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A. Chambolle * , **M. Novaga** †

Abstract

In this note, we provide simple proofs of consistency for two well-known algorithms for mean curvature motion, Almgren-Taylor-Wang's [1] variational approach, and Merriman-Bence-Osher's algorithm [3]. Our techniques, based on the same notion of strict sub- and superflows, also work in the (smooth) anisotropic case.

1 Introduction

The Mean curvature flow refers to the motion of a hypersurface $\Gamma(t) \subset \mathbb{R}^N$ whose normal velocity, at each point, is equal to (minus) its mean curvature. We will consider only compact hypersurfaces $\Gamma(t)$, that are the boundary of some evolving set $E(t)$ (bounded or unbounded). In this case, the motion is also known as the "area-diminishing" flow, and is in some sense the gradient flow of the perimeter of $E(t)$. It is well-known that this motion can be characterized in terms of the distance function to $\Gamma = \partial E$ [19, 2]. More precisely, if we define $d(x,t)$ as

$$d(x,t) \ := \ \mathrm{dist}(x, E(t)) - \mathrm{dist}(x, \mathbb{R}^N \setminus E(t))$$

(the signed distance function to $\partial E(t)$), then the exterior normal to E is given by ∇d whereas the curvature is Δd. On the other hand, the normal velocity of a point of the boundary is given, at each time, by $-\partial d/\partial t$, so that the evolution is characterized by

$$\frac{\partial d}{\partial t}(x,t) \ = \ \Delta d(x,t) \tag{1}$$

at any $x \in \partial E(t)$ (*i.e.*, (x,t) such that $d(x,t) = 0$).

The Mean curvature flow enjoys a comparison principle: if E, F are two (smooth) evolutions such that $E(t) \subseteq F(t)$ at some time t, then $E(s) \subseteq F(s)$ at any subsequent time $s > t$ as long as the flows are defined. This key property allows to define a generalized flow for nonsmooth surfaces, by comparison with smooth flows:

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basically, a generalized flow will be a flow such that any smooth flow starting inside remains inside while any smooth flow starting outside remains outside. The formal theory that provides such a generalization is known as the barrier theory and is initially due to De Giorgi [17, 9, 6]. The theory of viscosity solutions (which is also based on the comparison principle) defines the generalized flow as the zero sub- or superlevel set of a function u that solves an appropriate degenerate parabolic equation, and yields the same generalized flows as the barrier theory [7]. The generalized flow starting from a set E is usually unique, except when the "fattening" phenomenon occurs, which corresponds to the fattening of the level line $\{u = 0\}$ of the corresponding viscosity solution.

It is shown in [6] that a barrier solution can be characterized by comparison with appropriate sub- and superflow: in this case, a generalized flow will be characterized by the property that any smooth flow starting inside and evolving (strictly) *faster* than the Mean curvature flow remains inside, while a smooth flow starting outside and evolving (strictly) *slower* than the Mean curvature flow remains outside. The definition of a strict superflow of (1) is the following: $E(t)$ will be a strict superflow (on a small time interval $[t_0, t_1]$) iff its signed distance function satisfies

$$\frac{\partial d}{\partial t}(x,t) \; > \; \Delta d(x,t) \tag{2}$$

in a neighborhood of $\{d = 0\}$. A strict subflow is defined with the reverse inequality.

We show in this note that such a definition (which will be slightly adapted to cover non-isotropic cases) makes very easy the proof of convergence for two well-known approximation schemes for the Mean curvature flow, namely, the Almgren-Taylor-Wang [1] approach and the Merriman-Bence-Osher [3] approach. In both schemes, a time step $h > 0$ is fixed and a discrete-in-time evolution is defined, by providing a simple evolution operator $E \mapsto T_h E$ that approximates the evolution of a initial set E over a time interval of duration h. Given E_0, the discrete evolution $E_h(t)$ is simply $T_h^{[t/h]}(E_0)$ where $[\cdot]$ denotes the integer part. One then wants to know whether $E_h(t) \to E(t)$ as $h \to 0$, where $E(t)$ is the generalized evolution starting from E_0. The key to prove this convergence are the two properties of *monotonicity* and *consistency*. The operator T_h will be monotone if given any E, F with $E \subseteq F$, one has $T_h E \subseteq T_h F$. The notion of consistency we will use is based on our notion of strict super- and subflow: T_h will be consistent if, given any superflow E on $[t_0, t_1]$ and given $h > 0$ small enough, one has $E(t + h) \subseteq T_h E(t)$ for any $t \in [t_0, t_1 - h]$, while given any subflow, the same holds with the reverse inclusion. It follows from the theory of barriers that if T_h is monotone and consistent in the above-defined sense, then $\partial E_h(t)$ converges to $\partial E(t)$ as $h \to 0$ (in the Hausdorff sense), at any time, as long as the generalized flow $\partial E(t)$ is uniquely defined (*i.e.*, no fattening occurs).

In our cases, the set $T_h E(t)$ will be defined as a level set of some function u (depending on h and $E(t)$), satisfying some elliptic or parabolic equation, and it will be quite easy to build from a function d satisfying (2) a sub- or supersolution v of the same equation that will be compared to u, yielding a comparison of the level sets.

This note is organized as follows: in Section 2 we introduce the anisotropic curvature flow and we give a rigorous definition of the corresponding super and subflows. Then, in Section 3 we introduce the Merriman-Bence-Osher's scheme and we prove its consistency. In Section 4 we do the same for the Almgren-Taylor-Wang's algorithm. We observe that in this case, a result of consistency with smooth flows is already found in [1]. However, its proof is by far more complicated than ours.

2 Anisotropic curvature flow

We follow the definitions and notation in [8, 10]. Let us consider (ϕ, ϕ°) a pair of mutually polar, convex, one-homogeneous functions in $\mathbb{R}^N$ (i.e., $\phi^\circ(\xi) = \sup_{\phi(\eta)\le 1} \xi\cdot\eta$, $\phi(\eta) = \sup_{\phi^\circ(\xi)\le 1} \xi\cdot\eta$, see [25]). These are assumed to be locally finite, and, to simplify, even. The pair (ϕ, ϕ°) is referred as *the anisotropy* (the isotropic case corresponds to $\phi = \phi^\circ = |\cdot|$). The local finiteness implies that there is a constant $c > 1$ such that

$$c^{-1}|\eta| \le \phi(\eta) \le c|\eta| \quad \text{and} \quad c^{-1}|\xi| \le \phi^\circ(\xi) \le c|\xi|$$

for any η and ξ in $\mathbb{R}^N$. We refer to [8, 10] for the main properties of ϕ and ϕ°.

Being convex and 1-homogeneous, ϕ° (and ϕ) is also subadditive, so that the function $(x, y) \mapsto \phi(x - y)$ defines a distance, the "ϕ-distance". For $E \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$, we denote by $\mathrm{dist}^\phi(x, E) := \inf_{y\in E} \phi(x - y)$ the ϕ-distance of x to the set E, and by

$$d^\phi_E(x) \;:=\; \mathrm{dist}^\phi(x, E) \;-\; \mathrm{dist}^\phi(x, \mathbb{R}^N \setminus E)$$

the signed ϕ-distance to ∂E, negative in the interior of E and positive outside its closure. One easily checks that

$$|d^\phi_E(x) - d^\phi_E(y)| \;\le\; \phi(x - y) \;\le\; c|x - y|$$

for any $x, y \in \mathbb{R}^N$, so that (by Rademacher's theorem) d^ϕ_E is differentiable a.e. in $\mathbb{R}^N$. The former inequality shows moreover that $\nabla d^\phi_E(x)\cdot h \le \phi(h)$ for any $h \in \mathbb{R}^N$, if x is a point of differentiability: hence $\phi^\circ(\nabla d^\phi_E(x)) \le 1$. In this note we will always assume that ϕ and ϕ° are at least in $C^2(\mathbb{R}^N \setminus \{0\})$. In this case, one shows quite easily that d^ϕ_E is differentiable at each point x which has a unique ϕ-projection $y \in \partial E$ (solving $\min_{y\in\partial E} \phi(x - y)$). Then, $\nabla d^\phi_E(x)$ is given by $\nabla\phi((x - y)/d^\phi_E(x))$, so that $\phi^\circ(\nabla d^\phi_E(x)) = 1$. See [8, 10] for details.

The *Cahn-Hoffman* vector field n_ϕ is a vector field on ∂E given by $n_\phi(x) = \nabla\phi^\circ(\nu_E(x)) \;=\; \nabla\phi^\circ(\nabla d^\phi_E(x))$ a.e. on ∂E. Here, ν_E is the (Euclidean) exterior normal to ∂E. If E is smooth enough, then ∇d_E^ϕ does not vanish near ∂E so that one can define $n_\phi(x) = \nabla\phi^\circ(\nabla d_E^\phi(x))$ in a neighborhood of ∂E.

Then, we define the *ϕ-curvature* of ∂E by $\kappa_\phi = \operatorname{div} n_\phi$. The ϕ-curvature flow is an evolution $E(t)$ such that at each time, the velocity of $\partial E(t)$ is given by

$$V \;=\; -\kappa_\phi\, n_\phi\,, \tag{3}$$

where n_ϕ is the Cahn-Hoffman vector field and κ_ϕ is the ϕ-curvature. It is shown that, in some sense, it is the fastest way to diminish the anisotropic perimeter $\int_{\partial E}\phi^\circ(\nu_E)\,d\mathcal{H}^{N-1}$. If ϕ, ϕ° are merely Lipschitz (when, for instance, the *Wulff shape* $\{\phi \le 1\}$ is a convex polytope), then n_ϕ can be nonunique and the anisotropy is called *crystalline* [28, 8]. We refer to [15] for a proof of convergence of Merriman-Bence-Osher's scheme in the crystalline case.

The anisotropic variant of (1) is the following characterization of the anisotropic mean curvature flow: letting $d(x,t) = d^\phi_{E(t)}(x)$, the smooth set $E(t)$ evolves by anisotropic curvature if

$$\frac{\partial d}{\partial t}(x,t) \;=\; \operatorname{div}\nabla\phi^\circ(\nabla d(x,t))\,, \tag{4}$$

for any (x,t) with $d(x,t) = 0$. One therefore introduces the following definition of (strict) super- and subflows, which is simplified from [16]:

Definition 2.1. *Let $E(t) \subset \mathbb{R}^N$, $t \in [t_0,t_1]$. We say that $E(t)$ is a superflow of (4), if there exists a bounded open set $A \subset \mathbb{R}^N$, with $\bigcup_{t_0\le t\le t_1}\partial E(t)\times\{t\} \subset A\times[t_0,t_1]$, and $\delta > 0$, such that $d(x,t) = d_{E(t)}(x) \in C^1([t_0,t_1];C^2(A))$, and*

$$\frac{\partial d}{\partial t}(x,t) \;\ge\; \operatorname{div}\nabla\phi^\circ(\nabla d)(x,t)\;+\;\delta\,, \tag{5}$$

for any $x \in A$ and $t \in [t_0,t_1]$. We say that $E(t)$ is a subflow whenever $\delta < 0$ and the reverse inequality holds in (5).

Considering now a time discrete evolution scheme $E \mapsto T_hE$ (T_hE needs not be defined for all sets E, in our applications, it will be sufficient to define it for closed sets with compact boundary), parametrized by the time step $h > 0$, we introduce the following definition of consistency:

Definition 2.2. *The scheme T_h is consistent if and only if for any superflow $E(t)$, $t_0 \le t \le t_1$, in the sense of Definition 2.1, there exists h_0 such that if $h \le h_0$, then $T_hE(t) \supseteq E(t+h)$ for any $t \in [t_0, t_1-h]$, while for any subflow, the same holds with the reverse inclusion.*

This definition means that given a superflow, it will also go faster than the discretized evolutions, as soon as h is small enough. The following results follows from the theory of barriers (and the properties of the curvature flow), see [6, 7, 9, 16].

Proposition 2.3. *Assume T_h is a consistent scheme, in the sense of Definition 2.2 above, which is also monotone: for any $E, F \subset \mathbb{R}^N$, $E \subseteq F \Rightarrow T_hE \subseteq T_hF$. Let*

$E_0 \subset \mathbb{R}^N$ be a closed set with compact boundary such that the generalized anisotropic curvature flow $E(t)$ starting from E_0 is uniquely defined (no fattening). For any $t \geq 0$ let $E_h(t) := T^{[t/h]}E_0$. Then, for any t as long as $E(t)$ is not empty, $\partial E_h(t) \to \partial E(t)$ in the Hausdorff sense.

In the next sections, we prove consistency (and monotonicity), first for the (anisotropic) Merriman-Bence-Osher scheme, then for the Almgren-Taylor-Wang scheme, yielding, by Proposition 2.3, convergence to the generalized solution, when unique.

3 The Merriman-Bence-Osher algorithm

More than ten years ago, Merriman, Bence and Osher [3] proposed the following algorithm for the computation of the motion by mean curvature of a surface. Given a closed set $E \subset \mathbb{R}^N$, they let $T_h E = \{u(\cdot, h) \geq 1/2\}$, where u solves the heat equation with initial data $u(\cdot, 0) = \chi_E$, the characteristic function of E. They then conjectured that $E_h(t) := T_h^{[t/h]}E$ would converge to $E(t)$, where $E(t)$ is the (generalized) evolution by mean curvature starting from E.

The proof of convergence of this scheme was established by Evans [18], Barles and Georgelin [4]. Other proofs were given by H. Ishii [20] and Cao [12], where the heat equation was replaced by the convolution of χ_E with a more general symmetric kernel. Extensions and variants are found in [21, 27, 26, 29, 23].

As easily shown by formal asymptotic expansion, the natural anisotropic generalization of the Merriman-Bence-Osher algorithm is as follows. Given E a closed set with compact boundary in $\mathbb{R}^N$, we let $T_h(E) = \{x : u(x, h) \geq 1/2\}$ where $u(x,t)$ is the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) \;\in\; \mathrm{div}\,\Big(\phi^\circ(\nabla u)\partial\phi^\circ(\nabla u)\Big)(x,t) & t > 0,\ x \in \mathbb{R}^N\,, \\ u(\cdot, 0) \;=\; \chi_E & (t = 0)\,. \end{cases} \tag{6}$$

The funtion $u(x,t)$ is well defined and unique by classical results on contraction semigroups [11]: if E is compact, it corresponds to the flow in $L^2(\mathbb{R}^N)$ of the subdifferential of the functional $u \mapsto \int_{\mathbb{R}^N} \phi^\circ(\nabla u)^2/2\,dx$ if $u \in H^1(\mathbb{R}^N)$, and $+\infty$ otherwise. On the other hand, if $\mathbb{R}^N \setminus E$ is compact, one defines u by simply letting $u = 1 + v$ where v solves the same equation with initial data $\chi_E - 1$.

We first observe that the monotonicity of this scheme is obvious. Indeed, it follows from the comparision principle for equation (6)). We will show:

Proposition 3.1. *T_h, defined as above, is consistent in the sense of Definition 2.2.*

Proof. Let E be a superflow on $[t_0, t_1]$, in the sense of Definition 2.1, and let A be the associated neighborhood of $\partial E(t)$, $t \in [t_0, t_1]$.

Let $\gamma : \mathbb{R} \times [0, +\infty) \to [0, 1]$ be the solution of the 1D heat equation ($\partial\gamma/\partial\tau = \partial^2\gamma/\partial\xi^2$), with initial data the Heavyside function $\chi_{[0,+\infty)}$:

$$\gamma(\xi, \tau) \;=\; \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\xi} e^{-\frac{s^2}{4\tau}}\, ds\,.$$

One sees that it is self-similar: indeed, the change of variables $s' = s/\sqrt{\tau}$ yields

$$\gamma(\xi, \tau) \;=\; \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\frac{\xi}{\sqrt{\tau}}} e^{-\frac{s'^2}{4}}\, ds' \;=\; \gamma\left(\frac{\xi}{\sqrt{\tau}}, 1\right) \;=:\; \gamma_1\left(\frac{\xi}{\sqrt{\tau}}\right).$$

Fix $t < t_0$. The simplest idea would be to introduce the function $v(x, \tau) := \gamma(-d(x, t + \tau), \tau)$, defined in A for small τ. It satisfies $\{v(\cdot, \tau) \geq 1/2\} = E(t + \tau)$ and one has (using (5))

$$\frac{\partial v}{\partial \tau} \;=\; -\frac{\partial\gamma}{\partial\xi}\frac{\partial d}{\partial t} - \frac{\partial\gamma}{\partial\tau} \leq -\frac{\partial\gamma}{\partial\xi}\left(\operatorname{div}\nabla\phi^\circ(\nabla d) + \delta\right) - \frac{\partial\gamma}{\partial\tau}\,.$$

Also: $\nabla v = -(\partial\gamma/\partial\xi)\nabla d$, so that $\phi^\circ(\nabla v) = (\partial\gamma/\partial\xi)$ and $\nabla\phi^\circ(\nabla v) = -\nabla\phi^\circ(\nabla d)$, hence

$$\operatorname{div}\phi^\circ(\nabla v)\nabla\phi^\circ(\nabla v) \;=\; -\operatorname{div}\frac{\partial\gamma}{\partial\xi}\nabla\phi^\circ(\nabla d) \;=\; -\frac{\partial\gamma}{\partial\xi}\operatorname{div}\nabla\phi^\circ(\nabla d) - \frac{\partial^2\gamma}{\partial\xi^2}\,.$$

Here, we have used the fact that ϕ° is even and one-homogeneous, $\nabla\phi^\circ$ is odd and zero-homogeneous, $\phi^\circ(\nabla d) = 1$, and $\nabla d \cdot \nabla\phi^\circ(\nabla d) = \phi^\circ(\nabla d) = 1$ (by Euler's identity). Using $\partial\gamma/\partial\tau = \partial^2\gamma/\partial\xi^2$, we find:

$$\frac{\partial v}{\partial\tau} \;\leq\; \operatorname{div}\phi^\circ(\nabla v)\nabla\phi^\circ(\nabla v) - \delta\frac{\partial\gamma}{\partial\xi}\,.$$

Hence, v is a good candidate to be a subsolution of (6), with initial data $v(x, 0) = \chi_{E(t)}(x)$. If this were the case, we would get that $v \leq u$ (where u solves (6) with initial data $\chi_{E(t)}$), so that $\{v(\cdot, h) \geq 1/2\} \subseteq \{u(\cdot, h) \geq 1/2\}$, in other words, $E(t + h) \subseteq T_h E(t)$, which is our consistency. However, we cannot show that this v is less than u at the boundary of A (for instance), for $t \leq t+\tau \leq t+h$. This is why we define v in a slightly more complicated way: we let $v(x, \tau) := \gamma(-d(x, t + \tau) + \delta\tau, \tau) - \eta h$, where $\eta < \delta/\sqrt{2\pi}$ is fixed. Since now $\partial v/\partial\tau$ differs from the previous time derivative by $\delta\partial\gamma/\partial\xi$, one still has

$$\frac{\partial v}{\partial\tau} \;\leq\; \operatorname{div}\phi^\circ(\nabla v)\nabla\phi^\circ(\nabla v)\,. \tag{7}$$

at any $(x, \tau) \in A \times [0, h]$, hence v is a subsolution of (6). At $\tau = 0$, $v(x, 0) = \chi_{E(t)}(x) - \eta h < \chi_{E(t)}(x)$.

Let u solve (6) with initial data $\chi_{E(t)}$. First of all, we observe that since $d \in C^1([t_0, t_1]; C^2(A))$, $\partial E(t)$ is a C^2 compact hypersurface, continuous in time. Hence there exists $\rho > 0$, independent of t, such that each point $x \in \partial E(t)$, $E(t)$ satisfies an interior and exterior Wulff shape condition of radius ρ: there exist $z \in E(t)$ and $z' \notin E(t)$ with $\{\phi(\cdot - z) \leq \rho\} \subset E(t)$ and $\{\phi(\cdot - z') < \rho\} \cap E(t) = \emptyset$, while $\phi(x - z) = \phi(x - z') = \rho$. One may always assume that $\{|d(\cdot, s)| \leq \rho\} \subset A$ for all

$s \in [t_0, t_1]$. Let $B = \{|d(\cdot,t)| < \rho\}$. If h is small enough (independently of t), one also may assume that $|d(x,t+\tau) - d(x,t)| \le \rho/2$ in B for any $\tau \in [0,h]$, so that $\mathrm{dist}^{\phi}(\partial E(t+\tau), \partial B) \ge \rho/2$. We assume $h \le \rho/(4\delta)$. Let $x \in \partial B$ with $d(x,t) = \rho$: then $d(x,t+\tau) \ge \rho/2$ for any $\tau \in [0,h]$, so that $-d(x,t+\tau)+\delta\tau \le \delta h - \rho/2 \le -\rho/4$, and $v(x,\tau) \le \gamma(-\rho/4,\tau) - \eta h$ for any $\tau \in [0,h]$. Hence $v(x,\tau) \le \gamma_1(-\rho/(4\sqrt{\tau})) - \eta h \le \gamma_1(-\rho/(4\sqrt{h})) - \eta h$ which is negative if h is small enough. This shows that if h is small enough, $v(x,\tau) < 0 \le u(x,\tau)$ for any $\tau \le h$ and $x \in \partial B \cap \{d(\cdot,t) = \rho\}$.

If now $x \in \partial B$ with $d(x,t) = -\rho$, we use the fact that $u \ge w$, where w solves (6) with initial data $w_0 = \chi_{\{\phi(\cdot - x) \le \rho\}}$. One shows that $w(y,\tau) = U(\phi(y-x)/\rho, \tau/\rho^2)$ where $U(|x|,\tau) = \tilde{U}(x,\tau)$ and $\tilde{U}$ is the (radial) solution of the heat equation $\partial\tilde{U}/\partial t = \Delta\tilde{U}$ with initial datum χ_{B_1}, the characteristic function of the unit ball. It is well-known that

$$\tilde{U}(y,\tau) = \frac{1}{\sqrt{4\pi\tau}^N} \int_{\{|z|\le 1\}} \exp\left(-\frac{|y-z|^2}{4\tau}\right) dz$$

so that

$$U(0,\tau) \ = \ 1 - \frac{1}{\sqrt{4\pi}^N} \int_{\{|z|\ge 1/\sqrt{\tau}\}} \exp\left(-\frac{z^2}{4}\right) dz\,.$$

Hence, $u(x,\tau) \ge 1 - (1/\sqrt{4\pi}^N) \int_{\{|z|\ge\rho/\sqrt{\tau}\}} \exp(-z^2/4)\,dz \ge 1 - c\exp(-\rho/(4\sqrt{h}))$ for some constant $c > 0$, and any $\tau \in [0,h]$. Hence, for $\tau \in [0,h]$, $v(x,\tau) - u(x,\tau) \le c\exp(-\rho/(4\sqrt{h})) - \eta h$: clearly, this is negative if h is small enough (depending only on ρ). We have shown that v is below u on $\partial B \times [0,h]$, if h is small enough (uniformly in t).

By standard results on parabolic equations, we find that $v \le u$ on $B \times [0,h]$ and in particular $v(\cdot,h) \le u(\cdot,h)$ in B. Hence, $\{v(\cdot,h) \ge 1/2\} \subseteq \{u(\cdot,h) \ge 1/2\}$. Observe that $v(x,h) \ge 1/2$ iff $-d(x,t+h) + \delta h \ge (\gamma(\cdot,h))^{-1}(1/2 + \eta h) = \sqrt{2\pi}\eta h + o(h)$, that is, $d(x,t+h) \le (\sqrt{2\pi}\eta - \delta)h + o(h) =: \sigma_h$. If h is small enough, $\sigma_h > 0$, so that $x \in E(t+h) \Rightarrow d(x,t+h) \le \sigma_h \Leftrightarrow v(x,h) \ge 1/2$: we deduce $E(t+h) \subseteq T_h E(t)$, which was our claim. The proof of consistency with subflows is identical. □

See [15] for a proof of consistency and convergence which works in more general situations (namely, the crystalline case). See also K. Ishii [22]'s recent paper on an optimal estimate on the rate of convergence of Merriman-Bence-Osher's algorithm, in the isotropic case, where the proof of convergence is very close to ours.

4 The Almgren-Taylor-Wang algorithm

In Almgren, Taylor and Wang's paper [1], the transformation $T_h E$ is defined as *a* solution of

$$\min_{F \subseteq \mathbb{R}^N} P_\phi(F) \ + \ \frac{1}{h} \int_{F \triangle E} |d_E^{\phi}|(x)\,dx\,, \tag{8}$$

where now, $F \triangle E$ is the symmetric difference of the two sets F and E and $P_\phi(F)$ is the anisotropic perimeter. This is rigorously defined by $\int_{\mathbb{R}^N} \phi^\circ(D\chi_F)$, where the anisotropic total variation is given by

$$\int_{\mathbb{R}^N} \phi^\circ(Dv) \;:=\; \sup\left\{ \int_{\mathbb{R}^N} v(x) \mathrm{div}\, \psi(x)\, dx \;:\; \psi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N), \phi(\psi(x)) \le 1 \;\forall x \in \mathbb{R}^N \right\}.$$

The same approach to curvature motion has also been proposed by Luckhaus and Sturzenhecker [24], in the isotropic case.

It is shown in [14, 13, 5] that a *monotone* selection of $T_h E$ can be built in the following way: one fixes a bounded open set $\Omega \supset\supset E$, and one lets w be the (unique) minimizer of

$$\int_\Omega \phi^\circ(Dw) \;+\; \frac{1}{2h}(w(x) - d^\phi_E(x))^2\, dx\,, \tag{9}$$

then, $F = \{w \le 0\}$ is a solution of (8), as soon as the domain Ω is large enough. Clearly, letting $T_h E$ be this solution defines a monotone operator, since $E \subset E' \Rightarrow d^\phi_E \ge d^\phi_{E'}$ so that $w \ge w'$ (being w' the solution of (9) with E replaced with E'), and $T_h E \subset T_h E'$. On the other hand, it is also shown in [14, 13, 5] that this choice gives the largest solution, whereas $\{w < 0\}$ would be the smallest (yielding uniqueness, up to a negligible set, whenever $|\{w = 0\}| = 0$, which is "generically" true in some sense). The proof of consistency we will next give would also work with this second choice, yielding convergence of *any* selection of Almgren-Taylor-Wang's scheme to the generalized solution, when unique. We now show:

Proposition 4.1. *T_h, defined as above, is consistent in the sense of Definition 2.2.*

Proof. Let E be a superflow on $[t_0, t_1]$, in the sense of Definition 2.1, and let A be the associated neighborhood of $\partial E(t)$, $t \in [t_0, t_1]$.

Observe that as in the previous section, there exists $\rho > 0$ such that $\{d(\cdot, t) \le \rho\} \subset A$ at any time $t \in [t_0, t_1]$, and $\partial E(t)$ satisfies both an interior and exterior Wulff shape condition of radius ρ.

We fix $t \in [t_0, t_1)$, and let $B = \{d(\cdot, t) < \rho\}$. Consider $\psi : \mathbb{R} \to \mathbb{R}$ a smooth increasing function with $\psi(s) \ge s$ and $\psi(s) = s$ for $|s| \le \varepsilon/2$. We set, for $x \in B$, $v(x) := \psi(d(x, t + h))$. Then, from (5), it follows

$$\begin{aligned}\frac{v(x) - d_{E(t)}(x)}{h} \;\ge\; \frac{d(x, t+h) - d(x,t)}{h} \;&=\; \frac{1}{h}\int_0^h \frac{\partial d}{\partial t}(x, t+\tau)\, d\tau \\ &\ge\; \frac{1}{h}\int_t^{t+h} \mathrm{div}\, \nabla\phi^\circ(\nabla d)(x, t+\tau)\, d\tau \;+\; \delta\,.\end{aligned}$$

Let now ω be a modulus of continuity for $\mathrm{div}\, \nabla\phi^\circ(\nabla d)$ in $\{|d| \le \rho\}$: we find

$$\frac{v(x) - d_{E(t)}(x)}{h} \;\ge\; \mathrm{div}\, \nabla\phi^\circ(\nabla d)(x, t+h) \;+\; \delta \;-\; \omega(h).$$

Observe that for any $x \in B$ it holds $\nabla v(x) = \psi'(d(x, t+h))\nabla d(x, t+h)$, so that (recall that $\nabla\phi^\circ$ 0-homogeneous), $\nabla\phi^\circ(\nabla v(x)) = \nabla\phi^\circ(\nabla d(x, t+h))$ hence

$\operatorname{div} \nabla\phi^\circ(\nabla d)(x, t+h) = \operatorname{div} \nabla\phi^\circ(\nabla v)(x)$. Therefore, if h is small enough so that $\omega(h) \le \delta$, we get

$$\frac{v(x) - d_{E(t)}(x)}{h} \ \ge\ \operatorname{div} \nabla\phi^\circ(\nabla v)(x).$$

Let w solve (9), with $E = E(t)$. We will show that we may choose ψ in order to have $v \ge w$ on ∂B, so that v is a supersolution for the problem

$$\min\left\{ \int_B \phi^\circ(Du) \ +\ \frac{1}{2h}\int_B (u(x) - d_{E(t)}(x))^2\,dx \,:\, u = w \text{ on } \partial B \right\} \qquad (10)$$

(which is solved by w). We will deduce that $v \ge w$ in B, so that $\{w \le 0\} \supseteq \{v \le 0\} = \{d(\cdot, t+h) \le 0\}$, that is, $T_h(E(t)) \supseteq E(t+h)$.

First of all, d is uniformly continuous in time, so that if h is small enough, one has $d(x, t+h) \ge 3\rho/4$ if $d(x,t) = \rho$. If $M > \operatorname{diam}\Omega$, then one shows that $M \ge w$ in Ω. We may choose a function ψ with $\psi(3\rho/4) \ge M$, so that $v(x) \ge M \ge w(x)$ if $d(x,t) = \rho$.

On the other hand, since $E(t)$ satisfies an interior Wulff shape condition of radius ρ, one has $d_E^\phi \le \phi(\cdot - x) - \rho$ at any point $x \in \partial B$ with $d(x,t) = -\rho$. The analysis in [13, 16] shows that the solution of (9) with d_E^ϕ replaced with ϕ takes the value $2N\sqrt{h}/\sqrt{N+1}$ at the origin. We deduce that $w(x) \le 2N\sqrt{h}/\sqrt{N+1} - \rho$: hence, if h is small enough, we get $w(x) \le -3\rho/4$. We can choose ψ such that $\psi(s) \ge -3\rho/4$ for any s, so that $v(x) \ge w(x)$ if $d(x,t) = -\rho$. We conclude that $v \ge w$ on ∂B. Hence v is a supersolution for (10), which implies $T_{t,t+h}(E(t)) \supseteq E(t+h)$.

If $E(t)$ is a subflow, we can reproduce the same proof to show that $T_{t,t+h}(E(t)) \subseteq E(t+h)$. □

While a (much more difficult) proof of consistency with smooth flows is already found in Almgren, Taylor and Wang's paper [1], our proof is more easily adapted to other situations: in [16], we consider the case of a flow driven by anisotropic curvature with an additional time-dependent forcing term, possibly discontinuous.

References

[1] F. Almgren, J. E. Taylor, and L.-H. Wang, *Curvature-driven flows: a variational approach*, SIAM J. Control Optim. **31** (1993), no. 2, 387–438. MR 94h:58067

[2] L. Ambrosio and Soner H. M., *Level set approach to mean curvature flow in arbitrary codimension*, J. Differential Geom. **43** (1996), no. 4, 693–737.

[3] J. K. Bence B. Merriman and S. J. Osher, *Diffusion generated motion by mean curvature*, "Computational Crystal Growers Workshop", Selected Lectures in Math. Amer. Math. Soc., Providence, Rhode Island (J. E. Taylor, ed.), 1992.

[4] G. Barles and C. Georgelin, *A simple proof of convergence for an approximation scheme for computing motions by mean curvature*, SIAM J. Numer. Anal. **32** (1995), no. 2, 484–500. MR 96c:65140

[5] G. Bellettini, V. Caselles, A. Chambolle, and M. Novaga, *Crystalline mean curvature flow of convex sets*, Preprint Centro De Giorgi, Pisa, 2004.

[6] G. Bellettini and M. Novaga, *Minimal barriers for geometric evolutions*, J. Differential Equations **139** (1997), no. 1, 76–103. MR 98j:35083

[7] ______, *Comparison results between minimal barriers and viscosity solutions for geometric evolutions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), no. 1, 97–131. MR 99i:35081

[8] ______, *Approximation and comparison for nonsmooth anisotropic motion by mean curvature in* $\mathbf{R}^N$, Math. Models Methods Appl. Sci. **10** (2000), no. 1, 1–10. MR 2001a:53106

[9] G. Bellettini and M. Paolini, *Some results on minimal barriers in the sense of De Giorgi applied to driven motion by mean curvature*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) **19** (1995), 43–67. MR 97b:49038

[10] ______, *Anisotropic motion by mean curvature in the context of Finsler geometry*, Hokkaido Math. J. **25** (1996), no. 3, 537–566. MR 97i:53079

[11] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Publishing Co., Amsterdam, 1973, North-Holland Mathematics Studies, No. 5. Notas de Matemática (50). MR MR0348562 (50 #1060)

[12] F. Cao, *Partial differential equations and mathematical morphology*, J. Math. Pures Appl. (9) **77** (1998), no. 9, 909–941. MR 99m:35097

[13] V. Caselles and A. Chambolle, *Anisotropic curvature-driven flow of convex sets*, Preprint CMAP #528, Ecole Polytechnique, France, 2004.

[14] A. Chambolle, *An algorithm for mean curvature motion*, Interfaces Free Bound. **6** (2004), no. 2, 195–218. MR MR2079603

[15] A. Chambolle and M. Novaga, *Convergence of an algorithm for anisotropic mean curvature motion*, Preprint CMAP #572, Ecole Polytechnique, France, March 2005.

[16] ______, *Implicit time discretization of the mean curvature flow with a discontinuous forcing term*, Preprint CMAP #571, Ecole Polytechnique, France, March 2005.

[17] E. De Giorgi, *New ideas in calculus of variations and geometric measure theory*, Motion by mean curvature and related topics (Trento, 1992), de Gruyter, Berlin, 1994, pp. 63–69. MR MR1277391 (95f:49063)

[18] L. C. Evans, *Convergence of an algorithm for mean curvature motion*, Indiana Univ. Math. J. **42** (1993), no. 2, 533–557. MR 95d:58023

[19] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature. II*, Trans. Amer. Math. Soc. **330** (1992), no. 1, 321–332. MR 92f:58050

[20] H. Ishii, *A generalization of the Bence, Merriman and Osher algorithm for motion by mean curvature*, Curvature flows and related topics (Levico, 1994), GAKUTO Internat. Ser. Math. Sci. Appl., vol. 5, Gakkōtosho, Tokyo, 1995, pp. 111–127. MR 96m:35121

[21] H. Ishii, G. E. Pires, and P. E. Souganidis, *Threshold dynamics type approximation schemes for propagating fronts*, J. Math. Soc. Japan **51** (1999), no. 2, 267–308. MR 2000j:65016

[22] K. Ishii, *Optimal rate of convergence of the Bence-Merriman-Osher algorithm for motion by mean curvature*, Preprint #16 available at the *Control Theory and PDE* web site `http://cpde.iac.rm.cnr.it`, Nov. 2004.

[23] F. Leoni, *Convergence of an approximation scheme for curvature-dependent motions of sets*, SIAM J. Numer. Anal. **39** (2001), no. 4, 1115–1131 (electronic). MR 2002j:65093

[24] S. Luckhaus and T. Sturzenhecker, *Implicit time discretization for the mean curvature flow equation*, Calc. Var. Partial Differential Equations **3** (1995), no. 2, 253–271. MR 97e:65085

[25] R. T. Rockafellar, *Convex analysis*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997, Reprint of the 1970 original, Princeton Paperbacks. MR MR1451876 (97m:49001)

[26] S. J. Ruuth, *Efficient algorithms for diffusion-generated motion by mean curvature*, J. Comput. Phys. **144** (1998), no. 2, 603–625. MR MR1638032 (99f:73088)

[27] S. J. Ruuth and B. Merriman, *Convolution-generated motion and generalized Huygens' principles for interface motion*, SIAM J. Appl. Math. **60** (2000), no. 3, 868–890 (electronic). MR MR1740854 (2001c:35107)

[28] J. E. Taylor, J. W. Cahn, and C. A. Handwerker, *Geometric models of crystal growth*, Acta Metall. **40** (1992), 1443–1474.

[29] L. Vivier, *Convergence of an approximation scheme for computing motions with curvature dependent velocities*, Differential Integral Equations **13** (2000), no. 10-12, 1263–1288. MR MR1785707 (2001h:65121)