ECOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES APPLIQUÉES

UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 41 50. Fax: 01 69 33 30 11
http://www.cmap.polytechnique.fr/

Pricing Parisian Options

Céline LABART, Jérôme LELONG

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PRICING PARISIAN OPTIONS

C. LABART[1] AND J. LELONG[2]

Abstract. In this work, we propose to price Parisian options using Laplace transforms. Not only, do we compute the Laplace transforms of all the different Parisian options, but we also explain how to invert them numerically. We discuss the accuracy of the numerical inversion and present the evolution of the Greeks through a few graphs.

1. Introduction

With the development of stock exchanges around the world, more and more people have become interested in derivatives and especially in options. Standard options provide its owner with the right to buy or sell a number of stocks for a fixed amount of money at a given time, called the maturity time. There are more complex options, known under the name of exotic or also path-dependent options. These options are valuable only if the stock price has satisfied certain conditions before the maturity time, this is precisely this kind of options we are going to study. More precisely, we will deal with options that give their owners the right to buy (call options) or sell (put options) a number of stocks for a fixed amount of money (the strike) if the stock price has stayed below (or above) a certain level (the barrier) for a certain time (the option window) before the maturity time. This option is called a Parisian down-and-in option (or alternatively a Parisian up-and-in option). This is only one example of all the different Parisian options. Basically, we will only consider European style options, which means that one can only exercise his option at the maturity time. Parisian options are, to some extent, a kind of barrier options. One could influence the value of a barrier option by buying a lot of stocks or on the contrary by selling a lot of them. For instance, let us imagine that we own a lot of up-and-in barrier options which haven not been knocked in yet. If the maturity time is close, then we could be tempted to buy a lot of stocks to have the option knocked in. If we consider a Parisian up-and-in option, this is no longer possible since the asset price has to remain above the level for a much longer period (several days). Therefore, Parisian options can be seen as a guarantee against easy arbitrage.

As one will discover later on, there exist a lot of different Parisian options. There are two different ways of measuring the time spent above or below the barrier. Either, one only counts the time spent in a row and starts counting from 0 each time the stock price crosses the barrier, this type is referred to as the continuous Parisian options, or one adds the time spent below or above the barrier without resuming the counting from 0 each time the stock price crosses the barrier, these options are called cumulative Parisian options. In practice, these two kinds of Parisian options raise different questions about the paths of Brownian motion. Therefore, we will only stick to the continuous style options.

There already exist several studies on the Parisian Options. Basically, two techniques can be used to price Parisian options either Laplace transforms or partial differential equations. The Laplace transform technique was first introduced by Chesney et al. [3]. Schröder [9] and Hartley [6] have also tackled these options using Laplace transforms. The PDE method was developed by Haber et al. [5] and Wilmott [10].
In this article, we present a way of computing the prices of Parisian Options. The real issue in pricing options is to be able to hedge them. This can only be done if we are able to compute the prices at any time t smaller than the maturity time. The computation of the prices at time 0 requires to study a little of the excursion theory of Brownian motion. The most complex proofs will only be given in the Appendix. The pricing technique, we expose here, is based on Laplace transforms. In this work, we compute the Laplace transforms of all the different Parisian options and we also discuss in detail the accuracy of the numerical technique used to invert

[1]CMAP, Ecole Polytechnique, 91128 Palaiseau cedex, FRANCE. (labart@cmap.polytechnique.fr)
[2]CERMICS, Ecole Nationale des Ponts et Chaussées, Champs sur Marne, 77455 Marne La Vallée Cédex 2, FRANCE. (lelong@cermics.enpc.fr)
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the Laplace transform. The numerical inversion is based on Abate et al. [1].

The article is divided as follows. First of all in Section 2 we give some definitions concerning Brownian Motions and hitting time. We also explain how to write the price of such options in terms of hitting times. In Section 3, we explain how to compute the Laplace transform of the price of a Parisian Down Call at time 0. Section 4 is devoted to the computation of the Laplace transform of the prices of Parisian Up Calls still at time 0. Some parity relationships are given in Section 5 to deduce the prices of Parisian Puts. At that stage we are able to price any Parisian Options at time 0. In Section 6, we show how to compute the prices at some time t relying on the prices at time 0.
Then, in Section 6 we will expose an algorithm to invert numerically a Laplace transform and we will also discuss its accuracy and efficiency. This method is extremely accurate and fast compared with the PDE method.
To conclude this article we present a few graphs to try to better understand these options. We also give a few hedging simulations.
We have implemented in C the technique presented here. All the prices were computed using this program. The different graphs concerning the hedging of such options were generated using the C code we wrote.

A part of this work was done during an internship at TUDelft University in the Netherlands in 2003.

In tis article, we will use the following notations:

S	the process representing the asset price,
K	the strike,
T	the maturity of the option,
L	the barrier level for process S,
D	the option window,
x	the initial value of process S,
r	the interest rate,
δ	the dividend rate,
σ	the volatility,
k	$1/\sigma \ln(K/x)$,
b	$1/\sigma \ln(L/x)$ (*i.e.* the barrier level for the Brownian motion),
λ	the Laplace variable,
θ	$\sqrt{2\lambda}$,
d	$\frac{b-k}{\sqrt{D}}$,
m	$\frac{1}{\sigma}\left(r-\delta-\frac{\sigma^2}{2}\right)$.

2. Definitions

First, we will give a few definitions and notations used in the rest of the article. Then, we will present the features of such options. We only focus on the down-and-in and down-and-out calls in this section since the features of the other Parisian options can easily be deduceed from these two.

2.1. **Some notations.** Let us describe an excursion at (or away from) level L for an Itô process S. We define

$$g^S_{L,t} = \sup\{u \le t \mid S_u = L\},\ d^S_{L,t} = \inf\{u \ge t \mid S_u = L\}.$$

The trajectory of S between $g^S_{L,\,t}$ and $d^S_{L,\,t}$ is the excursion at level L, straddling time t.
Let $S = \{S_t,\ t \ge 0\}$ denote the price of the underlying asset. We suppose that under the risk neutral measure $\mathcal{Q}$, the dynamics of S is given by

$$dS_t = S_t((r-\delta)dt + \sigma dW_t),\ S_0 = x$$

where $W = \{W_t, t \ge 0\}$ is a $\mathcal{Q}$ Brownian motion and $x > 0$. It follows that

$$S_t = x \exp\left((r-\delta-\frac{\sigma^2}{2})t + \sigma W_t\right).$$

Let us introduce the following notations

$$m = \frac{1}{\sigma}\left(r-\delta-\frac{\sigma^2}{2}\right),\ b = \frac{1}{\sigma}\ln\left(\frac{L}{x}\right)$$

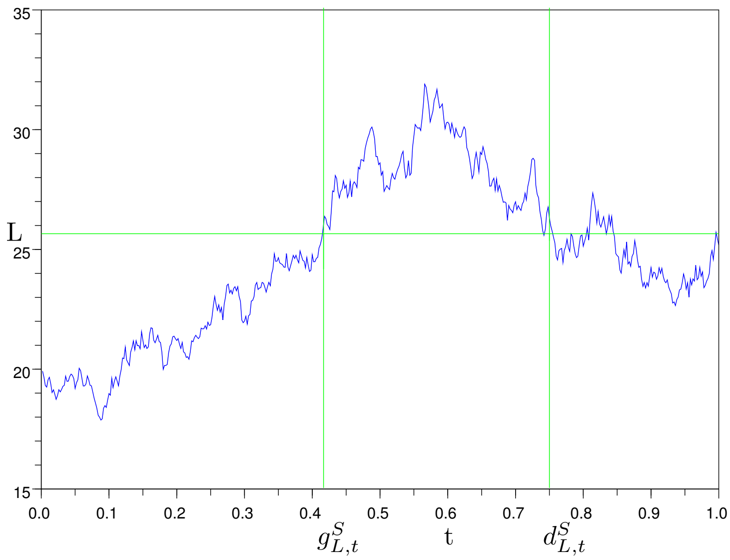

FIGURE 1. Dynamic of an asset

where L is the excursion level. Under $\mathcal{Q}$, the dynamics of the asset is given by $S_t = x \exp(\sigma(mt + W_t))$. From now on, we will consider that every option has a maturity time T. Relying on the Cameron-Martin-Girsanov theorem, we can introduce a new probability $\mathcal{P}$, which makes $Z = \{Z_t = W_t + mt, 0 \le t \le T\}$ a $\mathcal{P}$-Brownian motion and $\frac{d\mathcal{P}}{d\mathcal{Q}}_{|\mathcal{F}_T} = e^{mZ_T - \frac{m^2}{2}T}$. Thus, S rewrites $S_t = x\, e^{\sigma Z_t}$.

2.2. **The Parisian down-and-out call.** A down-and-out Parisian option becomes worthless if S reaches L and remains constantly below level L for a time interval longer than D before maturity time T, which is exactly the same as saying that Brownian motion Z makes an excursion below b older than D.
Let us introduce

$$\begin{aligned} T_b &= \inf\{t > 0 \mid Z_t = b\}, \\ g_t^b &= \sup\{u \le t \mid Z_u = b\}, \\ T_b^- &= \inf\{t > 0 \mid (t - g_t^b)\, \mathbf{1}_{\{Z_t < b\}} > D\}. \end{aligned}$$

One should notice that referring to the previous notations $g_t^b = g^S_{L,t}$.
The price of a down-and-out option at time 0 with payoff $\phi(S_T)$, in an arbitrage free model, is given by

$$e^{-rtT}\mathbb{E}_{\mathcal{Q}}\left(\phi(S_T)\mathbf{1}_{\{T_b^- > T\}}\right) = e^{-(r+\frac{m^2}{2})T}\mathbb{E}_{\mathcal{P}}\left(\mathbf{1}_{\{T_b^- > T\}}\phi(xe^{\sigma Z_T})e^{mZ_T}\right). \tag{1}$$

Let us denote by $PDOC(x, T; K, L; r, \delta)$ the value of a Parisian down-and-out call. From (1), we have

$$PDOC(x, T; K, L; r, \delta) = e^{-(r+\frac{1}{2}m^2)T}\mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- > T\}}(xe^{\sigma Z_T} - K)^+ e^{mZ_T}).$$

In many formulae involving a function Π of maturity T, as in (1), the discount factor $\exp\left[-(r + \frac{1}{2}m^2)T\right]$ appears. In order to give more concise formulae, we introduce the following notation:

$$* \,\Pi(T) = e^{(r+\frac{1}{2}m^2)T}\Pi(T). \tag{2}$$

Hence, we will compute the Laplace transform of $*\Pi$ rather than the one of Π. Any way the following obvious relation between their Laplace transforms hold

$$\widehat{\Pi}(\lambda) = \widehat{*\Pi}(\lambda + (r + \frac{1}{2}m^2)). \tag{3}$$

Since the functions Π we will consider will stand for option prices, they are bounded. This remark will enable us to state the accuracy of the numerical inversion in Section 7.

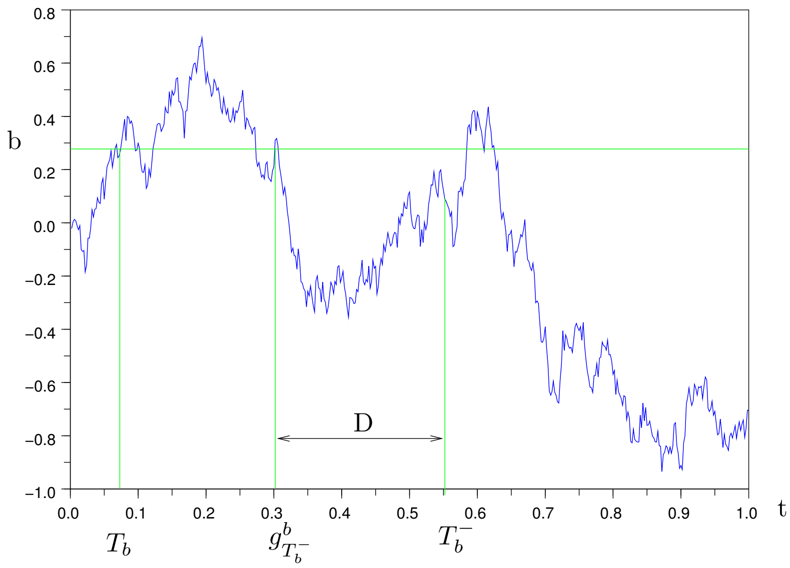

FIGURE 2. Excursion of Brownian Motion

Using notation (2), we obtain

$$*PDOC(x,T;K,L;r,\delta) = \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- > T\}}(xe^{\sigma Z_T} - K)^+ e^{mZ_T}).$$

2.3. **The Parisian down-and-in call.** The owner of a down-and-in option receives the pay-off if S makes an excursion below level L older than D before maturity time T, which is exactly the same as saying that Brownian motion Z makes an excursion below b older than D. The price of a down-and-in option at time 0 with payoff $\phi(S_T)$ is given by

$$e^{-rT}\mathbb{E}_{\mathcal{Q}}\left(\phi(S_T)\mathbf{1}_{\{T_b^- < T\}}\right) = e^{-(r+\frac{m^2}{2})T}\mathbb{E}_{\mathcal{P}}\left(\mathbf{1}_{\{T_b^- < T\}}\phi(xe^{\sigma Z_T})e^{mZ_T}\right). \tag{4}$$

Let us denote by $PDIC(x,T;K,L;r,\delta)$ the value of a Parisian down-and-in call. From (4), we have

$$PDIC(x,T;K,L;r,\delta) = e^{-(r+\frac{1}{2}m^2)T}\mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- < T\}}(xe^{\sigma Z_T} - K)^+ e^{mZ_T}). \tag{5}$$

Using notation (2), we obtain

$$*PDIC(x,T;K,L;r,\delta) = \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- < T\}}(xe^{\sigma Z_T} - K)^+ e^{mZ_T}). \tag{6}$$

The following scheme explains how to deduce the prices of the different kinds of Parisian options one from the others.

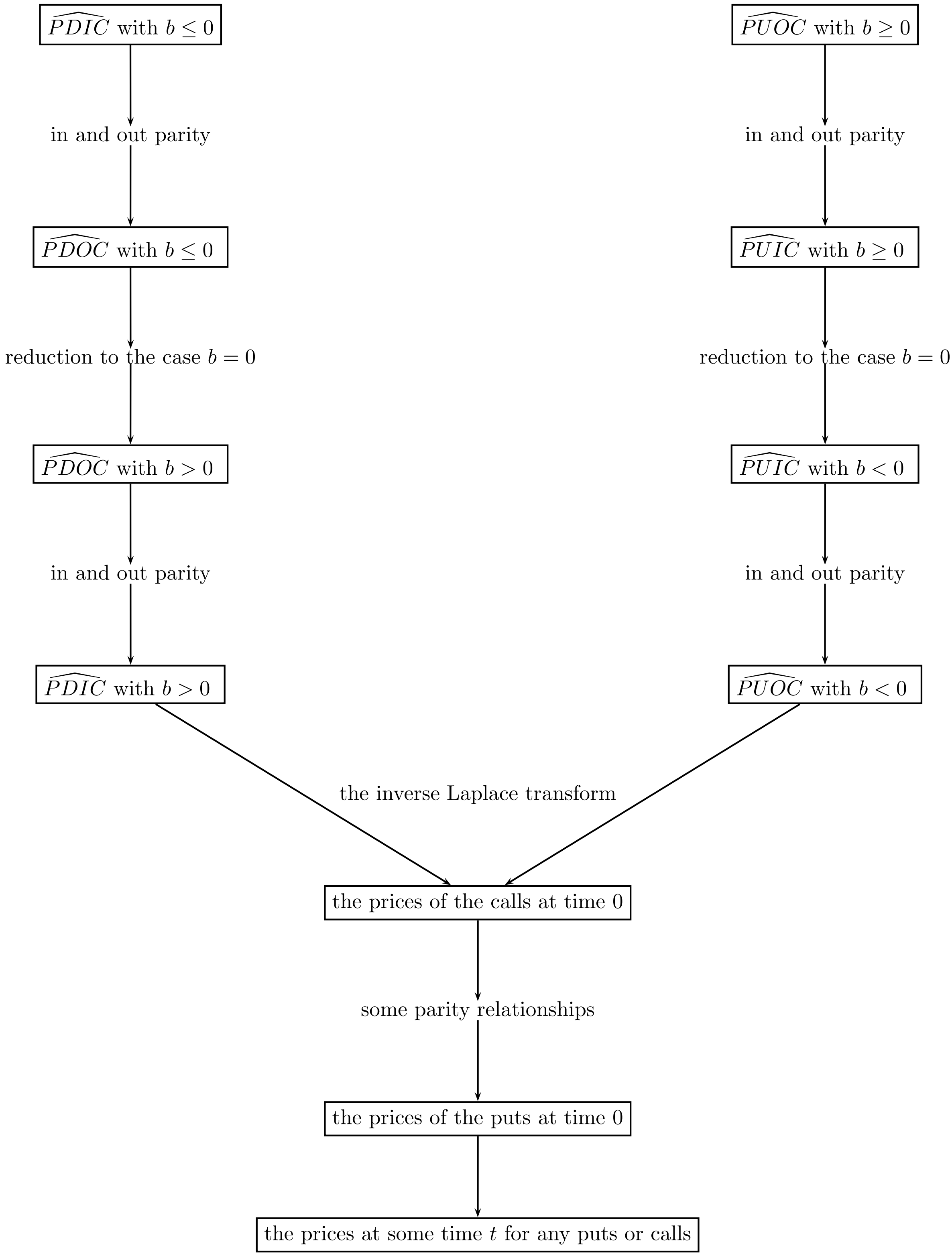

FIGURE 3. Organigram of how to deduce the prices one from the others

3. The Parisian Down Calls

As shown in the previous scheme, all the different prices are deduced from their Laplace transforms. Now, we will explain how to compute these Laplace transforms. In this section, we will only deal with down version of the calls. We will follow exactly the previous scheme to deduce step by step all the needed Laplace transforms.

3.1. **The valuation of a Parisian down-and-in call with $b \leq 0$.**

We want to compute $*PDIC(x,T;K,L;r,\delta)$. Let us denote by $\mathcal{F}_t = \sigma(Z_s, s \leq t)$ the natural filtration of Brownian motion $Z = \{Z_t; t \geq 0\}$. One notices that T_b^- is an $\mathcal{F}_t$-stopping time. We have

$$\begin{aligned} *\,PDIC(x,T;K,L;r,\delta) &= \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- < T\}}(xe^{\sigma Z_T} - K)^+ e^{mZ_T}), \\ &= \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- < T\}}\mathbb{E}_{\mathcal{P}}[(xe^{\sigma Z_T} - K)^+ e^{mZ_T} | \mathcal{F}_{T_b^-}]) \end{aligned}$$

and we can write

$$*PDIC(x,T;K,L;r,\delta) = \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- < T\}}\mathbb{E}_{\mathcal{P}}[xe^{\sigma(Z_T - Z_{T_b^-} + Z_{T_b^-})} - K)^+ e^{m(Z_T - Z_{T_b^-} + Z_{T_b^-})} | \mathcal{F}_{T_b^-}]).$$

$$\begin{aligned} &\mathbb{E}_{\mathcal{P}}\left[(xe^{\sigma(Z_T - Z_{T_b^-} + Z_{T_b^-})} - K)^+ e^{m(Z_T - Z_{T_b^-} + Z_{T_b^-})} | \mathcal{F}_{T_b^-}\right] = \\ &\qquad \mathbb{E}_{\mathcal{P}}\left[(xe^{\sigma(Z_T - Z_{T_b^-} + z)} - K)^+ e^{m(Z_T - Z_{T_b^-} + z)} | \mathcal{F}_{T_b^-}\right]_{|z = Z_{T_b^-}}. \end{aligned} \tag{7}$$

Let W_t denote $Z_{t+T_b^-} - Z_{T_b^-}$. Relying on the strong Markov property, W_t is independent of $\mathcal{F}_{T_b^-}$ and $W_{T-T_b^-} = Z_T - Z_{T_b^-}$. Let Y_t denote $(xe^{\sigma(W_{T-t}+z)} - K)^+ e^{m(W_{T-t}+z)}$, Y_t is independent of $\mathcal{F}_{T_b^-}$. Then, a well-known result on conditional expectations, states that $\mathbb{E}(Y_{T_b^-} | \mathcal{F}_{T_b^-}) = \mathbb{E}(Y_t)_{|t = T_b^-}$. So, we obtain

$$\begin{aligned} &\mathbb{E}_{\mathcal{P}}\left[(xe^{\sigma(Z_T - Z_{T_b^-} + Z_{T_b^-})} - K)^+ e^{m(Z_T - Z_{T_b^-} + Z_{T_b^-})} | \mathcal{F}_{T_b^-}\right] = \\ &\qquad \mathbb{E}_{\mathcal{P}}\left[(xe^{\sigma(W_{T-\tau} + z)} - K)^+ e^{m(W_{T-\tau} + z)}\right]_{|z = Z_{T_b^-},\, \tau = T_b^-}. \end{aligned} \tag{8}$$

$$\mathbb{E}_{\mathcal{P}}\left[(xe^{\sigma(WT-\tau+z)} - K)^+ e^{m(W_{T-\tau}+z)}\right] = \frac{1}{\sqrt{2\pi(T-\tau)}}\left(\int_{-\infty}^{\infty} e^{mu}(xe^{\sigma u} - K)^+ \exp\left(-\frac{(u-z)^2}{2(T-\tau)}\right) du\right).$$

So, we get

$$*PDIC(x,T;K,L;r,\delta) = \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- < T\}}\mathcal{P}_{T-T_b^-}(f_x)(Z_{T_b^-})),$$

with

$$f_x(z) = e^{mz}(e^{\sigma z} - K)^+,$$

and

$$\mathcal{P}_t(f_x)(z) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} f_x(u)\exp\left(-\frac{(u-z)^2}{2t}\right) du.$$

As recalled in *Appendix D*, the random variables $Z_{T_b^-}$ and T_b^- are independent. By denoting the law of $Z_{T_b^-}$ by $\nu(dz)$, we obtain

$$\begin{aligned} *\,PDIC(x,T;K,L;r,\delta) &= \int_{-\infty}^{\infty} \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- < T\}}\mathcal{P}_{T-T_b^-}(f_x)(z))\nu(dz), \\ &= \int_{-\infty}^{\infty} f_x(y) h_b(T,y) dy, \end{aligned} \tag{9}$$

where

$$h_b(t,y) = \int_{-\infty}^{\infty} \mathbb{E}_{\mathcal{P}}\left(\mathbf{1}_{\{T_b^- < t\}}\frac{\exp\left(-\frac{(z-y)^2}{2(t-T_b^-)}\right)}{\sqrt{2\pi(t-T_b^-)}}\right)\nu(dz). \tag{10}$$

Since we consider the case $b < 0$, we can use the following expression for the law of $Z_{T_b^-}$,as it is proved in *Appendix D*

$$\mathbb{P}(Z_{T_b^-} \in dx) = \frac{dx}{D}(b-x)\exp\left(-\frac{(x-b)^2}{2D}\right)\mathbf{1}_{\{x\le b\}}. \tag{11}$$

3.1.1. *The Laplace transform of* $*PDIC(x,T;K,L;r,\delta)$. We can calculate $*PDIC(x,T;K,L;r,\delta)$ by using a Laplace transform. Let $*\widehat{PDIC}(x,\lambda;K,L;r,\delta)$ denote the Laplace transform of $*PDIC(x,T;K,L;r,\delta)$ for any λ with $\mathcal{R}e(\lambda)$ large enough such as all the integrals discussed below are convergent. This condition implies that $m+\sigma-\sqrt{2\lambda}<0$. We have

$$\begin{aligned} *\widehat{PDIC}(x,\lambda;K,L;r,\delta) &= \int_0^\infty e^{-\lambda t}\int_{-\infty}^\infty f_x(y)h_b(t,y)dydt,\\ &= \int_{-\infty}^\infty f_x(y)\int_0^\infty e^{-\lambda t}h_b(t,y)dtdy. \end{aligned} \tag{12}$$

The Laplace transform of $h_b(T,y)$. We would like to compute

$$\widehat{h}_b(\lambda,y) = \int_0^\infty e^{-\lambda t}h_b(t,y)dt. \tag{13}$$

We know that :

$$h_b(t,y) = \int_{-\infty}^b \frac{b-z}{D}\exp\left(-\frac{(z-b)^2}{2D}\right)\mathbb{E}_{\mathcal{P}}\left(\mathbf{1}_{\{T_b^-<t\}}\frac{\exp\left(-\frac{(z-y)^2}{2(t-T_b^-)}\right)}{\sqrt{2\pi(t-T_b^-)}}\right)dz. \tag{14}$$

We can write

$$h_b(t,y) = \int_{-\infty}^b \frac{b-z}{D}\exp\left(-\frac{(z-b)^2}{2D}\right)\gamma(t,z-y)dz, \tag{15}$$

where

$$\gamma(t,x) = \mathbb{E}_{\mathcal{P}}\left(\mathbf{1}_{\{T_b^-<t\}}\frac{\exp\left(-\frac{x^2}{2(t-T_b^-)}\right)}{\sqrt{2\pi(t-T_b^-)}}\right), \tag{16}$$

and we have

$$\widehat{h}_b(\lambda,y) = \int_{-\infty}^b \frac{b-z}{D}\exp\left(-\frac{(z-b)^2}{2D}\right)\int_0^\infty e^{-\lambda t}\gamma(t,z-y)dt\,dz. \tag{17}$$

So, we need to compute the Laplace transform of $\gamma(t,x)$

$$\int_0^\infty e^{-\lambda t}\gamma(t,x)dt = \mathbb{E}_{\mathcal{P}}\left(\int_{T_b^-}^\infty e^{-\lambda t}\frac{\exp\left(-\frac{x^2}{2(t-T_b^-)}\right)}{\sqrt{2\pi(t-T_b^-)}}dt\right). \tag{18}$$

The change of variables $u = t - T_b^-$ gives

$$\int_0^\infty e^{-\lambda t}\gamma(t,x)dt = \mathbb{E}_{\mathcal{P}}(e^{-\lambda T_b^-})\int_0^\infty e^{-\lambda u}\frac{\exp\left(-\frac{x^2}{2u}\right)}{\sqrt{2\pi u}}du. \tag{19}$$

Using results from *Appendix A* and *B*, we get

$$\int_0^\infty e^{-\lambda t}\gamma(t,x)dt = \frac{\exp\left[-(|x|-b)\theta\right]}{\theta\psi(\theta\sqrt{D})}.$$

Thanks to (17), we can rewrite

$$\widehat{h}_b(\lambda,y) = \frac{e^{b\theta}}{D\theta\psi(\theta\sqrt{D})}\int_{-\infty}^b (b-z)\exp\left(-\frac{(z-b)^2}{2D}-|z-y|\theta\right)dz.$$

By changing variables $x = b - z$, we have

$$\widehat{h}_b(\lambda,y) = \frac{e^{b\theta}}{D\theta\psi(\theta\sqrt{D})}\int_0^\infty x\exp\left(-\frac{x^2}{2D}-|b-x-y|\theta\right)dx. \tag{20}$$

Let $K_{\lambda,D}(b-y)$ denote $\displaystyle\int_0^{+\infty} x\exp\left(-\frac{x^2}{2D}-|b-x-y|\theta\right)dx$.
valuation of $K_{\lambda,D}(b-y)$. Relying on the definition of $f_x(y)$, we know that y is always bigger than $\displaystyle\frac{1}{\sigma}\ln\left(\frac{K}{x}\right)$.
► Let us consider the case $K\geq L$. In this case we have $y-b\geq\frac{1}{\sigma}\ln(\frac{K}{L})$, then $y-b\geq 0$. So we get

$$K_{\lambda,D}(b-y)=\int_0^{\infty} x\exp\left(-\frac{x^2}{2D}+(b-x-y)\theta\right)dx$$

because $x\geq 0$ and $y-b\geq 0$.

$$\begin{aligned}K_{\lambda,D}(b-y) &= e^{(b-y)\theta}\int_0^{\infty} x\exp\left(-\frac{x^2}{2D}-x\theta\right)dx,\\ &= De^{(b-y)\theta}\psi(-\theta\sqrt{D}).\end{aligned}$$

From (20) we obtain

$$\widehat{h_b}(\lambda,y)=\frac{\psi(-\theta\sqrt{D})}{\psi(\theta\sqrt{D})}\frac{\exp[(2b-y)\theta]}{\theta}. \tag{21}$$

If we fill in (12) with the expression of $\widehat{h_b}(\lambda,y)$, we get

$$*\widehat{PDIC}(x,\lambda;K,L;r,\delta)=\frac{\psi(-\theta\sqrt{D})e^{2b\theta}}{\theta\psi(\theta\sqrt{D})}\int_{\frac{1}{\sigma}\ln(\frac{K}{x})}^{\infty} e^{-y\theta}e^{my}(xe^{\sigma y}-K)dy. \tag{22}$$

Let k denote $\displaystyle\frac{1}{\sigma}\ln\left(\frac{K}{x}\right)$.
We come up with the following formula for $*\widehat{PDIC}(x,\lambda;K,L;r,\delta)$.

$$\boldsymbol{*\widehat{PDIC}(x,\lambda;K,L;r,\delta)} = \boldsymbol{\frac{\psi(-\theta\sqrt{D})e^{2b\theta}}{\theta\psi(\theta\sqrt{D})}Ke^{(m-\theta)k}\left(\frac{1}{m-\theta}-\frac{1}{m+\sigma-\theta}\right)},$$
$$\text{for } \boldsymbol{K>L} \text{ and } \boldsymbol{x\geq L}.$$

► Let us consider the case $K\leq L$. In this case we have $k<b$. We also have

$$*\widehat{PDIC}(x,\lambda;K,L;r,\delta)=\frac{e^{2b\theta}}{\theta D\psi(\theta\sqrt{D})}\int_k^{+\infty} e^{my}(xe^{\sigma y}-K)K_{\lambda,D}(b-y)dy$$

where

$$K_{\lambda,D}(b-y)=\int_0^{+\infty} z\exp\left(-\frac{z^2}{2D}-|b-z-y|\theta\right)dz.$$

For $y\in[b,+\infty[$ we have $b-y\leq 0$. $K_{\lambda,D}(b-y)$ has already been computed in this case. For $y\in[k,b]$, we have $b-y\geq 0$. We have to compute $K_{\lambda,D}$ in such a case. Let a denote $b-y$, $a>0$.

$$\begin{aligned}K_{\lambda,D}(a) &= \int_0^{\infty} z\exp\left(-\frac{z^2}{2D}-\mid a-z\mid\theta\right)dz,\\ &= \underbrace{\int_0^{a} z\exp\left(-\frac{z^2}{2D}-(a-z)\theta\right)dz}_{\text{A}}+\underbrace{\int_a^{+\infty} z\exp\left(-\frac{z^2}{2D}+(a-z)\theta\right)dz}_{\text{B}}.\end{aligned}$$

▷ The valuation of B

$$\begin{aligned}\int_a^{+\infty} ze^{\left(-\frac{z^2}{2D}+(a-z)\theta\right)}dz &= e^{a\theta}\int_a^{+\infty} ze^{\left(-\frac{z^2}{2D}-z\theta\right)}dz,\\ &= e^{a\theta}\int_a^{+\infty} D\left(\frac{z}{D}+\theta-\theta\right)e^{\left(-\frac{z^2}{2D}-z\theta\right)}dz,\\ &= e^{a\theta}D\left[-e^{\left(-\frac{z^2}{2D}-z\theta\right)}\right]_a^{+\infty}-e^{a\theta}\theta D\int_a^{+\infty} e^{\left(-\frac{z^2}{2D}-z\theta\right)}dz,\\ &= e^{a\theta}De^{-\frac{a^2}{2D}-a\theta}-e^{a\theta}\theta D\int_a^{+\infty} e^{-\frac{1}{2}\left(\frac{z}{\sqrt{D}}+\theta\sqrt{D}\right)^2+\lambda D}dz,\\ &= De^{-\frac{a^2}{2D}}-e^{a\theta}\theta De^{\lambda D}\int_a^{+\infty} e^{-\frac{1}{2}\left(\frac{z}{\sqrt{D}}+\theta\sqrt{D}\right)^2}dz.\end{aligned}$$

By changing variables $u = \dfrac{z}{\sqrt{D}} + \theta\sqrt{D}$, we get

$$\begin{aligned} B &= De^{-\frac{a^2}{2D}} - e^{a\theta}\theta De^{\lambda D}\sqrt{D}\int_{\frac{a}{\sqrt{D}}+\theta\sqrt{D}}^{+\infty} e^{-\frac{1}{2}u^2}du, \\ &= De^{-\frac{a^2}{2D}} - e^{a\theta}\theta De^{\lambda D}\sqrt{2\pi D}(1-\mathcal{N}\left(\frac{a}{\sqrt{D}}+\theta\sqrt{D}\right). \end{aligned}$$

We finally obtain :

$$B = D\left[e^{-\frac{a^2}{2D}} - e^{a\theta}\theta\sqrt{2\pi D}e^{\lambda D}\left(1-\mathcal{N}\left(\frac{a}{\sqrt{D}}+\theta\sqrt{D}\right)\right)\right]. \tag{23}$$

▷ The valuation of A

$$\begin{aligned} \int_0^a z\exp\left(-\frac{z^2}{2D}-(a-z)\theta\right)dz &= e^{-a\theta}\int_0^a ze^{-\frac{z^2}{2D}+z\theta}dz, \\ &= e^{-a\theta}\int_0^a D(\frac{z}{D}+\theta-\theta)e^{-\frac{z^2}{2D}+z\theta}dz, \\ &= e^{-a\theta}D\left[-e^{-\frac{z^2}{2D}+z\theta}\right]_0^a + D\theta e^{-a\theta}\int_0^a e^{-\frac{z^2}{2D}+z\theta}dz, \\ &= -De^{-\frac{a^2}{2D}} + De^{-a\theta} + D\theta e^{\lambda D}e^{-a\theta}\int_0^a e^{-\frac{1}{2}(\frac{z}{\sqrt{D}}-\theta\sqrt{D})^2}dz, \\ &= -De^{-\frac{a^2}{2D}} + De^{-a\theta} + D\theta e^{\lambda D}e^{-a\theta}\sqrt{D}\int_{-\theta\sqrt{D}}^{\frac{a}{\sqrt{D}}-\theta\sqrt{D}} e^{-\frac{u^2}{2}}du. \end{aligned}$$

By changing variables $u = \dfrac{z}{\sqrt{D}}$, we get

$$A = D\left(e^{-a\theta} - e^{-\frac{a^2}{2D}} + \sqrt{2\pi D}\theta e^{\lambda D}e^{-a\theta}\left(\mathcal{N}\left(\frac{a}{\sqrt{D}}-\theta\sqrt{D}\right) - \mathcal{N}(-\theta\sqrt{D})\right)\right).$$

Finally, in the case $a = b - y \geq 0$ we get

$$\begin{aligned} K_{\lambda,D}(a) = D\Bigg[e^{-a\theta} + e^{\lambda D}\theta\sqrt{2\pi D}\Bigg(e^{-a\theta}\left[\mathcal{N}\left(\frac{a}{\sqrt{D}}-\theta\sqrt{D}\right) - \mathcal{N}(-\theta\sqrt{D})\right] \\ -e^{a\theta}\left[1-\mathcal{N}\left(\frac{a}{\sqrt{D}}+\theta\sqrt{D}\right)\right]\Bigg)\Bigg]. \end{aligned} \tag{24}$$

So, we find

$$\begin{aligned} &*\widehat{PDIC}(x,\lambda;K,L;r,\delta) = \\ &\frac{e^{2b\theta}}{\theta\psi(\theta\sqrt{D})}\Bigg[\int_k^b e^{my}(xe^{\sigma y}-K)\Bigg[e^{-(b-y)\theta} + \theta\sqrt{2\pi D}e^{\lambda D}\Bigg(e^{-(b-y)\theta}\left[\mathcal{N}\left(\frac{b-y}{\sqrt{D}}-\theta\sqrt{D}\right)-\mathcal{N}(-\theta\sqrt{D})\right] \\ &\quad -e^{(b-y)\theta}\left[1-\mathcal{N}(\frac{b-y}{\sqrt{D}}+\theta\sqrt{D})\right]\Bigg)\Bigg]dy + \int_b^{+\infty} e^{my}(xe^{\sigma y}-K)e^{(b-y)\theta}\psi(\theta\sqrt{D})dy\Bigg]. \end{aligned}$$

After doing long but not difficult computations we get, for $\mathbf{K} \leq \mathbf{L} \leq \mathbf{x}$,

$$\begin{aligned} *\widehat{PDIC}(x,\lambda;K,L) = \frac{e^{(m+\theta)b}}{\psi(\theta\sqrt{D})}\Bigg(\frac{2K}{m^2-\theta^2}\left[\psi(-\sqrt{D}m) + \sqrt{2\pi D}e^{\frac{Dm^2}{2}}m\mathcal{N}(-d-\sqrt{D}m)\right] \\ -\frac{2L}{(m+\sigma)^2-\theta^2}\left[\psi(-\sqrt{D}(m+\sigma)) + \sqrt{2\pi D}e^{\frac{D}{2}(m+\sigma)^2}(m+\sigma)\mathcal{N}\left(-d-\sqrt{D}(m+\sigma)\right)\right]\Bigg) \\ +\frac{Ke^{(m+\theta)k}}{\theta\psi(\theta\sqrt{D})}\left(\frac{1}{m+\theta}-\frac{1}{m+\sigma+\theta}\right)\left[\psi(-\theta\sqrt{D}) + \theta e^{\lambda D}\sqrt{2\pi D}\mathcal{N}(d-\theta\sqrt{D})\right] \\ +\frac{e^{\lambda D}\sqrt{2\pi D}}{\psi(\theta\sqrt{D})}Ke^{2b\theta}e^{(m-\theta)k}\mathcal{N}(-d-\theta\sqrt{D})\left(\frac{1}{m+\sigma-\theta}-\frac{1}{m-\theta}\right), \end{aligned} \tag{25}$$

where $d = \frac{b-k}{\sqrt{D}}$.

3.2. **The valuation of a Parisian down-and-out call with $b \leq 0$.** To find the valuation of a Parisian down-and-out call we can use the relation between
$*PDIC(x,T;K,L;r,\delta)$, $*PDOC(x,T;K,L;r,\delta)$ and the Black-Scholes price of an European call

$$*PDOC(x,T;K,L;r,\delta) = *BSC(x,T;K;r,\delta) - *PDIC(x,T;K,L;r,\delta),$$

where

$$*BSC(x,T;K;r,\delta) = \mathbb{E}_{\mathcal{P}}(e^{mZ_T}(xe^{\sigma Z_T} - K)^+).$$

Therefore, we obtain

$$*\widehat{PDOC}(x,\lambda;K,L;r,\delta) = *\widehat{BSC}(x,\lambda;K;r,\delta) - *\widehat{PDIC}(x,\lambda;K,L;r,\delta).$$

Now, we need to find the valuation of $*\widehat{BSC}(x,\lambda;K,L;r,\delta)$

$$\begin{aligned} *BSC(x,T;K;r,\delta) &= \mathbb{E}_{\mathcal{P}}(e^{mZ_T}(xe^{\sigma Z_T} - K)^+), \\ &= \int_{-\infty}^{+\infty} e^{mz}(xe^{\sigma z} - K)^+ \frac{1}{\sqrt{2\pi T}} e^{-\frac{z^2}{2T}} dz. \end{aligned}$$

$$*\widehat{BSC}(x,\lambda;K;r,\delta) = \int_{-\infty}^{+\infty} e^{mz}(xe^{\sigma z} - K)^+ \int_0^{+\infty} \frac{e^{-\lambda t}}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} dt\, dz.$$

Thanks to *Appendix B* we have

$$\int_0^{+\infty} \frac{e^{-\lambda t}}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} dt = \frac{e^{-|z|\theta}}{\theta}.$$

Then, we can write

$$\begin{aligned} *\widehat{BSC}(x,\lambda;K;r,\delta) &= \int_{-\infty}^{+\infty} e^{mz}(xe^{\sigma z} - K)^+ \frac{e^{-|z|\sqrt{2\lambda}}}{\theta} dz, \\ &= \int_{\frac{1}{\sigma}\ln(\frac{K}{x})}^{+\infty} e^{mz}(xe^{\sigma z} - K) \frac{e^{-|z|\theta}}{\theta} dz. \end{aligned}$$

3.2.1. *case $K \geq x$.* In this case, we can easily compute $*\widehat{BSC}(x,\lambda;K)$. Using the previous notations we have

$$*\widehat{BSC}(x,\lambda;K;r,\delta) = \int_k^{+\infty} e^{mz}(xe^{\sigma z} - K)\frac{e^{-|z|\theta}}{\theta} dz$$

and in this case $\frac{1}{\sigma}\ln\left(\frac{K}{x}\right) \geq 0$, so we get

$$\begin{aligned} *\widehat{BSC}(x,\lambda;K;r,\delta) &= \int_k^{+\infty} e^{mz}(xe^{\sigma z} - K)\frac{e^{-z\theta}}{\theta} dz, \\ &= \frac{1}{\theta}\int_k^{+\infty} e^{(m+\sigma-\theta)z} dz - \frac{K}{\theta}\int_k^{+\infty} xe^{(m-\theta)z} dz, \\ &= -\frac{K}{\theta}\frac{e^{(m-\theta)k}}{m+\sigma-\theta} + \frac{K}{\theta}\frac{e^{(m-\theta)k}}{m-\theta}. \end{aligned} \tag{26}$$

Then, we get the formula for the Laplace transform of the Black-Scholes call in the case $K \geq x$:

$$\boldsymbol{*\widehat{BSC}(x,\lambda;K;r,\delta) = \frac{K}{\theta}e^{(m-\theta)k}\left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta}\right)}, \text{ for } \boldsymbol{K \geq x}.$$

To obtain $*\widehat{PDOC}(x,\lambda;K,L;r,\delta)$ we only need to subtract $*\widehat{PDIC}(x,\lambda;K,L;r,\delta)$.

$$\begin{aligned} *\widehat{PDOC}(x,\lambda;K,L;r,\delta) &= \frac{K}{\theta}e^{(m-\theta)k}\left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta}\right) - \frac{\psi(-\theta\sqrt{D})e^{2b\theta}}{\theta\psi(\theta\sqrt{D})} \\ &\quad Ke^{(m-\theta)k}\left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta}\right), \\ &= \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta}\right)\frac{K}{\theta}e^{(m-\theta)k}\left[1 - \frac{e^{2b\theta}\psi(-\theta\sqrt{D})}{\psi(\theta\sqrt{D})}\right]. \end{aligned} \tag{27}$$

Furthermore,

$$\psi(-\theta\sqrt{D}) = \psi(\theta\sqrt{D}) - \theta\sqrt{2\pi D}e^{\lambda D}. \tag{28}$$

So, the following formula holds

$$*\widehat{\boldsymbol{PDOC}}(\boldsymbol{x},\boldsymbol{\lambda};\boldsymbol{K},\boldsymbol{L};\boldsymbol{r},\boldsymbol{\delta})=\left[\mathbf{1}-\boldsymbol{e^{2b\theta}}+\frac{\boldsymbol{\theta e^{2b\theta}\sqrt{2\pi D}e^{\lambda D}}}{\boldsymbol{\psi(\theta\sqrt{D})}}\right]\frac{\boldsymbol{K}}{\boldsymbol{\theta}}\boldsymbol{e^{(m-\theta)k}}$$
$$\left(\frac{\mathbf{1}}{\boldsymbol{m-\theta}}-\frac{\mathbf{1}}{\boldsymbol{m+\sigma-\theta}}\right)\text{ for } \boldsymbol{K\geq x\geq L}. \quad (29)$$

3.2.2. *case* $K\leq x$. In this case the integral has to be split.

$$\begin{aligned}
*\widehat{BSC}(x,\lambda;K;r,\delta) &= \int_{\frac{1}{\sigma}\ln(\frac{K}{x})}^{+\infty} e^{mz}(xe^{\sigma z}-K)\frac{e^{-|z|\theta}}{\theta}dz,\\
&= \int_k^0 e^{mz}(xe^{\sigma z}-K)\frac{e^{z\theta}}{\theta}dz \;+\; \int_0^{+\infty} e^{mz}(xe^{\sigma z}-K)\frac{e^{-z\theta}}{\theta}dz,\\
&= \frac{1}{\theta}\left(\int_k^0 xe^{(m+\sigma+\theta)z}-Ke^{(m+\theta)z}dz \;+\; \int_0^{+\infty} xe^{(m+\sigma-\theta)z}-Ke^{(m-\theta)z}dz\right),\\
&= \frac{1}{\theta}\left(\frac{x}{m+\sigma+\theta}-\frac{K}{m+\theta}-\frac{Ke^{(m+\theta)k}}{m+\sigma+\theta}+\frac{Ke^{(m+\theta)k}}{m+\theta}-\frac{x}{m+\sigma-\theta}+\frac{K}{m+\theta}\right),\\
&= \frac{2K}{m^2-\theta^2}-\frac{2x}{(m+\sigma)^2-\theta^2}+\frac{Ke^{(m+\theta)k}}{\theta}\left(\frac{1}{m+\theta}-\frac{1}{m+\sigma+\theta}\right). \qquad (30)
\end{aligned}$$

So, we get

$$*\widehat{\boldsymbol{BSC}}(\boldsymbol{x},\boldsymbol{\lambda};\boldsymbol{K};\boldsymbol{r},\boldsymbol{\delta})=\frac{\boldsymbol{2K}}{\boldsymbol{m^2-\theta^2}}-\frac{\boldsymbol{2x}}{\boldsymbol{(m+\sigma)^2-\theta^2}}+\frac{\boldsymbol{Ke^{(m+\theta)k}}}{\boldsymbol{\theta}}$$
$$\left(\frac{\mathbf{1}}{\boldsymbol{m+\theta}}-\frac{\mathbf{1}}{\boldsymbol{m+\sigma+\theta}}\right),\text{ for } \boldsymbol{K\leq x}. \quad (31)$$

Finally, we come up with the following formula for the valuation of a Parisian down-and-out call with $\boldsymbol{b\leq 0}$

► Case $\boldsymbol{K\geq L}$.

$$\begin{aligned}
&*\widehat{\boldsymbol{PDOC}}(\boldsymbol{x},\boldsymbol{\lambda};\boldsymbol{K},\boldsymbol{L};\boldsymbol{r},\boldsymbol{\delta})\\
&\quad= \frac{\boldsymbol{2K}}{\boldsymbol{m^2-\theta^2}}-\frac{\boldsymbol{2x}}{\boldsymbol{(m+\sigma)^2-\theta^2}}+\frac{\boldsymbol{Ke^{(m+\theta)k}}}{\boldsymbol{\theta}}\left(\frac{\mathbf{1}}{\boldsymbol{m+\theta}}-\frac{\mathbf{1}}{\boldsymbol{m+\sigma+\theta}}\right)\\
&\qquad -\frac{\boldsymbol{\psi(-\theta\sqrt{D})e^{2b\theta}}}{\boldsymbol{\theta\psi(\theta\sqrt{D})}}\boldsymbol{Ke^{(m-\theta)k}}\left(\frac{\mathbf{1}}{\boldsymbol{m-\theta}}-\frac{\mathbf{1}}{\boldsymbol{m+\sigma-\theta}}\right),\text{ for } \boldsymbol{x\geq K\geq L}. \qquad (32)
\end{aligned}$$

► Case $\boldsymbol{K\leq L}$.

$$\begin{aligned}
&*\widehat{\boldsymbol{PDOC}}(\boldsymbol{x},\boldsymbol{\lambda};\boldsymbol{K},\boldsymbol{L})=\\
&\frac{\boldsymbol{2K}}{\boldsymbol{m^2-\theta^2}}\left[\mathbf{1}-\frac{\boldsymbol{e^{(m+\theta)b}}}{\boldsymbol{\psi(\theta\sqrt{D})}}\left(\boldsymbol{\psi(-\sqrt{D}m)}+\boldsymbol{\sqrt{2\pi D}e^{\frac{Dm^2}{2}}m\mathcal{N}(-d-\sqrt{D}m)}\right)\right]\\
&-\frac{\mathbf{2}}{\boldsymbol{(m+\sigma^2)-\theta^2}}\left[\boldsymbol{x}-\frac{\boldsymbol{Le^{(m+\theta)b}}}{\boldsymbol{\psi(\theta\sqrt{D})}}\left(\boldsymbol{\psi(-\sqrt{D}(m+\sigma))}+\boldsymbol{\sqrt{2\pi D}e^{\frac{D(m+\sigma)^2}{2}}}\right.\right.\\
&\qquad\qquad\left.\left.\boldsymbol{(m+\sigma)\mathcal{N}(-d-\sqrt{D}(m+\sigma))}\right)\right]\\
&+\frac{\boldsymbol{Ke^{(m+\theta)k}}}{\boldsymbol{\theta}}\left(\frac{\mathbf{1}}{\boldsymbol{m+\theta}}-\frac{\mathbf{1}}{\boldsymbol{m+\theta+\sigma}}\right)\left[\mathbf{1}-\frac{\mathbf{1}}{\boldsymbol{\psi(\theta\sqrt{D})}}\left(\boldsymbol{\psi(-\theta\sqrt{D})}+\boldsymbol{\theta e^{\lambda D}\sqrt{2\pi D}\mathcal{N}(d-\theta\sqrt{D})}\right)\right]\\
&-\frac{\boldsymbol{e^{\lambda D}\sqrt{2\pi D}}}{\boldsymbol{\psi(\theta\sqrt{D})}}\boldsymbol{Ke^{2b\theta}e^{(m-\theta)k}\mathcal{N}(-d-\theta\sqrt{D})}\left(\frac{\mathbf{1}}{\boldsymbol{m-\theta+\sigma}}-\frac{\mathbf{1}}{\boldsymbol{m-\theta}}\right),
\end{aligned}$$

for $\boldsymbol{K\leq L\leq x}$.

3.3. **The valuation of a Parisian down-and-out call with** $b>0$**.**

3.3.1. *reduction to the case* $b=0$. If b is positive and $T_b^- \geq T \geq D$, then $T_b \leq D$.

Therefore, the discounted value of a down-and-out call in the case $L > x$ is

$$*PDOC(x,T;K,L;r,\delta) = \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- \geq T\}}\mathbf{1}_{\{T_b \leq D\}}[xe^{\sigma Z_T} - K]^+ e^{mZ_T}). \tag{33}$$

We can also write :

$$*PDOC(x,T;K,L;r,\delta) = \mathbb{E}_{\mathcal{P}}\left(\mathbb{E}_{\mathcal{P}}[\mathbf{1}_{\{T_b^- \geq T\}}\mathbf{1}_{\{T_b \leq D\}}[xe^{\sigma(Z_T - Z_{T_b}+b)} - K]^+ e^{m(Z_T - Z_{T_b}+b)} \mid \mathcal{F}_{T_b}]\right).$$

We have $\mathcal{F}_{T_b} = \{A \in \mathcal{A}, \forall t \geq 0, A\bigcap\{T_b \leq t\} \in \mathcal{F}_t\}$, then $\{T_b \leq D\} \in \mathcal{F}_{T_b}$, because

$$\{T_b \leq D\}\bigcap\{T_b \leq t\} = \{T_b \leq D \wedge t\}$$

and

$$\{T_b \leq D \wedge t\} \in \mathcal{F}_{t\wedge D} \subset \mathcal{F}_t.$$

Therefore $\mathbf{1}_{\{T_b \leq D\}}$ is $\mathcal{F}_{T_b}$-measurable.
So we get

$$*PDOC(x,T;K,L;r,\delta) = \mathbb{E}_{\mathcal{P}}\left(\mathbf{1}_{\{T_b \leq D\}}\mathbb{E}_{\mathcal{P}}[\mathbf{1}_{\{T_b^- - T_b \geq T - T_b\}}[xe^{\sigma Z_T - Z_{T_b}+b} - K]^+ e^{m(Z_T - Z_{T_b}+b)} \mid \mathcal{F}_{T_b}]\right).$$

Relying on the strong Markov property we can write that $T_b^- - T_b \overset{law}{=} T_0^-$.
Hence

$$*PDOC(x,T;K,L;r,\delta) = \mathbb{E}_{\mathcal{P}}\left(\mathbf{1}_{\{T_b \leq D\}}\mathbb{E}_{\mathcal{P}}[\mathbf{1}_{\{T_0^- \geq T - T_b\}}[xe^{\sigma(Z_T - Z_{T_b}+b)} - K]^+ e^{m(Z_T - Z_{T_b}+b)} \mid \mathcal{F}_{T_b}]\right).$$

Let W_t denote $Z_{T_b+t} - Z_{T_b}$, relying on the strong Markov property W_t is independent of $\mathcal{F}_{T_b}$.
Let Y_t denote $\mathbf{1}_{\{T_0^- \geq T-t\}}[xe^{\sigma(W_{T-t}+b)} - K]^+ e^{m(W_{T-t}+b)}$.

- Y_t is independent of $\mathcal{F}_{T_b}$,
- T_b is $\mathcal{F}_{T_b}$-measurable so we can write $\mathbb{E}[Y_{T_b}|\mathcal{F}_{T_b}] = \mathbb{E}[Y_t]_{|t=T_b}$.

Hence we have

$$\begin{aligned} *PDOC(x,T;K,L;r,\delta) &= \mathbb{E}[\mathbf{1}_{\{T_b \leq D\}}\mathbb{E}[Y_t]_{|t=T_b}], \\ &= \int_{-\infty}^{\infty} \mathbf{1}_{\{u \leq D\}}\mathbb{E}_{\mathcal{P}}[Y_u]\mu_b(du) \end{aligned}$$

where $\mu_b(du)$ is the law of T_b recalled in *Appendix A*. We get

$$*PDOC(x,T;K,L;r,\delta) = \int_0^D \mathbb{E}_{\mathcal{P}}\left(\mathbf{1}_{\{T_0^- \geq T-u\}}[xe^{\sigma(W_{T-u}+b)} - K]^+ e^{m(W_{T-u}+b)}\right)\mu_b(du).$$

So, we have

$$*PDOC(x,T;K,L;r,\delta) = e^{mb}\int_0^D \mathbb{E}_{\mathcal{P}}\left(\mathbf{1}_{\{T_0^- \geq T-u\}}[xe^{\sigma b}e^{\sigma W_{T-u}} - K]^+ e^{mW_{T-u}}\right)\mu_b(du).$$

As $b = \frac{1}{\sigma}\ln\left(\frac{L}{x}\right)$, we get

$$*PDOC(x,T;K,L;r,\delta) = Le^{mb}\int_0^D \mathbb{E}_{\mathcal{P}}\left(\mathbf{1}_{\{T_0^- \geq T-u\}}[e^{\sigma W_{T-u}} - K/L]^+ e^{mW_{T-u}}\right)\mu_b(du).$$

The price of a Parisian down-and-out call in the case $b > 0$ is given by

$$\boldsymbol{*PDOC(x,T;K,L;r,\delta) = Le^{mb}\int_0^D *PDOC^0(T-u;K/L;r,\delta)\mu_b(du)} \tag{34}$$

where

$$*PDOC^0(T;K;r,\delta) = \mathbb{E}_{\mathcal{P}}\left(\mathbf{1}_{\{T_0^- \geq T\}}[e^{\sigma Z_T} - K]^+ e^{mZ_T}\right).$$

3.3.2. *The Laplace transform of* $*PDOC(x,T;K,L;r,\delta)$. If we consider the Laplace transform of $*PDOC(x,T;K,L;r,\delta)$ with respect to T, we get

$$\begin{aligned}
*\widehat{PDOC}(x,\lambda;K,L;r,\delta) &= \int_0^{+\infty} e^{-\lambda t}\, Le^{mb}\int_0^D *PDOC^0(t-u;K/L;r,\delta)\mu_b(du)\mathbf{1}_{\{t-u>0\}}dt,\\
&= Le^{mb}\int_0^D \mu_b(du)\int_u^{+\infty} e^{-\lambda t} * PDOC^0(t-u;K/L;r,\delta)\,dt,\\
&\quad \text{we change variables } (v,u)=(t-u,u)\\
&= Le^{mb}\int_0^D \mu_b(du)e^{-\lambda u}\int_0^{+\infty} e^{-\lambda v} * PDOC^0(v;K/L;r,\delta)\,dv,\\
&= Le^{mb}\int_0^D \mu_b(du)e^{-\lambda u} *\widehat{PDOC}^0(\lambda;K/L;r,\delta).
\end{aligned}$$

If we compute $\int_0^D \mu_b(du)e^{-\lambda u}$, we find $e^{-\theta b}\mathcal{N}\left(\theta\sqrt{D}-\frac{b}{\sqrt{D}}\right)+e^{\theta b}\mathcal{N}\left(-\sqrt{D}\theta-\frac{b}{\sqrt{D}}\right)$ as proved in *Appendix A*, where $\mathcal{N}$ denotes the standard normal cumulative distribution. Finally, we come up with the following formula

$$\boldsymbol{*\widehat{PDOC}(x,\lambda;K,L;r,\delta) = L\left[e^{(m-\theta)b}\mathcal{N}\left(\sqrt{D}\theta-\frac{b}{\sqrt{D}}\right)+\right.}$$
$$\boldsymbol{\left.e^{(m+\theta)b}\mathcal{N}\left(-\sqrt{D}\theta-\frac{b}{\sqrt{D}}\right)\right] *\widehat{PDOC}^0(\lambda;K/L;r,\delta)}, \text{ for } \boldsymbol{L\geq x}. \quad (35)$$

► Case $K\geq L$. $*\widehat{PDOC}^0(\lambda;K/L)$ has already been computed in (29), and we had found

$$*\widehat{PDOC}^0(\lambda;K/L;r,\delta)=\frac{\sqrt{2\pi D}e^{\lambda D}}{\psi(\theta\sqrt{D})}\frac{K}{L}e^{(m-\theta)k}\left(\frac{1}{m-\theta}-\frac{1}{m+\sigma-\theta}\right), \text{ for } K>L.$$

Then, we now have an explicit formula for the Laplace transform of $*PDOC(x,T;K,L;r,\delta)$ when $K>L$.

$$\boldsymbol{*\widehat{PDOC}(x,\lambda;K,L;r,\delta) = \left[e^{(m-\theta)b}\mathcal{N}\left(\theta\sqrt{D}-\frac{b}{\sqrt{D}}\right)+e^{(m+\theta)b}\mathcal{N}\left(-\theta\sqrt{D}-\frac{b}{\sqrt{D}}\right)\right]}$$
$$\boldsymbol{\frac{\sqrt{2\pi D}e^{\lambda D}}{\psi(\theta\sqrt{D})}Ke^{(m-\theta)k}\left(\frac{1}{m-\theta}-\frac{1}{m+\sigma-\theta}\right)}, \text{ for } \boldsymbol{K\geq L\geq x}. \quad (36)$$

► Case $K\leq L$. In this case, we have

$$\begin{aligned}
&*\widehat{PDOC}^0(\lambda;K/L)=\\
&\frac{2K}{L(m^2-\theta^2)}\left[1-\frac{1}{\psi(\theta\sqrt{D})}\left(\psi(-\sqrt{D}m)+\sqrt{2\pi D}e^{\frac{Dm^2}{2}}m\mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}}-\sqrt{D}m\right)\right)\right]\\
&-\frac{2}{(m+\sigma^2)-\theta^2}\left[1-\frac{1}{\psi(\theta\sqrt{D})}\left(\psi(-\sqrt{D}(m+\sigma))\right.\right.\\
&\qquad\qquad\left.\left.+\sqrt{2\pi D}e^{\frac{D(m+\sigma)^2}{2}}(m+\sigma)\mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}}-\sqrt{D}(m+\sigma)\right)\right)\right]\\
&+\frac{Ke^{\frac{m+\theta}{\sigma}\ln(\frac{K}{L})}}{L\theta}\left(\frac{1}{m+\theta}-\frac{1}{m+\theta+\sigma}\right)\\
&\qquad\qquad\left[1-\frac{1}{\psi(\theta\sqrt{D})}\left(\psi(-\theta\sqrt{D})+\theta e^{\lambda D}\sqrt{2\pi D}\mathcal{N}\left(\frac{\ln(\frac{L}{K})}{\sigma\sqrt{D}}-\theta\sqrt{D}\right)\right)\right]\\
&-\frac{e^{\lambda D}\sqrt{2\pi D}}{\psi(\theta\sqrt{D})}\frac{K}{L}e^{\frac{m-\theta}{\sigma}\ln(\frac{K}{L})}\mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}}-\theta\sqrt{D}\right)\left(\frac{1}{m-\theta+\sigma}-\frac{1}{m-\theta}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&*\widehat{PDOC}(x,\lambda;K,L;r,\delta) = L\left(e^{(m-\theta)b}\mathcal{N}\left(\sqrt{D}\theta - \tfrac{b}{\sqrt{D}}\right) + e^{(m+\theta)b}\mathcal{N}\left(-\sqrt{D}\theta - \tfrac{b}{\sqrt{D}}\right)\right)\\
&\left\{\frac{2K}{L(m^2-\theta^2)}\left[1 - \frac{1}{\psi(\theta\sqrt{D})}\left(\psi(-\sqrt{D}m) + \sqrt{2\pi D}e^{\frac{Dm^2}{2}} m\mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \sqrt{D}m\right)\right)\right]\right.\\
&-\frac{2}{(m+\sigma^2)-\theta^2}\left[1 - \frac{1}{\psi(\theta\sqrt{D})}\left(\psi(-\sqrt{D}(m+\sigma))\right.\right.\\
&\qquad\qquad\left.\left. + \sqrt{2\pi D}e^{\frac{D(m+\sigma)^2}{2}}(m+\sigma)\mathcal{N}\left(\tfrac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \sqrt{D}(m+\sigma)\right)\right)\right]\\
&+\frac{Ke^{\frac{m+\theta}{\sigma}\ln(\frac{K}{L})}}{L\theta}\left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma}\right)\\
&\qquad\left[1 - \tfrac{1}{\psi(\theta\sqrt{D})}\left(\psi(-\theta\sqrt{D}) + \theta e^{\lambda D}\sqrt{2\pi D}\mathcal{N}(\tfrac{\ln(\frac{L}{K})}{\sigma\sqrt{D}} - \theta\sqrt{D})\right)\right]\\
&\left.-\frac{e^{\lambda D}\sqrt{2\pi D}}{\psi(\theta\sqrt{D})}\frac{K}{L}e^{\frac{m-\theta}{\sigma}\ln(\frac{K}{L})}\mathcal{N}(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \theta\sqrt{D})\left(\frac{1}{m-\theta+\sigma} - \frac{1}{m-\theta}\right)\right\},
\end{aligned}$$

for $K \leq L$ and $x \leq L$.

3.4. **The valuation of a Parisian down-and-in call with $b > 0$.** So far, we have managed to find explicit formulae for the Laplace transforms of the down-and-out call prices with $b > 0$. Now, we will use the relationships existing between down-and-out options and down-and-in options to compute the price of a down-and-in call in the case $b > 0$. In fact, the following formula holds

$$*\widehat{PDIC}(x,\lambda;K,L;r,\delta) = *\widehat{BSC}(x,\lambda,K,r,\delta) - *\widehat{PDOC}(x,\lambda;K,L;r,\delta)$$

where $*\widehat{PDOC}(x,\lambda;K,L;r,\delta)$ has already been computed above in the *Section 3.3.2* for $b > 0$ and $*\widehat{BSC}(x,\lambda,K,r,\delta)$ has been calculated in (31) and (27). We simply recall the formula

$$*\widehat{BSC}(x,\lambda,K,r,\delta) = \begin{cases} \dfrac{K}{\theta}e^{(m-\theta)k}\left(\dfrac{1}{m-\theta} - \dfrac{1}{m-\theta+\sigma}\right) & \text{if } K \geq x,\\ \dfrac{2K}{m^2-\theta^2} - \dfrac{2x}{(m+\sigma)^2-\theta^2} + \dfrac{K}{\theta}e^{(m+\theta)k}\left(\dfrac{1}{m+\theta} - \dfrac{1}{m+\theta+\sigma}\right) & \text{if } K \leq x. \end{cases}$$

If we put all the terms together we find the following formula
▶ Case $K \geq L$.

$$\begin{aligned}
&*\widehat{PDIC}(x,\lambda;K,L;r,\delta) =\\
&\qquad\frac{K}{\theta}e^{(m-\theta)k}\left(\frac{1}{m-\theta} - \frac{1}{m-\theta+\sigma}\right)\\
&\qquad-\left(e^{(m-\theta)b}\mathcal{N}\left(\theta\sqrt{D} - \frac{b}{\sqrt{D}}\right) + e^{(m+\theta)b}\mathcal{N}\left(-\theta\sqrt{D} - \frac{b}{\sqrt{D}}\right)\right)\\
&\qquad\frac{\sqrt{2\pi D}e^{\lambda D}}{\psi(\theta\sqrt{D})}Ke^{(m-\theta)k}\left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta}\right), \text{ for } K \geq L \geq x.
\end{aligned}$$

▶ Case $K \le L$.

$$
\begin{aligned}
&*\widehat{PDIC}(x,\lambda;K,L;r,\delta) = \frac{K}{\theta}e^{(m-\theta)k}\left(\frac{1}{m-\theta}-\frac{1}{m-\theta+\sigma}\right)\\
&-L\left(e^{(m-\theta)b}\mathcal{N}\left(\sqrt{D}\theta-\tfrac{b}{\sqrt{D}}\right)+e^{(m+\theta)b}\mathcal{N}\left(-\sqrt{D}\theta-\tfrac{b}{\sqrt{D}}\right)\right)\\
&\Bigg\{\frac{2K}{L(m^2-\theta^2)}\left[1-\frac{1}{\psi(\theta\sqrt{D})}\left(\psi(-\sqrt{D}m)+\sqrt{2\pi D}e^{\frac{Dm^2}{2}}m\mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}}-\sqrt{D}m\right)\right)\right]\\
&-\frac{2}{(m+\sigma^2)-\theta^2}\\
&\left[1-\frac{1}{\psi(\theta\sqrt{D})}\left(\psi(-\sqrt{D}(m+\sigma))+\sqrt{2\pi D}e^{\frac{D(m+\sigma)^2}{2}}(m+\sigma)\mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}}-\sqrt{D}(m+\sigma)\right)\right)\right]\\
&+\frac{Ke^{\frac{m+\theta}{\sigma}\ln(\frac{K}{L})}}{L\theta}\left(\frac{1}{m+\theta}-\frac{1}{m+\theta+\sigma}\right)\\
&\qquad\left[1-\frac{1}{\psi(\theta\sqrt{D})}\left(\psi(-\theta\sqrt{D})+\theta e^{\lambda D}\sqrt{2\pi D}\mathcal{N}\left(\frac{\ln(\frac{L}{K})}{\sigma\sqrt{D}}-\theta\sqrt{D}\right)\right)\right]\\
&-\frac{e^{\lambda D}\sqrt{2\pi D}}{\psi(\theta\sqrt{D})}\frac{K}{L}e^{\frac{m-\theta}{\sigma}\ln(\frac{K}{L})}\mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}}-\theta\sqrt{D}\right)\left(\frac{1}{m-\theta+\sigma}-\frac{1}{m-\theta}\right)\Bigg\},\ \text{for } x\le K\le L.
\end{aligned}
$$

$$
\begin{aligned}
&*\widehat{PDIC}(x,\lambda;K,L;r,\delta) = \frac{2K}{m^2-\theta^2}-\frac{2x}{(m+\sigma)^2-\theta^2}+\frac{K}{\theta}e^{(m+\theta)k}\left(\frac{1}{m+\theta}-\frac{1}{m+\theta+\sigma}\right)\\
&-L\left(e^{(m-\theta)b}\mathcal{N}\left(\sqrt{D}\theta-\tfrac{b}{\sqrt{D}}\right)+e^{(m+\theta)b}\mathcal{N}\left(-\sqrt{D}\theta-\tfrac{b}{\sqrt{D}}\right)\right)\\
&\Bigg\{\frac{2K}{L(m^2-\theta^2)}\left[1-\frac{1}{\psi(\theta\sqrt{D})}\left(\psi(-\sqrt{D}m)+\sqrt{2\pi D}e^{\frac{Dm^2}{2}}m\mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}}-\sqrt{D}m\right)\right)\right]\\
&-\frac{2}{(m+\sigma^2)-\theta^2}\Bigg[1-\frac{1}{\psi(\theta\sqrt{D})}\Bigg(\psi(-\sqrt{D}(m+\sigma))\\
&\qquad+\sqrt{2\pi D}e^{\frac{D(m+\sigma)^2}{2}}(m+\sigma)\mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}}-\sqrt{D}(m+\sigma)\right)\Bigg)\Bigg]\\
&+\frac{Ke^{\frac{m+\theta}{\sigma}\ln(\frac{K}{L})}}{L\theta}\left(\frac{1}{m+\theta}-\frac{1}{m+\theta+\sigma}\right)\\
&\qquad\left[1-\frac{1}{\psi(\theta\sqrt{D})}\left(\psi(-\theta\sqrt{D})+\theta e^{\lambda D}\sqrt{2\pi D}\mathcal{N}\left(\frac{\ln(\frac{L}{K})}{\sigma\sqrt{D}}-\theta\sqrt{D}\right)\right)\right]\\
&-\frac{e^{\lambda D}\sqrt{2\pi D}}{\psi(\theta\sqrt{D})}\frac{K}{L}e^{\frac{m-\theta}{\sigma}\ln(\frac{K}{L})}\mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}}-\theta\sqrt{D}\right)\left(\frac{1}{m-\theta+\sigma}-\frac{1}{m-\theta}\right)\Bigg\},\ \text{for } K\le x\le L.
\end{aligned}
$$

4. The Parisian Up Calls

This section will go exactly through the same points as the previous one but considering the Up calls instead of the Down ones this time. Once again the organisation of this section is based on the presentation scheme.

4.1. **The valuation of a Parisian Up-and-in call with $b \ge 0$.** The owner of an up-and-in option receives the pay-off if S makes an excursion above the level L older than D before the maturity time T, which is exactly the same as saying Brownian motion Z makes an excursion above b older than D. Using the previous notations we can write :

$$*PUIC(x,T;K,L;r,\delta) = \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^+<T\}}(xe^{\sigma Z_T}-K)^+e^{mZ_T}), \tag{37}$$

where

$$T_b^+ = \inf\{t>0|\mathbf{1}_{\{Z_t>b\}}(t-g_t^b)>D\}. \tag{38}$$

The computation of $*PUIC(x,T;K,L;r,\delta)$ for $b>0$ is exactly the same as the computation of $*PDIC(x,T;K,L;r,\delta)$ for $b<0$. We just have to find the law of T_b^+. We have

$$*PUIC(x,T;K,L;r,\delta)=\int_{-\infty}^{+\infty}\mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^+<T\}}\mathcal{P}_{T-T_b^+}(f_x)(z))\nu(dz), \tag{39}$$

where

- $f_x(z)=e^{mz}(xe^{\sigma z}-K)^+$,
- $\mathcal{P}_t(f_x)(z)=\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty}f_x(u)\exp\left(-\frac{(u-z)^2}{2t}\right)du$,
- $\nu(dz)$ is the law of $Z_{T_b^+}$.

We have

$$*PUIC(x,T;K,L;r,\delta)=\int_{-\infty}^{+\infty}f_x(y)h_b(T,y)dy, \tag{40}$$

where

$$h_b(t,y)=\int_{-\infty}^{\infty}\mathbb{E}_{\mathcal{P}}\left(\mathbf{1}_{\{T_b^+<t\}}\frac{\exp\left(-\frac{(z-y)^2}{2(t-T_b^+)}\right)}{\sqrt{2\pi(t-T_b^+)}}\right)\nu(dz). \tag{41}$$

Since we consider the case $b>0$, we can use the following expression for the law of $Z_{T_b^+}$,as it is proved in *Appendix D*

$$\mathbb{P}(Z_{T_b^+}\in dx)=\frac{dx}{D}(x-b)\exp\left(-\frac{(x-b)^2}{2D}\right)\mathbf{1}_{\{x\geq b\}}. \tag{42}$$

4.1.1. *The Laplace transform of $*PUIC(x,T;K,L;r,\delta)$.* We still have

$$*\widehat{PUIC}(x,\lambda;K,L;r,\delta)=\int_{-\infty}^{\infty}f_x(y)\int_0^{\infty}e^{-\lambda t}h_b(t,y)dtdy. \tag{43}$$

We would like to compute

$$\widehat{h_b}(\lambda,y)=\int_0^{\infty}e^{-\lambda t}h_b(t,y)dt. \tag{44}$$

We know that

$$h_b(t,y)=\int_b^{+\infty}\frac{z-b}{D}\exp\left(-\frac{(z-b)^2}{2D}\right)\mathbb{E}_{\mathcal{P}}\left(\mathbf{1}_{\{T_b^+<t\}}\frac{\exp\left(-\frac{(z-y)^2}{2(t-T_b^+)}\right)}{\sqrt{2\pi(t-T_b^+)}}\right)dz. \tag{45}$$

We can write

$$h_b(t,y)=\int_b^{+\infty}\frac{z-b}{D}\exp\left(-\frac{(z-b)^2}{2D}\right)\gamma(t,z-y)dz,$$

where

$$\gamma(t,x)=\mathbb{E}_{\mathcal{P}}\left(\mathbf{1}_{\{T_b^+<t\}}\frac{\exp\left(-\frac{x^2}{2(t-T_b^+)}\right)}{\sqrt{2\pi(t-T_b^+)}}\right)$$

and we have

$$\widehat{h_b}(\lambda,y)=\int_b^{+\infty}\frac{z-b}{D}\exp\left(-\frac{(z-b)^2}{2D}\right)\int_0^{\infty}e^{-\lambda t}\gamma(t,z-y)dt\,dz. \tag{46}$$

So, we need to compute the Laplace transform of $\gamma(t,x)$

$$\int_0^{\infty}e^{-\lambda t}\gamma(t,x)dt=\mathbb{E}_{\mathcal{P}}\left(\int_{T_b^+}^{\infty}e^{-\lambda t}\frac{\exp\left(-\frac{x^2}{2(t-T_b^+)}\right)}{\sqrt{2\pi(t-T_b^+)}}dt\right).$$

By changing variables $u=t-T_b^+$, we get

$$\int_0^{\infty}e^{-\lambda t}\gamma(t,x)dt=\mathbb{E}_{\mathcal{P}}(e^{-\lambda T_b^+})\int_0^{\infty}e^{-\lambda u}\frac{\exp\left(-\frac{x^2}{2u}\right)}{\sqrt{2\pi u}}du.$$

Using results from *Appendix* D and B, we come up with

$$\int_0^\infty e^{-\lambda t}\gamma(t,x)dt = \frac{\exp\left[-(|x|+b)\theta\right]}{\theta\psi(\theta\sqrt{D})}. \tag{47}$$

Thanks to (46) we can rewrite

$$\widehat{h_b}(\lambda,y) = \frac{e^{-b\theta}}{D\theta\psi(\theta\sqrt{D})}\int_0^\infty x\exp\left(-\frac{x^2}{2D}-|b+x-y|\theta\right)dx. \tag{48}$$

Let $K\mathbf{1}_{\lambda,D}(y-b)$ denote $\displaystyle\int_0^{+\infty} x\exp\left(-\frac{x^2}{2D}-|b+x-y|\theta\right)dx$.

4.1.2. *The valuation of* $K\mathbf{1}_{\lambda,D}(y-b)$. Let c denote $y-b$.
We have $K\mathbf{1}_{\lambda,D}(c) = \displaystyle\int_0^{+\infty} x\exp\left(-\frac{x^2}{2D}-|x-c|\theta\right)dx$.

▶ Case $K \geq L$. In such a case we have, for $y \in [k,+\infty[$, $y-b\geq 0$.
We can use the formula (24) to compute $K\mathbf{1}_{\lambda,D}(c)$. Then for $\widehat{h_b}(\lambda,y)$ we get :

$$\begin{aligned}\widehat{h_b}(\lambda,y) = \frac{e^{-b\theta}}{\theta\psi(\theta\sqrt{D})}\Bigg[e^{-(y-b)\theta} + \theta\sqrt{2\pi D}e^{\lambda D}\Bigg(e^{-(y-b)\theta}[\mathcal{N}(\frac{y-b}{\sqrt{D}}-\theta\sqrt{D})-\mathcal{N}(-\theta\sqrt{D})] \\ -e^{(y-b)\theta}\left(1-\mathcal{N}(\frac{y-b}{\sqrt{D}}+\theta\sqrt{D})\right)\Bigg)\Bigg].\end{aligned} \tag{49}$$

By plugging this result in(43) and by doing long but easy calculations we get:

$$\begin{aligned}\boldsymbol{*\widehat{PUIC}(x,\lambda;K,L;r,\delta)} &= \boldsymbol{e^{(m-\theta)b}\frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})}\left[\frac{2K}{m^2-\theta^2}e^{\frac{Dm^2}{2}}m\mathcal{N}(d+\sqrt{D}m)\right.} \\ &\quad \boldsymbol{\left.-\frac{2L}{(m+\sigma)^2-\theta^2}e^{\frac{D(m+\sigma)^2}{2}}(m+\sigma)\mathcal{N}(d+\sqrt{D}(m+\sigma))\right]} \\ &\quad \boldsymbol{+\frac{e^{-2b\theta}}{\psi(\theta\sqrt{D})}Ke^{(m+\theta)k}e^{\lambda D}\sqrt{2\pi D}\mathcal{N}(d-\theta\sqrt{D})\left(\frac{1}{m+\sigma+\theta}-\frac{1}{m+\theta}\right)} \\ &\quad \boldsymbol{+\frac{e^{(m-\theta)k}}{\theta\psi(\theta\sqrt{D})}K\left(\frac{1}{m-\theta}-\frac{1}{m+\sigma-\theta}\right)\left(\psi(-\theta\sqrt{D})+\theta\sqrt{2\pi D}e^{\lambda D}\mathcal{N}(d-\theta\sqrt{D})\right)},\end{aligned} \tag{50}$$

for $\boldsymbol{x \leq L \leq K}$.
▶ Case $K \leq L$. If $K \leq L$ we have $y-b\geq 0$ for $y \in [b,+\infty[$ and $y-b\leq 0$ for $y\in[k,b]$. So we get

$$\begin{aligned}*\widehat{PUIC}(x,\lambda;K,L;r,\delta) = \frac{e^{-b\theta}}{D\theta\psi(\theta\sqrt{D})}\left(\int_k^b e^{my}(xe^{\sigma y}-K)\int_0^{+\infty} z\exp\left(-\frac{z^2}{2D}-(z+b-y)\theta\right)dzdy\right. \\ \left.+\int_b^{+\infty} e^{my}(xe^{\sigma y}-K)\int_0^{+\infty} z\exp\left(-\frac{z^2}{2D}-|z+b-y|\theta\right)dzdy\right).\end{aligned}$$

After doing computations we get

$$\begin{aligned}\boldsymbol{*\widehat{PUIC}(x,\lambda;K,L;r,\delta)} &= \boldsymbol{\frac{e^{(m-\theta)b}}{\psi(\theta\sqrt{D})}\left[\frac{2K}{m^2-\theta^2}\psi(\sqrt{D}m)-\frac{2L}{(m+\sigma)^2-\theta^2}\psi(\sqrt{D}(m+\sigma))\right]} \\ &\quad \boldsymbol{+\frac{e^{-2b\theta}\psi(-\theta\sqrt{D})}{\theta\psi(\theta\sqrt{D})}Ke(m+\theta)k\left(\frac{1}{m+\theta}-\frac{1}{m+\theta+\sigma}\right)}, \text{ for } \boldsymbol{K\leq L} \text{ and } \boldsymbol{x\leq L}.\end{aligned} \tag{51}$$

4.2. **The valuation of a Parisian Up-and-out call with $b \geq 0$.** Thanks to the formula of $*\widehat{PUIC}(x, \lambda; K, L; r, \delta)$ we can find $*\widehat{PUOC}(x, \lambda; K, L; r, \delta)$. By using the relations between $*\widehat{PUIC}$ and $*\widehat{PUOC}$ and the Laplace transform of a Call when $x \leq K$ (which has been computed in *3.2.1*).
So, for $x \leq L \leq K$, we obtain

$$\begin{aligned}
&*\widehat{PUOC}(x, \lambda; K, L; r, \delta) = \frac{K}{\theta} e^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) \\
&- e^{(m-\theta)b} \frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})} \left[\frac{2K}{m^2-\theta^2} e^{\frac{Dm^2}{2}} m \mathcal{N}(d+\sqrt{D}m) \right. \\
&\qquad\qquad \left. - \frac{2L}{(m+\sigma)^2-\theta^2} e^{\frac{D(m+\sigma)^2}{2}} (m+\sigma) \mathcal{N}(d+\sqrt{D}(m+\sigma)) \right] \\
&- \frac{e^{-2b\theta}}{\psi(\theta\sqrt{D})} K e^{(m+\theta)k} e^{\lambda D} \sqrt{2\pi D} \mathcal{N}(d-\theta\sqrt{D}) \left(\frac{1}{m+\sigma+\theta} - \frac{1}{m+\theta} \right) \\
&- \frac{e^{(m-\theta)k}}{\theta\psi(\theta\sqrt{D})} K \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) \left(\psi(-\theta\sqrt{D}) + \theta\sqrt{2\pi D} e^{\lambda D} \mathcal{N}(d-\theta\sqrt{D}) \right),
\end{aligned}$$

and for $K \leq x \leq L$, we have

$$\begin{aligned}
&*\widehat{PUOC}(x, \lambda; K, L; r, \delta) = \\
&\qquad \frac{2K}{m^2-\theta^2} - \frac{2x}{(m+\sigma)^2-\theta^2} + \frac{K}{\theta} e^{(m+\theta)k} \left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma} \right) \\
&\qquad - \frac{e^{(m-\theta)b}}{\psi(\theta\sqrt{D})} \left[\frac{2K}{m^2-\theta^2} \psi(\sqrt{D}m) - \frac{2L}{(m+\sigma)^2-\theta^2} \psi(\sqrt{D}(m+\sigma)) \right] \\
&\qquad - \frac{e^{-2b\theta}\psi(-\theta\sqrt{D})}{\theta\psi(\theta\sqrt{D})} K e(m+\theta)k \left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma} \right).
\end{aligned} \tag{52}$$

Finally, for the case $x \leq K \leq L$ we get

$$\begin{aligned}
&*\widehat{PUOC}(x, \lambda; K, L; r, \delta) = \\
&\qquad \frac{K}{\theta} e^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) \\
&\qquad - \frac{e^{(m-\theta)b}}{\psi(\theta\sqrt{D})} \left[\frac{2K}{m^2-\theta^2} \psi(\sqrt{D}m) - \frac{2L}{(m+\sigma)^2-\theta^2} \psi(\sqrt{D}(m+\sigma)) \right] &(53) \\
&\qquad - \frac{e^{-2b\theta}\psi(-\theta\sqrt{D})}{\theta\psi(\theta\sqrt{D})} K e(m+\theta)k \left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma} \right). &(54)
\end{aligned}$$

4.3. **The valuation of a Parisian Up-and-out call with $b \leq 0$.** We proceed exactly the same way as for the case $b \geq 0$.
We have

$$*PUOC(x, T; K, L; r, \delta) = Le^{mb} \int_0^D *PUOC^0(1, T-u; K/L, 1; r, \delta) \mu_b(du),$$

and for its Laplace transform we get

$$*\widehat{PUOC}(x, T; K, L; r, \delta) = Le^{mb} \int_0^D \mu_b(du) e^{-\lambda u} *\widehat{PUOC}^0(1, \lambda; K/L, 1; r, \delta).$$

To compute $\int_0^D \mu_b(du) e^{-\lambda u}$, we can refer to *Appendix A*, but by plugging $-b$ instead of b. So we find

$$\int_0^D \mu_b(du) e^{-\lambda u} = e^{\theta b} \mathcal{N}(\theta\sqrt{D} + \frac{b}{\sqrt{D}}) + e^{-\theta b} \mathcal{N}(-\theta\sqrt{D} + \frac{b}{\sqrt{D}}).$$

Therefore, for $L \leq x$ we get

$$\boldsymbol{*\widehat{PUOC}(x,T;K,L;r,\delta) = L\left(e^{(m+\theta)b}\mathcal{N}(\theta\sqrt{D}+\frac{b}{\sqrt{D}})\right.}$$
$$\boldsymbol{\left.+e^{(m-\theta)b}\mathcal{N}(-\theta\sqrt{D}+\frac{b}{\sqrt{D}})\right) * \widehat{PUOC}^{0}(1,\lambda;\frac{K}{L},1;r,\delta).}$$

Depending on the relative position of K and L, one of the following formula for $*\widehat{PUOC}(x,T;K,L;r,\delta)$ holds.

▶ Case $K \geq L$.

$$\begin{aligned}
*\widehat{PUOC}^{0}(1,\lambda;\frac{K}{L},1;r,\delta) &= \frac{K}{L\theta}e^{\frac{m-\theta}{\sigma}\ln(\frac{K}{L})}\left(\frac{1}{m-\theta}-\frac{1}{m+\sigma-\theta}\right)\\
&-\frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})}\left[\frac{2K}{L(m^2-\theta^2)}e^{\frac{Dm^2}{2}}m\mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L})+\sqrt{D}m\right)\right.\\
&\left.-\frac{2}{(m+\sigma)^2-\theta^2}e^{\frac{D(m+\sigma)^2}{2}}(m+\sigma)\mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L})+\sqrt{D}(m+\sigma)\right)\right]\\
&-\frac{1}{\psi(\theta\sqrt{D})}\frac{K}{L}e^{\frac{m+\theta}{\sigma}\ln(\frac{K}{L})}e^{\lambda D}\sqrt{2\pi D}\mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L})-\theta\sqrt{D}\right)\left(\frac{1}{m+\sigma+\theta}-\frac{1}{m+\theta}\right)\\
&-\frac{e^{\frac{m-\theta}{\sigma}\ln(\frac{K}{L})}}{\theta\psi(\theta\sqrt{D})}\frac{K}{L}\left(\frac{1}{m-\theta}-\frac{1}{m+\sigma-\theta}\right)\left(\psi(-\theta\sqrt{D})+\theta\sqrt{2\pi D}e^{\lambda D}\mathcal{N}\left(\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L})-\theta\sqrt{D}\right)\right).
\end{aligned}$$

$$\boldsymbol{\begin{aligned}
*\widehat{PUOC}(x,T;K,L;r,\delta) &= L\left(e^{(m+\theta)b}\mathcal{N}\left(\theta\sqrt{D}+\frac{b}{\sqrt{D}}\right)+e^{(m-\theta)b}\mathcal{N}(-\theta\sqrt{D}+\frac{b}{\sqrt{D}})\right)\\
&\left\{\frac{K}{L\theta}e^{\frac{m-\theta}{\sigma}\ln(\frac{K}{L})}\left(\frac{1}{m-\theta}-\frac{1}{m+\sigma-\theta}\right)-\frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})}\left[\frac{2K}{L(m^2-\theta^2)}e^{\frac{Dm^2}{2}}m\right.\right.\\
&\mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L})+\sqrt{D}m\right)-\frac{2}{(m+\sigma)^2-\theta^2}e^{\frac{D(m+\sigma)^2}{2}}(m+\sigma)\\
&\left.\mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L})+\sqrt{D}(m+\sigma)\right)\right]-\frac{1}{\psi(\theta\sqrt{D})}\frac{K}{L}e^{\frac{m+\theta}{\sigma}\ln(\frac{K}{L})}e^{\lambda D}\sqrt{2\pi D}\\
&\mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L})-\theta\sqrt{D}\right)\left(\frac{1}{m+\sigma+\theta}-\frac{1}{m+\theta}\right)-\frac{e^{\frac{m-\theta}{\sigma}\ln(\frac{K}{L})}}{\theta\psi(\theta\sqrt{D})}\frac{K}{L}\\
&\left.\left(\frac{1}{m-\theta}-\frac{1}{m+\sigma-\theta}\right)\left(\psi(-\theta\sqrt{D})+\theta\sqrt{2\pi D}e^{\lambda D}\mathcal{N}\left(\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L})-\theta\sqrt{D}\right)\right)\right\}
\end{aligned}}$$

for $L \leq K$ and $L \leq x$.

▶ Case $K \leq L$.

$$\begin{aligned}
&*\widehat{PUOC}^{0}(1,\lambda;\frac{K}{L},1;r,\delta) =\\
&\quad\frac{2K}{L(m^2-\theta^2)}-\frac{2}{(m+\sigma)^2-\theta^2}+\frac{K}{L\theta}e^{\frac{m+\theta}{\sigma}\ln(\frac{K}{L})}\left(\frac{1}{m+\theta}-\frac{1}{m+\theta+\sigma}\right)\\
&\quad-\left[\frac{1}{\psi(\theta\sqrt{D})}\left(\frac{2K}{L(m^2-\theta^2)}\psi(\sqrt{D}m)-\frac{2}{(m+\sigma)^2-\theta^2}\psi(\sqrt{D}(m+\sigma))\right)\right]\\
&\quad-\frac{\psi(-\theta\sqrt{D})}{\theta\psi(\theta\sqrt{D})}\frac{K}{L}e^{\frac{m+\theta}{\sigma}\ln(\frac{K}{L})}\left(\frac{1}{m+\theta}-\frac{1}{m+\sigma+\theta}\right).
\end{aligned}$$

$$\begin{aligned}
&*\widehat{PUOC}(x,T;K,L;r,\delta) = \\
&\quad L\left(e^{(m+\theta)b}\mathcal{N}(\theta\sqrt{D}+\frac{b}{\sqrt{D}})+e^{(m-\theta)b}\mathcal{N}(-\theta\sqrt{D}+\frac{b}{\sqrt{D}})\right) \\
&\quad \left\{\frac{2K}{L(m^2-\theta^2)}-\frac{2}{(m+\sigma)^2-\theta^2}+\frac{K}{L\theta}e^{\frac{m+\theta}{\sigma}\ln(\frac{K}{L})}\left(\frac{1}{m+\theta}-\frac{1}{m+\theta+\sigma}\right)\right. \\
&\quad -\left[\frac{1}{\psi(\theta\sqrt{D})}\left(\frac{2K}{L(m^2-\theta^2)}\psi(\sqrt{D}m)-\frac{2}{(m+\sigma)^2-\theta^2}\psi(\sqrt{D}(m+\sigma))\right)\right] \\
&\quad \left.-\frac{\psi(-\theta\sqrt{D})}{\theta\psi(\theta\sqrt{D})}\frac{K}{L}e^{\frac{m+\theta}{\sigma}\ln(\frac{K}{L})}\left(\frac{1}{m+\theta}-\frac{1}{m+\sigma+\theta}\right)\right\}, \text{ for } K\le L\le x.
\end{aligned}$$

4.4. **The valuation of a Parisian Up-and-in call with** $b \le 0$**.** We will also use the relations between $*\widehat{PUIC}(x,\lambda;K,L;r,\delta)$ and $*\widehat{PUOC}(x,\lambda;K,L;r,\delta)$. We have

$$*\widehat{PUIC}(x,\lambda;K,L;r,\delta) = *\widehat{BSC}(x,\lambda,K,r,\delta) - *\widehat{PUOC}(x,\lambda;K,L;r,\delta)$$

where $*\widehat{PUOC}(x,\lambda;K,L;r,\delta)$ has already been computed above in *Section 4.3* for $b\le 0$ and $*\widehat{BSC}(x,\lambda,K,r,\delta)$ has been calculated in *Section 3.2.1*.
So we derive the three following formulae

$$\begin{aligned}
&*\widehat{PUIC}(x,\lambda;K,L;r,\delta) = \frac{K}{\theta}e^{(m-\theta)k}\left(\frac{1}{m-\theta}-\frac{1}{m+\sigma-\theta}\right) \\
&\quad -L\left(e^{(m+\theta)b}\mathcal{N}\left(\theta\sqrt{D}+\frac{b}{\sqrt{D}}\right)+e^{(m-\theta)b}\mathcal{N}\left(-\theta\sqrt{D}+\frac{b}{\sqrt{D}}\right)\right) \\
&\qquad\qquad \left(\frac{K}{L\theta}e^{\frac{m-\theta}{\sigma}\ln(\frac{K}{L})}\left(\frac{1}{m-\theta}-\frac{1}{m+\sigma-\theta}\right)\right. \\
&\quad -\frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})}\left[\frac{2K}{L(m^2-\theta^2)}e^{\frac{Dm^2}{2}}m\mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L})+\sqrt{D}m\right)\right. \\
&\quad \left.-\frac{2}{(m+\sigma)^2-\theta^2}e^{\frac{D(m+\sigma)^2}{2}}(m+\sigma)\mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L})+\sqrt{D}(m+\sigma)\right)\right] \\
&\quad -\frac{1}{\psi(\theta\sqrt{D})}\frac{K}{L}e^{\frac{m+\theta}{\sigma}\ln(\frac{K}{L})}e^{\lambda D}\sqrt{2\pi D}\mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L})-\theta\sqrt{D}\right)\left(\frac{1}{m+\sigma+\theta}-\frac{1}{m+\theta}\right) \\
&\quad -\frac{e^{\frac{m-\theta}{\sigma}\ln(\frac{K}{L})}}{\theta\psi(\theta\sqrt{D})}\frac{K}{L}\left(\frac{1}{m-\theta}-\frac{1}{m+\sigma-\theta}\right) \\
&\qquad\qquad \left.\left(\psi(-\theta\sqrt{D})+\theta\sqrt{2\pi D}e^{\lambda D}\mathcal{N}\left(\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L})-\theta\sqrt{D}\right)\right)\right), \text{ for } L\le x\le K,
\end{aligned}$$

$$
\begin{aligned}
&*\widehat{\boldsymbol{PUIC}}(\boldsymbol{x},\boldsymbol{\lambda};\boldsymbol{K},\boldsymbol{L};\boldsymbol{r},\boldsymbol{\delta}) = \\
&\qquad \frac{2K}{m^2-\theta^2} - \frac{2x}{(m+\sigma)^2-\theta^2} + \frac{K}{\theta} e^{(m+\theta)k}\left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma}\right) \\
&\qquad - L\left(e^{(m+\theta)b}\mathcal{N}(\theta\sqrt{D} + \frac{b}{\sqrt{D}}) + e^{(m-\theta)b}\mathcal{N}(-\theta\sqrt{D} + \frac{b}{\sqrt{D}})\right) \\
&\qquad \left\{\frac{K}{L\theta} e^{\frac{m-\theta}{\sigma}\ln(\frac{K}{L})}\left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta}\right)\right. \\
&\qquad - \frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})}\left[\frac{2K}{L(m^2-\theta^2)} e^{\frac{Dm^2}{2}} m\mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L}) + \sqrt{D}m\right)\right. \\
&\qquad \left. - \frac{2}{(m+\sigma)^2-\theta^2} e^{\frac{D(m+\sigma)^2}{2}}(m+\sigma)\mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L}) + \sqrt{D}(m+\sigma)\right)\right] \\
&\qquad - \frac{1}{\psi(\theta\sqrt{D})}\frac{K}{L} e^{\frac{m+\theta}{\sigma}\ln(\frac{K}{L})} e^{\lambda D}\sqrt{2\pi D}\mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}}\ln(\frac{K}{L}) - \theta\sqrt{D}\right)\left(\frac{1}{m+\sigma+\theta} - \frac{1}{m+\theta}\right) \\
&\qquad - \frac{e^{\frac{m-\theta}{\sigma}\ln(\frac{K}{L})}}{\theta\psi(\theta\sqrt{D})}\frac{K}{L}\left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta}\right) \\
&\qquad \left.\left(\psi(-\theta\sqrt{D}) + \theta\sqrt{2\pi D} e^{\lambda D}\mathcal{N}\left(\frac{1}{\sigma\sqrt{D}}\ln\left(\frac{K}{L}\right) - \theta\sqrt{D}\right)\right)\right\}, \text{ for } L \le K \le x,
\end{aligned}
$$

$$
\begin{aligned}
&*\widehat{\boldsymbol{PUIC}}(\boldsymbol{x},\boldsymbol{\lambda};\boldsymbol{K},\boldsymbol{L};\boldsymbol{r},\boldsymbol{\delta}) = \frac{2K}{m^2-\theta^2} - \frac{2x}{(m+\sigma)^2-\theta^2} \\
&\qquad + \frac{K}{\theta} e^{(m+\theta)k}\left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma}\right) - L\left(e^{(m+\theta)b}\mathcal{N}\left(\theta\sqrt{D} + \frac{b}{\sqrt{D}}\right)\right. \\
&\qquad \left. + e^{(m-\theta)b}\mathcal{N}\left(-\theta\sqrt{D} + \frac{b}{\sqrt{D}}\right)\right)\left\{\frac{2K}{L(m^2-\theta^2)} - \frac{2}{(m+\sigma)^2-\theta^2} + \right. \\
&\qquad \frac{K}{L\theta} e^{\frac{m+\theta}{\sigma}\ln(\frac{K}{L})}\left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma}\right) \\
&\qquad - \left[\frac{1}{\psi(\theta\sqrt{D})}\left(\frac{2K}{L(m^2-\theta^2)}\psi(\sqrt{D}m) - \frac{2}{(m+\sigma)^2-\theta^2}\psi(\sqrt{D}(m+\sigma))\right)\right] \\
&\qquad \left. - \frac{\psi(-\theta\sqrt{D})}{\theta\psi(\theta\sqrt{D})}\frac{K}{L} e^{\frac{m+\theta}{\sigma}\ln(\frac{K}{L})}\left(\frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta}\right)\right\}, \text{ for } K \le L \le x.
\end{aligned}
$$

5. Some parity relationships

Now we will explain how to find all the other prices by simply using the formulae we have established so far and some parity relationships.

Let us consider a Parisian Down and Out Put.

$$
PDOP(x,T;K,L,D,r,\delta) = \mathbb{E}\left(e^{mZ_T}(K - xe^{\sigma Z_T})^+ \mathbf{1}_{\{T_b^- > T\}}\right) e^{-\left(r+\frac{m^2}{2}\right)T}. \tag{55}
$$

One notices that the first time the Z Brownian motion makes below b an excursion longer than D is the same as the first time Brownian motion $-Z$ makes above $-b$ an excursion longer than D. Therefore, introducing the new Brownian motion $W = -Z$ we can rewrite

$$
\begin{aligned}
PDOP(x,T;K,L,D,r,\delta) &= \mathbb{E}\left(e^{-mW_T}(K - xe^{-\sigma W_T})^+ \mathbf{1}_{\{T_{-b}^+ > T\}}\right) e^{-\left(r+\frac{m^2}{2}\right)T}, \\
&= Kx\,\mathbb{E}\left(e^{-(m+\sigma)W_T}\left(\frac{1}{x}e^{\sigma W_T} - \frac{1}{K}\right)^+ \mathbf{1}_{\{T_{-b}^+ > T\}}\right) e^{-\left(r+\frac{m^2}{2}\right)T}. \qquad (56)
\end{aligned}
$$

Let us introduce $m' = -(m+\sigma)$, $\delta' = r$, $r' = \delta$ and $b' = -b$. With these relations we easily check that $m' = \frac{1}{\sigma}\left(r' - \delta' - \frac{\sigma^2}{2}\right)$ and that $r' + \frac{m'^2}{2} = r + \frac{m^2}{2}$. Moreover, we notice that the barrier L' corresponding to $b' = -b$ is $\dfrac{1}{L}$. Therefore, $\mathbb{E}\left(e^{-(m+\sigma)W_T}\left(\dfrac{1}{x}e^{\sigma W_T} - \dfrac{1}{K}\right)^+ \mathbf{1}_{\{T^+_{-b} > T\}}\right) e^{-\left(r+\frac{m^2}{2}\right)T}$ is in fact the price of a Up and Out Call $PUOC\left(\dfrac{1}{x}, T; \dfrac{1}{K}, \dfrac{1}{L}, D, \delta, r\right)$. Finally, we come up with the following relation

$$PDOP(x,T;K,L,D,r,\delta) = xK\ PUOC\left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r\right).$$

The same relation holds if we replace a call by a put and vice-versa and if we consider In options instead of Out ones.

$$\begin{aligned}
PUOP(x,T;K,L,D,r,\delta) &= xK\ PDOC\left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r\right),\\
PUIP(x,T;K,L,D,r,\delta) &= xK\ PDIC\left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r\right),\\
PDIP(x,T;K,L,D,r,\delta) &= xK\ PUIC\left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r\right).
\end{aligned}$$

In the previous sections we computed the price of all the Down Calls an Up Calls. From these relationships, we can deduce the prices of all the Parisian Puts. What we still have to find is how to invert the Laplace transform.

6. Prices at any time t

At this stage we can compute all the prices at time 0, but to be able to hedge such an option we besides need the prices at some time $t \le T$. So we will consider a Down-and-In option to show how the price at some time t can be deduced from the prices at time 0 of the Down-and-In options with different parameters. Relying on this example one can easily prove similar formulae for other options.

6.1. **Three different paths for the Brownian motion.** The price of a Parisian Down and In Call at time 0 is given by the formula (4). From this formula, we can deduce the price of a Down and In call at any time t.

$$PDIC(S_t, t; x, T; K, L, D, r, \delta) = e^{-r(T-t)}\mathbb{E}_{\mathcal{Q}}\left((xe^{\sigma(W_T+mT)} - K)^+ \mathbf{1}_{\{T_b^- \le T\}} | \mathcal{F}_t\right). \tag{57}$$

Now we can change the probability measure as we did at the beginning to make $Z = \{W_t + mt; t \ge 0\}$ a Brownian motion under the new probability we called $\mathcal{P}$, ($\mathbb{E}$ will from now on denote the expectation under the probability $\mathcal{P}$) . Then, we can write

$$\begin{aligned}
PDIC(S_t, t; x, T; K, L, D, r, \delta) &= e^{-r(T-t)}\frac{\mathbb{E}\left(e^{mZ_T - \frac{1}{2}m^2T}(xe^{\sigma Z_T} - K)^+ \mathbf{1}_{\{T_b^- \le T\}} | \mathcal{F}_t\right)}{e^{mZ_t - \frac{1}{2}m^2 t}},\\
&= e^{-r(T-t)}\frac{\mathbb{E}\left(e^{mZ_t}e^{m(Z_T-Z_t) - \frac{1}{2}m^2T}(xe^{\sigma Z_T} - K)^+ \mathbf{1}_{\{T_b^- \le T\}} | \mathcal{F}_t\right)}{e^{mZ_t - \frac{1}{2}m^2 t}},\\
&= e^{-(r+\frac{m^2}{2})(T-t)}\mathbb{E}\left(e^{m(Z_T-Z_t)}(xe^{\sigma Z_T} - K)^+ \mathbf{1}_{\{T_b^- \le T\}} | \mathcal{F}_t\right).
\end{aligned} \tag{58}$$

Let us introduce a few notations

$$T' = T - t \text{ and } b' = \frac{1}{\sigma}\ln\left(\frac{L}{S_t}\right), \tag{59}$$

$$T'_{b'} = \inf\{s > 0; Z_{t+s} - Z_t = b'\}. \tag{60}$$

In the case $Z_t < b$, we introduce D' the time Z has already spent in the excursion.

$$PDIC(S_t, t; x, T; K, L, D, r, \delta) = e^{-\left(r+\frac{m^2}{2}\right)T'}\mathbb{E}\left(e^{m(Z_T-Z_t)}(S_t e^{\sigma(Z_T-Z_t)} - K)^+ \mathbf{1}_{\{T_b^- \le T\}} | \mathcal{F}_t\right). \tag{61}$$

The indicator can be split up in several parts depending on which path you are on. On both paths the excursion has already started. On the red one, the excursion will not last long enough, so the asset still has to do an

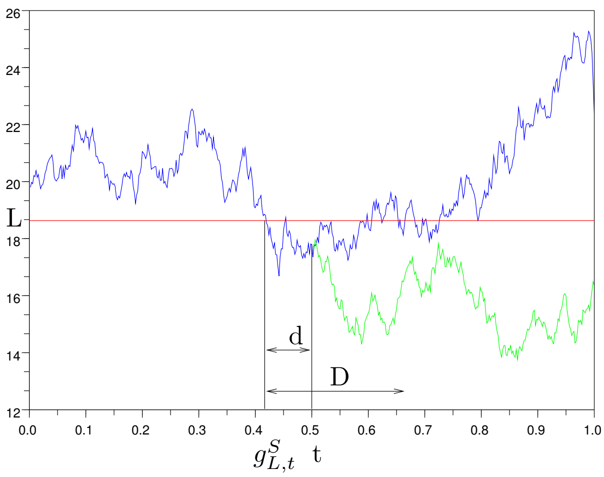

FIGURE 4. Possible evolutions of an asset price

entirely new excursion below L longer than D, whereas on the green path the process only has to remain below L for a time longer than $D-d$. All these remarks enable us to rewrite the indicator as follows

$$\mathbf{1}_{\{T_b^- \le T\}} = \mathbf{1}_{\{Z_t>b\}}\mathbf{1}_{\{T_{b'}^{'-} \le T'\}} + \mathbf{1}_{\{Z_t \le b\}} \left(\mathbf{1}_{\{T'_{b'} \ge D-D'\}} \mathbf{1}_{\{D-D' \le T'\}} + \mathbf{1}_{\{T'_{b'} < D-D\}} \mathbf{1}_{\{T_{b'}^{'-} \le T'\}} \right). \tag{62}$$

$$\begin{aligned}
&PDIC(S_t,t;x,T;K,L,D,r,\delta) \\
&= e^{-\left(r+\frac{m^2}{2}\right)T'} \left\{ \mathbb{E}\left(e^{m(Z_T-Z_t)} (S_t e^{\sigma(Z_T-Z_t)} - K)^+ \, \mathbf{1}_{\{Z_t>b\}} \mathbf{1}_{\{T_{b'}^{'-} \le T'\}} | \mathcal{F}_t \right), \right. \\
&\quad + \mathbb{E}\left(e^{m(Z_T-Z_t)} (S_t e^{\sigma(Z_T-Z_t)} - K)^+ \, \mathbf{1}_{\{Z_t \le b\}} \mathbf{1}_{\{T'_{b'} \ge D-d\}} \mathbf{1}_{\{D-d \le T-t\}} | \mathcal{F}_t \right), \\
&\quad \left. + \, \mathbb{E}\left(e^{m(Z_T-Z_t)} (S_t e^{\sigma(Z_T-Z_t)} - K)^+ \, \mathbf{1}_{\{Z_t \le b\}} \mathbf{1}_{\{T'_{b'} \le D-D'\}} \mathbf{1}_{\{T_{b'}^{'-} \le T-t\}} | \mathcal{F}_t \right) \right\}, \\
&= e^{-\left(r+\frac{m^2}{2}\right)T'} \left\{ \mathbf{1}_{\{Z_t>b\}} \mathbb{E}\left(e^{m(Z_T-Z_t)} (S_t e^{\sigma(Z_T-Z_t)} - K)^+ \, \mathbf{1}_{\{T_{b'}^{'-} \le T'\}} | \mathcal{F}_t \right), \right. \\
&\quad + \mathbf{1}_{\{Z_t \le b\}} \mathbf{1}_{\{D-D' \le T-t\}} \mathbb{E}\left(e^{m(Z_T-Z_t)} (S_t e^{\sigma(Z_T-Z_t)} - K)^+ \, \mathbf{1}_{\{T'_{b'} \ge D-d\}} | \mathcal{F}_t \right), \\
&\quad \left. + \, \mathbf{1}_{\{Z_t \le b\}} \mathbb{E}\left(e^{m(Z_T-Z_t)} (S_t e^{\sigma(Z_T-Z_t)} - K)^+ \, \mathbf{1}_{\{T'_{b'} \le D-D'\}} \mathbf{1}_{\{T_{b'}^{'-} \le T-t\}} | \mathcal{F}_t \right) \right\}.
\end{aligned}$$

$T'_{b'}$ and $T_{b'}^{'-}$ are both independent of $\mathcal{F}_t$, so we can write

$$\begin{aligned}
PDIC(S_t,t;x,T;K,L,D,r,\delta) &= e^{-\left(r+\frac{m^2}{2}\right)T'}\Big\{\mathbf{1}_{\{Z_t>b\}}\mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\mathbf{1}_{\{T_{b'}^-\le T'\}}\right) \\
&\quad +\mathbf{1}_{\{Z_t\le b\}}\mathbf{1}_{\{D-D'\le T'\}}\mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\mathbf{1}_{\{T_{b'}^-\ge D-D'\}}\right) \\
&\quad +\mathbf{1}_{\{Z_t\le b\}}\mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\mathbf{1}_{\{T_{b'}\le D-D'\}}\mathbf{1}_{\{T_{b'}^-\le T'\}}\right)\Big\}, \\
&= e^{-\left(r+\frac{m^2}{2}\right)T'}\Bigg\{\mathbf{1}_{\{Z_t>b\}}PDIC(S_t,T';K,L;r,\delta) \\
&\quad +\mathbf{1}_{\{Z_t\le b\}}\mathbf{1}_{\{D-D'\le T'\}}\underbrace{\mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\mathbf{1}_{\{T_{b'}\ge D-D'\}}\right)}_{(i)} \\
&\quad +\mathbf{1}_{\{Z_t\le b\}}\underbrace{\mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\mathbf{1}_{\{T_{b'}\le D-D'\}}\mathbf{1}_{\{T_{b'}^-\le T'\}}\right)}_{(ii)}\Bigg\}.
\end{aligned}$$

6.2. **The computation of the different expectations.** Let us calculate (i) in the case $D-D'\le T'$

$$\begin{aligned}
&\mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\mathbf{1}_{\{T_{b'}\ge D-D'\}}\right) \\
&\quad= \mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\right)-\mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\mathbf{1}_{\{T_{b'}\le D-D'\}}\right) \\
&\quad= *BSC(S_t,T';K,r,\delta)-\mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\mathbf{1}_{\{T_{b'}\le D-D'\}}\right)
\end{aligned} \tag{63}$$

The last expectation above can be computed by conditioning with respect to $\mathcal{F}_{T_{b'}}$ since $D-D'\le T'$.

$$\begin{aligned}
&\mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\mathbf{1}_{\{T_{b'}\le D-D'\}}\right) \\
&\quad= \mathbb{E}\left(\mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\mathbf{1}_{\{T_{b'}\le D-D'\}}|\mathcal{F}_{T_{b'}}\right)\right), \\
&\quad= \mathbb{E}\left(\mathbf{1}_{\{T_{b'}\le D-D'\}}\mathbb{E}\left(e^{m(Z_{T'}-Z_{T_{b'}})}e^{mb'}(S_te^{\sigma b'}e^{\sigma(Z_{T'}-Z_{T'_{b'}})}-K)^+|\mathcal{F}_{T_{b'}}\right)\right).
\end{aligned}$$

If $W_t=Z_{t+T'_b}-Z_{T_b-'}$ and Y_t denotes $e^{mW_{T'-t}}(Le^{\sigma W_{T'-t}}-K)^+$, then Y_t is independent of $\mathcal{F}_{T_{b'}}$ and $T_{b'}$ is $\mathcal{F}_{T_{b'}}$-measurable

$$\begin{aligned}
&\mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\mathbf{1}_{\{T_{b'}\le D-D'\}}\right) \\
&\quad= \mathbb{E}\left(\mathbf{1}_{\{T_{b'}\le D-D'\}}\mathbb{E}\left(e^{mW_{T'-\tau}}e^{mb'}(S_te^{\sigma b'}e^{\sigma W_{T'-\tau}}-K)^+\right)|_{\tau=T_{b'}}\right), \\
&\quad= \underbrace{\int_0^{D-D'}e^{mb'}\mathbb{E}(e^{mW_{T'-u}}(Le^{\sigma W_{T'-u}}-K)^+)\mu_{b'}(u)du}_{P(L,T')}.
\end{aligned} \tag{64}$$

Now, we will consider the Laplace transform of $P(L,T')$ with respect to T'

$$
\begin{aligned}
\widehat{P}(L,\lambda) &= \int_0^{+\infty} e^{-\lambda\tau}\int_0^{D-D'} e^{mb'}\mathbb{E}(e^{mW_{\tau-u}}(Le^{\sigma W_{\tau-u}}-K)^+)\mu_{b'}(u)du\,d\tau,\\
&= \int_0^{D-D'}\int_0^{+\infty} e^{-\lambda\tau}e^{mb'}\mathbb{E}(e^{mW_{\tau-u}}(Le^{\sigma W_{\tau-u}}-K)^+)d\tau\,\mu_{b'}(u)du,\\
&\quad \text{a change of variables } (u,\xi)=(u,\tau-u) \text{ gives}\\
&= \int_0^{D-D'}\int_0^{+\infty} e^{-\lambda u}e^{-\lambda\xi}e^{mb'}\mathbb{E}(e^{mW_\xi}(Le^{\sigma W_\xi}-K)^+)\mu_{b'}(u)du\,d\xi,\\
&\quad \text{relying on } \textit{Appendix A} \text{ we can write}\\
&= e^{mb'}\left\{e^{-\theta|b'|}\mathcal{N}\left(\theta\sqrt{D-D'}-\frac{|b'|}{\sqrt{D-D'}}\right)\right.\\
&\quad \left.+e^{\theta|b'|}\mathcal{N}\left(-\theta\sqrt{D-D'}-\frac{|b'|}{\sqrt{D-D'}}\right)\right\} * \widehat{BSC}(L,\lambda;K,r,\delta). \qquad (65)
\end{aligned}
$$

Let us now compute (ii). We can condition with respect to $\mathcal{F}_{T_{b'}}$ since $T_{b'}^-$ is bound to be bigger than $D-D'$ so $T_{b'}$ is almost surely smaller than T'

$$
\begin{aligned}
&\mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\,\mathbf{1}_{\{T_{b'}\le D-D'\}}\mathbf{1}_{\{T_{b'}^-\le T'\}}\right)\\
&\quad= \mathbb{E}\left(\mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\,\mathbf{1}_{\{T_{b'}\le D-D'\}}\mathbf{1}_{\{T_{b'}^-\le T'\}}|\mathcal{F}_{T_{b'}}\right)\right),\\
&\quad= e^{mb'}\mathbb{E}\left(\mathbf{1}_{\{T_{b'}\le D-D'\}}\mathbb{E}\left(e^{m(Z_{T'}-Z_{T'_{b'}})}(Le^{\sigma(Z_{T'}-Z_{T_{b'}})}-K)^+\,\mathbf{1}_{\{T_0^-\le T'-T_{b'}\}}|\mathcal{F}_{T_{b'}}\right)\right). \qquad (66)
\end{aligned}
$$

If $W_t = Z_{t+T'_b}-Z_{T'_b}$ and Y_t denotes $e^{mW_{T'-t}}(Le^{\sigma W_{T'-t}}-K)^+\,\mathbf{1}_{\{T_0^-\le T'-t\}}$, Y_t is independent of $\mathcal{F}_{T_{b'}}$ and $T_{b'}$ is $\mathcal{F}_{T_{b'}}$-measurable. So $\mathbb{E}(Y_t|\mathcal{F}_{T_{b'}})=\mathbb{E}(Y_t)|_{t=T_{b'}}$ and therefore we can write

$$
\begin{aligned}
&\mathbb{E}\left(e^{mZ_{T'}}(S_te^{\sigma Z_{T'}}-K)^+\,\mathbf{1}_{\{T_{b'}\le D-D'\}}\mathbf{1}_{\{T_{b'}^-\le T'\}}\right)\\
&\quad= e^{mb'}\mathbb{E}\left(\mathbf{1}_{\{T_{b'}\le D-D'\}}\mathbb{E}\left(e^{mW_{T'-u}}(Le^{\sigma W_{T'-u}}-K)^+\,\mathbf{1}_{\{T_0^-\le T'-u\}}\right)|_{u=T_{b'}}\right),\\
&\quad= \underbrace{\int_0^{D-D'} e^{mb'}\mathbb{E}\left(e^{mW_{T'-u}}(Le^{\sigma W_{T'-u}}-K)^+\,\mathbf{1}_{\{T_0^-\le T'-u\}}\right)\mu_{b'}(u)du}_{Q(L,T')}. \qquad (67)
\end{aligned}
$$

Let us consider the Laplace transform of $Q(L,T')$ with respect to T'.

$$
\begin{aligned}
\widehat{Q}(L,\lambda) &= \int_0^{+\infty} e^{-\lambda\tau}\int_0^{D-D'} e^{mb'}\mathbb{E}\left(e^{mW_{\tau-u}}(Le^{\sigma W_{\tau-u}}-K)^+\,\mathbf{1}_{\{T_0^-\le\tau-u\}}\right)\mu_{b'}(u)du\,d\tau,\\
&= Le^{mb'}\int_0^{+\infty} e^{-\lambda\tau}\int_0^{D-D'} *PDIC^0(1,\tau-u,\frac{K}{L},1,D,r,\delta)d\tau\,\mu_{b'}(u)du,\\
&= Le^{mb'}\int_0^{D-D'}\mu_{b'}e^{-\lambda u}du*\widehat{PDIC}^0(1,\lambda,\frac{K}{L},1,D,r,\delta). \qquad (68)
\end{aligned}
$$

Finally we obtain

$$
\begin{aligned}
&\boldsymbol{*\widehat{PDIC}(S_t,t;x,T;K,L,D,r,\delta)}\\
&\quad \boldsymbol{= \mathbf{1}_{\{Z_t>b\}}*\widehat{PDIC}(S_t,T',K,L,D,r,\delta)+\mathbf{1}_{\{Z_t\le b\}}\mathbf{1}_{\{D-D'\le T'\}}}\\
&\quad \boldsymbol{\left(Le^{mb'}\int_0^{D-D'}\mu_{b'}e^{-\lambda u}du\left(*\widehat{PDIC}^0(1,\lambda,\frac{K}{L},1,D,r,\delta)\right.\right.}\\
&\quad \boldsymbol{\left.\left.-*\widehat{BSC}(1,\lambda,\frac{K}{L},r,\delta)\right)+*\widehat{BSC}(S_t,T',K,r,\delta)\right).} \qquad (69)
\end{aligned}
$$

If we compute $*\widehat{PUIC}(S_t, t; x, T; K, L, D, r, \delta)$ we get exactly the same result by changing $*\widehat{PDIC}^0(1, \lambda, \frac{K}{L}, 1, D, r, \delta)$ into $*\widehat{PUIC}^0(1, \lambda, \frac{K}{L}, 1, D, r, \delta)$ in the previous formula.

If one wants to value the Put Options, one can rely on the parity relationships given in the previous section and then use again the price of the Calls at time t.

7. The inverse Laplace transform

This part is devoted to explaining how we compute the inverse Laplace transform of a function f, and how we can use the Euler summation to get an accurate approximation of f.

7.1. **An analytic formula for the inverse.** Let us consider a function f integrable over $\mathbb{R}^+$, assume that $f(x) = 0$ if $x < 0$. We will need a few notations

$$\mathcal{L}(f)(z) = \widehat{f}(z) = \int_0^{+\infty} f(t)e^{-zt}dt \text{ for } z \in \mathbb{C} \text{ with } \mathcal{R}e(z) > 0, \tag{70}$$

$$\mathcal{F}(f)(\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)e^{-i\xi t}dt \text{ for } \xi \in \mathbb{R}. \tag{71}$$

One straightaway notices that since $f(x) = 0$ if $x < 0$ the following equality holds

$$\mathcal{L}(f)(\sigma + i\xi) = 2\pi\mathcal{F}(f(\cdot)e^{-\sigma\cdot})(\xi) \text{ for } \sigma > 0. \tag{72}$$

In our case f denotes an option price so it is bounded, therefore $f(t)e^{-\sigma t}$ is square integrable for all positive σ. Since the Fourier Transform is one-to-one on the set of square integrable functions, we have

$$f(t)e^{-\sigma t} = \mathcal{F}^{-1}\left(\frac{1}{2\pi}\mathcal{L}(f)(\sigma + i\cdot)\right)(t)$$

So, using the inverse of operator $\mathcal{F}$, we obtain

$$\begin{aligned} f(t) &= e^{\sigma t}\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{L}(f)(\sigma + i\xi)e^{i\xi t}d\xi, \\ &\quad \text{let us change variable } u = \sigma + i\xi \qquad (73) \\ &= \frac{1}{2i\pi}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{ut}\mathcal{L}(f)(u)du. \qquad (74) \end{aligned}$$

$$\begin{aligned} f(t) &= \frac{1}{2i\pi}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{ut}\widehat{f}(u)du, \\ &= \frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{+\infty} \widehat{f}(\sigma + iu)(\cos(ut) + i\sin(ut))du, \\ &\quad \text{f has real values so only the real part of the integral is worth taking into account} \\ &= \frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{+\infty} \mathcal{R}e(\widehat{f}(\sigma + iu)\cos(ut)) - \mathcal{I}m(\widehat{f}(\sigma + iu)\sin(ut)) \qquad (75) \end{aligned}$$

Moreover, we notice that

$$\begin{aligned}
\int_{-\infty}^{+\infty} \mathcal{I}m(\widehat{f}(\sigma+iu)\sin(ut))du &= \int_{-\infty}^{+\infty}\int_{0}^{+\infty} f(z)e^{\sigma z}\sin(uz)\sin(ut)dzdu, \\
&= \int_{-\infty}^{+\infty}\int_{0}^{+\infty} f(z)e^{\sigma z}\frac{\cos(u(z-t))-\cos(u(z+t))}{2}dzdu, \\
&\quad \text{this function is even with respect to } u\text{, so we can write} \\
&= 2\int_{0}^{+\infty} \mathcal{R}e\left(\int_{0}^{+\infty} f(z)e^{\sigma z}e^{iuz}\frac{e^{iut}+e^{-iut}}{2}dz\right)du, \\
&= 2\int_{0}^{+\infty} \mathcal{R}e\left(\int_{0}^{+\infty} f(z)e^{\sigma z}e^{iuz}cos(ut)dz\right)du, \\
&= 2\int_{0}^{+\infty} \mathcal{R}e\left(\widehat{f}(\sigma+iu)\cos(ut)\right)du. \qquad (76)
\end{aligned}$$

If we put all the terms together we find the following expression for f

$$\boldsymbol{f(t) = \frac{2e^{\sigma t}}{\pi}\int_{0}^{+\infty} \mathcal{R}e\left(\widehat{f}(\sigma+iu)\cos(ut)\right)du.} \qquad (77)$$

The only remaining problem is to compute numerically this non finite integral. We numerically evaluate the integral above by means of the trapezoidal rule. If we use a step size h, then this gives : for any $h \in \mathbb{R}$

$$f_h(t) = \frac{he^{\sigma t}}{\pi}\widehat{f}(\sigma) + \frac{2he^{\sigma t}}{\pi}\sum_{n=1}^{\infty}\mathcal{R}e\left(\widehat{f}(\sigma+inh)\right)\cos(nht). \qquad (78)$$

Let us change the variable as following $h = \dfrac{\pi}{2t}$ and $\sigma = \dfrac{A}{2t}$ to get a new expression

$$f_h(t) = \frac{e^{A/2}}{2t}\widehat{f}\left(\frac{A}{2t}\right) + \frac{e^{A/2}}{t}\sum_{n=1}^{\infty}\mathcal{R}e\left(\widehat{f}\left(\frac{A+2in\pi h}{2t}\right)\right)(-1)^n. \qquad (79)$$

Now, we would like to measure how well $f_h(t)$ approximates $f(t)$. To do so we need to establish the so-called Poisson summation formula.

7.2. **The Poisson summation formula.** First, one will notice that in any case $f(t)$ is positive, continuous, bounded by the initial value of the asset and $f(t) = 0$ if $t < 0$. Let us introduce

$$g(x) = f\left(t + \frac{2\pi x}{h}\right)e^{-b\left(t+\frac{2\pi x}{h}\right)}, \text{ where b is a positive constant.}$$

Relying on the definition of f, it is straightforward that $\sum_{n=-\infty}^{+\infty} g(n+x)$ converges uniformly for any real x. So we can define $G(x) = \sum_{n=-\infty}^{+\infty} g(n+x)$. Thanks to the uniform convergence, G is continuous with period 1 and therefore equal to its Fourier series expansion

$$G(x) = \sum_{n=-\infty}^{+\infty} a_n e^{2i\pi nx}.$$

where

$$
\begin{aligned}
a_n &= \int_0^1 G(x) e^{-2i\pi n x} dx, \\
&= \int_0^1 \sum_{k=-\infty}^{+\infty} g(k+x) e^{-2i\pi n x} dx, \\
& \text{because the series converges uniformly we can interchange the summation and the integral} \\
&= \sum_{k=-\infty}^{+\infty} \int_0^1 g(k+x) e^{-2i\pi n x} dx, \\
& \text{let us change variable } u = x + k \\
&= \sum_{k=-\infty}^{+\infty} \int_k^{k+1} g(u) e^{-2i\pi n u} du, \\
&= \int_{-\infty}^{+\infty} g(u) e^{-2i\pi n u} du.
\end{aligned}
$$

Then, we come up with the following expression for G

$$
\begin{aligned}
G(x) &= \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(u) e^{-2i\pi n u} du \, e^{2i\pi n x}, \\
&= \sum_{n=-\infty}^{+\infty} 2\pi \mathcal{F}(g)(2\pi n) \, e^{2i\pi n x}, \\
& \text{let us put } x = 0 \text{, we get the Poisson summation formula} \\
\sum_{n=-\infty}^{+\infty} g(n) &= 2\pi \sum_{n=-\infty}^{+\infty} \mathcal{F}(g)(2\pi n), \\
& \text{plugging in the definition of } g \text{, we get} \\
\sum_{n=-\infty}^{+\infty} f\left(t + \frac{2\pi n}{h}\right) e^{-b\left(t+\frac{2\pi n}{h}\right)} &= 2\pi \sum_{n=-\infty}^{+\infty} \mathcal{F}\left(f\left(t + \frac{2\pi \cdot}{h}\right) e^{-b\left(t+\frac{2\pi \cdot}{h}\right)}\right)(2\pi n). \qquad (80)
\end{aligned}
$$

Let us try to compute the Fourier transform

$$
\begin{aligned}
\mathcal{F}\left(f\left(t + \tfrac{2\pi \cdot}{h}\right) e^{-b\left(t+\frac{2\pi \cdot}{h}\right)}\right)(\xi) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f\left(t + \frac{2\pi x}{h}\right) e^{-b\left(t+\frac{2\pi x}{h}\right)} e^{-i\xi x} dx, \\
& \text{Let us change variable } u = t + \frac{2\pi x}{h} \\
&= \tfrac{h}{4\pi^2} \int_{-\infty}^{+\infty} f(u) e^{-bu} e^{-\frac{i\xi h}{2\pi}(u-t)} du, \\
&= \frac{h}{2\pi} e^{\frac{i\xi h t}{2\pi}} \mathcal{F}\left(f(\cdot) e^{-b\cdot}\right)\left(\frac{h\xi}{2\pi}\right), \\
&= \frac{h}{4\pi^2} e^{\frac{i\xi t h}{2\pi}} \widehat{f}\left(b + \frac{ih\xi}{2\pi}\right).
\end{aligned}
$$

Now, if we put this expression back into (80) and define $h = \pi/t$ and $b = A/2t$, we get the following formula

$$\begin{aligned}
\sum_{-\infty}^{+\infty} f((1+2n)t)e^{-(1+2n)A/2} &= \sum_{-\infty}^{+\infty} \frac{1}{2t} e^{in\pi} \widehat{f}\left(\frac{A+2i\pi n}{2t}\right), \\
&\text{f is positive so we get} \\
\sum_{0}^{+\infty} f((1+2n)t)e^{-nA} &= \sum_{-\infty}^{+\infty} \frac{e^{A/2}}{2t} (-1)^n \widehat{f}\left(\frac{A+2i\pi n}{2t}\right), \\
&\text{f has real values so only the real part of the summation is worth taking into account} \\
\sum_{0}^{+\infty} f((1+2n)t)e^{-nA} &= \sum_{-\infty}^{+\infty} \frac{e^{A/2}}{2t} (-1)^n \mathcal{R}e\left(\widehat{f}\left(\frac{A+2i\pi n}{2t}\right)\right), \\
&\text{it is easy to check that the real part of } \widehat{f}\left(\tfrac{A+2i\pi n}{2t}\right) \text{ is even with respect to n} \\
\sum_{0}^{+\infty} f((1+2n)t)e^{-nA} &= \frac{e^{A/2}}{2t} \widehat{f}\left(\frac{A}{2t}\right) + \sum_{0}^{+\infty} \frac{e^{A/2}}{t} (-1)^n \mathcal{R}e\left(\widehat{f}\left(\frac{A+2i\pi n}{2t}\right)\right).
\end{aligned}$$

We can deduce the value of f from this expression

$$\boldsymbol{f(t) = \frac{e^{A/2}}{2t} \widehat{f}\left(\frac{A}{2t}\right) + \sum_{1}^{+\infty} \frac{e^{A/2}}{t} (-1)^n \mathcal{R}e\left(\widehat{f}\left(\frac{A+2i\pi n}{2t}\right)\right) - \sum_{1}^{+\infty} f((1+2n)t)e^{-nA}.} \tag{81}$$

If we compare with expression (79), we notice that the error made if we approach f by f_h is bounded by $\sum_{1}^{+\infty} f((1+2n)t)e^{-nA}$. As recalled above, f is bounded by the initial value of the stock price So so the error is bounded by $So\dfrac{e^{-A}}{1-e^{-A}}$. So is about 100, and if we take $A = 18.4$ the error is smaller than 10^{-6}. One could be tempted to increase the value of A, unfortunately it is not so simple as you discover later on in *section 7*.
The remaining problem is to numerically compute (79), which involves a non-finite summation and then we would have a rather good approximation of f. We could simply truncate the summation and try to measure how well it converges. Let us suppose we approach f_h by

$$\frac{e^{A/2}}{2t} \widehat{f}\left(\frac{A}{2t}\right) + \frac{e^{A/2}}{t} \sum_{k=1}^{n} \mathcal{R}e\left(\widehat{f}\left(\frac{A+2in\pi h}{2t}\right)\right)(-1)^n. \tag{82}$$

and let us find an upper bound for the error.
S,o we would like to bound $\displaystyle\sum_{k=n+1}^{\infty} \mathcal{R}e\left(\widehat{f}\left(\frac{A+2in\pi h}{2t}\right)\right)(-1)^n$.

$$\begin{aligned}
\sum_{k=n+1}^{\infty} \mathcal{R}e\left(\widehat{f}\left(\frac{A+2in\pi h}{2t}\right)\right)(-1)^n &= \sum_{k=n+1}^{\infty} \int_0^{+\infty} f(z)\cos\left(\frac{in\pi h}{t}\right) e^{-Az/2t}(-1)^n, \\
\left| \sum_{k=n+1}^{\infty} \mathcal{R}e\left(\widehat{f}\left(\frac{A+2in\pi h}{2t}\right)\right)(-1)^n \right| &\leq \sum_{k=n+1}^{\infty} \int_0^{+\infty} So \left|\cos\left(\frac{in\pi h}{t}\right)\right| e^{-Az/2t}, \\
&\quad f \text{ is bounded by } So \\
&\leq \frac{2tSo}{A} \sum_{k=n+1}^{\infty} \frac{1}{1+\frac{4\pi^2 k^2}{A}}.
\end{aligned}$$

Since we have

$$\int_n^{n+1} \frac{1}{1+ax^2}dx \leq \frac{1}{1+an^2} \leq \int_{n-1}^{n} \frac{1}{1+ax^2}dx$$

and we know that $\displaystyle\int_{n-1}^{\infty} \frac{1}{1+ax^2}dx$ is equivalent to $\dfrac{1}{an}$ we get

$$\sum_{n=N}^{+\infty} \frac{1}{1+an^2} \sim \frac{1}{a\,N}.$$

So, we come up with the following upper bound

$$\mid \sum_{k=n+1}^{\infty} \mathcal{R}e\left(\widehat{f}\left(\frac{A+2in\pi h}{2t}\right)\right)(-1)^n \mid \leq \frac{tSoA}{2\pi^2}\frac{1}{n}.$$

Therefore, if we want an accuracy up to 10^{-6}, we need 1.10^7 terms in the summation which is rather huge. For this reason, we will present in the next section a way to accelerate the convergence of the series.

7.3. **The Euler summation.** Let $s_n(t)$ be an approximation of $f_h(t)$, the infinite series is truncated to n terms,

$$s_n(t) = \frac{e^{A/2}}{2t}\widehat{f}\left(\frac{A}{2t}\right) + \frac{e^{A/2}}{t}\sum_{k=1}^{n}(-1)^k a_k(t),$$

where

$$a_k(t) = \mathcal{R}e\left(\widehat{f}\left(\frac{A+2i\pi k}{2t}\right)\right).$$

We apply Euler summation to m terms after an initial n, so that the Euler sum (approximation to (79)) is

$$E(m,n,t) = \sum_{k=0}^{m} C_k^m 2^{-m} s_{n+k}(t), \tag{83}$$

(83) is the binomial average of the terms $s_n,\ s_{n+1}, ..., s_{n+m}$.
We will prove that $E(m,n,t)$ goes to s when n goes to $+\infty$.
In (83), the sum of the weights is equal to 1. So we have :

$$\min_{k\in[0,\ m]} s_{n+k} \leq E(m,n,t) \leq \max_{k\in[0,\ m]} s_{n+k}.$$

So, when n goes to $+\infty$ $\min_{k\in[0,\ m]} s_{n+k}$ goes to s, as well as $\max_{k\in[0,\ m]} s_{n+k}$. Then

$$\lim_{n\to+\infty} E(m,n,t) = s.$$

Now, we are interested in how fast E goes to s when n goes to ∞. To estimate the error associated with Euler summation, we suggest to use the difference of successive terms, i.e., $E(m,n+1,t) - E(m,n,t)$.

$$\begin{aligned}
E(m,n+1,t) - E(m,n,t) &= \sum_{k=0}^{m} C_m^k 2^{-m}(-1)^{n+1+k} a_{n+1+k}(t),\\
&= 2^{-m}\sum_{k=0}^{m} C_m^k (-1)^{n+1+k}\mathcal{R}e\left(\int_0^{+\infty} e^{-\left(\frac{A}{2t}+\frac{k+n+1}{t}\pi i\right)s} f(s)ds\right),\\
&= 2^{-m}(-1)^{n+1}\int_0^{+\infty} e^{-\frac{As}{2t}}\mathcal{R}e\left(e^{-\frac{(n+1)\pi i s}{t}}\sum_{k=0}^{m} C_m^k(-1)^k e^{-\frac{k\pi i s}{t}}\right) f(s)ds,\\
&= 2^{-m}(-1)^{n+1}\int_0^{+\infty} e^{-\frac{As}{2t}}\mathcal{R}e\left(e^{-\frac{(n+1)\pi i s}{t}}(1-e^{-\frac{\pi i s}{t}})\right) f(s)ds.
\end{aligned}$$

So, we can bound the difference between $E(m,n+1,t) - E(m,n,t)$,

$$\mid E(m,n+1,t) - E(m,n,t) \mid \leq \frac{So}{2^m}\int_0^{+\infty} e^{-\frac{As}{2t}}\left|\cos\left(\frac{(n+1)\pi s}{t}\right) - \cos\left(\frac{(n+2)\pi s}{t}\right)\right| ds.$$

By changing variables $x = \frac{As}{2t}$, we obtain

$$\begin{aligned}
\mid E(m,n+1,t) - E(m,n,t) \mid &\leq \frac{So}{2^m}\int_0^{+\infty} e^{-x}\left|\cos\left(\frac{2(n+1)\pi x}{A}\right) - \cos\left(\frac{2(n+2)\pi x}{A}\right)\right|\frac{2t}{A}dx,\\
&\leq \frac{2tSo}{A2^m}\int_0^{+\infty} e^{-x}\left|\cos\left(\frac{2(n+1)\pi x}{A}\right)\right| + e^{-x}\left|\cos\left(\frac{2(n+2)\pi x}{A}\right)\right| dx.
\end{aligned}$$

Furthermore, we have

$$\int_0^{+\infty} e^{-x}\left|\cos\left(\frac{2\pi x(n+1)}{A}\right)\right| dx = \frac{A^2}{A^2+4\pi^2(n+1)^2}.$$

Then, we get

$$\begin{aligned} \mid E(m,n+1,t) - E(m,n,t) \mid &\leq \frac{2tSo}{A2^m}\left(\frac{A^2}{A^2+4\pi^2(n+1)^2} + \frac{A^2}{A^2+4\pi^2(n+2)^2}\right), \\ &\leq \frac{AtSo}{2^m\pi^2n^2}. \end{aligned}$$

Now, we can bound the difference $\mid s - E(m,n,t) \mid$.

$$\mid s - E(m,n,t) \mid \leq \frac{AtSo}{2^m\pi^2}\sum_{n=N}^{+\infty}\frac{1}{n^2}.$$

Since we have

$$\int_n^{n+1}\frac{1}{x^2}dx \leq \frac{1}{n^2} \leq \int_{n-1}^{n}\frac{1}{x^2}dx,$$

we get

$$\frac{1}{N} \leq \sum_{n=N}^{+\infty}\frac{1}{n^2} \leq \frac{1}{N-1}.$$

Then,

$$\mid s - E(m,n,t) \mid \leq \frac{AtSo}{2^m\pi^2}\frac{1}{n}.$$

To have an 10^{-6} error, if we choose $n = 15$, we need $m = 23$. The number of iterations is really small compared with the previous result we got for n (we needed 10^7 terms). The improvement is tremendously significant. However one must have noticed that most prices involved the function ψ or at least $\mathcal{N}$ and we only have an approximation with a 10^{-7} precision on the computation of these functions. This means that we are not able to compute the exact value of the Laplace transforms at a given point but only an approximation. We will now try to measure the consequences of such an approximation.

7.4. **Accuracy of the numerical inversion.** A polynomial approximation of the cumulative normal distribution is described in Lamberton and Lapeyre [7].

$$\mathcal{N}(x) = 1 - \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}(b_1t + b_2t^2 + b_3t^3 + b_4t^4 + b_5t^5), \text{ for } x > 0$$

where

$$\begin{aligned} b_1 &= 0.319381530, \\ b_2 &= -0.356563782, \\ b_3 &= 1.781477937, \\ b_4 &= -1.821255978, \\ b_5 &= 1.330274429, \\ p &= 0.2316419, \\ t &= \frac{1}{1+px}. \end{aligned}$$

The approximation is accurate up to 10^{-6} and pretty fast.

If one remembers the definition of the function $\psi(x) = 1 + x\sqrt{2\pi}e^{\frac{x^2}{2}}\mathcal{N}(x)$, one straightaway understands that the accuracy of *psi* decreases as quickly as x increases. The purpose is then to reduce as well as possible the absolute value of the argument of the function ψ. As we will show it this implies to decrease the value of A which is in contradiction with what we have found just above (*c.f.* 7.2 where we have discussed the error due to the trapezoidal approximation of the integral).

7.5. **The accuracy of ψ.**

$$\begin{aligned} \psi(x) - \psi_a(x) &= \sqrt{2\pi}xe^{\frac{x^2}{2}}(\mathcal{N}(x) - \mathcal{N}_a(x)), \\ |\psi(x) - \psi_a(x)| &\leq = \sqrt{2\pi}|x|e^{\frac{x^2}{2}}10^{-7}. \end{aligned}$$

We will now measure the error made at the first order on the most often appearing factor in the Laplace transforms of the prices and creating errors on them

$$\widehat{B} = \frac{\psi(-\theta\sqrt{D})e^{2b\theta}}{\theta\psi(\theta\sqrt{D})}Ke^{(m-\theta)k}\left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta}\right), \text{ when } B \leq 0.$$

$$
\begin{aligned}
\widehat{B_a} &= \frac{\psi_a(-\theta\sqrt{D})e^{2b\theta}}{\theta\psi_a(\theta\sqrt{D})} K e^{(m-\theta)k}\left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta}\right), \\
|\widehat{B} - \widehat{B_a}| &\leq 10^{-7}\left|\frac{\theta\sqrt{2\pi D}e^{2b\theta}e^{2\lambda}}{\theta\psi(\theta\sqrt{D})} K e^{(m-\theta)k}\left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta}\right)\right|, \\
& \quad \text{we know that } \lambda \in \left\{\frac{A}{2T} + \frac{ik\pi}{T} \,; k \in [|0,m|]\right\} \\
&\leq 10^{-7} K e^{\frac{A}{2}}\sqrt{2\pi D}\left|\frac{e^{2b\theta}}{\psi(\theta\sqrt{D})} e^{(m-\theta)k}\frac{\sigma}{2\lambda}\right|, \\
&\leq 10^{-7} K e^{\frac{A}{2}}\sqrt{2\pi D}\left|\frac{e^{2b\theta}}{\psi(\theta\sqrt{D})} e^{(m-\theta)k}\frac{\sigma}{2\lambda}\right|, \\
&\leq 10^{-7} K e^{\frac{A}{2}}\sqrt{2\pi D}\left|\frac{1}{\psi(\theta\sqrt{D})} e^{(m-\theta)k}\frac{\sigma}{2\lambda}\right| \quad \text{because } b \leq 0.
\end{aligned}
$$

If we assume that $\frac{2}{3} < \frac{K}{x} < \frac{3}{2}$, $D < 0.5$, and we calculate how the error is transformed by the numerical inversion we use for the Laplace transform we get

$$|B - B_a| \leq 1.5\, 10^{-4}, \text{ for } A = 13.8.$$

The two other errors are bounded by 10^{-6} and $e^{-A} = 10^{-5}$, we are sure that the prices will be accurate up to 10^{-4}.

The terms only involving the function ψ in their denominators are much more accurate since one notices that considering the values of λ, the relative error on ψ is smaller than 10^{-7}.

8. A few graphs

After all these technical computations, we would like to present Parisian options through a few graphs. First, we can compare Parisian options with normal barrier options. Then, we will try to understand a bit more about the different Greeks of Parisian options.

8.1. **A comparison with standard barrier options.** We would like to plot a graph of the evolution of a down and in Call when D decreases up to 0, with the following parameters:
strike 100
maturity 1 year
barrier 75
volatility 0.2
interest rate 0.05
dividend rate 0

We can see that when D goes to 0, the down and in Call price goes to a down and in barrier price.

8.2. **Hedging.** Options become interesting for trading companies as soon as they can be hedged, which means that they are able to find a replicating strategy based on the Black-Scholes' theory. The underlying hypothesis of Black-Scholes' model is a continuous hedging, nevertheless this is not applicable in a real world. This discrete time hedging creates an error, the burning issue of hedging such options is to reduce the error as close as possible to zero. How well can replicate such an option? We will try to give some answers to these problems. First we will present some graphs showing the evolution of the Greeks with respect to the initial value. One will realise how much they differ from standard European options. Then we will generate a stock price over one year and hedge it to see how the hedging error evolves and also the number of stocks you should own.

8.2.1. *A glimpse of the different Greeks.* Let us consider a down-and-in Call with the following parameter $K = 60$, $l = 80$, $D = 30$ days, $\sigma = 0.3$, $r = 0.045$ and $\delta = 0$. The graphs below show the evolution of the different Greeks with respect to the initial value.
We notice that there are huge variations in the graphs when the initial stock price is close to the barrier. For instance the delta suddenly drops down whereas the initial stock price tends to the barrier. This sensibility of the delta around the barrier let us think that it may be difficult to hedge such an option. Now we will simulate a stock path over one year and hedge an option.

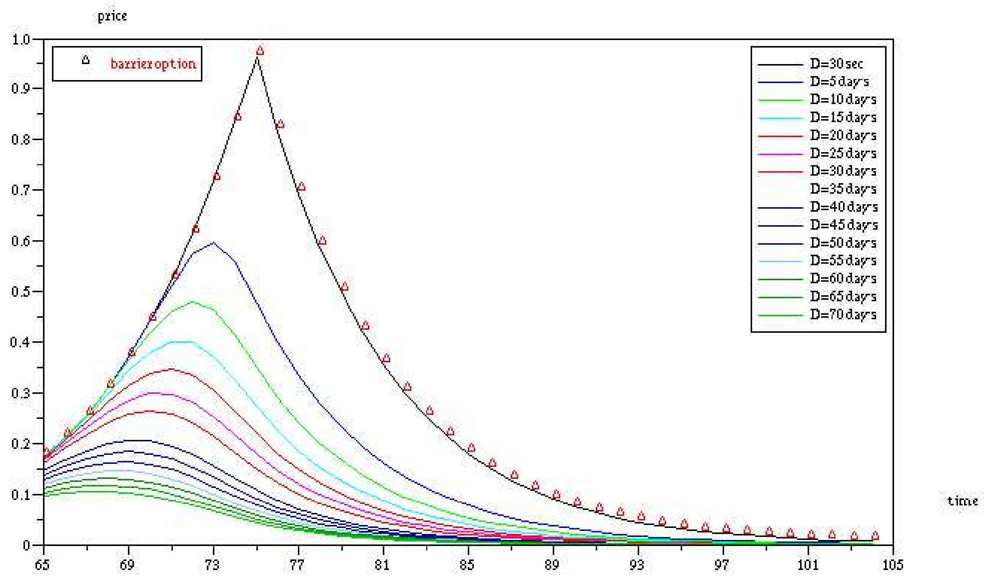

FIGURE 5. Evolution of the price with respect to the window length

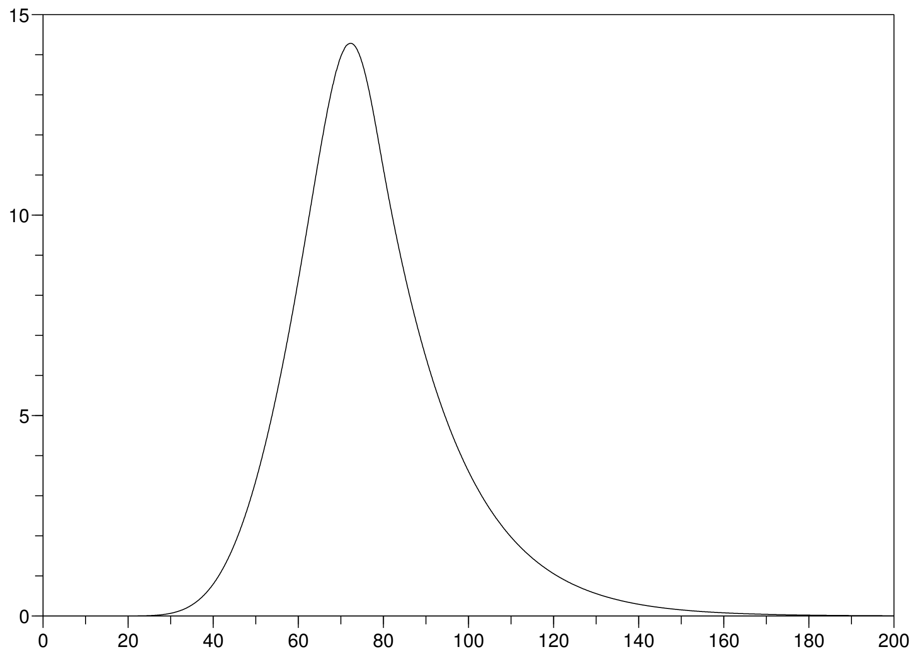

FIGURE 6. Price of a down-and-in call

8.2.2. *Hedging simulation.* We still consider a down-and-in call with parameters $S_0 = 82$, $K = 60$, $l = 80$, $D = 30$ days, $\sigma = 0.3$, $r = 0.045$, $\delta = 0$ and a drift $\mu = 0.1$. Now we are going to simulate the stock path over the year with three steps per day. The graphs below show the results we obtain with our program within five seconds.

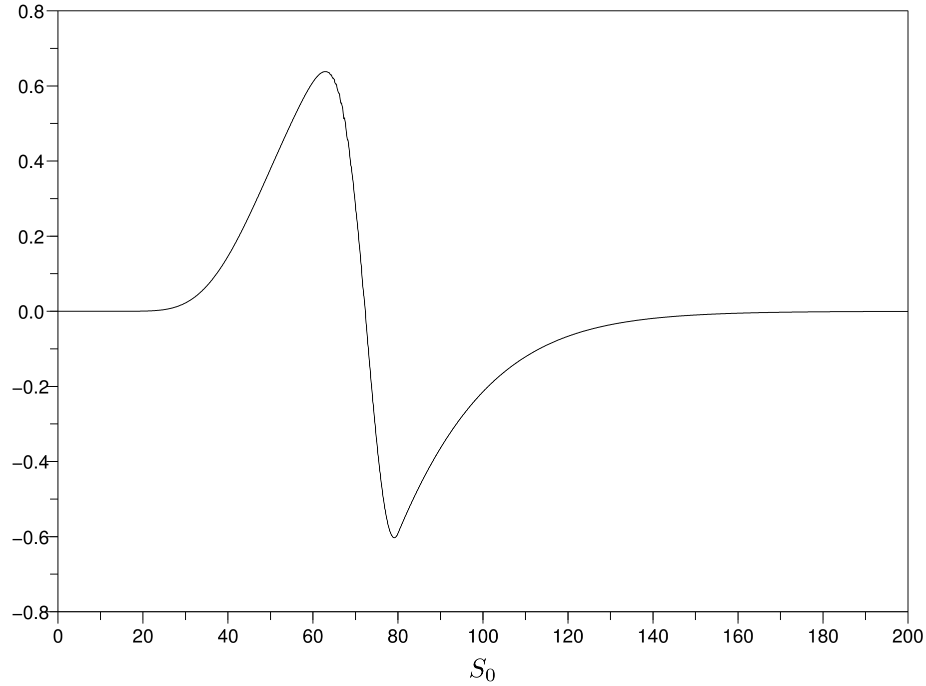

FIGURE 7. Delta of a down-and-in call

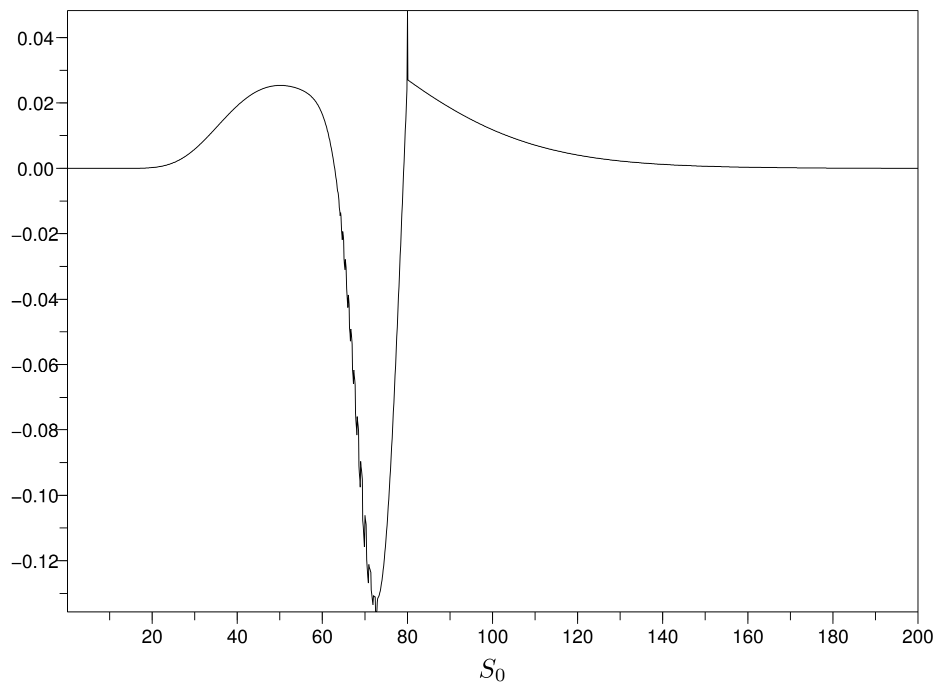

FIGURE 8. Gamma of a down-and-in call

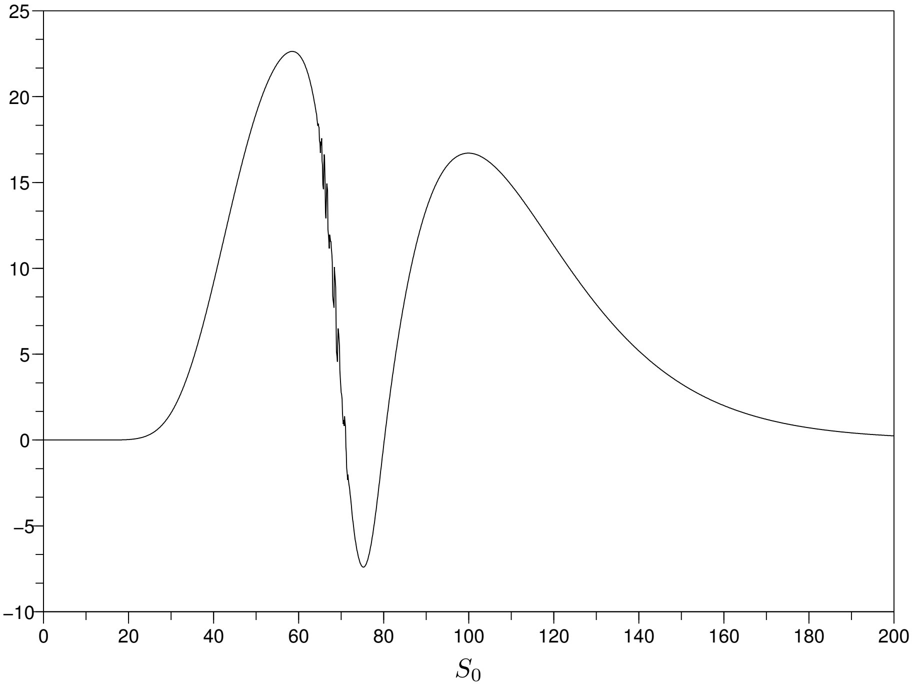

FIGURE 9. Vega of a down-and-in call

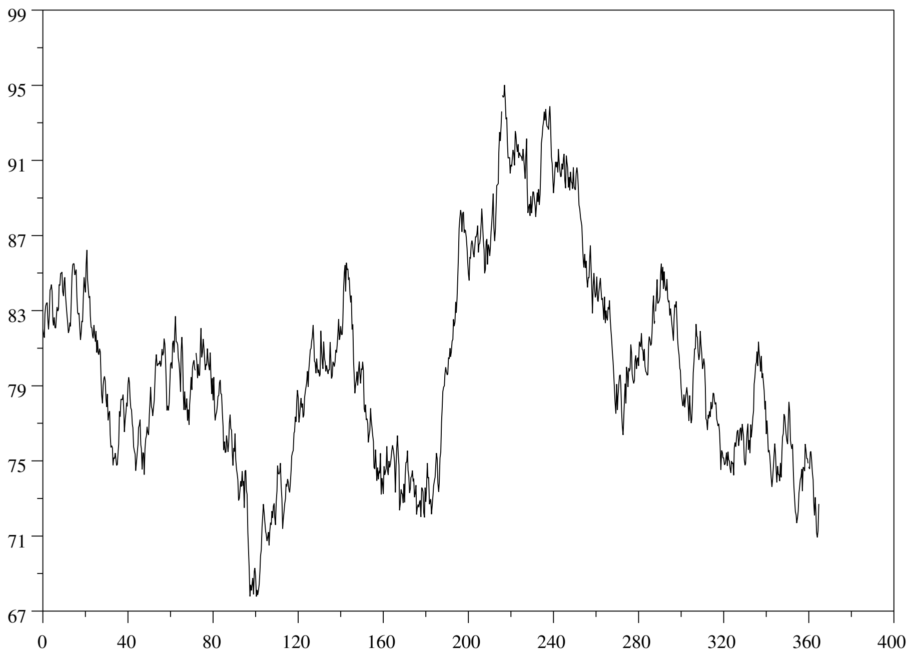

FIGURE 10. Evolution of the stock price

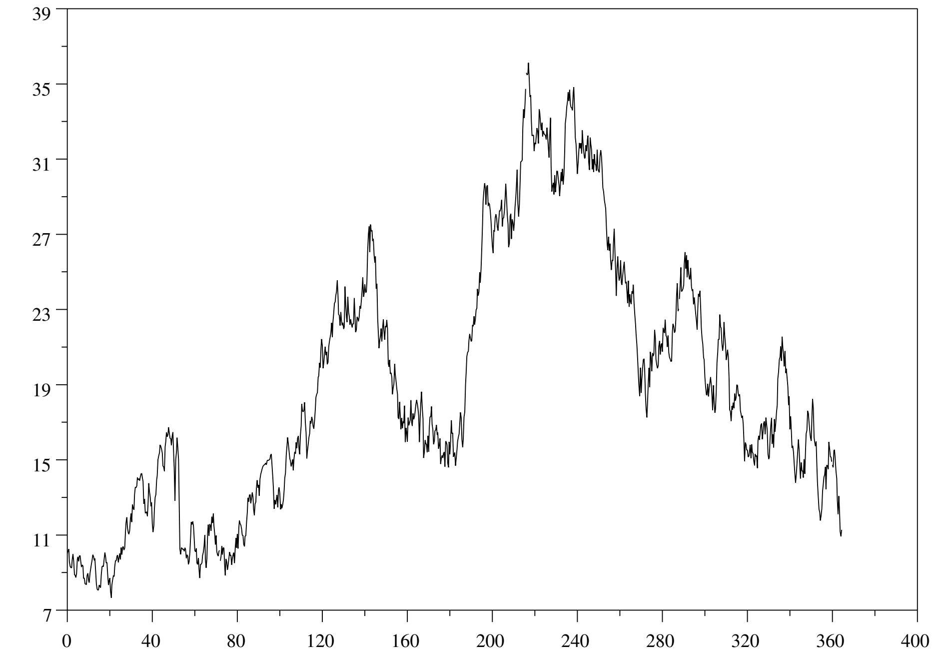

Figure 11. Evolution of the option price

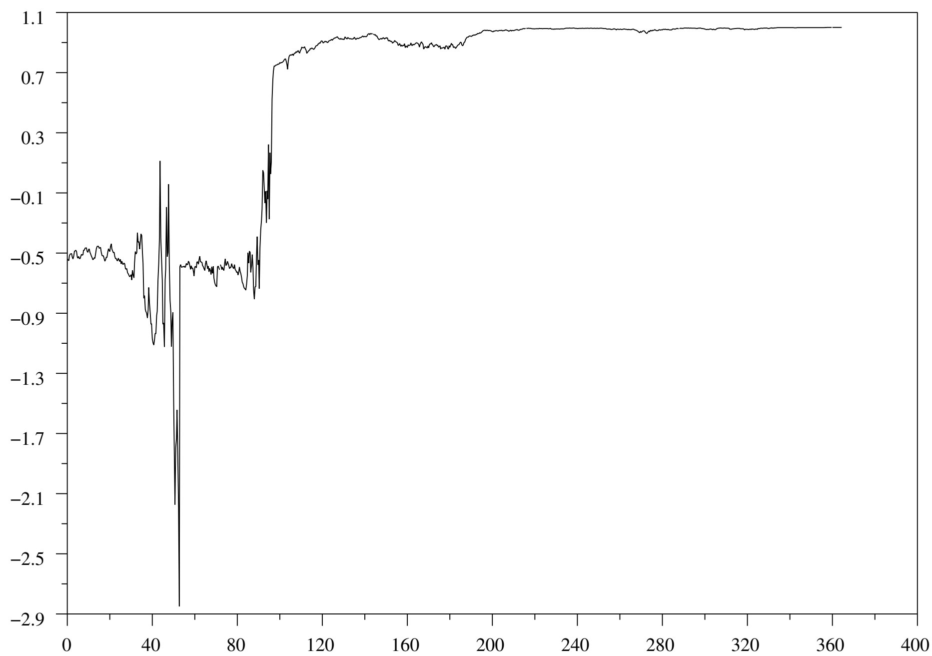

Figure 12. Evolution of the delta

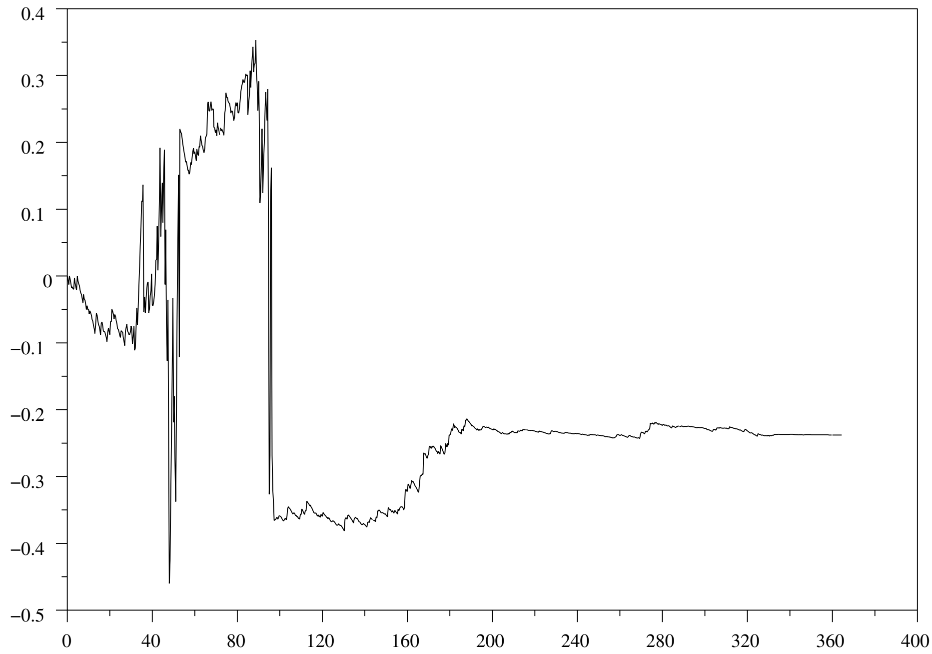

Figure 13. Evolution of the hedging error

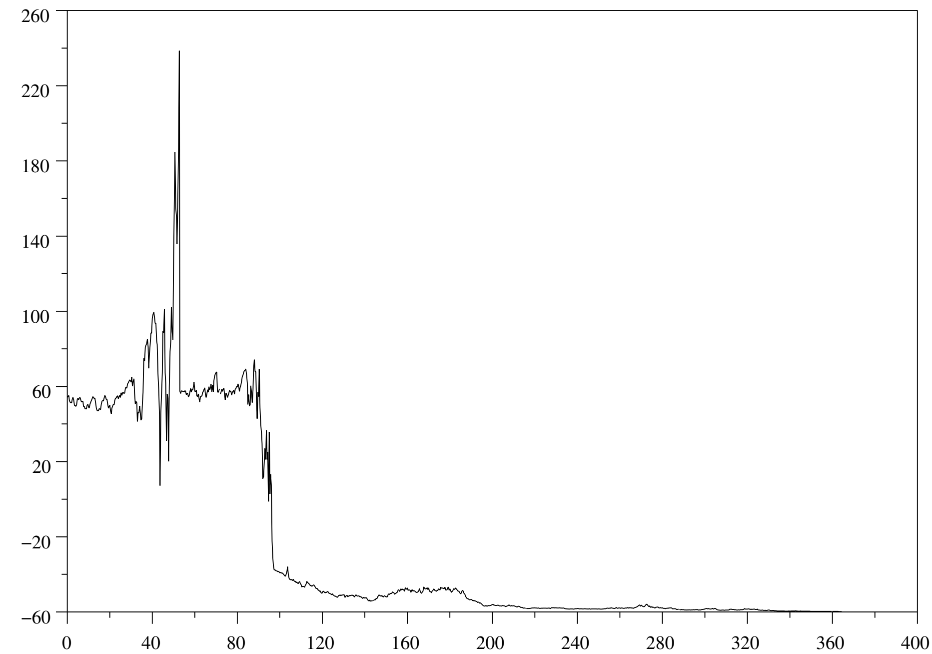

Figure 14. Evolution of the bank account

It is pretty clear that when the underlying asset crosses the barrier or is about to cross the barrier whereas the excursion has not been completes yet, the delta varies very sharply. These huge variations tend to increase the hedging error. As a matter of fact, when the asset is in a excursion and the option has not been knocked yet, the delta varies very quickly. As soon as the option has been knocked, the Greeks become smoother. This volatility in the delta graph makes it pretty hard to hedge such options. The more often the stock price crosses the barrier before being knocked, the bigger is the hedging error.

Appendix A. The valuation of $\int_0^D \mu_b(du)e^{-\lambda u}$ in the case $b > 0$

We already know that $\mu_b(du) = \dfrac{|\, b \,|}{\sqrt{2\pi u^3}} e^{\left(\frac{-b^2}{2u}\right)} du$.

$$
\begin{aligned}
\int_0^D e^{-\lambda u}\mu_b(du) &= \int_0^D e^{-\lambda u}\frac{b}{\sqrt{2\pi u^3}}e^{\frac{-b^2}{2u}}du \\
&\quad \text{with a change of variable } t = \tfrac{1}{\sqrt{u}} \text{ we get,} \\
&= \int_{1/\sqrt{D}}^{+\infty} b\sqrt{\frac{2}{\pi}}\, e^{\frac{-\lambda}{t^2}} e^{\frac{-b^2t^2}{2}} dt, \\
&= \int_{1/\sqrt{D}}^{+\infty} b\sqrt{\frac{2}{\pi}}\, e^{\frac{-\lambda}{t^2}} e^{\frac{-b^2t^2}{2}} dt, \\
&\quad \text{let } \theta \text{ denote } \sqrt{2\lambda} \\
&= \int_{1/\sqrt{D}}^{+\infty} b\sqrt{\frac{2}{\pi}} \exp\left(\frac{-\theta b}{2}\left(\frac{1}{(\sqrt{\frac{b}{\theta}}t)^2} + (\sqrt{\frac{b}{\theta}}t)^2\right)\right)dt, \\
&\quad \text{let's change variable again } u = \sqrt{\tfrac{b}{\theta}}t \\
&= \int_{\frac{\sqrt{b}}{\sqrt{\theta D}}}^{+\infty} \sqrt{\frac{2b\theta}{\pi}} \exp\left(\frac{-\theta b}{2}\left(\frac{1}{u^2}+u^2\right)\right)du, \\
&= \int_{\frac{\sqrt{b}}{\sqrt{\theta D}}}^{+\infty} \sqrt{\frac{2b\theta}{\pi}} \exp\left(\frac{-\theta b}{2}\left(\frac{1}{u}-u\right)^2\right)e^{-\theta b}du, \\
&\quad \text{a new change of variable } v = \tfrac{1}{u} - u \text{ gives} \\
&= \sqrt{\frac{b\theta}{2\pi}}e^{-\theta b}\int_{-\infty}^{\sqrt{\frac{\theta D}{b}}-\frac{\sqrt{b}}{\sqrt{\theta D}}} e^{\frac{-\theta b}{2}v^2}\left(1 - \frac{v}{\sqrt{v^2+4}}\right)dv, \\
&\quad \text{one more change of variable } u = \sqrt{\theta b}v \text{ provides the following expression} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\theta b}\int_{-\infty}^{\theta\sqrt{D}-\frac{b}{\sqrt{D}}} e^{-u^2/2}\left(1-\frac{u}{\sqrt{u^2+4\theta b}}\right)du, \\
&\quad \text{a last change of variable } v = \sqrt{u^2+4\theta b} \text{ ends the calculation} \\
&= e^{-\theta b}\mathcal{N}\left(\theta\sqrt{D}-\tfrac{b}{\sqrt{D}}\right) + \tfrac{1}{\sqrt{2\pi}}e^{-\theta b}\int_{\theta\sqrt{D}+\frac{b}{\sqrt{D}}}^{+\infty} e^{-\frac{v^2-4\theta b}{2}}dv, \\
&= e^{-\theta b}\mathcal{N}\left(\theta\sqrt{D}-\frac{b}{\sqrt{D}}\right) + e^{\theta b}\mathcal{N}\left(-\theta\sqrt{D}-\frac{b}{\sqrt{D}}\right).
\end{aligned}
$$

If we let D go to infinity, we can deduce the Laplace transform of T_b, for any real b

$$\mathbb{E}[e^{-\lambda T_b}] = e^{-\theta|b|}.$$

Appendix B. The valuation of $\int_0^{+\infty} e^{-\lambda u}\frac{\exp\left(-\frac{x^2}{2u}\right)}{\sqrt{2\pi u}}du$

Once again we introduce $\theta = \sqrt{2\lambda}$.
A change of variable $u = \frac{|x|t^2}{\theta}$ straightforward gives the new expression

$$
\begin{aligned}
\int_0^{+\infty} e^{-\lambda u}\frac{\exp\left(-\frac{x^2}{2u}\right)}{\sqrt{2\pi u}}du &= \int_0^{+\infty}\sqrt{\frac{2\,|\,x\,|}{\pi\theta}}\exp\left(-\frac{\theta\,|\,x\,|}{2}\left(\frac{1}{t^2}+t^2\right)\right)dt, \\
&= \sqrt{\frac{2\,|\,x\,|}{\pi\theta}}e^{-\theta|x|}\int_0^{+\infty}\exp\left(-\frac{\theta\,|\,x\,|}{2}\left(\frac{1}{t}-t\right)^2\right)dt.
\end{aligned}
$$

Once again, we can use the change of variable $s = u - \frac{1}{u}$, which maps $[0, +\infty[$ into $]-\infty, +\infty[$ and we have $du = \frac{ds}{2}\left(1 + \frac{s}{\sqrt{s^2+4}}\right)$. The second of the last term is odd, so its integral over $\mathbb{R}$ cancels and we get

$$\sqrt{\frac{|x|}{2\pi\theta}} e^{-\theta|x|} \int_{-\infty}^{+\infty} \exp\left(-\frac{\theta\,|x|}{2} s^2\right) ds.$$

So finally we obtain

$$\boldsymbol{\int_0^{+\infty} e^{-\lambda u} \frac{\exp\left(-\frac{x^2}{2u}\right)}{\sqrt{2\pi u}} du = \frac{1}{\theta} e^{-\theta|x|}.} \tag{84}$$

Appendix C. The Brownian meander

In this section, we only recall some useful results on excursion theory and Brownian meander. To find the proofs of the results announced, one can refer to Revuz and Yor [8] or Chung [4] for instance.
We denote by $g_t = \sup\{s \leq t; Z_s = 0\}$ the left extremity of the excursion straddling time t. We define $(\mathcal{F}^+_{g_t}, t \geq))$ the slow Brownian filtration as $\mathcal{F}^+_{g_t} = \mathcal{F}^-_{g_t} \bigvee \sigma(\text{sgn}(Z_t))$. $\mathcal{F}^-_{g_t}$ is the $\sigma-$algebra generated by the random variables X_{g_t}, where X is a predictable process for the natural filtration of Brownian motion Z.
We denote by $g = g_1 = \sup\{s \leq 1; Z_s = 0\}$ the left extremity of the excursion straddling time 1.
The Brownian meander is defined as process

$$m = \left\{ m_u = \frac{1}{\sqrt{1-g}} |Z_{g+u(1-g)}|; \quad u \leq 1 \right\}. \tag{85}$$

It is known that process m is independent of $\mathcal{F}^+_g$. The law of m_1 is given by

$$\mathbb{P}(m_1 \in dx) = x e^{-\frac{x^2}{2}} \mathbf{1}_{x>0}\, dx. \tag{86}$$

To find the law of m_1 we begin to calculate $\mathbb{P}(m_1 \leq \lambda)$

$$\begin{aligned} \mathbb{P}(m_1 \leq \lambda) &= \mathbb{P}(\tfrac{1}{\sqrt{1-g}} |Z_1| \leq \lambda) \\ &= \mathbb{E}[\mathbf{1}_{\{\frac{1}{\sqrt{1-g}}|Z_1| \leq \lambda\}}]. \end{aligned}$$

Thanks to formula (116), we can write

$$\mathbb{P}(m_1 \leq \lambda) = \int_{s=0}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{1}_{\{\frac{|x|}{\sqrt{1-s}} \leq \lambda\}} \mathbf{1}_{\{s \leq 1\}} \frac{|x|}{2\pi\sqrt{s(1-s)^3}} e^{-\frac{x^2}{2(1-s)}} dx ds.$$

For $\lambda \leq 0$, $\mathbb{P}(m_1 \leq \lambda) = 0$. From now on, we assume $\lambda \geq 0$. So, we get

$$\begin{aligned} \mathbb{P}(m_1 \leq \lambda) &= \int_{s=0}^{1} \int_{x=-\lambda\sqrt{1-s}}^{+\lambda\sqrt{1-s}} \frac{|x|}{2\pi\sqrt{s(1-s)^3}} e^{-\frac{x^2}{2(1-s)}} dx ds, \\ &= 2\int_0^1 \int_0^{+\lambda\sqrt{1-s}} \frac{x}{2\pi\sqrt{s(1-s)^3}} e^{-\frac{x^2}{2(1-s)}} dx ds, \\ &= 2\int_0^1 \left[-\frac{1}{2\pi\sqrt{s(1-s)}} e^{-\frac{x^2}{2(1-s)}} \right]_{x=0}^{\lambda\sqrt{1-s}} ds, \\ &= 2\int_0^1 \frac{1}{2\pi\sqrt{s(1-s)}} (1 - e^{-\frac{\lambda^2}{2}}) ds, \\ &= \frac{(1 - e^{-\frac{\lambda^2}{2}})}{\pi} \int_0^1 \frac{1}{\sqrt{s(1-s)}} ds. \end{aligned}$$

Now, we have to evaluate $\int_{s=0}^{1} \frac{1}{\sqrt{s(1-s)}} ds$.
The change of variables $s = \cos^2\theta$ gives

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{s(1-s)}} ds &= \int_{-\frac{\pi}{2}}^{0} -2\frac{\sin\theta\cos\theta}{|\sin\theta\cos\theta|} d\theta, \\
&= -2\int_{-\frac{\pi}{2}}^{0} \operatorname{sgn}(\sin(2\theta)) d\theta, \\
&= \pi.
\end{aligned}$$

So, we obtain

$$\mathbb{P}(m_1 \leq \lambda) = (1 - e^{-\frac{\lambda^2}{2}}).$$

For any $\lambda \in \mathbb{R}$, we have

$$\mathbb{P}(m_1 \in d\lambda) = \mathbf{1}_{\lambda \geq 0} \lambda e^{-\frac{\lambda^2}{2}} d\lambda. \tag{87}$$

Using Brownian scaling again, we can derive exactly the same results if we consider the excursion straddling time t instead of 1. Namely, we define

$$m^{(t)} = \left\{ m_u^{(t)} = \frac{1}{\sqrt{t - g_t}} |Z_{g_t + u(t-g_t)}|;\ u \leq 1 \right\}, \tag{88}$$

which is a Brownian meander independent of the σ-field $\mathcal{F}^+_{g_t}$. In particular, the law of $m^{(t)}$ does not depend on t.

Moreover, these results still hold if we consider a $\mathcal{F}$-stopping time instead of t. The remark is definitely essential as far as we are concerned and makes it possible to compute the law of $(T_b^-, Z_{T_b^-})$, as we do it in appendix D.

C.1. **The Azema martingale.** We now introduce the so-called Azema martingale $\mu_t = \operatorname{sgn}(Z_t)\sqrt{(t - g_t)}$, which is a $\mathcal{F}^+_{g_t}$-martingale (see Azéma and Yor [2] for a detailed study on the μ martingale). We have

$$\mathbb{E}(\exp(\lambda Z_t - \frac{1}{2}\lambda^2 t) | \mathcal{F}^+_{g_t}) = \mathbb{E}(\exp(\lambda m_1^{(t)} \mu_t - \frac{1}{2}\lambda^2 t) | \mathcal{F}^+_{g_t})$$

and, from the independence property we have just recalled, we get

$$\mathbb{E}(\exp(\lambda Z_t - \frac{1}{2}\lambda^2 t) | \mathcal{F}^+_{g_t}) = \exp(-\frac{1}{2}\lambda^2 t)\psi(\lambda\mu_t), \tag{89}$$

where

$$\psi(z) = \mathbb{E}(\exp(z m_1)) = \int_0^{+\infty} x \exp\left(zx - \frac{1}{2}x^2\right) dx$$

Appendix D. The law of $(T_b^-, Z_{T_b^-})$ and $(T_b^+, Z_{T_b^+})$

D.1. **Case $b = 0$.** In this case, we denote $T^- = T_0^-$.
For any $t > 0$, we have

$$\begin{aligned}
\{T^- \leq t\} &= \{\exists u \leq t;\ u - g_u \geq D \text{ and } \operatorname{sgn}(Z_u) = -1\}, \\
&= \bigcup_{u \leq t} (\{u - g_u \geq D\} \cap \{\operatorname{sgn}(Z_u) = -1\}).
\end{aligned} \tag{90}$$

Since each term composing the union belongs to $\mathcal{F}_t$, the random variable T^- is an $(\mathcal{F}_t)_t$ stopping time.
As recalled in Appendix C, we can use the definition of $m_u^{(t)}$ for $t = T^-$. Hence, process $m^{(T^-)}$ defined by

$$m^{(T^-)} = \left\{ m_u^{T^-} = \frac{1}{\sqrt{T^- - g_{T^-}}} |Z_{g_{T^-} + u(T^- - g_{T^-})}|; u \leq 1 \right\}$$

is a Brownian meander independent of $\mathcal{F}^+_{g_{T^-}}$.
As $g_{T^-} + D = T^-$, $\frac{1}{\sqrt{D}} Z_{T^-} = -m_1^{(T^-)}$, because Z_{T^-} is negative. Thus, Z_{T^-} is independent of $\mathcal{F}^+_{g_{T^-}}$ and we can deduce the law of Z_{T^-} from equation (87).

$$\boldsymbol{\mathbb{P}(Z_{T^-} \in dx) = -\frac{x}{D} \exp\left(-\frac{x^2}{2D}\right) \mathbf{1}_{x<0} dx.} \tag{91}$$

Moreover as T^- is $\mathcal{F}^+_{g_{T^-}}$-measurable, it comes out that Z_{T^-} and T^- are independent.

Using equation (89), process $\{\psi(-\lambda\mu_t)\exp(-\frac{1}{2}\lambda^2 t), t \geq 0\}$, is a $\mathcal{F}^+_{g_t}$ martingale for any $\lambda > 0$. By applying the optimal stopping time theorem at T^-, we obtain

$$\mathbb{E}(\psi(-\lambda\mu_{T^-})\exp(-\frac{1}{2}\lambda^2 T^-)) = \psi(0) = 1. \tag{92}$$

Since $\mu_{T^-} = -\sqrt{D}$, we get

$$\boldsymbol{\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T^-)) = \frac{1}{\psi(\lambda\sqrt{D})}.} \tag{93}$$

Similarly, $\frac{1}{\sqrt{D}} Z_{T^+} = m_1^{(T^-)}$ and the law of Z_{T^+} is given by

$$\boldsymbol{\mathbb{P}(Z_{T^+} \in dx) = \frac{x}{D}\exp\left(-\frac{x^2}{2D}\right)1_{x>0}dx.} \tag{94}$$

With exactly the same method, we find

$$\boldsymbol{\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T^+)) = \frac{1}{\psi(\lambda\sqrt{D})}.} \tag{95}$$

D.2. **Case** $b < 0$**.** This case study can be reduced to the previous one, with the help of the stopping time T_b. We can write $T_b^- = T_b + T^-(W)$, with

$$\begin{aligned} T_0^-(W) &= \inf\{t \geq 0; \mathbf{1}_{W_t \leq 0}(t - g_t^W) \geq D\} \stackrel{law}{=} T_0^-, \\ W &= \{W_t = Z_{T_b+t} - b; t \geq 0\}, \\ g_t^W &= \sup\{u \leq t; W_u = 0\}. \end{aligned}$$

Moreover using the strong Markov property it is clear that T_b and $T_0^-(W)$ are independent.

$$\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b^-)) = \mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b))\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_0^-(W))).$$

As $\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b)) = \exp(-|b|\lambda)$, we get

$$\boldsymbol{\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b^-)) = \frac{\exp(b\lambda)}{\psi(\lambda\sqrt{D})}.} \tag{96}$$

Now, we are trying to find the law of $Z_{T_b^-}$

$$\begin{aligned} \mathbb{P}(Z_{T_b^-} \in dx) &= \mathbb{P}(Z_{T_b^- - T_b} o\, \theta_{T_b} \in dx), \\ &= \mathbb{E}[\mathbf{1}_{Z_{T_b^- - T_b} o\, \theta_{T_b} \in dx}], \\ &= \mathbb{E}\left[\mathbb{E}[\mathbf{1}_{\{Z_{T_b^- - T_b} o\, \theta_{T_b} \in dx\}} | \mathcal{F}_{T_b}]\right], \\ &= \mathbb{E}\left[\mathbb{E}^b[\mathbf{1}_{\{Z_{T_b^- - T_b} \in dx\}} | \mathcal{F}_{T_b}]\right], \\ &= \mathbb{E}\left[\mathbb{E}^b[\mathbf{1}_{\{Z_{T_b^- - T_b} \in dx\}}]\right], \\ &= \mathbb{E}\left[\mathbb{E}^b[\mathbf{1}_{\{Z_{T^-} \in dx\}}]\right], \\ &= \mathbb{E}\left[\mathbb{P}^b[Z_{T^-} \in dx]\right], \\ &= \mathbb{E}\left[\mathbb{P}[Z_{T^-} + b \in dx]\right], \\ &= \mathbb{E}\left[\mathbb{P}[Z_{T^-} \in (dx - b)]\right], \\ &= \mathbb{P}[Z_{T^-} \in (dx - b)]. \end{aligned}$$

Finally we obtain

$$\boldsymbol{\mathbb{P}(Z_{T_b^-} \in dx) = \frac{b - x}{D}\exp\left(-\frac{(x-b)^2}{2D}\right)1_{\{x<b\}}dx.} \tag{97}$$

D.3. **Case $b > 0$.** If $b > 0$, we can write $T_b^+ = T_b + T_0^+(W)$ with

$$\begin{aligned} T_0^+(W) &= \inf\{t \geq 0; \mathbf{1}_{W_t \geq 0}(t - g_t^W) \geq D\} \stackrel{law}{=} T_0^+, \\ W &= \{W_t = Z_{T_b + t} - b; t \geq 0\}, \\ g_t^W &= \sup\{u \leq t; W_u = 0\}. \end{aligned}$$

It follows, from the independence of T_b and $T_0^+(W)$ by using the strong Markov property, that

$$\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b^+)) = \mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b))\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_0^+(W))).$$

As $\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b)) = \exp(-|b|\lambda)$, we get

$$\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b^-)) = \frac{\exp(-b\lambda)}{\psi(\lambda\sqrt{D})}. \tag{98}$$

The law of $Z_{T_b^+}$ can be computed in the same way as the law of $Z_{T_b^-}$

$$\mathbb{P}(Z_{T_b^+} \in dx) = \mathbb{P}[Z_{T^+} \in (dx - b)].$$

Finally, we have

$$\mathbb{P}(Z_{T_b^+} \in dx) = \frac{x-b}{D} \exp\left(-\frac{(x-b)^2}{2D}\right) \mathbf{1}_{\{x > b\}} dx. \tag{99}$$

Appendix E. Around Brownian Motion

Let us consider a standard Brownian motion $W = \{W_t; t \geq 0\}$. First of all, we recall two results on the joint law of the Brownian motion and its extrema. A proof can be found in Revuz and Yor [8].

E.1. **Law of $(W_t, \sup_{0 \leq u \leq t} W_u)$.**

$$\mathbb{P}(W_t \in dx, \sup_{0 \leq u \leq t} W_u \in dy) = \mathbf{1}_{\{0 \leq y\}} \mathbf{1}_{\{x \leq y\}} \frac{2(2y - x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2y-x)^2}{2t}\right) dx dy. \tag{100}$$

E.2. **Law of $(W_t, \inf_{0 \leq u \leq t} W_u)$.**

$$\mathbb{P}(W_t \in dx, \inf_{0 \leq u \leq t} W_u \in dy) = \mathbf{1}_{\{y \leq 0\}} \mathbf{1}_{\{y \leq x\}} \frac{2(2y - x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2y-x)^2}{2t}\right) dx dy \tag{101}$$

We try to compute

$$\begin{aligned} &\mathbb{P}(W_t \leq \lambda, \inf_{0 \leq u \leq t} W_u \leq \mu) \\ &= \mathbb{P}(-W_t \geq -\lambda, \sup_{0 \leq u \leq t} -W_u \geq -\mu), \\ &\qquad -W \text{ is also a standard Brownian motion, so we can write } W_t \text{ instead of } -W_t \\ &= \mathbb{P}(W_t \geq -\lambda, \sup_{0 \leq u \leq t} W_u \geq -\mu), \\ &= \mathbb{P}(W_t \geq -\lambda) - \mathbb{P}(\sup_{0 \leq u \leq t} W_u \leq -\mu) + \mathbb{P}(W_t \leq -\lambda, \sup_{0 \leq u \leq t} W_u \leq -\mu). \end{aligned} \tag{102}$$

Differentiating with respect to to μ and λ and using equation (100) give the result announced above.

E.3. **Hitting time.** The purpose is to find the law of T_b, which is defined as following :

$$T_b = \inf\{t \geq 0 \mid W_t = b\}.$$

Case $b > 0$. We want to calculate $\mathbb{P}(T_b \leq x)$ to find the law of T_b.
We have :

$$\begin{aligned} \mathbb{P}(T_b \leq x) &= \mathbb{P}\left(\inf\{t \geq 0 \mid W_t = b\} \leq x\right), \\ &= \mathbb{P}\left(\sup_{0 \leq u \leq x} W_u \geq b\right). \end{aligned} \tag{103}$$

Since we know the law of $\sup\limits_{0 \leq u \leq x} W_u$, which is

$$\mathbf{1}_{\{0 \leq y\}} \frac{2}{\sqrt{2\pi x}} e^{\left(\frac{-y^2}{2x}\right)} dy. \tag{104}$$

we obtain

$$\begin{aligned} \mathbb{P}(T_b \leq x) &= \int_{-\infty}^{\infty} \mathbf{1}_{\{y \geq b\}} \mathbf{1}_{\{0 \leq y\}} \frac{2}{\sqrt{2\pi x}} e^{\left(\frac{-y^2}{2x}\right)} dy, \\ &= \int_{b}^{\infty} \frac{2}{\sqrt{s\pi x}} e^{\left(\frac{-y^2}{2x}\right)} dy, \\ &= \int_{\frac{b}{\sqrt{x}}}^{\infty} \frac{2}{\sqrt{2\pi}} e^{\left(\frac{-u^2}{2}\right)} du. \end{aligned} \tag{105}$$

To find the law of T_b we just have to take the derivative with respect to x. Finally we find :

$$\mathbb{P}(T_b \in dx) = \frac{b}{\sqrt{2\pi x^3}} e^{\left(\frac{-b^2}{2x}\right)} dx. \tag{106}$$

Case $b < 0$.

$$\begin{aligned} \mathbb{P}(T_b \leq x) &= \mathbb{P}\left(\inf\{t \geq 0 \mid B_t = b\} \leq x\right), \\ &= \mathbb{P}\left(\inf_{0 \leq u \leq x} W_u \leq b\right), \\ &= \mathbb{P}\left(\sup_{0 \leq u \leq x} -W_u \geq -b\right). \end{aligned}$$

W and $-W$ follow the same law, $-b > 0$ so we can use what we have found in the first case, and we get :

$$\mathbb{P}(T_b \in dx) = \frac{-b}{\sqrt{2\pi x^3}} e^{-\frac{b^2}{2x}} dx. \tag{107}$$

For any b, we have

$$\boldsymbol{\mathbb{P}(T_b \in dx) = \frac{\mid b \mid}{\sqrt{2\pi x^3}} e^{-\frac{b^2}{2x}} dx.} \tag{108}$$

E.4. **Excursion.** Let g_t denote the last time before t that W hit the level 0.

$$g_t = \sup\{u \leq t \mid W_u = 0\}. \tag{109}$$

The purpose is to find the law of (g_t, W_t). Let $\mathbb{P}^x$ denote the probability starting at level x. The probability starting at the level 0 is simply denoted by $\mathbb{P}$.
First we would like to calculate $\mathbb{P}^x(W_t \in dy, T_0 > t)$, with $x > 0$ and $y > 0$.

$$\mathbb{P}^x(W_t \in dy, T_0 > t) = \mathbb{P}^x(W_t \in dy) - \mathbb{P}^x(W_t \in dy, T_0 < t). \tag{110}$$

Using the reflexion principle, we can stop the Brownian motion at time T_0 and reflect the rest of the trajectory. So it is the same for the Brownian motion issued from x to cross 0 before time t and to end up in the neighbourhood of y as to end up in the neighbourhood of $-y$. Thanks to the almost sure continuity of the Brownian motion paths we can drop the condition that the Brownian motion has hit 0 before time t. So we come up with the following equality

$$\mathbb{P}(W_{t-T_0^x} \in -dy, T_0^x < t) = \mathbb{P}^x(W_t \in -dy). \tag{111}$$

So, putting all the different terms together and using the law of the Brownian motion at time t, we come up with the following formula :

$$\mathbb{P}^x(W_t \in dy, T_0 > t) = \frac{1}{\sqrt{2\pi t}} \left(e^{-(x-y)^2/2t} - e^{-(x+y)^2/2t} \right) \mathbf{1}_{xy>0}\, dy. \tag{112}$$

Now, we can try to compute the law of (g_t, W_t). Let's calculate $\mathbb{P}(W_t \in dy, g_t \le s)$. If $t < s$, then g_t is always smaller than s because g_t is bounded by t, so the probability does not depend on s anymore. Thus, its partial differential with respect to s is identically null. Now we assume that $s \le t$, $y > 0$.

$$\begin{aligned}
\mathbb{P}(W_t \in dy, g_t \le s,) &= \mathbb{E}(\mathbf{1}_{\{W_t \in dy,\ g_t \le s\}}), \\
&= \mathbb{E}(\mathbb{E}(\mathbf{1}_{\{W_t \in dy,\ W_u \neq 0\ \forall u \in [s,t]\}} \mid \mathcal{F}_s)), \\
&= \mathbb{E}(\mathbb{E}(\mathbf{1}_{\{W_{t-s}\circ\theta_s \in dy,\ W_u\circ\theta_s \neq 0\ \forall u \in [0,t-s]\}} \mid \mathcal{F}_s)), \\
&\quad \text{Relying on the Markov property, we may write} \\
&= \mathbb{E}(\mathbb{E}^{W_s}(\mathbf{1}_{\{W_{t-s} \in dy,\ W_u \neq 0\ \forall u \in [0,t-s]\}})), \\
&\quad \text{we calculated the second expectation above, so we get} \\
&= \mathbb{E}\left(\frac{1}{\sqrt{2\pi(t-s)}} \left(e^{-(W_s-y)^2/2(t-s)} - e^{-(W_s+y)^2/2(t-s)} \right) dy \right), \\
&= \int_0^\infty dx \frac{1}{\sqrt{2\pi s}} e^{-x^2/2s} \frac{1}{\sqrt{2\pi(t-s)}} \left(e^{-(x-y)^2/2(t-s)} - e^{-(x+y)^2/2(t-s)} \right) dy, \\
&= \sqrt{\frac{s(t-s)}{t}} \int_{y\frac{s}{t(t-s)}}^\infty e^{-z^2/2} dz e^{-y^2/2t}.
\end{aligned} \tag{113}$$

Finally, we only have to differentiate with respect to s to come up with the formula of the density of (g_t, W_t).

$$\mathbb{P}(W_t \in dy, g_t \in ds) = \frac{y}{2\pi\sqrt{s(t-s)^3}} \exp\left(-\frac{y^2}{2(t-s)} \right) \mathbf{1}_{s \le t}\, ds dy. \tag{114}$$

If we assume that $y < 0$ then, since W and $-W$ follow the same law, we can write

$$\mathbb{P}(W_t \in dy, g_t \le s) = \mathbb{P}(W_t \in -dy, g_t \le s), \tag{115}$$

which enables us to refer to the previous case and the final formula for the law of the couple (g_t, W_t) is given by

$$\mathbb{P}(W_t \in dy, g_t \in ds) = \frac{|y|}{2\pi\sqrt{s(t-s)^3}} \exp\left(-\frac{y^2}{2(t-s)} \right) \mathbf{1}_{s \le t}\, ds\, dy. \tag{116}$$

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