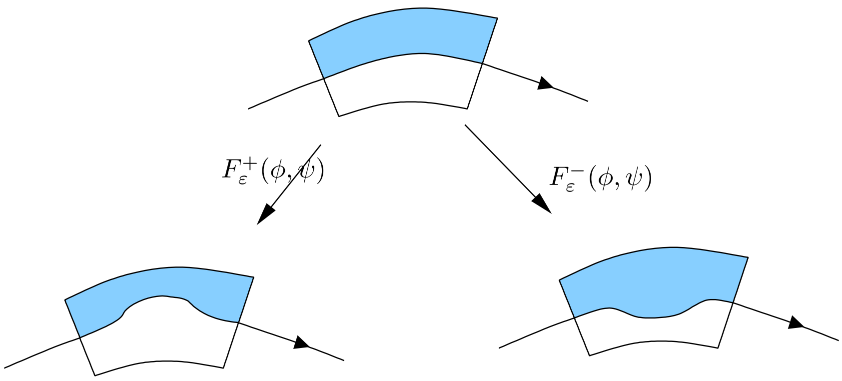

Fig. 9. Action of $F_\varepsilon^\pm(\phi,\psi)$ on $\mathbb{R}^2$.

We now define three different subsets A_1, A_2 and A_3 of S^1 such that $S^1 = A_1 \cup A_2 \cup A_3$ by

$$A_1 = U \cup \psi^{-1}(V^c);\ A_2 = {A_1}^c \cap (H_U \circ \psi)^{-1}(\mathbb{R}_*);\ A_3 = {A_1}^c \cap (H_U \circ \psi)^{-1}(0).$$

Moreover, the intersection between two of these subsets is empty. Finally, we define two deformations $S_\varepsilon^+(\phi,\psi)$ and $S_\varepsilon^-(\phi,\psi)$ from S^1 into $\mathbb{R}^2$ by

- For all $x \in A_1$,
$$S_\varepsilon^\pm(\phi,\psi) = \psi(x);$$
- For all $x \in A_2$,
  If $H_U(\psi(x)) > 0$, we set
$$S_\varepsilon^+(\phi,\psi)(x) = F_\varepsilon^+(\phi,\psi) \circ \psi(x),$$
$$S_\varepsilon^-(\phi,\psi)(x) = \psi(x).$$
  If $H_U(\psi(x)) < 0$, we set
$$S_\varepsilon^+(\phi,\psi)(x) = \psi(x),$$
$$S_\varepsilon^-(\phi,\psi)(x) = F_\varepsilon^-(\phi,\psi) \circ \psi(x).$$
- For all $x \in A_3$, we have $(\Pi_U \circ \psi(x), x) \in K^*(\psi) \subset K^*(\varphi)$. We set $\Lambda = \Lambda_\varphi(\Pi_U \circ \psi(x), x)$.
  If $P_\varphi(\Lambda) = +1$, we set
$$S_\varepsilon^+(\phi,\psi)(x) = F_\varepsilon^+(\phi,\psi) \circ \psi(x),$$
$$S_\varepsilon^-(\phi,\psi)(x) = \psi(x).$$
  If $P_\varphi(\Lambda) = -1$, we set
$$S_\varepsilon^+(\phi,\psi)(x) = \psi(x),$$
$$S_\varepsilon^-(\phi,\psi)(x) = F_\varepsilon^-(\phi,\psi) \circ \psi(x).$$

The deformation $S^+(\phi, \psi)$ is obtained from the deformation ψ by pushing away from $\psi(K)$ the element x of $S^1 \setminus K$ such that $\psi(x)$ is just "located on the top" of $\psi(K)$. The deformation $S^-(\phi, \psi)$ is obtained in the same way, but acts on the point x of $S^1 \setminus K$ "located on the bottom" of $\psi(K)$. Hence, the deformation $S^+(\phi, S^-(\phi, \psi))$ is such that $\psi(K)$ has no double points. Performing this transformation for a partition of the unity ϕ_k of S^1, will enable us to construct an embedding close to ψ. It remains to carry out this reasoning rigorously.

Lemma 3. *For all $\varphi \in \mathcal{A}_G$, for all $\psi \in \mathcal{B}_G(\varphi)$, for all $\phi \in \mathcal{C}^1(S^1; [0, 1])$ such that the restriction of ψ to the support of ϕ is injective, $F_\varepsilon^\pm$ converges to the identity in $\mathbb{R}^2$ for the $\mathcal{C}^1$ topology.*

Lemma 4. *For all $\varphi \in \mathcal{A}_G$, $\psi \in \mathcal{B}_G(\varphi)$, ϕ such that the restriction of ψ to the support of ϕ is injective, we have, for all $x \in S^1$,*

$$\begin{aligned} & S_\varepsilon^\mu(\phi, \psi) = F_\varepsilon^\mu(\phi, \psi) \circ \psi, && \text{on a neighborhood of } x \\ \text{or } & S_\varepsilon^\mu(\phi, \psi) = \psi && \text{on a neighborhood of } x. \end{aligned}$$

Proposition 16. *For all $\varphi \in \mathcal{A}_G$, $\psi \in \mathcal{B}_G$ and $\phi \in \mathcal{C}^1(S^1; [0, 1])$, such that the restriction of ψ to the support of ϕ is injective, $S_\varepsilon^\pm(\phi, \psi)$ is an immersion and*

$$S_\varepsilon^\pm(\phi, \psi) \xrightarrow{\varepsilon \to 0} \psi \text{ in } \mathcal{C}^1(S^1; \mathbb{R}^2).$$

Lemma 5. *For all $\varphi \in \mathcal{A}_G$, $\psi \in \mathcal{B}_G(\varphi)$ and $\phi \in \mathcal{C}^1(S^1; [0, 1])$, such that the restriction of ψ to the support of ϕ is injective, then*

$$K^*(S_\varepsilon^\pm(\phi, \psi)) \subset K^*(\psi).$$

Lemma 6. *For all $\varphi \in \mathcal{A}_G$, $\psi \in \mathcal{B}_G(\varphi)$ and $\phi \in \mathcal{C}^1(S^1; [0, 1])$, such that the restriction of ψ to the support of ϕ is injective, we have*

$$n_x(S_\varepsilon^\pm(\phi, \psi)).n_y(S_\varepsilon^\pm(\phi, \psi)) = O_\varphi(\Lambda_\varphi(x, y)),$$

for all $(x, y) \in K^*(S^\pm(\phi, \psi))$.

Lemma 7. *For all $\varphi \in \mathcal{A}_G$, $\psi \in \mathcal{B}_G$ and $\phi \in \mathcal{C}^1(S^1; [0, 1])$, such that the restriction of ψ to the support of ϕ is injective, we have for all $(x, y) \in K^*(S^\pm(\phi, \psi))$,*

$$\operatorname{sign}(h_{S_\varepsilon^\pm(\phi, \psi)}) \neq -P_\varphi(\Lambda_\varphi(x, y))$$

on a neighborhood of (x, y).

Proposition 17. *For all $\varphi \in \mathcal{A}_G$, $\psi \in \mathcal{B}_G$ and $\phi \in \mathcal{C}^1(S^1; [0, 1])$, such that the restriction of ψ to the support of ϕ is injective, we have*

$$S_\varepsilon^\pm(\phi, \psi) \in \mathcal{B}_G(\varphi).$$

Proposition 18. *For all $\varphi \in \mathcal{A}_G$, $\psi \in \mathcal{B}_G$ and $\phi \in \mathcal{C}^1(S^1;[0,1])$, such that the restriction of ψ to the support K of ϕ is injective. Let $\varepsilon^+ > 0$ and $\varepsilon^- > 0$ small enough for $\widetilde{\psi} = S^-_{\varepsilon^-}(\phi, S^+_{\varepsilon^+}(\phi,\psi))$ to be correctly defined, we have*

$$\widetilde{\psi}(K) \cap \widetilde{\psi}(K^c) = \emptyset.$$

Proposition 19. *For all $\varphi \in \mathcal{A}_G$, $\mathcal{B}_G$ is included in the $\mathcal{C}^1$-closure of the embeddings.*

Proof (Proposition 15). The Proposition follows from the fact that φ belongs to $\mathcal{B}_G(\varphi)$ and Proposition 19.

Proof (Proposition 19). Let $\varphi \in \mathcal{A}_G$. There exists a partition of the unity $(U_k, \phi_k)_{k=1,\dots,N}$ such that for all k, $\phi_k \in \mathcal{C}^\infty(S^1;[0,1])$, U_k is an open subset of S^1, the restriction of the mapping φ to U_k is injective, the support K_k of ϕ_k is included in U_k and $\sum_k \phi_k = 1$.

Let $\psi \in \mathcal{B}_G(\varphi)$ and $\varepsilon > 0$. We define the family $(\psi_k)_{k=0,\dots,N}$ of elements of $\mathcal{B}_G(\varphi)$ by

$$\begin{cases} \psi_0 = \psi \\ \psi_{k+1} = S^+_{\varepsilon_k^+}(\phi_{k+1}, S^-_{\varepsilon_k^-}(\phi_{k+1},\psi)), \end{cases}$$

where ε_k^+ and ε_k^- are small enough positive reals. The sequence ψ_k is, by Proposition 17, well defined. Moreover, from Proposition 16, we can choose them such that

$$\|\psi_{k+1} - \psi_k\|_{\mathcal{C}^1} < \varepsilon/N. \tag{37}$$

For all k, from Lemma 5,

$$K^*(\psi_{k+1}) \subset K^*(\psi_k).$$

Thus,

$$K^*(\psi_N) \subset \bigcap_{k=0}^{N} K^*(\psi_k)$$

and

$$\bigcup_{k=0}^{N} K^*(\psi_k)^c \subset K^*(\psi_N)^c. \tag{38}$$

As the restriction of φ on K_k is injective for all k, we have

$$K^*(\varphi)^c \supset \bigcup_{k=1}^{N} K_k \times K_k. \tag{39}$$

Moreover, from Proposition 18,

$$K^*(\psi_k)^c \supset K_k \times {K_k}^c. \tag{40}$$

From (38), (39) and (40), we deduce that

$$\begin{aligned} K^*(\psi_N)^c \supset \cup\left(\bigcup_{k=0}^N K^*(\psi_k)^c\right) \supset \left(\bigcup_{k=1}^N K_k \times K_k\right) \cup \left(\bigcup_{k=1}^N K_k \times {K_k}^c\right) \\ = S^1 \times S^1, \end{aligned}$$

and $K^*(\psi_N) = \emptyset$. The immersion ψ_N is injective, thus it is an embedding. Furthermore, from (37), we have $\|\psi - \psi_N\| < \varepsilon$.

Proof (Proposition 18). We claim that for all elements φ, ψ and ϕ which fulfill the assumptions of the Proposition, for all ε small enough and for $\mu = \pm 1$,

$$K^*(S_\varepsilon^\mu(\phi,\psi)) \cap K \times K^c \subset K^*(\psi) \cap \{(x,y) \in K^*(\varphi) \ : \ P_\varphi(\Lambda_\varphi(x,y)) = -\mu\}. \tag{41}$$

Thus, by applying this relation successively to $\mu = -1$ and $\mu = +1$, we get

$$\begin{aligned} &K^*(S_{\varepsilon^-}^-(\phi, S_{\varepsilon^+}^+(\phi,\psi))) \cap K \times K^c \\ \subset &K^*(S_{\varepsilon^+}^+(\phi,\psi)) \cap K \times K^c \cap \{(x,y) \in K^*(\varphi) \ : \ P_\varphi(\Lambda_\varphi(x,y)) = +1\} \\ \subset &K^*(\psi) \cap \{(x,y) \in K^*(\varphi) \ : \ P_\varphi(\Lambda_\varphi(x,y)) = -1\} \\ &\cap \{(x,y) \in K^*(\varphi) \ : \ P_\varphi(\Lambda_\varphi(x,y)) = +1\} = \emptyset, \end{aligned}$$

which is nothing more than the required result.

It remains to prove relation (41). We consider the case $\mu = 1$. The other case ($\mu = -1$) can be treated in the same way. From Lemma 5 and Proposition 17,

$$K^*(S_\varepsilon^+(\phi,\psi)) \subset K^*(\psi) \subset K^*(\varphi). \tag{42}$$

Let $(x,y) \in K^*(S_\varepsilon^+(\phi,\psi)) \cap K \times K^c$. Assume that

$$P_\varphi(\Lambda_\varphi(x,y)) \neq -1,$$

that is $P_\varphi(\Lambda_\varphi(x,y)) = 1$. As $(x,y) \in K^*(\psi)$, we get from the definition of S_ε^+ that

$$\begin{aligned} S_\varepsilon^+(\phi,\psi)(y) &= F_\varepsilon^+(\phi,\psi)(\psi(y)) \\ &= F_\varepsilon^+(\phi,\psi)(\psi(x)) \neq \psi(x) \end{aligned}$$

and

$$S_\varepsilon^+(\phi,\psi)(x) = \psi(x).$$

Therefore, $S_\varepsilon^+(\phi,\psi)(x) \neq S_\varepsilon^+(\phi,\psi)(y)$ and $(x,y) \notin K^*(S_\varepsilon^+(\phi,\psi))$. Therefore, for all $(x,y) \in K^*(S_\varepsilon^+(\phi,\psi))$, $P_\varphi(\Lambda_\varphi(x,y)) = -1$ and with (42), we obtain the expected relation (41).

Proof (Proposition 17). The proposition follows from Proposition 16 and Lemmas 5 ,6 and 7.

Proof (Lemma 3). Let us recall that

$$F_\varepsilon^{\pm}(\phi,\psi)(x) = \begin{cases} g_\psi \circ T_\varepsilon^{\pm}(\phi) \circ g_{\psi}{}^{-1}_{|U\times]-r;r[} & \text{if } x \in g_\psi(U\times]-r;r[) \\ x & \text{if } x \notin g_\psi(U\times]-r;r[) \end{cases}$$

where

$$T_\varepsilon^{\pm}(\phi)(x,t) = (x, t \pm \varepsilon^2 \phi(x)\alpha(\frac{t}{\varepsilon})).$$

We have

$$D_{(x,t)}(T_\varepsilon^{\pm}(\phi)) = \begin{pmatrix} 1 & 0 \\ \varepsilon^2\dot{\phi}(x)\alpha\left(\frac{T}{\varepsilon}\right) & 1 \pm \varepsilon\phi(x)\dot{\alpha}\left(\frac{T}{\varepsilon}\right) \end{pmatrix}.$$

Thus,

$$D_{(x,t)}(T_\varepsilon^{\pm}(\phi)) \xrightarrow{\varepsilon\to 0} I_2$$

for the $\mathcal{C}^0$ topology. The conclusion follows from

$$D_x F_\varepsilon^{\pm}(\phi,\psi) = \begin{cases} Dg_\psi \circ D(T_\varepsilon^{\pm}(\phi)) \circ Dg_{\psi}{}^{-1}_{|U\times]-r;r[} & \text{si } x \in g_\psi(U\times]-r;r[) \\ x & \text{otherwise.} \end{cases}.$$

Proof (Lemma 4). Let $\mu = \pm 1$. We can check that $S_\varepsilon^\mu(\phi,\psi) = \psi$ on the open set $U \cup \psi^{-1}(g_\psi(K \times [-\varepsilon/2, \varepsilon/2]^c)$, ($K$ is the support of ϕ). As $A_1 \subset U \cup \psi^{-1}(g_\psi(K \times [-\varepsilon/2, \varepsilon/2]^c)$, the Lemma is true for every element $x \in A_1$. Moreover, let us remark that

$$A_2 = \{x \in S^1; x \in \psi^{-1}(V) \text{ and } H_U(\psi(x)) \neq 0\}.$$

Then, A_2 is an open set and the Lemma is also true for every $x \in A_2$ as the sign of $H_U(\psi(x)$ is constant on each connected component of A_2.

It remains to study the case $x \in A_3$. As $H_U(\psi(x)) = 0$ and $x \notin U$, $(\Pi_U \circ \psi(x), x) \in K^*(\psi) \subset K^*(\varphi)$. Let $\Lambda = \Lambda_\varphi(\Pi_U \circ \psi(x), x)$. There exists a neighborhood U_x of x in S^1 such that

$$\Pi_U \circ \psi(U_x) \times U_x \subset W_\psi(\Lambda),$$

where $W_\psi(\Lambda)$ is the open set introduce in the definition of $\mathcal{B}_G$ (see 36). For all $\overline{x} \in U_x$, we have two different cases:
First case $H_U \circ \psi \neq 0$.
In this case, $\text{sign}(H_U \circ \psi(\overline{x})) = \text{sign}(h_x(\Pi_U \circ \psi(\overline{x}), \psi(\overline{x}))) = P_\varphi(\Lambda)$. Hence,

$$S_\varepsilon^\mu(\phi,\psi)(\bar{x}) = \begin{cases} F_\varepsilon^\mu \circ \psi(\bar{x}) & \text{if } \mu = P_\varphi(\Lambda), \\ \psi(\bar{x}) & \text{otherwise.} \end{cases} \tag{43}$$

Second Case $H_U \circ \psi(\overline{x}) = 0$.
In this case, $(\Pi_U \circ \psi(\overline{x}), \overline{x}) \in K^*(()\varphi)$. Let $\Lambda' = \Lambda_\varphi(\Pi_U \circ \psi(\overline{x}), x)$. If $\Lambda' = \Lambda$, we have (43). Otherwise, without loss of generality, we can assume that $\overline{x} > x$ (this has a meaning as x and $\overline{x}$ belong to U_x which is diffeomorphic to an open interval of $\mathbb{R}$). We set

$$\widetilde{x} = \min\{z \in U_x; H_U(\psi([z, \overline{x})) = 0\} > x.$$

The element $(\Pi_U(\widetilde{x}), \widetilde{x})$ belongs to Λ'. Moreover, for all neighborhood $W \subset U_x$ of $\widetilde{x}$, there exists $y \in W$ such that $H_U \circ \psi(y) \neq 0$ and $(\Pi_U \circ \psi(y), y) \in W_\psi(\Lambda') \cap W_\psi(\Lambda)$. Hence,

$$P_\varphi(\Lambda') = \text{sign}(h_\varphi(\Pi_U \circ \psi(y), \psi(y))) = P_\varphi(\Lambda).$$

Finally, we get (43) once again and the proof is complete.

Proof (Propositon 16). It follows obviously from Lemmas 4 and 3.

Proof (Lemma 5). We set $V^+ = g_\psi(U \times]0, r[)$. From the definition of $S_\varepsilon^+(\phi, \psi)$, we deduce that

$$\begin{aligned} S_\varepsilon^+(\phi,\psi)(x) \in V^+ &\Rightarrow S_\varepsilon^+(\phi,\psi)(x) = F_\varepsilon^+(\phi,\psi) \circ \psi(x), \\ S_\varepsilon^+(\phi,\psi)(x) \notin V^+ &\Rightarrow S_\varepsilon^+(\phi,\psi)(x) = \psi(x). \end{aligned}$$

One can easily check that in all of these cases,

$$S_\varepsilon^+(\phi,\psi)(x) = S_\varepsilon^+(\phi,\psi)(y) \Rightarrow \psi(x) = \psi(y).$$

Performing the same reasoning for $S_\varepsilon^-(\phi,\psi)$, we obtain that $K^*(S_\varepsilon^\pm(\phi,\psi)) \subset K^*(\psi)$.

Proof (Lemma 6). Let $\mu = \pm 1$. We set $\widetilde{\psi} = S_\varepsilon^\mu(\phi,\psi)$. Let $(x,y) \in K^*(\widetilde{\psi})$. From Lemma 4,

$$\begin{aligned} &\widetilde{\psi} = F_\varepsilon^\mu(\phi,\psi) \circ \psi, && \text{on a neighborhood of } x \\ \text{or } &\widetilde{\psi} = \psi && \text{on a neighborhood of } x. \end{aligned}$$

et

$$\begin{aligned} &\widetilde{\psi} = F_\varepsilon^\mu(\phi,\psi) \circ \psi, && \text{on a neighborhood of } y \\ \text{or } &\widetilde{\psi} = \psi && \text{on a neighborhood of } y. \end{aligned}$$

It is easy to check that the following cases cover all possible situations

1. $\widetilde{\psi} = \psi$ on a neighborhood of x and y.
2. $\widetilde{\psi} = \psi$ on a neighborhood of x, $\widetilde{\psi} = F_\varepsilon^\mu \circ \psi$ on a neighborhood of y and $\psi(x) = \psi(y) \in \psi(U)$, $y \notin U$, $P_\varphi(\Lambda_\varphi(\Pi_U \circ \psi(y), y)) = \mu$.
   (a) $x \in U$.
   (b) $x \notin U$ and $P_\varphi(\Lambda_\varphi(\Pi_U \circ \psi(x), x)) = -\mu$.
3. $\widetilde{\psi} = \psi$ on a neighborhood of y, $\widetilde{\psi} = F_\varepsilon^\mu \circ \psi$ on a neighborhood of x and $\psi(x) = \psi(y) \in \psi(U)$, $x \notin U$, $P_\varphi(\Lambda_\varphi(\Pi_U \circ \psi(x), x)) = \mu$.
   (a) $y \in U$.
   (b) $y \notin U$ and $P_\varphi(\Lambda_\varphi(\Pi_U \circ \psi(y), y)) = -\mu$.
4. $\widetilde{\psi} = F_\varepsilon^\mu \circ \psi$ on a neighborhood of x and y, $\psi(x) = \psi(y) \in g_\psi(U \times]-r; r[)$, $y \notin U$, $x \notin U$ et $P_\varphi(\Lambda_\varphi(\Pi_U \circ \psi(x), x)) = P_\varphi(\Lambda_\varphi(\Pi_U \circ \psi(y), y)) = \mu$.

Cases 2 and 3 are equivalent up to swapping x and y.
Case 1
In this case, $n_x(\widetilde{\psi}).n_y(\widetilde{\psi}) = n_x(\psi).n_y(\psi)$ and the conclusion follows from the fact that $\psi \in \mathcal{B}_G(\varphi)$.
Case 2 and 3
Let us consider the case 2. We have

$$\dot{\widetilde{\psi}}(y) = D_{\psi(y)}F^\mu_\varepsilon(\phi,\psi).\dot{\psi}(y)$$

with

$$D_{\psi(y)}F^\mu_\varepsilon(\phi,\psi) = Dg_\psi \circ D_{(z,t)}(T^\mu_\varepsilon(\phi)) \circ D_{\psi(y)}g^{-1}_{\psi_{|U\times]-r;r[}}.$$

and $(z,t) = (g_{\psi_{|U\times]-r,r[}})^{-1}(\psi(y))$. As $\psi(y)$ belongs to $\psi(U)$, we have

$$t = H_U(\psi(y)) = 0.$$

Hence,

$$\widetilde{\psi}(y) = g_\psi(z, \varepsilon^2\mu\phi(z)).$$

Moreover, $\widetilde{\psi}(y) = \widetilde{\psi}(x) = \psi(x) \in U$. Then, $H_U(\widetilde{\psi}(y)) = 0$ and $\phi(z) = 0$. As ϕ is a positive function of class $\mathcal{C}^1$, we deduce that $\dot{\phi}(z) = 0$. Let us recall that

$$D_{(z,t)}(T^\mu_\varepsilon(\phi)) = \begin{pmatrix} 1 & 0 \\ \varepsilon^2\dot{\phi}(z)\alpha\left(\frac{t}{\varepsilon}\right) & 1 \pm \varepsilon\phi(z)\dot{\alpha}\left(\frac{t}{\varepsilon}\right) \end{pmatrix}.$$

Thus, in the present case, we have

$$D_{(z,t)}(T^\mu_\varepsilon(\phi)) = I_2.$$

Finally,

$$\dot{\widetilde{\psi}}(y) = \dot{\psi}(y),$$

and $n_y(\widetilde{\psi}) = n_y(\psi)$. We conclude as in the previous case.
Case 4
We have

$$\dot{\widetilde{\psi}}(x) = D_{\psi(x)}F^\mu_\varepsilon(\phi,\psi).\dot{\psi}(x) = |\dot{\psi}(x)|D_{\psi(x)}F^\mu_\varepsilon(\phi,\psi).\tau_x\psi,$$

and

$$\dot{\widetilde{\psi}}(y) = D_{\psi(x)}F^\mu_\varepsilon(\phi,\psi).\dot{\psi}(y) = |\dot{\psi}(y)|D_{\psi(x)}F^\mu_\varepsilon(\phi,\psi).\tau_y\psi.$$

Now $\tau_x(\psi) = O_\varphi(\Lambda_\varphi(x,y))\tau_y(\psi)$, therefore

$$|\dot{\psi}(y)|\dot{\widetilde{\psi}}(x) = O_\varphi(\Lambda_\varphi(x,y))|\dot{\psi}(x)|\dot{\widetilde{\psi}}(y),$$

and

$$n_x(\widetilde{\psi}).n_y(\widetilde{\psi}) = O_\varphi(\Lambda_\varphi(x,y)).$$

Proof (Lemma 7). Let $\mu = \pm 1$. We set $\widetilde{\psi} = S^{\mu}_{\varepsilon}(\phi, \psi)$. Let $(x, y) \in K^*(\widetilde{\psi})$. We consider the same case than in the proof of Lemma 6.

Case 1

We have $h_{\widetilde{\psi}} = h_{\psi}$ on a neighborhood of (x, y). The conclusion follows from the fact that $\psi \in \mathcal{B}_G(\varphi)$.

Case 2 and 3

In this case, $\psi(x) = \psi(y) = \widetilde{\psi}(x) = \widetilde{\psi}(y) \in g_{\psi}(U\times]-r, r[)$. Furthermore, we set $z = \Pi_U(\psi(x))$ (Note that in the case 2a, $z = x$ and in the case 3a, $z = y$). As $\widetilde{\psi}$ is an immersion, and as $\psi(z) = \widetilde{\psi}(x)$, we can choose connected neighborhoods $\widetilde{W}_x$ of $(x, 0)$ and W_z of $(z, 0)$ in the normal bundle $S^1 \times \mathbb{R}$ such that

$$g_{\widetilde{\psi}}(\widetilde{W}_x) = g_{\psi}(W_z) \subset g_{\psi}(U\times] - r, r[).$$

Moreover, W_z can be chosen such that $W_z \cap S^1 \times \mathbb{R}^*_+$ and $W_z \cap S^1 \times \mathbb{R}^*_-$ are connected. We introduce the integer $s = \pm 1$ define by

$$s = \begin{cases} P_{\varphi}(\Lambda_{\varphi}(z, y)) & \text{if } z \neq y \\ -P_{\varphi}(\Lambda_{\varphi}(z, x)) & \text{if } z = y \end{cases}$$

On can check that in every case, by the definition of $S^{\mu}_{\varepsilon}(\phi, \psi)$,

$$g_{\psi}(W_z \cap S^1 \times \mathbb{R}^*_s) \cap g_{\widetilde{\psi}}(\widetilde{W}_x \cap S^1 \times \{0\}) = \emptyset.$$

As $g_{\psi}(W_z \cap S^1 \times \mathbb{R}^*_s)$ is connected and included in $g_{\widetilde{\psi}}(\widetilde{W}_x)$, we deduce from the previous relation that $g_{\psi}(W_z \cap S^1 \times \mathbb{R}^*_s)$ is included in one of the connected components of $g_{\widetilde{\psi}}(\widetilde{W}_x \cap S^1 \times \mathbb{R}^*)$. In particular, $g_{\psi}(W_z \cap S^1 \times \mathbb{R}^*_s)$ is included in $g_{\widetilde{\psi}}(\widetilde{W}_x \cap S^1 \times \mathbb{R}^*_+)$ or $g_{\widetilde{\psi}}(\widetilde{W}_x \cap S^1 \times \mathbb{R}^*_-)$. We are now going to find out in which of these sets $g_{\psi}(W_z \cap S^1 \times \mathbb{R}^*_s)$ is included. Using the differentiability of g, we obtain that

$$P_{\mathbb{R}} \circ (g_{\widetilde{\psi}_{|U\times]-r,r[}})^{-1} \circ g_{\psi}(z, t) = n_x(\widetilde{\psi}).n_z(\psi)|\dot{\psi}(z)||\dot{\widetilde{\psi}}(x)|^{-1} t + o(t).$$

Moreover, as seen in the proof of Lemma 6, we have $n_x(\widetilde{\psi}) = n_x(\psi)$. Therefore, for t small enough such that $\mathrm{sign}(t) = s$, we obtain that

$$g_{\psi}(z, t) \in g_{\widetilde{\psi}}(\widetilde{W}_x \times \mathbb{R}^*_{sn_x(\psi).n_z(\psi)}).$$

As $(z, t) \in W_z \cap S^1 \times \mathbb{R}^*_s$, we deduce that

$$g_{\psi}(W_z \cap S^1 \times \mathbb{R}^*_s) \subset g_{\widetilde{\psi}}(\widetilde{W}_x \times \mathbb{R}^*_{sn_x(\psi).n_z(\psi)}).$$

Furthermore, $g_{\psi_{|W_z}}$ and $g_{\widetilde{\psi}_{|\widetilde{W}_x}}$ are diffeomorphisms thus,

$$g_{\psi}(W_z \cap S^1 \times \mathbb{R}_s) \subset g_{\widetilde{\psi}}(\widetilde{W}_x \times \mathbb{R}_{sn_x(\psi).n_z(\psi)}). \tag{44}$$

It is now easy to compute the sign of $h_{\widetilde{\psi}}$ on a neighborhood of (x,y). Let $(\overline{x},\overline{y})$ be close to (x,y). By the definition of s and $S^{\mu}_{\varepsilon}(\phi,\psi)$, $\widetilde{\psi}(\overline{y})$ belongs to $g_{\psi}(W_z \cap S^1 \times \mathbb{R}_s)$. From (44), we deduce that

$$\widetilde{\psi}(\overline{y}) \in g_{\widetilde{\psi}}(\widetilde{W}_x \times \mathbb{R}_{s(n_x(\psi).n_y(\psi))})$$

and that

$$\operatorname{sign}(h_{\widetilde{\psi}}(\overline{x},\overline{y})) = \operatorname{sign}(h_{\widetilde{\psi}}(x,\overline{y})) \neq -s(n_x(\psi).n_z(\psi)).$$

It remains to show that in every particular case,

$$s(n_x(\psi).n_z(\psi)) = P_{\varphi}(\Lambda_{\varphi}(x,y))$$

and the proof will be complete.

Case 2a

We have $z = x$ and $s = P_{\varphi}(\Lambda_{\varphi}(z,y))$, thus

$$s(n_x(\psi).n_z(\psi)) = P_{\varphi}(\Lambda_{\varphi}(x,y)).$$

Case 2b and 3b

We have $n_x(\psi).n_z(\psi) = O_{\varphi}(\Lambda_{\varphi}(x,z))$ and

$$s = P_{\varphi}(\Lambda_{\varphi}(z,y)) = \mu = -P_{\varphi}(\Lambda_{\varphi}(z,x)).$$

The Propositions 12 and 13 imply respectively

$$s(n_x(\psi).n_z(\psi)) = P_{\varphi}(\Lambda_{\varphi}(x,z)),$$

and

$$s(n_x(\psi).n_z(\psi)) = P_{\varphi}(\Lambda_{\varphi}(x,y)).$$

Case 3a

We have $z = y$ and $s = -P_{\varphi}(\Lambda_{\varphi}(z,x)) = -P_{\varphi}(\Lambda_{\varphi}(y,x))$. Furthermore,

$$n_x(\psi).n_z(\psi) = n_x(\psi).n_y(\psi) = O_{\varphi}(\Lambda_{\varphi}(y,x)).$$

Then, using Proposition 12 we get

$$s((n_x(\psi).n_z(\psi)) = P_{\varphi}(\Lambda_{\varphi}(x,y)).$$

Case 4

As $F^{\mu}_{\varepsilon} \circ \psi(x) = \widetilde{\psi}(x)$, there exist W_x and $\widetilde{W}_x$ neighborhoods of $(x,0)$ in the normal bundle $S^1 \times \mathbb{R}$ such that $F^{\mu}_{\varepsilon} \circ g_{\psi|W_x}$ and $g_{\widetilde{\psi}|\widetilde{W}_x}$ are diffeomorphisms on the same image and

$$F^{\mu}_{\varepsilon} \circ g_{\psi|W_x} = g_{\widetilde{\psi}|\widetilde{W}_x}$$

on $W_x \cap (S^1 \times \{0\}) = \widetilde{W}_x \cap (S^1 \times \{0\})$. Moreover, as $\det(D(F^{\mu}_{\varepsilon} \circ g_{\psi})) = \det(g_{\widetilde{\psi}}) > 0$, thence

$$F^{\mu}_{\varepsilon} \circ g_{\psi}(W_x \cap S^1 \times \mathbb{R}_+) = g_{\widetilde{\psi}}(\widetilde{W}_x \cap S^1 \times \mathbb{R}_+) \tag{45}$$

and

$$F_\varepsilon^\mu \circ g_\psi(W_x \cap S^1 \times \mathbb{R}_-) = g_{\widetilde{\psi}}(\widetilde{W}_x \cap S^1 \times \mathbb{R}_-). \tag{46}$$

For all $(\bar{x}, \bar{y})$ in the neighborhood of (x, y), we have

$$h_\psi(\bar{x}, \bar{y}) \neq -P_\varphi(\Lambda_\varphi(x, y)),$$

that is

$$\psi(\bar{y}) \in g_\psi(W_x \cap S^1 \times \mathbb{R}_{P_\varphi(\Lambda_\varphi(x,y))}).$$

From (45) and (46), we deduce that

$$\begin{aligned}\widetilde{\psi}(\overline{y}) = F_\varepsilon^\mu \circ \psi(\overline{y}) \in F_\varepsilon \circ g_\psi(W_x \cap S^1 \times \mathbb{R}_{P_\varphi(\Lambda_\varphi(x,y))}) \\ = g_{\widetilde{\psi}}(W_x \cap S^1 \times \mathbb{R}_{P_\varphi(\Lambda_\varphi(x,y))}).\end{aligned}$$

Thence,

$$h_{\widetilde{\psi}}(\bar{x}, \bar{y}) = h_{\widetilde{\psi}}(x, \overline{y}) \neq -P_\varphi(\Lambda_\varphi(x, y)),$$

and the proof is complete.

6.2. Partial equivalence between the geometrical and algebraic definition of the admissible set of deformations

Let φ be an immersion of S^1 into $\mathbb{R}^2$. The winding number of φ is defined as

$$\sharp\varphi = \deg(\tau(\varphi)), \tag{47}$$

where $\tau(\varphi)$ is the mapping from S^1 into S^1 which maps every element $x \in S^1$ to the unitary vector $\dot{\varphi}(x)/|\dot{\varphi}(x)|$.

Proposition 20. $\mathcal{A}_\phi(j_{S^1}) \cap \mathrm{Imm}(S^1; \mathbb{R}^2) = \{\varphi \in \mathcal{A}_G; \sharp\varphi = +1\}$.

Proposition 21. *The set of the embeddings isotopic to j_{S^1} is dense for the $\mathcal{C}^1$topology in $\mathcal{A}_\phi(j_{S^1}) \cap \mathrm{Imm}(S^1; \mathbb{R}^2)$.*

A straightforward corollary is that

Corollary 1. *Every immersion of $\mathcal{A}_\phi(j_M)$ belongs to $\mathcal{A}(j_M)$, that is*

$$\mathcal{A}_\phi(j_M) \cap \mathrm{Imm}(S^1; \mathbb{R}^2) = \mathcal{A}(j_M) \cap \mathrm{Imm}(S^1; \mathbb{R}^2).$$

The proof is split into two lemmas

Lemma 8. *For all immersion $\varphi \in \mathcal{A}_\phi(j_{S^1})$, $\sharp\varphi = +1$.*

Lemma 9. *For all immersion $\varphi \in \mathcal{A}_\phi(j_{S^1})$, $\varphi \in \mathcal{A}_G$.*

Proof (Proposition 20). From Lemmas 8 and 9, we have

$$\mathcal{A}_\phi(j_{S^1}) \cap \mathrm{Imm}(S^1;\mathbb{R}^2) \subset \{\varphi \in \mathcal{A}_G : \sharp\varphi = +1\}.$$

It remains to prove the converse inclusion. Let $\varphi \in \mathcal{A}_G$, such that $\sharp\varphi = +1$. From Proposition 15, for all $\varepsilon > 0$, there exists an embedding ψ_ε such that

$$\|\varphi - \psi_\varepsilon\|_{\mathcal{C}^1} \leq \varepsilon.$$

As the mapping which maps every $\varphi \in \mathrm{Imm}(S^1;\mathbb{R}^2)$ to its winding number $\sharp\varphi$ is continuous for the $\mathcal{C}^1$ topology, for ε small enough, we have

$$\sharp\psi_\varepsilon = \sharp\varphi = +1.$$

Moreover, it is well-known that immersions of S^1 into $\mathbb{R}^2$ with the same winding number are isotopic. As $\sharp j_{S^1} = +1$, ψ_ε is isotopic to j_{S^1}. Thus, the deformation φ belongs to the $\mathcal{C}^0-$closure of the embeddings isotopic to j_{S^1} and it follows from Proposition 2 that $\varphi \in \mathcal{A}_\phi(j_{S^1})$.

Proof (Proposition 21). From Proposition 15, the set of embeddings is dense in $\mathcal{A}_G$ for the $\mathcal{C}^1$ topology. It is easy to deduce that the set of embeddings isotopic to j_{S^1} is dense for the $\mathcal{C}^1$ topology to $\{\varphi \in \mathcal{A}_G \ : \ \sharp\varphi = +1\}$, which is equal (Proposition 20) to $\mathcal{A}_\phi(j_{S^1}) \cap \mathrm{Imm}(S^1;\mathbb{R}^2)$.

Proof (Lemma 8). Let $\varphi \in \mathcal{A}_\phi(j_{S^1}) \cap \mathrm{Imm}(S^1;\mathbb{R}^2)$. The winding number $\sharp\varphi$ of φ (see definition (47)) could also be defined as the degree of the mapping

$$\tau(\varphi, h) = \frac{\varphi(x+h) - \varphi(x)}{|\varphi(x+h) - \varphi(x)|},$$

for h small enough. In other words,

$$\sharp\varphi = \int_{S^1} \tau(\varphi, h)^*(\phi_{S^1}).$$

As $\phi_{S^1} = {j_{S^1}}^*(\phi_{\mathbb{R}^2_*})$, we have

$$\begin{aligned} \sharp\varphi &= \int_{S^1} \tau(\varphi, h)^* \circ {j_{S^1}}^*(\phi_{\mathbb{R}^2_*}) \\ &= \int_{S^1} (j_{S^1} \circ \tau(\varphi, h))^*(\phi_{\mathbb{R}^2_*}). \end{aligned}$$

We set

$$\begin{aligned} \gamma_h : S^1 &\to S^1 \times S^1 - K(\varphi) \\ x &\mapsto (x+h, x). \end{aligned}$$

The function $d_\varphi \circ \gamma_h = \varphi(x+h) - \varphi(x)$ is homotopic to $j_{S^1} \circ \tau(\varphi, h)$. Therefore,

$$\begin{aligned}\sharp\varphi &= \int_{S^1} (d_\varphi \circ \gamma_h)^*(\phi_{\mathbb{R}^2_*}) \\ &= \int_{S^1} \gamma_h^* \circ d_\varphi^*(\phi_{\mathbb{R}^2_*})\end{aligned}$$

As φ belongs to $\mathcal{A}_\phi(j_{S^1})$, we obtain that

$$\sharp\varphi = \int_{S^1} \gamma_h^* \circ d_{j_1}^*(\phi_{\mathbb{R}^2_*}) = \sharp j_{S^1} = 1.$$

We split the proof of the Lemma 9 in two parts:

Lemma 10. *For all $\varphi \in \mathcal{A}_\phi(j_{S^1}) \cap \mathrm{Imm}(S^1;\mathbb{R}^2)$, Proposition $\mathcal{P}_1(\varphi)$ is true, that is: For all $\Lambda \in \mathcal{E}(\varphi)$, there exists $(x,y) \in \Lambda$ such that for every neighborhood V of (x,y),*

$$h_{\varphi|V} \neq 0.$$

Lemma 11. *For all $\varphi \in \mathcal{A}_\phi(j_{S^1}) \cap \mathrm{Imm}(S^1;\mathbb{R}^2)$, Proposition $\mathcal{P}_2(\varphi)$ is true, that is: For all $\Lambda \in \mathcal{E}(\varphi)$, there exists $V_\varphi(\Lambda)$, neighborhood of Λ suh that $h_{\varphi|V_\varphi(\Lambda)} \geq 0$ or $h_{\varphi|V_\varphi(\Lambda)} \leq 0$.*

Proof (Lemma 10). Let us assume that the lemma is false, that is that there exists $\varphi \in \mathcal{A}_\phi \cap \mathrm{Imm}(S^1;\mathbb{R}^2)$ such that $\mathcal{P}_1(\varphi)$ is false. For the sake of brevity, we will denote P_x instead of $P_x(\varphi)$. There exists $\Lambda \in \mathcal{E}(\varphi)$, such that for all element (x,y) of Λ, there exists small enough connected neighborhoods U_x of x and U_y of y such that

$$h_{|U_x \times U_y} = 0.$$

Hence, we have

$$\begin{aligned}K^*(\varphi) \cap (U_x \times U_y) &= \{(\overline{x},\overline{y}) \in U_x \times U_y \ ; \ h(\overline{x},\overline{y}) = 0 \text{ and } \overline{x} = P_{\overline{x}}(\varphi(\overline{y}))\} \\ &= \{(\overline{x},\overline{y}) \in U_x \times U_y \ ; \ \overline{x} = P_{\overline{x}}(\varphi(\overline{y}))\}.\end{aligned}$$

Up to choosing U_y small enough, we can assume that $P_x(\varphi(U_y)) \subset U_x$. Thus,

$$K^*(\varphi) \cap (U_x \times U_y) = \{(\overline{x},\overline{y}) \ ; \ \overline{x} = P_x(\varphi(\overline{y})), \overline{y} \in U_y\}.$$

This set is connected and

$$K^*(\varphi) \cap (U_x \times U_y) = \Lambda \cap (U_x \times U_y).$$

In particular,

$$\Lambda \cap (U_x \times U_y) = \{(\overline{x},\overline{y}) \ ; \ \overline{x} = P_x(\varphi(\overline{y})), \overline{y} \in U_y\}. \tag{48}$$

As the deformation φ belongs to the ϕ-admissible set, it has no transversal self-intersection. In particular, $\dot{\varphi}(x)$ and $\dot{\varphi}(y)$ are collinear vectors. Hence, $d(P_x \circ \varphi)/dy = \dot{\varphi}(x).\dot{\varphi}(y) \neq 0$. The mapping $P_x \circ \varphi$ is of maximal rank on a neighborhood of y, and the connected component Λ is locally diffeomorphic to $]-1,1[$. It is a compact submanifold of $S^1 \times S^1$ without boundary and of dimension one. It follows that Λ is nothing else but a circle. We set j_Λ the injection of Λ in $S^1 \times S^1 \setminus \triangle(S^1)$.
Let $U^*(0,\pi)$ be the open ball of $\mathbb{R}^2$ of radius π, centered on the origin , from which one has removed its center. The manifold $S^1 \times S^1 \setminus \triangle(S^1)$ is diffeomorphic to $U^*(0,\pi)$ by the mapping

$$\begin{aligned} \Theta : U^*(0,\pi) &\to S^1 \times S^1 \setminus \triangle(S^1) \\ re^{i\theta} &\mapsto (\theta - r, r + \theta), \end{aligned}$$

where S^1 is identified to $\mathbb{R}/2\pi\mathbb{Z}$. We recall that θ is the mapping from $\mathbb{R}^2_*$ to S^1 defined by $\theta(x) = x/|x|$.
The mapping $\Theta^{-1} \circ j_\Lambda$ is an embedding of S^1 into $\mathbb{R}^2$. From the Theorem of Jordan-Brouwer, the winding number of the curve $\Theta^{-1} \circ j_\Lambda$ around the origin could be either zero or ± 1,

$$\int_{S^1} (\theta \circ \Theta^{-1} \circ j_\Lambda)^*(\phi_{S^1}) = \begin{cases} \pm 1 \\ 0 \end{cases}$$

We denote by π_1 and π_2 the projections of $S^1 \times S^1 \setminus \triangle(S^1)$ onto its first and second coordinate. The mappings π_2 and $\theta \circ \Theta^{-1}$ are homotopic, thus we have

$$\int_{S^1} (\pi_2 \circ j_\Lambda)^*(\phi_{S^1}) = \begin{cases} \pm 1 \\ 0 \end{cases}. \tag{49}$$

From (48), $\pi_2 \circ j_\Lambda$ is an immersion of S^1 into S^1, thus,

$$\int_{S^1} (\pi_2 \circ j_\Lambda)^*(\phi_{S^1}) = \deg(\pi_2 \circ j_\Lambda) = \pm\, \mathrm{Card}((\pi_2 \circ j_\Lambda)^{-1}(y)).$$

From (49), we conclude that

$$\mathrm{Card}((\pi_2 \circ j_\Lambda)^{-1}(y)) = +1,$$

and that $\pi_2 \circ j_\Lambda$ is a diffeomorphism. The same conclusion holds for $\pi_1 \circ j_\Lambda$.
Let $x_0 \in S^1$. We define the sequence (x_k) by

$$x_{k+1} = (\pi_1 \circ j_\Lambda) \circ (\pi_2 \circ j_\Lambda)^{-1}(x_k).$$

For all integer k, we have $\varphi(x_{k+1}) = \varphi(x_k) = \varphi(x_0)$. As the cardinal of $\varphi^{-1}(0)$ is finite, there exist a minimal integer $N > 0$ and an integer k such that $x_k = x_{k+N}$. Up to applying k times the mapping $(\pi_2 \circ j_\Lambda) \circ (\pi_1 \circ j_\Lambda)^{-1}$ to this last equation, we can assume that $k = 0$. Once again, we consider S^1 as $\mathbb{R}/2\pi\mathbb{Z}$. For simplicity, we will not do the distinction between an element of $\mathbb{R}$ an its class in $\mathbb{R}/2\pi\mathbb{Z}$. We choose $x_k \in \mathbb{R}$ such that $x_{k+1} > x_k$ and

$|x_{k+1} - x_k| \leq 2\pi$. Let us denote by c_k the mappings from $[0,1]$ into S^1 defined by

$$c_0(t) = tx_0 + (1-t)x_1$$
$$c_{k+1} = \left((\pi_1 \circ j_\Lambda) \circ (\pi_2 \circ j_\Lambda)^{-1}\right) \circ (c_k).$$

As $(\pi_1 \circ j_\Lambda) \circ (\pi_2 \circ j_\Lambda)^{-1}$ is a diffeomorphism from S^1 into S^1 which preserves the orientation, it is clear that the mapping c_k is homotopic to the mapping which maps t to $tx_k + (1-t)x_{k+1}$. Furthermore, $\sum_{k=0}^{N-1} c_k$ is a cycle, that is

$$\sum_0^{N-1} \int_0^1 c_k^*(df) = 0,$$

for any mapping $f : S^1 \to \mathbb{R}$. Thus, there exists an integer p such that for every close form α,

$$\sum_{k=0}^{N-1} \int_0^1 c_k^*(\alpha) = p \int_{S^1} \alpha.$$

and p is nothing but the number of integers k, $0 \leq k < N$ such that x_0 belongs to $[x_k, x_{k+1}[$.
Let $\tau = \dot{\varphi}/|\varphi| : S^1 \to S^1$. The chain $\tau_\#(c_k)$ is a cycle. From

$$\sharp\varphi = \int_{S^1} \tau^*(\phi_{S^1}),$$

we deduce that

$$p\sharp\varphi = \sum_{k=0}^{N-1} \int_0^1 c_k^* \circ \tau^*(\phi_{S^1}) \tag{50}$$

For all integers k, we set

$$d_k = (\pi_2 \circ j_\Lambda)^{-1}(c_k).$$

We have

$$\tau \circ c_k = (\tau \circ (\pi_2 \circ j_\Lambda)) \circ d_k.$$

As $(\varphi \circ \pi_2)_{|\Lambda} = (\varphi \circ \pi_1)_{|\Lambda}$, we get

$$\tau \circ c_k = (\tau \circ (\pi_1 \circ j_\Lambda)) \circ d_k,$$

that is,

$$\tau \circ c_k = \tau \circ c_{k+1}$$

From (50), it follows that

$$p\sharp\varphi = N \int_{S^1} (\tau \circ c_0)^*(\phi_{S^1}).$$

As φ is an element of $\mathcal{A}_\phi \cap \mathrm{Imm}(S^1; \mathbb{R}^2)$, by Lemma 8, we have $\sharp\varphi = 1$. Hence,

$$p = N \int_{S^1} (\tau \circ c_0)^*(\phi_{S^1}).$$

Moreover, p is an integer such that $0 < p \leq N$, hence,

$$\int_{S^1} (\tau \circ c_0)^*(\phi_{S^1}) = 1$$

and $p = N$. It follows that, for all k, we have $x_0 \in [x_k, x_{k+1}[$. In particular,

$$x_0 \in [x_{N-1}, x_N[.$$

As $|x_{N-1} - x_N| \leq 2\pi$ and $x_0 = x_N$, we have $|x_{N-1} - x_N| = 2\pi$ and $x_{N-1} = x_0$. We deduce that

$$(\pi_1 \circ j_\Lambda)^{-1}(x_0) = (\pi_2 \circ j_\Lambda)^{-1}(x_0).$$

Hence, (x_0, x_0) belongs to Λ. This could not hold as $\Lambda \in S^1 \times S^1 \setminus \triangle(S^1)$.

Proof (Lemma 11). Let $\varphi \in \mathcal{A}_\phi(j_{S^1}) \cap \mathrm{Imm}(S^1; \mathbb{R}^2)$. We assume that $\mathcal{P}_2(\varphi)$ is false. For simplicity, we will note P_x instead of $P_x(\varphi)$ and h instead of h_φ. As $\mathcal{P}_2(\varphi)$ is false, there exists $\Lambda \in \mathcal{E}(\varphi)$, such that, if V_Λ is the connected component of $V_{K^*(\varphi)}$ (see definition (28)), there exists $(x_+, y_+) \in V_\Lambda$, $(x_-, y_-) \in V_\Lambda$ with $h(x_+, y_+) > 0$ and $h(x_-, y_-) < 0$. As V_Λ is connected, there exists $j = (j_1, j_2) : [0,1] \to V_\Lambda$ such that $j(0) = (x_-, y_-)$ and $j(1) = (x_+, y_+)$. Let $\varepsilon > 0$ be a small real. We define the mapping Γ from $[0,1] \times [-\varepsilon, \varepsilon]$ by

$$\Gamma(t, s) = \left(P_{j_1(t)}(\varphi \circ j_2(t)) + s, j_2(t)\right).$$

We denote by γ the restriction of Γ to the boundary of $[0,1] \times [-\varepsilon, \varepsilon]$ which is homeomorphic to S^1. The mapping γ is a contractible loop in $S^1 \times S^1 \setminus \Delta(S^1)$ with values in $U = d_\varphi^{-1}(\mathbb{R}^2_*)$, and

$$\int_{S^1} \gamma^* \circ d^*_{j_{S^1}}(\phi_{\mathbb{R}^2_*}) = 0.$$

As φ belongs to $\mathcal{A}_\phi(j_{S^1})$, we have

$$\int_{S^1} \gamma^* \circ d^*_\varphi(\phi_{\mathbb{R}^2_*}) = \int_{S^1} \gamma^* \circ d^*_{j_{S^1}}(\phi_{\mathbb{R}^2_*}) = 0.$$

Let ψ be close to φ, such that the intersections between φ and ψ are transverse. We set

$$\begin{aligned} d_{\varphi,\psi} : S^1 \times S^1 &\to \mathbb{R}^2 \\ (x, y) &\mapsto \varphi(x) - \psi(y). \end{aligned}$$

We have

$$\int_{S^1} (d_{\varphi,\psi} \circ \gamma)^*(\phi_{\mathbb{R}^2_*}) = \int_{S^1} (d_\varphi \circ \gamma)^*(\phi_{\mathbb{R}^2_*}) = 0. \tag{51}$$

As the intersection between φ and ψ is transverse, $d_{\varphi,\psi}|_\Gamma^{-1}(0)$ is a finite set of points endowed with a sign. One can show (as in the proof of Proposition 5) that

$$\int_{S^1} (d_{\varphi,\psi} \circ \gamma)^*(\phi_{\mathbb{R}^2_*}) = \int_{S^1} (d_{\varphi,\psi} \circ \partial\Gamma)^*(\phi_{\mathbb{R}^2_*}) = \sum_{(x,y) \in d_{\varphi,\psi}|_\Gamma^{-1}(0)} s_{\varphi,\psi}(x, y),$$

where $s_{\varphi,\psi}(x,y)$ is the orientation of the element (x,y) of $d_{\varphi,\psi}|_{\Gamma}^{-1}(0)$. From (51), it follows that the cardinal of $d_{\varphi,\psi}|_{\Gamma}^{-1}(0)$ is even.

$$\text{Card}\left(d_{\varphi,\psi}|_{\Gamma}^{-1}(0)\right) = 0 \mod 2. \tag{52}$$

Moreover, on the one hand

$$d_{\varphi,\psi}|_{\Gamma}^{-1}(0) = \Big\{(x,y) \in S^1 \times S^1 \ : \ x = (P_{j_1(t)}(\varphi \circ j_2(t)) + s,\ y = j_2(t),$$
$$x = P_{j_1(t)}(\psi \circ j_2(t)),\ H_{j_1(t)}(\psi \circ j_2(t)) = 0, \text{ with } (t,s) \in [0,1] \times [-\varepsilon,\varepsilon]\Big\}$$

For ψ close enough to φ, we have

$$d_{\varphi,\psi}|_{\Gamma}^{-1}(0) = \Big\{(x,y) \in S^1 \times S^1 \ : \ y = j_2(t), x = P_{j_1(t)}(\psi \circ j_2(t)),$$
$$H_{j_1(t)}(\psi \circ j_2(t)) = 0, \text{ with } (t,s) \in [0,1] \times [-\varepsilon,\varepsilon]\Big\}.$$

Hence,

$$\text{Card}\left(d_{\varphi,\psi}|_{\Gamma}^{-1}(0)\right) = \text{Card}\left(\left\{t \in [0,1] \ : \ H_{j_1(t)}(\psi \circ j_2(t)) = 0\right\}\right).$$

Furthermore,

$$\text{sign}(H_{j_1(0)}(\psi \circ j_2(0)) = \text{sign}(H_{j_1(0)}(\varphi \circ j_2(0)) = \text{sign}(h(x_-, y_-)) < 0$$

and on the other hand

$$\text{sign}(H_{j_1(1)})(\psi \circ j_2(1)) > 0.$$

Thus, the function $H_{j_1(t)}(\psi \circ j_2(t))$ has an odd number of roots and the cardinal of $d_{\varphi,\psi}|_{\Gamma}^{-1}(0)$ is odd, whereas we have already proved that it is even (52).

6.3. Proof of the partial equivalence between the minimization problem and the Euler-Lagrange equations

In order to be able to prove the partial equivalence between the minimization problem and the Euler-Lagrange equation stated in Proposition 10, we will first prove the following lemmas:

Lemma 12. *If φ is a regular immersion, solution of the minimization problem ($\mathcal{P}_\phi$), then*

$$\sum_{y \in \varphi^{-1}(z)} \left(\frac{d(DW(\dot{\varphi}))}{dx}(y) + f(y)\right) |\dot{\varphi}(y)|^{-1} = 0 \textit{ for all } z \in \text{Im}(\varphi).$$

Lemma 13. *If φ is a regular immersion, solution of the minimization problem ($\mathcal{P}_\phi$), then*

$$\left(f(x) + \frac{dDW(\dot{\varphi}(x))}{dx}\right).\tau_x = 0,$$

where $\tau_x = \frac{\dot{\varphi}(x)}{|\dot{\varphi}(x)|}$.

Proof (lemma 13). Let $u \in \mathcal{C}^2(S^1;\mathbb{R}^2)$. We set $\gamma(\varepsilon) = \varphi(x + \varepsilon u(x))$. As, from Proposition 3, $\gamma(\varepsilon) \in \mathcal{A}_G$, for ε small enough, we have

$$I(\gamma(\varepsilon)) \geq I(\gamma).$$

Furthermore, $I \circ \gamma$ is differentiable and

$$(I \dot{\circ} \gamma) = \int_{S^1} DW(\dot{\varphi}(x)).\frac{d(\dot{\varphi}(x)u(x))}{dx} - f(x).\dot{\varphi}(x)u(x)dx.$$

Thus, the former inequality implies that

$$\int_{S^1} DW(\dot{\varphi}(x)).\frac{d(\dot{\varphi}(x)u(x))}{dx} - f(x).\dot{\varphi}(x)u(x)dx = 0$$

and

$$\int_{S^1}\left(\frac{dDW(\dot{\varphi}(x))}{dx}.\dot{\varphi} + f.\dot{\varphi}\right)u(x)dx = 0.$$

Proof (lemma 12). Let $\varphi \in \mathcal{C}^2(S^1;\mathbb{R}^2)$ be an immersion, solution of the minimization problem ($\mathcal{P}_\phi$). Let $z \in \mathrm{Im}(\varphi)$ and

$$\{x_0, \ldots, x_N\} = \varphi^{-1}(z).$$

There exists U_0 a neighborhood of x_0 in S^1 such that

$$\begin{aligned} g \,:\, U_0 \times]{-r}, r[&\rightarrow g_\varphi(U_0 \times]{-r}, r[) = V_0 \\ x &\mapsto g_\varphi(x) \end{aligned}$$

is a diffeomorphism. Let $\phi : S^1 \rightarrow \mathbb{R}^2$ be a regular function with support included in U_0. We set

$$\begin{aligned} F(\phi) \,:\, S^1 \times \mathbb{R} &\rightarrow \mathbb{R}^2 \\ (x,h) &\mapsto \phi(x)\alpha(h/r), \end{aligned}$$

where α is a regular function from $\mathbb{R}$ to $\mathbb{R}$, whose support is included in $]-1/2, 1/2[$ such that $\alpha([-1/4, 1/4]) = 1$.

The support of $F(\phi)$ is included in $U_0 \times]-r, r[$ and the mapping

$$\begin{aligned} G(\phi) \,:\, \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ x &\mapsto \begin{cases} (F(\phi) \circ g^{-1})(x) & \text{if } x \in V_0, \\ 0 & \text{if } x \notin V_0 \end{cases} \end{aligned}$$

is regular. For $t > 0$ small enough, $\mathrm{Id} + tG(\phi)$ is a orientation-preserving diffeomorphism of $\mathbb{R}^2$. Thus, from proposition 4,

$$\gamma(t) = (\mathrm{Id} + tG(\phi)) \circ \varphi \in \mathcal{A}_\phi(j_M).$$

It follows that

$$I(\gamma(t)) \geq I(\varphi)$$

and

$$(I \dot{\circ} \gamma)(0) \geq 0.$$

The computation of the derivative of $I \circ \gamma$ is easy and we get

$$\begin{aligned} 0 &\leq \int_{S^1} DW(\dot{\varphi})).\left(\frac{dG(\phi)\circ\varphi}{dx}(x)\right)dx - \int_{S^1} f(x).G(\phi)\circ\varphi(x)dx \\ &= -\int_{S^1} \frac{d(DW(\dot{\varphi}))}{dx}.G(\phi)\circ\varphi(x)dx - \int_{S^1} f(x).G(\phi)\circ\varphi(x)dx \end{aligned}$$

There exists a family $(U_k)_{k=1,\ldots,N}$ of neighborhoods of the points x_k such that for every k,

$$P_{S^1} \circ \left(g_{|U_0 \times]-r/4, r/4[}\right)^{-1} \circ \varphi_{|U_k}$$

is a diffeomorphism onto its image.

As S^1 is compact, there exists a neighborhood W of $\varphi(x_0)$ in $\mathbb{R}^2$ such that

$$g_\varphi^{-1}(W) \subset \bigcup_{k=0}^{N} U_k.$$

We can assume that the support of ϕ is included in $U_0 \cap g_\varphi^{-1}(W)$. Then,

$$\begin{aligned} 0 &\geq \int_{S^1} \left(\frac{d(DW(\dot{\varphi}))}{dx} + f\right).G(\phi)\circ\varphi(x)dx \\ &= \int_{g^{-1}(W)} \left(\frac{d(DW(\dot{\varphi}))}{dx} + f\right).G(\phi)\circ\varphi(x)dx \\ &= \sum_{k=0}^{N} \int_{U_k} \left(\frac{d(DW(\dot{\varphi}))}{dx} + f\right).G(\phi)\circ\varphi(x)dx. \end{aligned}$$

Using the expression of $G(\phi)$, we get

$$\sum_{k=0}^{N} \int_{U_k} \left(\frac{d(DW(\dot{\varphi}))}{dx} + f\right).\phi(P_{S^1} \circ g^{-1}(\varphi(x))\alpha(P_{\mathbb{R}} \circ g^{-1}(\varphi(x))/r)dx \leq 0. \tag{53}$$

For all $x \in U_0$, $P_{\mathbb{R}} \circ g^{-1}(\varphi(x)) = 0$, and

$$\alpha(P_{\mathbb{R}} \circ g^{-1}(\varphi(x))/r) = 1. \tag{54}$$

Furthermore, for all k and $y_k \in U_k$,

$$\varphi(y_k) \in g(U_0 \times]-r/4, r/4[),$$

and

$$P_{\mathbb{R}} \circ g^{-1}(\varphi(y_k)) \in]-r/4, r/4[.$$

Thus, we have once again

$$\alpha(P_{\mathbb{R}} \circ g^{-1}(\varphi(y_k)) = 1. \tag{55}$$

From (53), (54) and (55), we deduce that

$$\sum_{k=0}^{N} \int_{U_k} \left(\frac{d(DW(\dot{\varphi}))}{dx} + f \right) .\phi(P_{S^1} \circ g^{-1}(\varphi(x))dx \leq 0.$$

On each open set U_k, we perform the change of variables induced by

$$s_k = P_{S^1} \circ g^{-1} \circ \varphi_{|U_k}.$$

We obtain that

$$\sum_{k=0}^{N} \int_{U_0} \left(\frac{d(DW(\dot{\varphi}))}{dx}(s_k^{-1}(y)) + f(s_k^{-1}(y)) \right) .\phi(y)(J_k(s_k^{-1}(y)))^{-1}dy \leq 0,$$

where $J_k(z) = |\dot{s}_k(z)|$. As this inequality holds for all ϕ with compact support in the neighborhood $U_0 \cap g_{\varphi}^{-1}(W)$ of x_0, we deduce that

$$\sum_{k=0}^{N} \left(\frac{d(DW(\dot{\varphi}))}{dx}(x_k) + f(x_k)) \right) (J_k(x_k))^{-1} = 0.$$

Furthermore, as $J_k(x_k) = |\dot{\varphi}(x_0)|^{-1}|\dot{\varphi}(x_k)|$, we have

$$\sum_{k=0}^{N} \left(\frac{d(DW(\dot{\varphi}))}{dx}(x_k)) + f(x_k) \right) |\dot{\varphi}(x_k)|^{-1}|\dot{\varphi}(x_0)| = 0,$$

and

$$\sum_{k=0}^{N} \left(\frac{d(DW(\dot{\varphi}))}{dx}(x_k)) + f(x_k) \right) |\dot{\varphi}(x_k)|^{-1} = 0$$

as claimed.

We are now in a position to prove our result of partial equivalence between the minimization problem and the Euler-Lagrange equations.

Proof (Proposition 10). Let φ be a solution of class $\mathcal{C}^1$ of the minimization problem $(\mathcal{P}_\phi)$. Let $z \in \mathrm{Im}(\varphi)$, n a normal vector to the image of φ at z and $(x_0, \ldots, x_N)$ the family obtained by proposition 14. We recall that

$$\varphi^{-1}(z) = \{x_0, \ldots, x_N\}$$

and that for every $k < l$,

$$P_\varphi(\Lambda(x_k, x_l)) = n.n_{x_k}.$$

Let $m \in \{0, \dots, N-1\}$. We set $\mu = n.n_{x_m}$. There exists a neighborhood U_m of x_m in S^1 and a real $r_0 > 0$ such that

$$\begin{aligned} g : U_m \times]-r_0, r_0[&\to g_\varphi(U_m \times]-r_0, r_0[) \\ x &\mapsto g_\varphi(x) \end{aligned}$$

is a diffeomorphism. There exists a family of neighborhoods (U_k) of x_k in S^1 ($k \in \{0, \dots, N\}$, $k \neq m$) such that for all k

$$P_{S^1} \circ \left(g_{|U_m \times]-r,r[}\right)^{-1} \circ \varphi_{|U_k}$$

is a diffeomorphism onto its image and such that for all k and $l \in \{0, \dots N\}$,

$$U_k \cap U_l = \emptyset. \tag{56}$$

As S^1 is a compact set and as $\varphi^{-1}(\varphi(x_m)) = \{x_0, \dots, x_N\}$, there exists a neighborhood W of $\varphi(x_m)$ in $\mathbb{R}^2$ such that

$$\varphi^{-1}(W) \subset \bigcup_{k=0}^{N} U_k.$$

There exists $V \subset \cap_{k=0}^N (P_{S^1} \circ g^{-1} \circ \varphi)(U_k)$, neighborhood of x_m and $\varepsilon \leq r$ such that

$$g_\varphi(V \times]-\varepsilon \times \varepsilon[) \subset W.$$

Let $\phi \in \mathcal{C}_0^\infty(V; \mathbb{R}^+)$ such that

$$\|\phi\|_{\mathcal{C}^0} \leq \varepsilon^{-1} \|\alpha\|_{\mathcal{C}^1}^{-1},$$

where α is defined as in the proof of Proposition 15. The mapping

$$\gamma(t) = S_\varepsilon^\mu(t\phi, \varphi)$$

is well defined for all $t < 1$. By applying the same procedure as the one used in the proof of Lemma 4, we can prove that

$$\gamma(t)(x) = \begin{cases} F_\varepsilon^\mu(t\phi, \varphi) \circ \varphi(x) & \text{if } x \in \bigcup_{k>m} U_k, \\ \varphi(x) & \text{otherwise.} \end{cases}$$

We define

$$\begin{aligned} H : \textstyle\bigcup_{k=0}^N U_k &\to \mathbb{R} \\ x &\mapsto P_{\mathbb{R}}\left(g^{-1}(\varphi(x))\right) \end{aligned}$$

and

$$\begin{aligned} \Pi : \textstyle\bigcup_{k=0}^N U_k &\to U_m \\ x &\mapsto P_{S^1}\left(g^{-1}(\varphi(x))\right). \end{aligned}$$

From the definition of F_ε^μ, we have

$$\gamma(t)(x) = \begin{cases} g_\varphi\left(\Pi(x), H(x) + \mu\varepsilon^2\phi(\Pi(x))\alpha(H(x)/\varepsilon)\right) & \text{if } x \in \bigcup_{k>m} U_k, \\ \varphi(x) & \text{otherwise.} \end{cases}$$

A straightforward computation leads to

$$\dot\gamma(0)(x) = \begin{cases} D_{g_{|V}^{-1}\circ\varphi(x)} g_\varphi.(0, \mu\varepsilon^2\phi(\Pi(x))\alpha(H(x)/\varepsilon) & \text{if } x \in \bigcup_{k>m} U_k, \\ 0 & \text{otherwise.} \end{cases}$$

Using the expression of the differential of g_φ, we get

$$\dot\gamma(0)(x) = \mu\varepsilon^2\phi(\Pi(x))\alpha(H(x)/\varepsilon)\begin{pmatrix} -\dot\varphi_2(\Pi(x)+H(x)) \\ \dot\varphi_1(\Pi(x)+H(x)) \end{pmatrix} \tag{57}$$

if x belongs to $\bigcup_{k>m} U_k$, and $\dot\gamma(0)(x) = 0$ otherwise. From Propositions 17 and 19, $\gamma(t) \in \mathcal{B}_G$ and $\gamma(t)$ belongs to the $\mathcal{C}^1$-closure of the set of embeddings. Furthermore, as $\sharp\gamma(t) = \sharp\varphi = +1$, every embedding close $\gamma(t)$ for the $\mathcal{C}^1$ topology is isotopic to the reference embedding j_{S^1}, as its winding number is equal to $+1$. Hence, $\gamma(t)$ belongs to the set of embeddings isotopic to j_{S^1} and $\gamma(t) \in \mathcal{A}_\phi(j_{S^1})$ is an admissible deformation (by Proposition 2). As φ is a minimizer of I on the set of admissible deformations, we have

$$I(\gamma(t)) \geq I(\gamma(0)).$$

By differentiation, we deduce that

$$\int_{S^1}\left(\frac{dDW(\dot\varphi)}{dx}(y) + f(y)\right).\dot\gamma(0)(y)dy \leq 0.$$

Using the expression (57) of $\dot\gamma(0)$ and the property (56), we get

$$\sum_{k>m}\int_{U_k}\left(\frac{dDW(\dot\varphi)}{dx}(y_k) + f(y_k)\right)\cdot\begin{pmatrix} -\dot\varphi_2(\Pi(y_k)+H(y_k)) \\ \dot\varphi_1(\Pi(y_k)+H(y_k)) \end{pmatrix}$$
$$\phi(\Pi(y_k))\alpha(H(x)/\varepsilon)\mu\, dy_k \leq 0.$$

On each open set U_k, the mapping $\Pi\circ\varphi$ is a diffeomorphism onto its image. Thus, one can perform the change of variable $x = \Pi\circ\varphi(y_k)$ on each open set U_k. We get

$$\sum_{k>m}\int_{\Pi\circ\varphi(U_k)}\left(\frac{dDW(\dot\varphi))}{dx}(y_k) + f(y_k)\right)\cdot\begin{pmatrix} -\dot\varphi_2(x+H(y_k)) \\ \dot\varphi_1(x+H(y_k)) \end{pmatrix}$$
$$\phi(x)\alpha(H(y_k)/\varepsilon)\left(J_{m,k}(y_k)\right)^{-1}\mu\, dx \leq 0,$$

where $J_{m,k}(y) = |D_y\Pi\circ\varphi|$ and $y_k(x) = (\Pi\circ\varphi_{|U_k})^{-1}(x)$. As the support of ϕ is included in $V \subset \bigcup_{k=0}^N(\Pi\circ\varphi)(U_k)$, we have

$$\sum_{k>m}\int_{V}\left(\frac{dDW(\dot\varphi))}{dx}(y_k) + f(y_k)\right)\cdot\begin{pmatrix} -\dot\varphi_2(x+H(y_k)) \\ \dot\varphi_1(x+H(y_k)) \end{pmatrix}$$
$$\phi(x)\alpha(H(y_k)/\varepsilon)\left(J_{m,k}(y_k)\right)^{-1}\mu\, dx \leq 0.$$

As this inequality is true for every test function $\phi \in \mathcal{C}_0^\infty(V;\mathbb{R}^+)$ small enough, one gets that

$$\sum_{k>m} \left(\frac{dDW(\dot\varphi))}{dx}(y_k) + f(y_k) \right) \cdot \begin{pmatrix} -\dot\varphi_2(x+H(y_k)) \\ \dot\varphi_1(x+H(y_k)) \end{pmatrix}$$
$$\alpha(H(y_k)/\varepsilon)\,(J_{m,k}(y_k))^{-1}\,\mu \leq 0.$$

We apply this inequality to $x = x_m$. We have $y_k(x_m) = x_k$, $J_{m,k}(x_k) = |\dot\varphi(x_k)|/|\dot\varphi(x_m)|$, $H(x_k) = 0$, and $\alpha(0) = 1$. Thus,

$$\sum_{k>m} \left(\frac{dDW(\dot\varphi))}{dx}(x_k) + f(x_k) \right) \cdot \begin{pmatrix} -\dot\varphi_2(x+H(x_k)) \\ \dot\varphi_1(x+H(x_k)) \end{pmatrix} \frac{|\dot\varphi(x_m)|}{|\dot\varphi(x_k)|} n.n_{x_m} \leq 0.$$

We have proved that for every $m \in \{0,\dots,N-1\}$,

$$\boxed{\sum_{k>m} \left(\frac{dDW(\dot\varphi))}{dx}(x_k) + f(x_k) \right) .n|\dot\varphi(x_k)|^{-1} \leq 0.} \tag{58}$$

For all $-1 \leq m \leq N$, we set

$$\lambda_m = -\sum_{k>m} \left(\frac{dDW(\dot\varphi))}{dx}(x_k) + f(x_k) \right) .n|\dot\varphi(x_k)|^{-1} \geq 0,$$

From Lemma 12,

$$\lambda_{-1} = \sum_{k=0}^{N} \left(\frac{dDW(\dot\varphi))}{dx}(x_k) + f(x_k) \right) .n|\dot\varphi(x_k)|^{-1} = 0.$$

By definition, $\lambda_N = 0$. For all $0 < m < N-1$, we have

$$\lambda_{m-1} - \lambda_m = \left(\frac{dDW(\dot\varphi)}{dx}(x_m) + f(x_m) \right) .n|\dot\varphi(x_m)|^{-1},$$

or as well

$$\left(\frac{dDW(\dot\varphi)}{dx}(x_m) + f(x_m) \right) .n = |\dot\varphi(x_m)|(\lambda_{m-1} - \lambda_m).$$

Moreover, we have from Lemma 13,

$$\left(\frac{dDW(\dot\varphi)}{dx}(x_m) + f(x_m) \right) .\tau_{x_m} = 0.$$

We conclude that

$$\frac{dDW(\dot\varphi)}{dx}(x_m) + f(x_m) = |\dot\varphi(x_m)|(\lambda_{m-1} - \lambda_m)n,$$

which completes the proof.

7. Conclusion

The modeling presented in this article allows us to consider contacts between elastic bodies of dimension m moving in $\mathbb{R}^n$, for any $m \leq n$. Even if, for shells, there is no equivalence between the minimization problem and the Euler-Lagrange equations, our modeling could be applied in such cases. We have already performed numerical simulations, in the case $n = 2$. The method will be presented in a forthcoming article. For higher dimension ($n = 3$ and $\dim(M) = 2$ or 3), similar numerical methods could be applied. However, it leads to high-dimensional problems of dimension 4. To solve them, one will have to use some adaptive methods, which could be difficult to carry out. We are currently working on another modeling, for which the equivalence between the minimization problem and the Euler-Lagrange equation could be proved, even for shells. To this end, one must add another condition to the set of admissible deformations to forbid deformations with degenerate intersections.

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