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LOCALIZATION FOR THE SCHRÖDINGER EQUATION IN A LOCALLY PERIODIC MEDIUM

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ABSTRACT. We study the homogenization of a Schrödinger equation in a locally periodic medium. For the time and space scaling of semi-classical analysis we consider well-prepared initial data that are concentrated near a stationary point (with respect to both space and phase) of the energy, i.e. the Bloch cell eigenvalue. We show that there exists a localized solution which is asymptotically given as the product of a Bloch wave and of the solution of an homogenized Schrödinger equation with quadratic potential.

Key words: Homogenization, localization, Bloch waves, Schrödinger.

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1. INTRODUCTION

We study the homogenization of the following Schrödinger equation

$$
\left\{ \begin{array}{ll} \dfrac{i}{\varepsilon}\dfrac{\partial u_\varepsilon}{\partial t} - \operatorname{div}\left(A\left(x, \dfrac{x}{\varepsilon}\right) \nabla u_\varepsilon \right) + \dfrac{1}{\varepsilon^2} c\left(x, \dfrac{x}{\varepsilon}\right) u_\varepsilon = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ u_\varepsilon(0,x) = u_\varepsilon^0(x) & \text{in } \mathbb{R}^N \end{array} \right. \tag{1.1}
$$

where the unknown $u_\varepsilon(t,x)$ is a complex-valued function. The coefficients $A(x,y)$ and $c(x,y)$ are real and sufficiently smooth bounded functions defined for $x \in \mathbb{R}^N$ (the macroscopic variable) and $y \in \mathbb{T}^N$ (the microscopic variable in the unit torus). The period ε is a small positive parameter which is intended to go to zero. Furthermore the matrix A is symmetric, uniformly positive definite. Of course the usual Schrödinger equation is recovered when $A \equiv Id$ but, since there is no additional difficulty, we keep the general form of equation (1.1) in the sequel (which can be interpreted as introducing a non flat locally periodic metric).

The scaling of (1.1) is that of semi-classical analysis (see e.g. [5], [8], [10], [11], [12], [13], [14], [18], [19]): if the period is rescaled to 1, it amounts to look at large, time and space, variables of order ε^{-1}. At least in the case when $A \equiv Id$ and $c(x,y) = c_0(x)+c_1(y)$, there is a well-known theory for the asymptotic limit of (1.1) when ϵ goes to zero. By using WKB asymptotic expansion or the notion of semi-classical measures (or Wigner transforms) the homogenized problem is in some sense the Liouville transport equation for a classical particle which is the limit of the wave function u_ε. In other words, for an initial data living in the n-th Bloch band and under some technical assumptions on the Bloch spectral cell problem (1.4), the semi-classical limit of (1.1) is given by the dynamic of the following Hamiltonian system in the phase space $(x,\theta) \in \mathbb{R}^N \times \mathbb{T}^N$

$$\left\{ \begin{array}{l} \dot{x} = \nabla_\theta \lambda_n(x,\theta) \\ \dot{\theta} = -\nabla_x \lambda_n(x,\theta) \end{array} \right. \tag{1.2}$$

where the Hamiltonian $\lambda_n(x,\theta)$ is precisely the n-th Bloch eigenvalue of (1.4) (see [8], [10], [11], [12], [13], [14], [18], [19] for more details).

Our approach to (1.1) is different since we consider special initial data that are monochromatic, have zero group velocity and zero applied force. Namely the initial data is concentrating at a point (x^n,θ^n) of the phase space where $\nabla_\theta \lambda_n(x^n,\theta^n) = \nabla_x \lambda_n(x^n,\theta^n) = 0$. In such a case, the previous Hamiltonian system (1.2) degenerates (its solution is constant) and is unable to describe the precise dynamic of the wave function u_ε. We exhibit another limit problem which is again a Schrödinger equation with quadratic potential. In other words we build a sequence of approximate solutions of (1.1) which are the product of a Bloch wave and of the solution of an homogenized Schrödinger equation. Furthermore, if the full Hessian tensor of the Bloch eigenvalue $\lambda_n(x,\theta)$ is positive definite at (x^n,θ^n), we prove that all the eigenfunctions of an homogenized Schrödinger equation are exponentially decreasing at infinity. In other words, we exhibit a localization phenomenon for (1.1) since we build a sequence of approximate solutions that decay exponentially fast away from x^n. The root of this localization phenomenon is the macroscopic modulation (i.e. with respect to x) of the periodic coefficients which is similar in spirit to the randomness that causes Anderson's localization (see [9] and references therein).

Let us describe more precisely the type of well-prepared initial data that we consider. For a given point $(x^n,\theta^n) \in \mathbb{R}^N \times \mathbb{T}^N$ and a given function $v^0 \in H^1(\mathbb{R}^N)$ we take

$$u_\varepsilon^0(x) = \psi_n\Big(x^n, \frac{x}{\varepsilon}, \theta^n\Big) e^{2i\pi \frac{\theta^n \cdot x}{\varepsilon}} v^0\Big(\frac{x - x^n}{\sqrt{\varepsilon}}\Big) \tag{1.3}$$

where $\psi_n(x,y,\theta)$ is a so-called Bloch eigenfunction, solution of the following Bloch spectral cell equation

$$-(\mathrm{div}_y + 2i\pi\theta)(A(x,y)(\nabla_y + 2i\pi\theta)\psi_n) + c(x,y) = \lambda_n(x,\theta)\psi_n \qquad \text{in } \mathbb{T}^N, \tag{1.4}$$

corresponding to the n-th eigenvalue or energy level λ_n. The Bloch wave ψ_n is periodic with respect to y but v^0 is not periodic, so $v^0\left(\frac{x-x^n}{\sqrt{\varepsilon}}\right)$ means that the initial data is concentrated around x^n with a support of asymptotic size $\sqrt{\varepsilon}$. The Bloch frequency $\theta^n \in \mathbb{T}^N$, the localization point $x^n \in \mathbb{R}^N$ and the energy level n are chosen such that $\lambda_n(x^n, \theta^n)$ is simple and $\nabla_x \lambda_n(x^n, \theta^n) = \nabla_\theta \lambda_n(x^n, \theta^n) = 0$.

Our main result (Theorem 3.2) shows that the solution of (1.1) is approximately given by

$$u_\varepsilon(t,x) \approx \psi_n\left(x^n, \frac{x}{\varepsilon}, \theta^n\right) e^{i\frac{\lambda_n(x^n,\theta^n)t}{\varepsilon}} e^{2i\pi\frac{\theta^n\cdot x}{\varepsilon}} v\left(t, \frac{x-x^n}{\sqrt{\varepsilon}}\right), \tag{1.5}$$

where v is the unique solution of the homogenized Schrödinger equation

$$\begin{cases} i\dfrac{\partial v}{\partial t} - \operatorname{div}\left(A^*\nabla v\right) + \operatorname{div}(vB^*z) + c^*v + vD^*z\cdot z = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ v(0,z) = v^0(z) & \text{in } \mathbb{R}^N \end{cases} \tag{1.6}$$

where c^* is a constant coefficient and A^*, B^*, D^* are constant matrices defined by

$$A^* = \frac{1}{8\pi^2}\nabla_\theta\nabla_\theta\lambda_n(x^n,\theta^n)\,,\ B^* = \frac{1}{2i\pi}\nabla_\theta\nabla_x\lambda_n(x^n,\theta^n)\,,\ D^* = \frac{1}{2}\nabla_x\nabla_x\lambda_n(x^n,\theta^n)\,.$$

In Proposition 3.4 we show that the homogenized problem (1.6) is well-posed since the underlying operator is self-adjoint. Furthermore, under the additional assumption that the Hessian tensor $\nabla\nabla\lambda_n(x^n,\theta^n)$ (with respect to both variables x and θ) is positive definite, we prove that (1.6) admits a countable number of eigenvalues and eigenfunctions which all decay exponentially at infinity (see Proposition 3.5). In such a case, formula (1.5) defines a family of approximate (exponentially) localized solutions of (1.1).

Let us indicate that the case of the first eigenvalue (ground state) $n = 1$ with $\theta^1 = 0$ was already studied in [3] (for the spectral problem rather than the evolution equation). The case of purely periodic coefficients (i.e. that depend only on y and not on x) is completely different and was studied in [4]. Indeed, in this latter case there is no localization effect and one proves that, for a longer time scale (of order ε^{-1} with respect to (1.1)), the homogenized limit is again a Schrödinger equation without the drift and quadratic potential in (1.6).

2. Preliminaries

In the present section we give our main assumptions, set some notation and a few preliminary results needed in the proof of the main results in Section 3.

We first assume that the coefficients $A_{ij}(x,y)$ and $c(x,y)$ are real, bounded, and Carathéodory functions (measurable with respect to y and continuous in x), which are periodic with respect to y. In other words, they belong to $C_b\left(\mathbb{R}^N; L^\infty(\mathbb{T}^N)\right)$. Furthermore, the tensor

$A(x,y)$ is symmetric uniformly coercive. Under these assumptions, it is well-known that, for any values of the parameters $\theta \in \mathbb{T}^N$ and $x \in \mathbb{R}^N$, the cell problem (1.4) defines a compact self-adjoint operator on $L^2(\mathbb{T}^N)$ which admits a countable sequence of real increasing eigenvalues $\{\lambda_n(x,\theta)\}_{n\geq 1}$ (repeated with their multiplicity) with corresponding eigenfunctions $\{\psi_n(x,\theta,y)\}_{n\geq 1}$ normalized by

$$||\psi_n(x,\theta,\cdot)||_{L^2(\mathbb{T}^N)} = 1\,.$$

Our main assumptions are:

Hypothesis H1. There exist $x^n \in \mathbb{R}^N$ and $\theta^n \in \mathbb{T}^N$ such that

$$\text{(2.1)}\quad \begin{cases} (i)\ \lambda_n(x^n,\theta^n) \text{ is a simple eigenvalue,} \\ (ii)\ (x^n,\theta^n) \text{ is a critical point of } \lambda_n(x,\theta), i.e.\ \nabla_x\lambda_n(x^n,\theta^n) = \nabla_\theta\lambda_n(x^n,\theta^n) = 0. \end{cases}$$

Hypothesis H2. The coefficients $A(x,y)$ and $c(x,y)$ are of class C^2 with respect to the variable x in a neighborhood of $x = x^n$.

Then we set:

$$A_{1,h}(y) := \frac{\partial A}{\partial x_h}(x^n,y)\,,\quad A_{2,lh}(y) := \frac{\partial^2 A}{\partial x_l \partial x_h}(x^n,y)\,,\quad \text{for } l,h = 1,\dots,N\,.$$

Similar notation is used to denote the derivatives of the function c with respect to the x-variable. With an abuse of notation we further set

$$A(y) := A(x^n,y)\,,\quad \lambda_n := \lambda_n(x^n,\theta^n)\,,\quad \psi_n(y) := \psi_n(x^n,y,\theta^n)\,,$$

and analogous notation holds for all derivatives of ψ_n and λ_n with respect to the x-variable and the θ-variable evaluated at $x = x^n$ and $\theta = \theta^n$. Without loss of generality we will assume in the sequel that $x^n = 0$.

Notation. For any function $\rho(y)$ defined on $\mathbb{T}^N$ we set

$$\rho^\varepsilon(z) := \rho(z/\sqrt{\varepsilon})$$

where $z := \sqrt{\varepsilon}y \equiv x/\sqrt{\varepsilon}$. In the sequel the symbols div_y and ∇_y will stand for the divergence and gradient operators which act with respect to the y-variable while div and ∇ will indicate the divergence and gradient operators which act with respect to the z-variable. Finally throughout this paper the Einstein summation convention is used.

Under assumption (2.1)-(i) it is a classical matter to prove that the n-th eigencouple of (1.4) is smooth with respect to the variable θ in a neighborhood of $\theta = \theta^n$ (see [16]) and has the same differentiability property as the coefficients with respect to the variable x. Introducing the unbounded operator $\mathbb{A}_n(x,\theta)$ defined on $L^2(\mathbb{T}^N)$ by

$$\mathbb{A}_n(x,\theta)\psi = -(\mathrm{div}_y + 2i\pi\theta)\Big(A(x,y)(\nabla_y + 2i\pi\theta)\psi\Big) + c(x,y)\psi - \lambda_n(x,\theta)\psi,$$

it is easy to differentiate (1.4). Denoting by $(e_k)_{1\leq k\leq N}$ the canonical basis of $\mathbb{R}^N$, the first derivatives satisfy

$$\begin{aligned}\mathbb{A}_n(x,\theta)\frac{\partial \psi_n}{\partial \theta_k} = \;& 2i\pi e_k A(x,y)(\nabla_y + 2i\pi\theta)\psi_n \\ &+(\mathrm{div}_y + 2i\pi\theta)\left(A(x,y)2i\pi e_k\psi_n\right) + \frac{\partial \lambda_n}{\partial \theta_k}(x,\theta)\psi_n\,,\end{aligned} \tag{2.2}$$

$$\begin{aligned}\mathbb{A}_n(x,\theta)\frac{\partial \psi_n}{\partial x_l} = \;& (\mathrm{div}_y + 2i\pi\theta)\Big(\frac{\partial A}{\partial x_l}(x,\theta)(\nabla_y + 2i\pi\theta)\psi\Big) \\ &-\frac{\partial c}{\partial x_l}(x,y)\psi_n + \frac{\partial \lambda_n}{\partial x_l}(x,\theta)\psi_n\,.\end{aligned} \tag{2.3}$$

Similar formulas hold for second order derivatives. By integrating the cell equations for the second order derivatives against ψ_n we obtain the following formulas that will be useful in the sequel (their proofs are safely left to the reader).

Lemma 2.1. *Assume that assumptions* **H1** *and* **H2** *hold true. Then the following equalities hold:*

$$\begin{aligned}&\int_{\mathbb{T}^N}\frac{1}{2\pi i}\Big[A_{1,h}(\nabla_y + 2i\pi\theta^n)\frac{\partial \psi_n}{\partial \theta_k}\cdot(\nabla_y - 2i\pi\theta^n)\bar\psi_n + c_{1,h}\,\frac{\partial \psi_n}{\partial \theta_k}\bar\psi_n\Big]\,dy \\ &+\int_{\mathbb{T}^N}\Big[A_{1,h}e_k\psi_n\cdot(\nabla_y - 2i\pi\theta^n)\bar\psi_n + Ae_k\frac{\partial \psi_n}{\partial x_h}\cdot(\nabla_y - 2i\pi\theta^n)\bar\psi_n\Big]\,dy \\ &-\int_{\mathbb{T}^N}\Big[e_k\bar\psi_n A_{1,h}\cdot(\nabla_y + 2i\pi\theta^n)\psi_n + e_k\bar\psi_n A\cdot(\nabla_y + 2i\pi\theta^n)\frac{\partial \psi_n}{\partial x_h}\Big]\,dy \\ &-\frac{1}{2i\pi}\frac{\partial^2\lambda_n}{\partial x_h\partial\theta_k} = 0\,,\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\int_{\mathbb{T}^N}\Big[A_{2,lh}(\nabla_y + 2i\pi\theta^n)\psi_n\cdot(\nabla_y - 2i\pi\theta^n)\bar\psi_n + \Big(c_{2,lh} - \frac{\partial^2\lambda_n}{\partial x_l\partial x_h}\Big)|\psi_n|^2\Big]\,dy \\ &+\int_{\mathbb{T}^N}\Big[A_{1,h}(\nabla_y + 2i\pi\theta^n)\frac{\partial \psi_n}{\partial x_l}\cdot(\nabla_y - 2i\pi\theta^n)\bar\psi_n + c_{1,h}\,\frac{\partial \psi_n}{\partial x_l}\bar\psi_n\Big]\,dy \\ &+\int_{\mathbb{T}^N}\Big[A_{1,l}(\nabla_y + 2i\pi\theta^n)\frac{\partial \psi_n}{\partial x_h}\cdot(\nabla_y - 2i\pi\theta^n)\bar\psi_n + c_{1,l}\,\frac{\partial \psi_n}{\partial x_h}\bar\psi_n\Big]\,dy = 0\,,\end{aligned} \tag{2.5}$$

$$
\begin{aligned}
(2.6) \quad & \int_{\mathbb{T}^N} \left[2i\pi e_k A(y)(\nabla_y + 2i\pi\theta^n)\frac{\partial \psi_n}{\partial \theta_l}\bar{\psi}_n - \left(A(y) 2i\pi e_k \frac{\partial \psi_n}{\partial \theta_l} \right)(\nabla_y - 2i\pi\theta^n)\bar{\psi}_n \right] dy \\
& + \int_{\mathbb{T}^N} \left[2i\pi e_l A(y)(\nabla_y + 2i\pi\theta^n)\frac{\partial \psi_n}{\partial \theta_k}\bar{\psi}_n - \left(A(y) 2i\pi e_l \frac{\partial \psi_n}{\partial \theta_k} \right)(\nabla_y - 2i\pi\theta^n)\bar{\psi}_n \right] dy \\
& - \int_{\mathbb{T}^N} \left[4\pi^2 e_k A(y) e_l |\psi_n|^2 + 4\pi^2 e_l A(y) e_k |\psi_n|^2 \right] dy \\
& + \frac{\partial^2 \lambda_n}{\partial \theta_l \partial \theta_k}(\theta^n) = 0 \, .
\end{aligned}
$$

We now give the variational formulations of the above cell problems, rescaled at size ε.

Lemma 2.2. *Assume that assumptions* **H1** *and* **H2** *hold true and let $\varphi(z)$ be a smooth compactly supported function defined from $\mathbb{R}^N$ into $\mathbb{C}$. Then the following equalities hold:*

$$
(2.7) \quad \int_{\mathbb{R}^N} \left[A^\varepsilon(\nabla_y + 2i\pi\theta^n)\psi_n^\varepsilon \cdot (\sqrt{\varepsilon}\nabla - 2i\pi\theta^n)\bar{\varphi}(z) + (c^\varepsilon - \lambda_n^\varepsilon)\psi_n^\varepsilon \bar{\varphi} \right] dz = 0 \, ,
$$

$$
\begin{aligned}
(2.8) \quad & \int_{\mathbb{R}^N} \left[A^\varepsilon(\nabla_y + 2i\pi\theta^n)\frac{\partial \psi_n^\varepsilon}{\partial \theta_k^n} \cdot (\sqrt{\varepsilon}\nabla - 2i\pi\theta^n)\bar{\varphi} + (c^\varepsilon - \lambda_n^\varepsilon)\frac{\partial \psi_n^\varepsilon}{\partial \theta_k^n}\bar{\varphi} \right] dz \\
& + \int_{\mathbb{R}^N} \left[-2\pi i e_k \cdot A^\varepsilon(\nabla_y + 2i\pi\theta^n)\psi_n^\varepsilon \bar{\varphi} + A^\varepsilon \, 2\pi i e_k \psi_n^\varepsilon \cdot (\sqrt{\varepsilon}\nabla - 2i\pi\theta^n)\bar{\varphi} \right] dz = 0 \, ,
\end{aligned}
$$

$$
\begin{aligned}
(2.9) \quad & \int_{\mathbb{R}^N} \left[A^\varepsilon(\nabla_y + 2i\pi\theta^n)\frac{\partial \psi_n^\varepsilon}{\partial x_h} \cdot (\sqrt{\varepsilon}\nabla - 2i\pi\theta^n)\bar{\varphi} + (c^\varepsilon - \lambda_n^\varepsilon)\frac{\partial \psi_n^\varepsilon}{\partial x_h}\bar{\varphi} \right] dz \\
& + \int_{\mathbb{R}^N} \left[A_{1,h}^\varepsilon(\nabla_y + 2i\pi\theta^n)\psi_n^\varepsilon \cdot (\sqrt{\varepsilon}\nabla - 2i\pi\theta^n)\bar{\varphi} + c_{1,h}^\varepsilon \, \psi_n^\varepsilon \bar{\varphi} \right] dz = 0 \, .
\end{aligned}
$$

Proof. Formula (2.7) follows straightforwardly from equation (1.4) while (2.8)-(2.9) are consequences of (2.2)-(2.3). □

Finally we recall the notion of two-scale convergence introduced in [1], [17] (that will be used with $\delta = \sqrt{\varepsilon}$).

Proposition 2.3. *Let f_δ be a sequence uniformly bounded in $L^2(\mathbb{R}^N)$.*

(1) *There exists a subsequence, still denoted by f_δ, and a limit $f_0(x,y) \in L^2(\mathbb{R}^N \times \mathbb{T}^N)$ such that f_δ* two-scale converges *weakly to f_0 in the sense that*

$$
\lim_{\delta \to 0} \int_{\mathbb{R}^N} f_\delta(x)\phi(x, x/\varepsilon)\, dx = \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} f_0(x,y)\phi(x,y)\, dx\, dy
$$

for all functions $\phi(x,y) \in L^2\left(\mathbb{R}^N; C(\mathbb{T}^N)\right)$.

(2) *Assume further that* f_δ *two-scale converges weakly to* f_0 *and that*

$$\lim_{\delta\to 0} \|f_\delta\|_{L^2(\mathbb{R}^N)} = \|f_0\|_{L^2(\mathbb{R}^N\times\mathbb{T}^N)}.$$

Then f_δ *is said to two-scale converge* strongly *to its limit* f_0 *in the sense that, if* f_0 *is smooth enough, e.g.* $f_0 \in L^2\left(\mathbb{R}^N; C(\mathbb{T}^N)\right)$, *we have*

$$\lim_{\delta\to 0} \int_{\mathbb{R}^N} |f_\delta(x) - f_0(\, x, x/\delta\,)|^2 \, dx = 0.$$

(3) *Assume that* $\delta\nabla f_\delta$ *is also uniformly bounded in* $L^2(\mathbb{R}^N)^N$. *Then there exists a subsequence, still denoted by* f_δ, *and a limit* $f_0(x,y) \in L^2(\mathbb{R}^N; H^1(\mathbb{T}^N))$ *such that* f_δ *two-scale converges to* $f_0(x,y)$ *and* $\delta\nabla f_\delta$ *two-scale converges to* $\nabla_y f_0(x,y)$.

3. Main results

We begin by recalling the usual a priori estimates for the solution of the Schrödinger equation (1.1) which hold true since the coefficients are real. They are obtained by multiplying the equation successively by $\overline{u}_\varepsilon$ and $\frac{\partial \overline{u}_\varepsilon}{\partial t}$, and integrating by parts.

Lemma 3.1. *There exists* $C > 0$ *independent of* ε *such that the solution of* (1.1) *satisfies*

$$||u_\varepsilon||_{L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^N))} = ||u_\varepsilon^0||_{L^2(\mathbb{R}^N)}\,,$$
$$\varepsilon||\nabla u_\varepsilon||_{L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^N))} \le C\Big(||u_\varepsilon^0||_{L^2(\mathbb{R}^N)} + \varepsilon||\nabla u_\varepsilon^0||_{L^2(\mathbb{R}^N)}\Big)\,.$$

Theorem 3.2. *Assume that assumptions* **H1** *and* **H2** *hold true and that the initial data* u_ε^0 *is of the form* (1.3). *Then the solution of* (1.1) *can be written as*

$$u_\varepsilon(t,x) = e^{i\frac{\lambda_n t}{\varepsilon}} e^{2i\pi\frac{\theta^n\cdot x}{\varepsilon}} v_\varepsilon\Big(t, \frac{x-x^n}{\sqrt{\varepsilon}}\Big)\,, \tag{3.1}$$

where $v_\varepsilon(t,z)$ *two-scale converges strongly to* $\psi_n(y)v(t,z)$, *i.e.*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \left| v_\varepsilon(t,z) - \psi_n\left(\frac{z}{\sqrt{\varepsilon}}\right) v(t,z)\right|^2 dz = 0, \tag{3.2}$$

uniformly on compact time intervals in $\mathbb{R}^+$, *and* v *is the unique solution of the homogenized Schrödinger equation*

$$\begin{cases} i\dfrac{\partial v}{\partial t} - \operatorname{div}\left(A^*\nabla v\right) + \operatorname{div}(vB^*z) + c^*v + vD^*z\cdot z = 0 & \text{in } \mathbb{R}^N\times\mathbb{R}^+ \\ v(0,z) = v^0(z) & \text{in } \mathbb{R}^N \end{cases} \tag{3.3}$$

where

$$A^* = \frac{1}{8\pi^2}\nabla_\theta\nabla_\theta\lambda_n(x^n,\theta^n)\,,\;\; B^* = \frac{1}{2i\pi}\nabla_\theta\nabla_x\lambda_n(x^n,\theta^n)\,,\;\; D^* = \frac{1}{2}\nabla_x\nabla_x\lambda_n(x^n,\theta^n)\,,$$

and c^* *is given by*

$$c^* = \int_{\mathbb{T}^N} \Big[A(\nabla_y + 2i\pi\theta^n)\psi_n \cdot \frac{\partial \bar{\psi}_n}{\partial x_k} e_k - A(\nabla_y - 2i\pi\theta^n)\frac{\partial \bar{\psi}_n}{\partial x_k} \cdot \psi_n \, e_k - A_{1,k}(\nabla_y - 2i\pi\theta^n)\bar{\psi}_n \cdot \psi_n e_k \Big] \, dy \,.$$

Remark 3.3. Notice that even if the tensor A^* might be non-coercive, the homogenized problem (3.3) is well posed. Indeed the operator $\mathbb{A}^* : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ defined by

$$\mathbb{A}^* \varphi = -\mathrm{div}\,(A^* \nabla \varphi) + \mathrm{div}(\varphi B^* z) + c^* \varphi + \varphi D^* z \cdot z \tag{3.4}$$

is self-adjoint (see Proposition 3.4) and therefore by using semi-group theory (see *e.g.* [6] or Chapter X in [20]), one can show that there exists a unique solution in $C(\mathbb{R}^+; L^2(\mathbb{R}^N))$, although it may not belong to $L^2(\mathbb{R}^+; H^1(\mathbb{R}^N))$.

The next result establishes the conservation of the L^2-norm for the solution v of the homogenized equation (3.3) and the self-adjointness of the operator $\mathbb{A}^*$.

Proposition 3.4. *Let* $v \in C(\mathbb{R}^+; L^2(\mathbb{R}^N))$ *be solution to* (3.3). *Then*

$$||v(t,\cdot)||_{L^2(\mathbb{R}^N)} = ||v^0||_{L^2(\mathbb{R}^N)} \quad \forall\, t \in \mathbb{R}^+ \,. \tag{3.5}$$

Moreover the operator $\mathbb{A}^*$ *defined in* (3.4) *is self-adjoint.*

Proof. We multiply the equation (3.3) by $\bar{v}$ and take the imaginary part to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |v|^2 \, dz = \mathrm{Im}\left(\int_{\mathbb{R}^N} v B^* z \cdot \nabla \bar{v} - c^* |v|^2 \, dz \right) . \tag{3.6}$$

After integrating by parts one finds that the right hand side of (3.6) equals

$$-\Big(\frac{1}{2i} \mathrm{tr}\, B^* + \mathrm{Im} c^* \Big) \int_{\mathbb{R}^N} |v|^2 \, dz$$

and therefore (3.5) is proved as soon as we show that

$$\frac{1}{2i}\mathrm{tr}\, B^* + \mathrm{Im} c^* = 0 \,. \tag{3.7}$$

In order to do this we first rewrite the coefficients c^* and B^* in a suitable form. Denoting by $\langle \cdot , \cdot \rangle$ the Hermitian inner product in $L^2(\mathbb{T}^N)$ and using equation (2.2) we write

$$c^* = \frac{1}{2i\pi} \langle \mathbb{A}_n \frac{\partial \psi_n}{\partial \theta_k}, \frac{\partial \psi_n}{\partial x_k} \rangle - \int_{\mathbb{T}^N} A_{1,k}(\nabla_y - 2i\pi\theta^n)\bar{\psi}_n \cdot \psi_n e_k \, dy \,, \tag{3.8}$$

while by equations (2.2)-(2.4) it follows that

$$\begin{aligned} \frac{1}{2i\pi}\frac{\partial^2 \lambda_n}{\partial x_h \partial \theta_k} = & - \frac{1}{2i\pi} \overline{\langle \mathbb{A}_n \frac{\partial \psi_n}{\partial \theta_k}, \frac{\partial \psi_n}{\partial x_h} \rangle} - \frac{1}{2i\pi} \overline{\langle \mathbb{A}_n \frac{\partial \psi_n}{\partial x_h}, \frac{\partial \psi_n}{\partial \theta_k} \rangle} \\ & + 2i \mathrm{Im} \int_{\mathbb{T}^N} A_{1,h}(\nabla_y - 2i\pi\theta^n)\bar{\psi}_n \cdot \psi_n e_k \, dy \,. \end{aligned} \tag{3.9}$$

By formulae (3.8)-(3.9) it is readily seen that equality (3.7) holds true.

In order to prove the self-adjointness of the operator $\mathbb{A}^*$, one first checks that $\mathbb{A}^*$ is symmetric, which easily follows by (3.7) and the fact that $\overline{B}^* = -B^*$, and then observes that up to addition of a multiple of the identity the operator $\mathbb{A}^*$ is monotone (see *e.g.* [7], Chapter VII). $\square$

In the next proposition we will denote by $\nabla\nabla\lambda_n$ the Hessian matrix of the function $\lambda_n(x,\theta)$ evaluated at the point (x^n,θ^n), namely

$$\nabla\nabla\lambda_n = \begin{pmatrix} \nabla_x\nabla_x\lambda_n & \nabla_\theta\nabla_x\lambda_n \\ \nabla_\theta\nabla_x\lambda_n & \nabla_\theta\nabla_\theta\lambda_n \end{pmatrix}(x^n,\theta^n)\,.$$

Proposition 3.5. *Assume that the matrix $\nabla\nabla\lambda_n$ is positive definite. Then there exists an orthonormal basis $\{\varphi_n\}_{n\geq 1}$ of eigenfunctions of $\mathbb{A}^*$; moreover for each n there exists a real constant $\gamma_n > 0$ such that*

$$e^{\gamma_n|z|}\varphi_n\,,\; e^{\gamma_n|z|}\nabla\varphi_n \in L^2(\mathbb{R}^N)\,. \tag{3.10}$$

Proof. Up to shifting the spectrum of the operator $\mathbb{A}^*$, we may assume that $\mathrm{Re}(c^*) = 0$. In order to prove the existence of an orthonormal basis of eigenfunctions we introduce the inverse operator of $\mathbb{A}^*$, denoted by G^*

$$G^* : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$$
$$f \to \varphi \text{ unique solution in } H^1(\mathbb{R}^N) \text{ of}$$
$$\mathbb{A}^*\varphi = f \quad \text{ in } \mathbb{R}^N \tag{3.11}$$

and we show that G^* is compact. Indeed multiplication of (3.11) by $\bar\varphi$ yields

$$\int_{\mathbb{R}^N} [A^*\nabla\varphi\cdot\nabla\bar\varphi - iB^*\mathrm{Im}(\varphi z\cdot\nabla\bar\varphi) + D^*z\cdot z|\varphi|^2]\,dz = \int_{\mathbb{R}^N} f\bar\varphi\,dz\,. \tag{3.12}$$

Upon defining the $2N$-dimensional vector-valued function Φ

$$\Phi := \begin{pmatrix} 2i\pi z\varphi \\ \nabla\varphi \end{pmatrix}$$

we rewrite (3.12) in agreement with this block notation

$$\int_{\mathbb{R}^N} \frac{1}{8\pi^2}\nabla\nabla\lambda_n\Phi\cdot\overline{\Phi}\,dz = \int_{\mathbb{R}^N} f\bar\varphi\,dz\,.$$

By the positivity assumption on the matrix $\nabla\nabla\lambda_n$ it follows that there exists a positive constant c_0 such that

$$c_0\Big(||\nabla\varphi||^2_{L^2(\mathbb{R}^N)} + ||z\varphi||^2_{L^2(\mathbb{R}^N)}\Big) \leq ||f||_{L^2(\mathbb{R}^N)}||\varphi||_{L^2(\mathbb{R}^N)}\,,$$

which implies by a standard argument

$$||\varphi||^2_{L^2(\mathbb{R}^N)} + ||\nabla\varphi||^2_{L^2(\mathbb{R}^N)} + ||z\varphi||^2_{L^2(\mathbb{R}^N)} \leq C||f||^2_{L^2(\mathbb{R}^N)},$$

from which we deduce the compactness of G^* in $L^2(\mathbb{R}^N)$-strong. Thus there exists an infinite countable number of eigenvalues for $\mathbb{A}^*$.

We are left to prove the exponential decay of the eigenfunctions (this fact is quite standard, see *e.g.* [2]). Let φ_n be an eigenfunction and let σ_n be the associated eigenvalue

$$\mathbb{A}^*\varphi_n = \sigma_n\varphi_n \,. \tag{3.13}$$

Let $R_0 > 0$ and $\rho \in C^\infty(\mathbb{R})$ be a real function such that $0 \le \rho \le 1$, $\rho(s) = 0$ for $s \le R_0$ and $\rho(s) = 1$ for $s \ge R_0 + 1$ and for every positive integer k define $\rho_k \in C^\infty(\mathbb{R}^N)$ in the following way

$$\rho_k(z) := \rho(|z| - k).$$

We now multiply (3.13) by $\bar{\varphi}_n\rho_k^2$ to get

$$\int_{\mathbb{R}^N} \rho_k^2 \left(A^*\nabla\varphi_n \cdot \nabla\bar{\varphi}_n - iB^*\mathrm{Im}(\varphi_n z \cdot \nabla\bar{\varphi}_n) + D^*z \cdot z|\varphi_n|^2 - \sigma_n|\varphi_n|^2\right) dz =$$

$$\int_{\mathbb{R}^N} \left(\rho_k|\varphi_n|^2 B^*z \cdot \nabla\rho_k - 2\rho_k\, \bar{\varphi}_n A^*\nabla\varphi_n \cdot \nabla\rho_k\right) dz \,. \tag{3.14}$$

Next remark that since the left hand side of (3.14) is real the right hand side must be also real and therefore it is equal to

$$\int_{\mathbb{R}^N} -2\rho_k\, \mathrm{Re}(\bar{\varphi}_n A^*\nabla\varphi_n) \cdot \nabla\rho_k \, dz \,. \tag{3.15}$$

Let B_k denote the ball of radius $R_0 + k$ and center $z = 0$ and observe that the support of $\nabla\rho_k$ is contained in $B_{k+1} \setminus B_k$. Then putting up together (3.14) and (3.15) and using again the positive definiteness of the matrix $\nabla\nabla\lambda_n$ we obtain for R_0 sufficiently large ($\sqrt{R_0} > \sigma_n$ does the job)

$$||\varphi_n||^2_{H^1(\mathbb{R}^N\setminus B_{k+1})} \le c_1\Big(||\varphi_n||^2_{H^1(\mathbb{R}^N\setminus B_k)} - ||\varphi_n||^2_{H^1(\mathbb{R}^N\setminus B_{k+1})}\Big)$$

where c_1 is a positive constant independent of k. Thus we deduce that

$$||\varphi_n||^2_{H^1(\mathbb{R}^N\setminus B_{k+1})} \le \left(\frac{c_1}{1+c_1}\right)^k ||\varphi_n||^2_{H^1(\mathbb{R}^N\setminus B_0)} \,. \tag{3.16}$$

Upon defining a positive constant $\gamma_0 > 0$ by

$$\left(\frac{c_1}{1+c_1}\right)^k = e^{-2\gamma_0(k+R_0)}$$

it is finally seen that (3.16) implies the estimate (3.10) for any exponent $0 < \gamma_n < \gamma_0$. □

Proof of Theorem 3.2. We rescale the space variable by introducing

$$z = \frac{x}{\sqrt{\varepsilon}} \,,$$

and define the sequence v_ε by

$$v_\varepsilon(t, z) := e^{-i\frac{\lambda_n t}{\varepsilon}} e^{-2i\pi\frac{\theta^n \cdot x}{\varepsilon}} u_\varepsilon(t, x) \,. \tag{3.17}$$

By the a priori estimates of Lemma 3.1 it follows that $v_\varepsilon(t,z)$ satisfies

$$||v_\varepsilon||_{L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^N))} + \sqrt{\varepsilon}||\nabla v_\varepsilon||_{L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^N))} \le C,$$

and applying the compactness of two-scale convergence (see Proposition 2.3), up to a subsequence, there exists a limit $v^*(t,z,y) \in L^2\left(\mathbb{R}^+ \times \mathbb{R}^N; H^1(\mathbb{T}^N)\right)$ such that v_ε and $\sqrt{\varepsilon}\nabla v_\varepsilon$ two-scale converge to v^* and $\nabla_y v^*$, respectively. Similarly, by definition of the initial data, $v_\varepsilon(0,z)$ two-scale converges to $\psi_n(y)v^0(z)$.

Although v_ε is the unknown which will pass to the limit in the sequel, it is simpler to write an equation for another function, namely

$$w_\varepsilon(t,z) := e^{2i\pi\frac{\theta^n\cdot z}{\sqrt{\varepsilon}}} v_\varepsilon(t,z) = e^{-i\frac{\lambda_n t}{\varepsilon}} u_\varepsilon(t,x)\,. \tag{3.18}$$

By (3.18) it follows that

$$\nabla w_\varepsilon = e^{2i\pi\frac{\theta^n\cdot z}{\sqrt{\varepsilon}}}\Big(\nabla + 2i\pi\frac{\theta^n}{\sqrt{\varepsilon}}\Big)v_\varepsilon\,, \tag{3.19}$$

and it can be checked that the new unknown w_ε solves the following equation

(3.20)

$$\begin{cases} i\dfrac{\partial w_\varepsilon}{\partial t} - \operatorname{div}[A\left(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon}\right)\nabla w_\varepsilon] + \dfrac{1}{\varepsilon}[c(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon}) - \lambda_n]w_\varepsilon = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ w_\varepsilon(0,z) = u^0_\varepsilon(\sqrt{\varepsilon}z) & \text{in } \mathbb{R}^N \end{cases}$$

where the differential operators div and ∇ act with respect to the new variable z.

First step. We multiply the equation (3.20) by the complex conjugate of

$$\varepsilon\phi\Big(t, z, \frac{z}{\sqrt{\varepsilon}}\Big)e^{2i\pi\frac{\theta^n\cdot z}{\sqrt{\varepsilon}}}$$

where $\phi(s,z,y)$ is a smooth test function defined on $\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{T}^N$, with compact support in $\mathbb{R}^+\times\mathbb{R}^N$. Since this test function has compact support (fixed with respect to ε), the effect of the non-periodic variable in the coefficients is negligible for sufficiently small ε. Therefore we can replace the value of each coefficient at $(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon})$ by its Taylor expansion of order two about the point $(0, z/\sqrt{\varepsilon})$. Integrating by parts and using (3.18) and (3.19) yields

$$\begin{aligned} &- i\varepsilon\int_0^{+\infty}\!\!\int_{\mathbb{R}^N} v_\varepsilon\frac{\partial\bar{\phi}^\varepsilon}{\partial t}\,dt\,dz - i\varepsilon\int_{\mathbb{R}^N} v_\varepsilon(0,z)\bar{\phi}\Big(0,z,\frac{z}{\sqrt{\varepsilon}}\Big)\,dz \\ &+ \int_0^{+\infty}\!\!\int_{\mathbb{R}^N}[A^\varepsilon + A^\varepsilon_{1,h}\sqrt{\varepsilon}z_h + \tfrac{1}{2}A^\varepsilon_{2,lh}\,\varepsilon z_l z_h + o(\varepsilon)](\sqrt{\varepsilon}\nabla + 2i\pi\theta^n)v_\varepsilon\cdot(\sqrt{\varepsilon}\nabla - 2i\pi\theta^n)\bar{\phi}^\varepsilon\,dz\,dt \\ &+ \int_0^{+\infty}\!\!\int_{\mathbb{R}^N}[c^\varepsilon + c^\varepsilon_{1,h}\sqrt{\varepsilon}z_h + \tfrac{1}{2}c^\varepsilon_{2,lh}\,\varepsilon z_l z_h + o(\varepsilon) - \lambda_n]\,v_\varepsilon\bar{\phi}^\varepsilon\,dz\,dt = 0. \end{aligned}$$

Passing to the two-scale limit we get the variational formulation of

$$-(\operatorname{div}_y + 2i\pi\theta^n)\Big(A(y)(\nabla_y + 2i\pi\theta^n)v^*\Big) + c(y)v^* = \lambda_n v^* \quad \text{in } \mathbb{T}^N.$$

The simplicity of λ_n implies that there exists a scalar function $v(t,z) \in L^2\left(\mathbb{R}^+ \times \mathbb{R}^N\right)$ such that

$$v^*(t,z,y) = v(t,z)\psi_n(y). \tag{3.21}$$

Second step. We multiply (3.20) by the complex conjugate of

$$\Psi_\varepsilon(t,z) = e^{2i\pi\theta^n\cdot\frac{z}{\sqrt{\varepsilon}}}\left[\psi_n^\varepsilon\phi(t,z) + \sqrt{\varepsilon}\sum_{k=1}^N\left(\frac{1}{2i\pi}\frac{\partial\psi_n^\varepsilon}{\partial\theta_k}\frac{\partial\phi}{\partial z_k}(t,z) + z_k\frac{\partial\psi_n^\varepsilon}{\partial x_k}\phi(t,z)\right)\right],$$

where $\phi(t,z)$ is a smooth test function with compact support in $\mathbb{R}^+ \times \mathbb{R}^N$. We first look at those terms of the equation involving time derivatives:

$$\begin{aligned}\int_0^{+\infty}\!\!\int_{\mathbb{R}^N} i\frac{\partial w_\varepsilon}{\partial t}\bar{\Psi}_\varepsilon\,dt\,dz = \\ \int_0^{+\infty}\!\!\int_{\mathbb{R}^N} -iv_\varepsilon\left[\bar{\psi}_n^\varepsilon\frac{\partial\bar{\phi}}{\partial t} + \sqrt{\varepsilon}\sum_{k=1}^N\left(-\frac{1}{2i\pi}\frac{\partial\bar{\psi}_n^\varepsilon}{\partial\theta_k}\frac{\partial^2\bar{\phi}}{\partial t\partial z_k} + z_k\frac{\partial\bar{\psi}_n^\varepsilon}{\partial x_k}\frac{\partial\bar{\phi}}{\partial t}\right)\right]dt\,dz \\ -i\int_{\mathbb{R}^N} v_\varepsilon(0,z)\left[\bar{\psi}_n^\varepsilon\bar{\phi}(0,z) + \sqrt{\varepsilon}\sum_{k=1}^N\left(-\frac{1}{2i\pi}\frac{\partial\bar{\psi}_n^\varepsilon}{\partial\theta_k}\frac{\partial\bar{\phi}}{\partial z_k}(0,z) + z_k\frac{\partial\bar{\psi}_n^\varepsilon}{\partial x_k}\bar{\phi}(0,z)\right)\right]dz\,.\end{aligned} \tag{3.22}$$

Passing to the two-scale limit in (3.22) and recalling the normalization $\int_{\mathbb{T}^N}|\psi_n|^2\,dy = 1$ we find

$$-i\int_0^{+\infty}\!\!\int_{\mathbb{R}^N} v\frac{\partial\bar{\phi}}{\partial t}\,dz\,dt - i\int_{\mathbb{R}^N} v^0\bar{\phi}(0,z)\,dz\,. \tag{3.23}$$

We further decompose Ψ_ε as follows

$$\Psi_\varepsilon = \Psi_\varepsilon^1 + \Psi_\varepsilon^2\cdot z \quad \text{with} \quad \Psi_\varepsilon^2 = \sqrt{\varepsilon}e^{2i\pi\theta^n\cdot\frac{z}{\sqrt{\varepsilon}}}\sum_{k=1}^N\frac{\partial\psi_n^\varepsilon}{\partial x_k}\phi(t,z)e_k.$$

Getting rid of all terms multiplied by $o(\varepsilon)$ and taking into account (3.18) and (3.19) we next pass to the limit in the remaining terms of (3.20) multiplied by $\bar{\Psi}_\varepsilon$. The computation is similar to [4] but it involves new terms since ψ_n and its derivatives also depend on x.

We first look at those terms which are of zero order with respect to z, namely

$$\begin{aligned}
(3.24)\quad & \int_0^{+\infty}\!\!\int_{\mathbb{R}^N} \Big[A^\varepsilon \nabla w_\varepsilon \cdot (\nabla \bar\Psi^1_\varepsilon + \bar\Psi^2_\varepsilon) + \frac{1}{\varepsilon}(c^\varepsilon - \lambda_n) w_\varepsilon \bar\Psi^1_\varepsilon\Big]\, dz\, dt \\
&= \int_0^{+\infty}\!\!\int_{\mathbb{R}^N} \Big[\frac{1}{\varepsilon} A^\varepsilon \Big(\sqrt{\varepsilon}\nabla + 2i\pi\theta^n\Big) v_\varepsilon \cdot (\nabla_y - 2i\pi\theta^n)\bar\psi^\varepsilon_n \bar\phi + \frac{1}{\varepsilon}(c^\varepsilon - \lambda_n)\bar\psi^\varepsilon_n v_\varepsilon \bar\phi\Big]\, dz\, dt \\
&- \frac{1}{2i\pi}\int_0^{+\infty}\!\!\int_{\mathbb{R}^N} \Big[\frac{1}{\sqrt{\varepsilon}} A^\varepsilon \Big(\sqrt{\varepsilon}\nabla + 2i\pi\theta^n\Big) v_\varepsilon \cdot (\nabla_y - 2i\pi\theta^n)\frac{\partial \bar\psi^\varepsilon_n}{\partial\theta_k}\frac{\partial\bar\phi}{\partial z_k} \\
&\qquad\qquad + \frac{1}{\sqrt{\varepsilon}}(c^\varepsilon - \lambda_n) v_\varepsilon \frac{\partial \bar\psi^\varepsilon_n}{\partial\theta_k}\frac{\partial\bar\phi}{\partial z_k}\Big]\, dz\, dt \\
&+ \int_0^{+\infty}\!\!\int_{\mathbb{R}^N} \frac{1}{\sqrt{\varepsilon}} A^\varepsilon \Big(\sqrt{\varepsilon}\nabla + 2i\pi\theta^n\Big) v_\varepsilon \cdot \bar\psi^\varepsilon_n \nabla\bar\phi\, dz\, dt \\
&+ \int_0^{+\infty}\!\!\int_{\mathbb{R}^N} -\frac{1}{2\pi i} A^\varepsilon \Big(\sqrt{\varepsilon}\nabla + 2i\pi\theta^n\Big) v_\varepsilon \cdot \frac{\partial \bar\psi^\varepsilon_n}{\partial\theta_k}\nabla\frac{\partial\bar\phi}{\partial z_k}\, dz\, dt \\
&+ \int_0^{+\infty}\!\!\int_{\mathbb{R}^N} A^\varepsilon \Big(\sqrt{\varepsilon}\nabla + 2i\pi\theta^n\Big) v_\varepsilon \cdot \frac{\partial \bar\psi^\varepsilon_n}{\partial x_k}\bar\phi\, e_k\, dz\, dt\,.
\end{aligned}$$

Using equation (2.7) with $\varphi = v_\varepsilon \bar\phi$ and equation (2.8) with $\varphi = v_\varepsilon \dfrac{\partial\bar\phi}{\partial z_k}$ we rewrite the first two integrals in the right hand side of (3.24) as follows

$$\begin{aligned}
& \int_0^{+\infty}\!\!\int_{\mathbb{R}^N} -\frac{1}{\sqrt{\varepsilon}} A^\varepsilon (\nabla_y - 2i\pi\theta^n)\bar\psi^\varepsilon_n \cdot v_\varepsilon \nabla\bar\phi\, dz\, dt \\
&+ \int_0^{+\infty}\!\!\int_{\mathbb{R}^N} \Big[\frac{1}{2i\pi} A^\varepsilon (\nabla_y - 2i\pi\theta^n)\frac{\partial \bar\psi^\varepsilon_n}{\partial\theta_k} \cdot v_\varepsilon \nabla\frac{\partial\bar\phi}{\partial z_k} + \frac{1}{\sqrt{\varepsilon}} A^\varepsilon e_k \cdot v_\varepsilon \frac{\partial\bar\phi}{\partial z_k}(\nabla_y - 2i\pi\theta^n)\bar\psi^\varepsilon_n \\
&\qquad - \frac{1}{\sqrt{\varepsilon}} A^\varepsilon \bar\psi^\varepsilon_n e_k \cdot \Big(\sqrt{\varepsilon}\nabla + 2i\pi\theta^n\Big)\Big(v_\varepsilon \frac{\partial\bar\phi}{\partial z_k}\Big)\Big]\, dz\, dt\,.
\end{aligned}$$

Combining the above terms with the other terms in (3.24) and passing to the two-scale limit in (3.24) yields

(3.25)

$$\begin{aligned}
& \int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\int_{\mathbb{T}^N} \Big[\frac{1}{2i\pi} A\psi_n (\nabla_y - 2i\pi\theta^n)\frac{\partial\bar\psi_n}{\partial\theta_k} - \frac{1}{2i\pi} A\frac{\partial\bar\psi_n}{\partial\theta_k}(\nabla_y + 2i\pi\theta^n)\psi_n - A|\psi_n|^2 e_k\Big] \\
&\qquad \cdot v\nabla\frac{\partial\bar\phi}{\partial z_k}\, dy\, dz\, dt \\
&+ \int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\int_{\mathbb{T}^N} A(\nabla_y + 2i\pi\theta^n)\psi_n \cdot \frac{\partial\bar\psi_n}{\partial x_k} v\bar\phi\, e_k\, dy\, dz\, dt\,.
\end{aligned}$$

By equation (2.6) it can be seen that the first integral of (3.25) equals

$$\int_0^{+\infty}\!\!\int_{\mathbb{R}^N} A^*\nabla v\nabla\bar\phi\,dz\,dt\,. \tag{3.26}$$

We now focus on those terms which are linear in z:

$$\begin{aligned}
&\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\Big[A^\varepsilon\nabla w_\varepsilon\cdot(\nabla\bar\Psi^2_\varepsilon z)+\frac{1}{\varepsilon}(c^\varepsilon-\lambda_n)w_\varepsilon\bar\Psi^2_\varepsilon z+A^\varepsilon_{1,k}\sqrt{\varepsilon}z_k\nabla w_\varepsilon\cdot(\nabla\bar\Psi^1_\varepsilon+\bar\Psi^2_\varepsilon)\\
&\qquad\qquad+\frac{1}{\sqrt{\varepsilon}}c^\varepsilon_{1,k}z_kw_\varepsilon\bar\Psi^1_e\Big]\,dz\,dt\\
&=\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\Big[\frac{1}{\sqrt{\varepsilon}}A^\varepsilon\Big(\sqrt{\varepsilon}\nabla+2i\pi\theta^n\Big)v_\varepsilon\cdot(\nabla_y-2i\pi\theta^n)\frac{\partial\bar\psi^\varepsilon_n}{\partial x_k}\,\bar\phi z_k+\frac{1}{\sqrt{\varepsilon}}(c^\varepsilon-\lambda_n)v_\varepsilon\frac{\partial\bar\psi^\varepsilon_n}{\partial x_k}\,\bar\phi z_k\Big]dz\,dt\\
&+\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\Big[\frac{1}{\sqrt{\varepsilon}}A^\varepsilon_{1,k}\Big(\sqrt{\varepsilon}\nabla+2i\pi\theta^n\Big)v_\varepsilon\cdot(\nabla_y-2i\pi\theta^n)\bar\psi^\varepsilon_n\,\bar\phi z_k+\frac{1}{\sqrt{\varepsilon}}c^\varepsilon_{1,k}v_\varepsilon\bar\psi^\varepsilon_n\,\bar\phi\,z_k\Big]\,dz\,dt\\
&+\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\Big[A^\varepsilon(\sqrt{\varepsilon}\nabla+2i\pi\theta^n)v_\varepsilon\cdot\frac{\partial\bar\psi^\varepsilon_n}{\partial x_k}\nabla\bar\phi\,z_k+A^\varepsilon_{1,k}(\sqrt{\varepsilon}\nabla+2i\pi\theta^n)v_\varepsilon\cdot\bar\psi^\varepsilon_n\nabla\bar\phi\,z_k\Big]\,dz\,dt\\
&-\frac{1}{2i\pi}\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\Big[A^\varepsilon_{1,h}(\sqrt{\varepsilon}\nabla+2i\pi\theta^n)v_\varepsilon\cdot(\nabla_y-2i\pi\theta^n)\frac{\partial\bar\psi_n}{\partial\theta_k}\frac{\partial\bar\phi}{\partial z_k}\,z_h+c^\varepsilon_{1,h}v_\varepsilon\frac{\partial\bar\psi_n}{\partial\theta_k}\frac{\partial\bar\phi}{\partial z_k}\,z_h\Big]\,dz\,dt\\
&+\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\Big[\sqrt{\varepsilon}A^\varepsilon_{1,h}(\sqrt{\varepsilon}\nabla+2i\pi\theta^n)v_\varepsilon\cdot\Big(-\frac{1}{2i\pi}\frac{\partial\bar\psi^\varepsilon_n}{\partial\theta_k}\nabla\frac{\partial\bar\phi}{\partial z_k}+\frac{\partial\bar\psi^\varepsilon_n}{\partial x_k}\bar\phi\,e_k\Big)z_h\Big]\,dz\,dt\,.
\end{aligned} \tag{3.27}$$

By equation (2.9) with $\varphi=v_\varepsilon\bar\phi z_k$ it can be seen that the sum of the first two integrals in the right hand side of (3.27) gives

$$-\int_0^{+\infty}\int_{\mathbb{R}^N}A^\varepsilon(\nabla_y-2i\pi\theta^n)\frac{\partial\bar\psi^\varepsilon_n}{\partial x_k}\cdot v_\varepsilon\nabla(\bar\phi z_k)+A^\varepsilon_{1,k}(\nabla_y-2i\pi\theta^n)\bar\psi^\varepsilon_n\cdot v_\varepsilon\nabla(\bar\phi z_k)\Big)\,dz\,dt\,. \tag{3.28}$$

Therefore passing to the two-scale limit in (3.27) we find

$$\begin{aligned}
&-\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\int_{\mathbb{T}^N}\Big[A(\nabla_y-2i\pi\theta^n)\frac{\partial\bar\psi_n}{\partial x_k}\cdot v\psi_n\bar\phi\,e_k+A_{1,k}(\nabla_y-2i\pi\theta^n)\bar\psi_n\cdot v\psi_n\bar\phi\,e_k\Big]\,dy\,dz\,dt\\
&-\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\int_{\mathbb{T}^N}\Big[A(\nabla_y-2i\pi\theta^n)\frac{\partial\bar\psi_n}{\partial x_k}\cdot v\psi_nz_k\nabla\bar\phi+A_{1,k}(\nabla_y-2i\pi\theta^n)\bar\psi_n\cdot v\psi_nz_k\nabla\bar\phi\Big]\,dy\,dz\,dt\\
&+\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\int_{\mathbb{T}^N}\Big[A(\nabla_y+2i\pi\theta^n)\psi_n\cdot v\frac{\partial\bar\psi_n}{\partial x_k}z_k\nabla\bar\phi+A_{1,k}(\nabla_y+2i\pi\theta^n)\psi_n\cdot v\bar\psi_nz_k\nabla\bar\phi\Big]\,dy\,dz\,dt\\
&-\frac{1}{2i\pi}\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\int_{\mathbb{T}^N}\Big[A_{1,h}(\nabla_y+2i\pi\theta^n)\psi_n\cdot(\nabla_y-2i\pi\theta^n)\frac{\partial\bar\psi_n}{\partial\theta_k}vz_h\frac{\partial\bar\phi}{\partial z_k}\\
&\qquad\qquad+c_{1,h}\psi_n\frac{\partial\bar\psi_n}{\partial\theta_k}vz_h\frac{\partial\bar\phi}{\partial z_k}\Big]\,dy\,dz\,dt\,.
\end{aligned} \tag{3.29}$$

By equation (2.4) it follows that the last integral in (3.29) is equal to

(3.30)

$$\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\int_{\mathbb{T}^N}\Big[A_{1,h}\psi_n e_k\cdot(\nabla_y-2i\pi\theta^n)\bar{\psi}_n + A\psi_n e_k\cdot(\nabla_y-2i\pi\theta^n)\frac{\partial\bar{\psi}_n}{\partial x_h}\psi_n\Big]vz_h\frac{\partial\bar{\phi}}{\partial z_k}\;dy\,dz\,dt$$
$$-\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\int_{\mathbb{T}^N}\Big[A_{1,h}\bar{\psi}_n e_k\cdot(\nabla_y+2i\pi\theta^n)\psi_n + A\frac{\partial\bar{\psi}_n}{\partial x_h}e_k\cdot(\nabla_y+2i\pi\theta^n)\psi_n\Big]vz_h\frac{\partial\bar{\phi}}{\partial z_k}\;dy\,dz\,dt$$
$$-\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\int_{\mathbb{T}^N}\frac{1}{2i\pi}\frac{\partial^2\lambda_n}{\partial x_h\partial\theta_k}|\psi_n|^2 vz_h\frac{\partial\bar{\phi}}{\partial z_k}\;dy\,dz\,dt\,.$$

Next notice that the first and the second line of (3.30) cancel out with the second and the third line of (3.29) respectively and therefore (3.29) reduces to

(3.31)

$$-\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\int_{\mathbb{T}^N}\Big[A(\nabla_y-2i\pi\theta^n)\frac{\partial\bar{\psi}_n}{\partial x_k}\cdot v\psi_n\bar{\phi}\,e_k + A_{1,k}(\nabla_y-2i\pi\theta^n)\bar{\psi}_n\cdot v\psi_n\bar{\phi}\,e_k\Big]\;dy\,dz\,dt$$
$$-\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\frac{1}{2i\pi}\frac{\partial^2\lambda_n}{\partial x_h\partial\theta_k}v\frac{\partial\bar{\phi}}{\partial z_k}\,z_h\,dz\,dt\,.$$

Finally we consider all quadratic in z terms:

$$\frac{1}{2}\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\Big[A^\varepsilon_{2,lh}\,\varepsilon z_l z_h\nabla w_\varepsilon\cdot(\nabla\bar{\Psi}^1_\varepsilon+\bar{\Psi}^2_\varepsilon)+c^\varepsilon_{2,lh}z_l z_h w_\varepsilon\bar{\Psi}^1_e\Big]\,dz\,dt$$
$$+\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\Big[A^\varepsilon_{1,k}\sqrt{\varepsilon}z_k\nabla w_\varepsilon\cdot(z\nabla\bar{\Psi}^2_\varepsilon)+\frac{1}{\sqrt{\varepsilon}}c^\varepsilon_{1,k}z_k w_\varepsilon z\cdot\bar{\Psi}^2_\varepsilon\Big]\,dz\,dt$$
$$=\frac{1}{2}\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}A^\varepsilon_{2,lh}\,\sqrt{\varepsilon}z_l z_h\Big(\sqrt{\varepsilon}\nabla+2i\pi\theta^n\Big)v_\varepsilon\cdot\Big[\frac{1}{\sqrt{\varepsilon}}(\nabla_y-2i\pi\theta^n)\bar{\psi}^\varepsilon_n\bar{\phi}+\bar{\psi}^\varepsilon_n\nabla\bar{\phi}\Big]\,dz\,dt$$
$$-\frac{1}{2}\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}A^\varepsilon_{2,lh}\,\sqrt{\varepsilon}z_l z_h\Big(\sqrt{\varepsilon}\nabla+2i\pi\theta^n\Big)v_\varepsilon$$
$$\cdot\Big[\frac{1}{2\pi i}\nabla_y\frac{\partial\bar{\psi}^\varepsilon_n}{\partial\theta_k}\frac{\partial\bar{\phi}}{\partial z_k}+\sqrt{\varepsilon}\Big(\frac{1}{2i\pi}\frac{\partial\bar{\psi}^\varepsilon_n}{\partial\theta_k}\nabla\frac{\partial\bar{\phi}}{\partial z_k}+e_k\frac{\partial\bar{\psi}^\varepsilon_n}{\partial x_k}\bar{\phi}\Big)\Big]\,dz\,dt$$
$$+\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}A^\varepsilon_{1,h}\,z_h\Big(\sqrt{\varepsilon}\nabla+2i\pi\theta^n\Big)v_\varepsilon\cdot\Big[z_k(\nabla_y-2i\pi\theta^n)\frac{\partial\bar{\psi}^\varepsilon_n}{\partial x_k}\bar{\phi}+\sqrt{\varepsilon}z_k\frac{\partial\bar{\psi}^\varepsilon_n}{\partial x_k}\nabla\bar{\phi}\Big]\;dz\,dt$$
$$+\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\frac{1}{2}c^\varepsilon_{2,lh}z_l z_h v_\varepsilon\Big(\bar{\psi}^\varepsilon_n\bar{\phi}-\sqrt{\varepsilon}\frac{1}{2i\pi}\frac{\partial\bar{\psi}^\varepsilon_n}{\partial\theta_k}\frac{\partial\bar{\phi}}{\partial z_k}\Big)\;dz\,dt$$
$$+\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}c^\varepsilon_{1,h}z_h v_\varepsilon z_k\frac{\partial\bar{\psi}^\varepsilon_n}{\partial x_k}\bar{\phi}\,dz\,dt$$

which give on passing to the two-scale limit

$$\begin{aligned} \frac{1}{2}\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\int_{\mathbb{T}^N} \Big[A_{2,lh}(\nabla_y + 2i\pi\theta^n)\psi_n \cdot (\nabla_y - 2i\pi\theta^n)\bar{\psi}_n + c_{2,lh}\psi_n\bar{\psi}_n\Big] v\bar{\phi}\, z_l z_h \, dy\, dz\, dt \\ + \int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\int_{\mathbb{T}^N} \Big[A_{1,h}(\nabla_y + 2i\pi\theta^n)\psi_n \cdot (\nabla_y - 2i\pi\theta^n)\frac{\partial\bar{\psi}_n}{\partial x_k} + c_{1,h}\psi_n\frac{\partial\bar{\psi}_n}{\partial x_k}\Big] v\bar{\phi}\, z_h z_k \, dy\, dz\, dt \end{aligned} \tag{3.32}$$

Now using the equation (2.5) we find that (3.32) reduces itself to

$$\int_0^{+\infty}\!\!\int_{\mathbb{R}^N} \frac{1}{2}\frac{\partial^2\lambda_n}{\partial x_l \partial x_h}\, v\bar{\phi}\, z_l z_h \, dz\, dt\,. \tag{3.33}$$

Summing up together (3.23), (3.25), (3.26), (3.31) and (3.33) yields the weak formulation of (3.3). By uniqueness of the solution of the homogenized problem (3.3), we deduce that the entire sequence v_ε two-scale converges weakly to $\psi_n(y)v(t,x)$.

It remains to prove the strong two-scale convergence of v_ε. By Lemma 3.1 we have

$$||v_\varepsilon(t)||_{L^2(\mathbb{R}^N)} = ||u_\varepsilon(t)||_{L^2(\mathbb{R}^N)} = ||u^0_\varepsilon||_{L^2(\mathbb{R}^N)} \to ||\psi_n v^0||_{L^2(\mathbb{R}^N\times\mathbb{T}^N)} = ||v^0||_{L^2(\mathbb{R}^N)}$$

by the normalization condition of ψ_n. From the conservation of energy of the homogenized equation (3.3) we have

$$||v(t)||_{L^2(\mathbb{R}^N)} = ||v^0||_{L^2(\mathbb{R}^N)},$$

and thus we deduce the strong convergence from Proposition 2.3. □

Remark 3.6. As usual in periodic homogenization [1], [5], the choice of the test function Ψ_ε, in the proof of Theorem 3.2, is dictated by the formal two-scale asymptotic expansion that can be obtained for the solution w_ε of (3.20), namely

$$w_\varepsilon(t,z) \approx e^{2i\pi\theta^n\cdot\frac{z}{\sqrt{\varepsilon}}}\Big[\psi_n\Big(\frac{z}{\sqrt{\varepsilon}}\Big)v(t,z) + \sqrt{\varepsilon}\sum_{k=1}^N\Big(\frac{1}{2i\pi}\frac{\partial\psi_n}{\partial\theta_k}\Big(\frac{z}{\sqrt{\varepsilon}}\Big)\frac{\partial v}{\partial z_k}(t,z) + z_k\frac{\partial\psi_n}{\partial x_k}\Big(\frac{z}{\sqrt{\varepsilon}}\Big)v(t,z)\Big)\Big]$$

where v is the homogenized solution of (3.3). Actually the homogenized equation that one gets by the asymptotic expansion method is

$$i\frac{\partial v}{\partial t} - \operatorname{div}(A^*\nabla v) + B^*\nabla v\cdot z + \bar{c}^* v + vD^*z\cdot z = 0\,, \tag{3.34}$$

which apparently differs from (3.3) by the following zero-order term

$$(\operatorname{tr}(\nabla_\theta\nabla_x\lambda_n) - 4\pi\operatorname{Im}(c^*))\, v\,.$$

By virtue of (3.7) the above term vanishes, so that both formulae (3.34) and (3.3) are equivalent.

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