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Piecewise rigidity

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PIECEWISE RIGIDITY

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Abstract. In this paper we provide a Liouville type theorem in the framework of fracture mechanics, and more precisely in the theory of SBV deformations for cracked bodies. We prove the following rigidity result: if $u \in SBV(\Omega, \mathbb{R}^N)$ is a deformation of Ω whose associated crack J_u has finite energy in the sense of Griffith's theory (i.e., $\mathcal{H}^{N-1}(J_u) < \infty$), and whose approximate gradient ∇u is almost everywhere a rotation, then u is a collection of an at most countable family of rigid motions. In other words, the cracked body does not store elastic energy if and only if all its connected components are deformed through rigid motions. In particular, global rigidity can fail only if the crack disconnects the body.

Contents

1. Introduction

A classical rigidity result in nonlinear elasticity, due to Liouville, states that if an elastic body is deformed in such a way that its deformation gradient is pointwise a rotation, then the body is indeed subject to a rigid motion. If the body is supposed to be hyperelastic with an elastic energy density $\mathcal{W}$ defined on a *natural* reference configuration Ω, a standard assumption for $\mathcal{W}$ which comes from its *frame indifference* is that $\mathcal{W}$ is minimized exactly on the set of rotations $SO(3)$. Hence the rigidity result implies that the body doesn't store elastic energy if and only if it is deformed through a rigid motion.

From a mathematical viewpoint, Liouville's Theorem can be stated as follows: if $\Omega \subseteq \mathbb{R}^N$ is open and connected, $u \in C^\infty(\Omega; \mathbb{R}^N)$ is such that $\nabla u(x) \in \mathrm{SO}(N)$ for every $x \in \Omega$, then $u = a + R \cdot x$ for some $a \in \mathbb{R}$ and $R \in \mathrm{SO}(N)$. The assumption on the regularity of u has been fairly weakened, and now the same rigidity result is available for deformations in the class of Sobolev maps (see Yu. Reshetnyak [17]). In this case the deformation gradient is defined only almost everywhere in Ω, so that the assumption for rigidity is $\nabla u(x) \in \mathrm{SO}(N)$ for a.e. $x \in \Omega$.

A quantitative rigidity estimate has been provided recently by Friesecke, James and Müller [13], in order to derive nonlinear plates theories from three dimensional elasticity. They proved that if Ω is connected and with Lipschitz boundary, there exists a constant C depending only on Ω and N such that for every $u \in W^{1,2}(\Omega, \mathbb{R}^N)$

$$\min_{R \in \mathrm{SO}(N)} \|\nabla u - R\|_{L^2(\Omega)} \le C \|\mathrm{dist}(\nabla u, \mathrm{SO}(N))\|_{L^2(\Omega)}. \tag{1.1}$$

As a consequence, if the deformation gradient is close to rotations (in L^2), then it is in fact close to a unique rotation. Estimate (1.1) is indeed true in L^p for every $1 < p < +\infty$ (see [10]).

The aim of this paper is to discuss the problem of rigidity in the framework of fracture mechanics, that is for bodies that can not only deform elastically, but also be cracked along surfaces where the deformation becomes discontinuous. The class of admissible deformations that we consider, in this setting, will be the space of *special functions of bounded variation* $SBV(\Omega;\mathbb{R}^N)$ (see Section 2 for a precise definition). Given $u \in SBV(\Omega;\mathbb{R}^N)$, the approximate gradient ∇u (which exists at almost every point of Ω) takes into account the elastic part of the deformation, while the jump set J_u represents a crack in the reference configuration. The set J_u is rectifiable, that is, it can be covered (up to a $\mathcal{H}^{N-1}$–negligible set) by a countable number of C^1 submanifolds of $\mathbb{R}^N$. So J_u is, in some sense, a $(N-1)$–dimensional surface.

In the context of SBV deformations, we cannot expect a rigidity result as for elastic deformations, because a crack can divide the body into two parts, each of one subject to a different rigid deformation. We prove that this is essentially the only way rigidity can be violated, provided the crack J_u has "finite energy" (which, in the framework of Griffith's theory, means that its total $(N-1)$–dimensional surface is finite). If the body is not suitably divided by a crack in several components, then rigidity as in the elastic case holds.

In order to formulate our result, we need some notions from geometric measure theory in order to make precise the notion of a partition Ω in connection with SBV deformations. We refer to Section 2 for more details. We say that a partition $(E_i)_{i\in\mathbb{N}}$ of Ω is a *Caccioppoli partition* if $\sum_{i\in\mathbb{N}} P(E_i,\Omega) < +\infty$, where $P(E_i,\Omega)$ denotes the perimeter of E_i in Ω. Given a rectifiable set $K \subset \Omega$, we say that a Caccioppoli partition $(E_i)_{i\in\mathbb{N}}$ of Ω is subordinated to K if (up to a $\mathcal{H}^{N-1}$–negligible set) the reduced boundary $\partial^* E_i$ of E_i is contained in K for every $i \in \mathbb{N}$. We say that $\Omega \setminus K$ is *indecomposable* if the only Caccioppoli partition subordinated to K is the trivial one, i.e., $E_0 = \Omega$.

The main rigidity result of the paper is the following Liouville's type theorem for SBV- deformations.

Theorem 1.1. *Let $u \in SBV(\Omega,\mathbb{R}^N)$ such that $\mathcal{H}^{N-1}(J_u) < +\infty$ and $\nabla u(x) \in SO(N)$ for a.e. $x \in \Omega$. Then u is a collection of an at most countable family of rigid deformations, i.e., there exists a Caccioppoli partition $(E_i)_{i\in\mathbb{N}}$ subordinated to J_u such that*

$$u = \sum_{i\in\mathbb{N}} (K_i x + b_i)\mathbf{1}_{E_i}(x),$$

where $K_i \in SO(N)$ and $b_i \in \mathbb{R}^N$ (and, as a consequence, $J_u \subseteq \cup_{i\in\mathbb{N}}\partial^* E_i$). In particular, if $\Omega \setminus J_u$ is indecomposable, then u is a rigid deformation, i.e., $u(x) = a + R \cdot x$ for some $a \in \mathbb{R}^N$ and $R \in SO(N)$ (hence, $J_u = \emptyset$).

Let us observe that the assumption that $\mathcal{H}^{N-1}(J_u)$ is finite is essential in this result. Indeed, it has been shown by Alberti [1, 5] that any N–dimensional L^1 vector field can be the gradient of a suitable SBV function, so that the rigidity clearly fails if one just assumes $\nabla u(x) \in \mathrm{SO}(N)$ for a.e. $x \in \Omega$.

In the context of fracture mechanics, Theorem 1.2 implies the following fact. Assume that the density of the elastic energy stored in the cracked body is represented by a function $\mathcal{W}$ vanishing exactly on $\mathrm{SO}(N)$. Then a deformation u of class SBV does not store elastic energy if and only if the crack J_u divides Ω in several subbodies, each of one subject to a rigid motion. If J_u is not enough to create subbodies of Ω, then u is a rigid motion for the entire body (and there is no jump J_u at all). In this respect, the space SBV seems to be appropriate for the study of elastic properties of cracked hyperelastic bodies.

The main difficulty to prove Theorem 1.1 is that the differential constraint $\operatorname{curl} \nabla u = 0$, valid for every Sobolev function, does not hold in general for SBV functions, because ∇u is only a part of the distributional derivative of u. However we prove that if $u \in SBV(\Omega; \mathbb{R}^N)$ with $\nabla u \in L^\infty(\Omega; M^{N\times N})$ then $\operatorname{curl} \nabla u$ is a measure, which is absolutely continuous with respect to $\mathcal{H}^{N-1} \llcorner J_u$. This result (up to our knowledge, new and interesting on its own), combined with the quantitative rigidity estimate (1.1) is enough to obtain our rigidity result.

The set of rotations in $\mathbb{R}^N$ can be replaced by any compact set of matrices $\mathcal{K} \subseteq M^{N\times N}$ which satisfy a L^p-quantitative rigidity estimate for $1 < p < \frac{N}{N-1}$, i.e., there exists $C > 0$ depending on N and p such that, for every $u \in W^{1,p}(\Omega, \mathbb{R}^N)$,

$$\min_{K \in \mathcal{K}} \|\nabla u - K\|_{L^p(\Omega)} \leq C \|\operatorname{dist}(\nabla u, \mathcal{K})\|_{L^p(\Omega)}. \tag{1.2}$$

Theorem 1.1 is obtained as a particular case of the following rigidity result.

Theorem 1.2 (The rigidity result). *Let $\mathcal{K} \subseteq M^{N\times N}$ be a compact set such that the quantitative rigidity estimate (1.2) holds for some $p \in (1, N/(N-1))$. Let $u \in SBV(\Omega, \mathbb{R}^N)$ be such that $\mathcal{H}^{N-1}(J_u) < +\infty$ and $\nabla u(x) \in \mathcal{K}$ for a.e. $x \in \Omega$. Then there exists a Caccioppoli partition $(E_i)_{i\in\mathbb{N}}$ of Ω subordinated to J_u such that*

$$u = \sum_{i\in\mathbb{N}} (K_i x + b_i) \mathbf{1}_{E_i}(x),$$

where $K_i \in \mathcal{K}$ and $b_i \in \mathbb{R}^N$ (and, as a consequence, $J_u \subseteq \cup_{i\in\mathbb{N}} \partial^* E_i$). In particular if $\Omega \setminus J_u$ is undecomposable, then $u = Kx + b$ for some $K \in \mathcal{K}$, $b \in \mathbb{R}^N$ (hence, $J_u = \emptyset$).

In order to prove Theorem 1.2, the key point is to show that ∇u is a piecewise constant function that can jump only on J_u, i.e., $\nabla u \in SBV(\Omega, M^{N\times N})$ with $\nabla(\nabla u) = 0$ and $J_{\nabla u} \subseteq J_u$: this implies that ∇u is constant on a Caccioppoli partition subordinated to J_u, and hence that u is affine on the same partition.

In order to establish that ∇u is piecewise constant with jumps on J_u, we use an approximation based on a covering argument inspired by [13]. First of all we split our domain in a disjoint union of small cubes Q_h of size h. On many of these cubes, $\mathcal{H}^{N-1}(J_u \cap Q_h)$ will be small, showing that $\operatorname{curl} \nabla u$ is close to zero in Q_h. A Helmholtz' type estimate for L^1 vector fields with curl-measure shows then that ∇u is close in L^p to the gradient ∇w_h of a Sobolev function, which by the quantitative rigidity estimate (1.2) is close in L^p to a unique matrix $K(Q_h) \in \mathcal{K}$. We show that ∇u is approximated by the piecewise constant functions ψ_h such that $\psi_h = w_h$ on Q_h. The sequence $(\psi_h)_{h\in\mathbb{N}}$ has a uniformly bounded total variation which is controlled by $\operatorname{curl} \nabla u$ and so by $\mathcal{H}^{N-1} \llcorner J_u$: we prove this, as in [13], by using again the quantitative rigidity estimate on the union of neighboring cubes. An application of Ambrosio's compactness theorem for SBV functions [2, 3, 4] is then enough to get the conclusion.

Let us mention that a local version of Liouville Theorem on sets of finite perimeter, for Lipschitz mappings, was already given in [12]. There, Dolzmann and Müller prove that if $u : \Omega \to \mathbb{R}^N$ is in $W^{1,\infty}(\Omega; \mathbb{R}^N)$, $\det \nabla u \geq c > 0$, and $\nabla u \in \mathrm{SO}(N)$ for a.e. $x \in E$, where E is a subset of Ω with finite perimeter, then $\nabla u \mathbf{1}_E \in BV(\Omega)$, and $D(\nabla u \mathbf{1}_E) \llcorner (\Omega \setminus \partial^* E) = 0$. (So that the thesis of Theorem 1.1 holds inside E.)

Rigidity results in the spirit of Liouville's Theorem play also an important role in order to understand possible microstructures arising in elastic bodies. The problem of microstructures can be stated mathematically in the following way: given a set of matrices $\mathcal{K} \subseteq M^{N\times N}$, find Lipschitz mappings $u : \Omega \to \mathbb{R}^N$ such that $\nabla u(x) \in \mathcal{K}$ for a.e. $x \in \Omega$. $\mathcal{K}$ is said to be rigid if it doesn't admit nontrivial microstructures, i.e., if the only maps $u \in W^{1,\infty}(\Omega)$ such that $\nabla u(x) \in \mathcal{K}$ for a.e. $x \in \Omega$ are affine.

An example of rigid set of matrices is provided by a famous result by Ball and James [6]: $\mathcal{K} = \{K_1, K_2\}$ is rigid if and only if $\operatorname{rank}(K_1 - K_2) \geq 2$. In this case, Ball and James proved that rigidity holds also in the stronger sense of *approximate solutions*: for every sequence $(u_h)_{h\in\mathbb{N}}$ of equi-Lipschitz functions such that $\operatorname{dist}(\nabla u_h, \mathcal{K}) \to 0$ in measure, then either $\operatorname{dist}(\nabla u_h, K_1) \to 0$, or $\operatorname{dist}(\nabla u_h, K_2) \to 0$ in measure.

Theorem 1.2 can be used to infer a similar result in the framework of the discontinuous deformations of class SBV. The quantitative rigidity estimate we need to apply our arguments has been recently provided by De Lellis and Székelyhidi [11]: they prove that if $\mathcal{K} \subseteq M^{N\times N}$ is a finite set of matrices which is rigid for approximate solutions, then the quantitative rigidity estimate (1.2) holds for any $p \in (1, +\infty)$ provided that Ω is Lipschitz-regular. As a consequence, we can deduce the following result.

Theorem 1.3. *Let $\mathcal{K} := \{K_1,\, K_2\}$, with $\operatorname{rank}(K_1 - K_2) \geq 2$. Let $u \in SBV_\infty(\Omega, \mathbb{R}^N)$ such that $\nabla u(x) \in \mathcal{K}$ for a.e. $x \in \Omega$. Then there exists a Caccioppoli partition $(E_i)_{i\in\mathbb{N}}$ of Ω subordinated to J_u such that*

$$u = \sum_{i\in\mathbb{N}} (K_i x + b_i)\mathbf{1}_{E_i}(x),$$

where $K_i \in \mathcal{K}$ and $b_i \in \mathbb{R}^N$ (and, as a consequence, $J_u \subseteq \cup_{i\in\mathbb{N}}\partial^* E_i$). In particular if $\Omega \setminus J_u$ is indecomposable, then $u = Kx + b$ for some $K \in \mathcal{K}$ and $b \in \mathbb{R}^N$ (hence, $J_u = \emptyset$).

The rigidity result with respect to approximate solutions by Ball and James has been generalized to the case where $\mathcal{K}$ consists of three matrices without any rank-1 connection in [18], so that Theorem 1.3 still holds in this case. For completeness, let us say that if $\mathcal{K}$ consists of four matrices without any rank-1 connection, rigidity can fail for approximate solutions for a suitable choice of the involved matrices (see [19], [20]), while $\mathcal{K}$ is always rigid with respect to exact solutions (see [9]). The case $N = 5$ is nicely illustrated in [14] by a non-rigid five point configuration without any rank-1 connection.

The paper is organized as follows. In Section 2 we recall some facts from geometric measure theory and from the theory of SBV spaces. In Section 3, we show that the curl of a function u that satisfies the assumptions of the rigidity theorem is a Radon measure absolutely continuous with respect to $\mathcal{H}^{N-1} \llcorner J_u$. Section 4 is devoted to the statement and proof of some Helmholtz type estimates in cubes. The proof of Theorem 1.2 is then given in Section 5.

2. Notations and preliminaries

In this section we recall the definition of the space SBV and some facts from geometric measure theory that will be used throughout the paper. We refer to [5] for further details.

The space SBV. Let Ω be an open set in $\mathbb{R}^N$. We say that $u \in BV(\Omega; \mathbb{R}^N)$ if $u \in L^1(\Omega; \mathbb{R}^N)$, and its distributional derivative Du is a vector-valued Radon measure on Ω. We say that $u \in SBV(\Omega; \mathbb{R}^N)$ if $u \in BV(\Omega; \mathbb{R}^N)$ and its distributional derivative can be represented as

$$Du(A) = \int_A \nabla u(x)\, dx + \int_{A\cap J_u} (u^+(x) - u^-(x)) \otimes \nu_x \, d\mathcal{H}^{N-1}(x),$$

where ∇u denotes the approximate gradient of u, J_u denotes the set of approximate jumps of u, u^+ and u^- are the traces of u on J_u, ν_x is the normal to J_u at x, and $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. The symbol $\otimes$ denotes the tensorial product of vectors: $(a \otimes b)_{ij} = a_i b_j$ for every $a, b \in \mathbb{R}^N$.

Note that if $u \in SBV(\Omega;\mathbb{R}^N)$, then the singular part of Du is concentrated on J_u which is a countably $\mathcal{H}^{N-1}$-rectifiable set: there exists a set E with $\mathcal{H}^{N-1}(E) = 0$ and a sequence $(M_i)_{i\in\mathbb{N}}$ of C^1-submanifolds of $\mathbb{R}^N$ such that $J_u \subseteq E \cup \bigcup_{i\in\mathbb{N}} M_i$.

We set, for $d \geq 1$,

$$SBV_\infty(\Omega;\mathbb{R}^d) := \{u \in SBV(\Omega;\mathbb{R}^d) : \nabla u \in L^\infty(\Omega; M^{d\times N}),\ \mathcal{H}^{N-1}(J_u) < +\infty\}. \tag{2.1}$$

and, as usual, $SBV_\infty(\Omega) := SBV_\infty(\Omega;\mathbb{R})$ whenever $d = 1$.

Piecewise constant functions and Caccioppoli partitions. Let Ω be an open set in $\mathbb{R}^N$, and let $E \subseteq \Omega$. We say that E has finite perimeter in Ω if $\mathbf{1}_E \in SBV(\Omega)$. The set of jumps of $\mathbf{1}_E$ is denoted by $\partial^* E$ and is called the reduced boundary of E: the derivative of $\mathbf{1}_E$ is concentrated on $\partial^* E$, and its total variation is given by $\mathcal{H}^{N-1} \llcorner \partial^* E$. The perimeter of E in Ω is given by $\mathcal{H}^{N-1}(\partial^* E)$.

We say that a partition $(E_i)_{i\in\mathbb{N}}$ of Ω is a *Caccioppoli partition* if $\sum_{i\in\mathbb{N}} \mathcal{H}^{N-1}(\partial^* E) < +\infty$. Given a rectifiable set $K \subset \Omega$, we say that a Caccioppoli partition $(E_i)_{i\in\mathbb{N}}$ of Ω is subordinated to K if (up to a $\mathcal{H}^{N-1}$–negligible set) the reduced boundary $\partial^* E_i$ of E_i is contained in K for every $i \in \mathbb{N}$. We say that $\Omega \setminus K$ is *indecomposable* if the only Caccioppoli partition subordinated to K is the trivial one, i.e., $E_0 = \Omega$.

Caccioppoli partitions are naturally associated to piecewise constant functions, i.e., functions $u \in SBV(\Omega;\mathbb{R}^N)$ such that $\nabla u = 0$ a.e. on Ω. These functions are said piecewise contant in Ω because they are indeed constant on the subsets E_i of a Caccioppoli partition of Ω. More precisely (see [5, Theorem 4.23]) there exists a Caccioppoli partition $(E_i)_{i\in\mathbb{N}}$ of Ω such that

$$u = \sum_{i\in\mathbb{N}} b_i \mathbf{1}_{E_i}, \tag{2.2}$$

with $b_i \neq b_j$ for $i \neq j$. Notice that if K is a rectifiable set in Ω such that $\Omega \setminus K$ is indecomposable, then a piecewise constant function u in Ω with $J_u \subseteq K$ is necessarily constant on Ω.

3. $\operatorname{curl} \nabla u$ IS A MEASURE FOR $u \in SBV_\infty(\Omega)$

In this section, we show that the curl of a function u that satisfies the assumptions of the theorem is in fact a measure, estimated with $\mathcal{H}^{N-1} \llcorner J_u$.

Theorem 3.1. *Let $u \in SBV_\infty(\Omega)$. Then $\operatorname{curl} \nabla u$ is a measure μ concentrated on J_u such that*

$$|\mu| \leq c \|\nabla u\|_\infty \mathcal{H}^{N-1} \llcorner J_u.$$

In this statement, the constant c depends on the dimension N. However, we conjecture that the optimal constant is $2\sqrt{2}$ (considering the Frobenius norm for matrices).

Remark 3.2. Clearly, if $u \in SBV_\infty(\Omega;\mathbb{R}^d)$ is a vector-valued function ($d \geq 2$), then the result still holds (with the same constant c if the norm on tensors is still the Euclidean norm of the associated matrix).

Proof. Let $u \in SBV_\infty(\Omega)$. We have: $u \in L^1(\Omega)$, $L := \|\nabla u\|_\infty < +\infty$, and $\mathcal{H}^{N-1}(J_u) < +\infty$. The distribution $\operatorname{curl} \nabla u$ is formally equal to the matrix $(\partial_i(\partial_j u) - \partial_j(\partial_i u))_{1\leq i,j\leq N}$ and is defined by

$$\langle \operatorname{curl} \nabla u, \varphi \rangle \ = \ \sum_{i,j=1}^N \int_\Omega \partial_i u(x) \partial_j (\varphi_{i,j} - \varphi_{j,i})(x)\, dx\,,$$

for any $\varphi \in C_c^\infty(\Omega; M^{N\times N})$. The thesis of the theorem is local, so that it is enough to prove that if $Q \subset\subset \Omega$ is a hypercube in Ω, then for any $\varphi \in C_c^\infty(Q; M^{N\times N})$, one has

$$\sum_{i,j=1}^{N} \int_Q \partial_i u(x)\partial_j(\varphi_{i,j} - \varphi_{j,i})(x)\,dx \;\le\; c\,\|\nabla u\|_\infty \|\varphi\|_\infty \mathcal{H}^{N-1}(J_u \cap Q)\,. \tag{3.1}$$

Without loss of generality, we may assume that $Q = (0,1)^N$. We will approximate u in Q with a piecewise smooth function, jumping only on facets of smaller hypercubes. This will be done using a simplified variant of the discretization/reinterpolation technique presented in [7, 8], and inspired from [16].

Step 1. Consider the set $J = J_u \cap Q$. Denote by $(e_i)_{i=1}^N$ the canonical basis of $\mathbb{R}^N$ ($e_i = (\delta_{i,j})_{j=1}^N$). One easily shows that for any i, the set

$$J_i^\varepsilon := \{-te_i + x \,:\, t \in [0,\varepsilon], x \in J\}$$

is Lebesgue-measurable. Indeed, up to a $\mathcal{H}^{N-1}$–negligible set $\mathcal{N}$, J is a countable union of compact sets: hence J_i^ε is the union of $[-\varepsilon e_i, 0] + \mathcal{N}$, which has Lebesgue measure zero, and of a countable union of compact sets. We have the estimate

$$|J_i^\varepsilon| \;\le\; \varepsilon \mathcal{H}^{N-1}(J)\,,$$

which can be derived in several ways, and more precisely one can show

$$|J_i^\varepsilon| \;\le\; \varepsilon \int_J |\nu_i(x)|\,d\mathcal{H}^{N-1} \tag{3.2}$$

where $\nu(x) = (\nu_1(x), \dots, \nu_N(x))$ is the normal to J at x, defined for $\mathcal{H}^{N-1}$-a.a. $x \in J$. For $y \in (0,1)^N$, we now also define the discrete binary variable $l^y_{\varepsilon,i}(k) := \mathbf{1}_{J_i^\varepsilon}(\varepsilon y + k)$, for any $k \in \varepsilon\mathbb{Z}^N \cap Q$.

One shows that for any $i = 1, \dots, N$

$$\int_{(0,1)^N} \varepsilon^{N-1} \sum_{k\in\varepsilon\mathbb{Z}^N\cap Q} l^y_{\varepsilon,i}(k)\,dy \;=\; \varepsilon^{-1} \int_{(0,\varepsilon)^n} \sum_{k\in\varepsilon\mathbb{Z}^N\cap Q} \mathbf{1}_{J_i^\varepsilon}(y+k)\,dy \;=\; \varepsilon^{-1}|J_i^\varepsilon|\,.$$

Hence using (3.2) and $\sum_{i=1}^N |\nu_i| \le \sqrt{N}$,

$$\int_{(0,1)^N} \varepsilon^{N-1} \sum_{i=1}^{N} \sum_{k\in\varepsilon\mathbb{Z}^N\cap Q} l^y_{\varepsilon,i}(k)\,dy \;\le\; \sqrt{N}\mathcal{H}^{N-1}(J).$$

Using Fatou's lemma, we deduce

$$\int_{(0,1)^N} \left(\liminf_{\varepsilon\to 0} \varepsilon^{N-1} \sum_{i=1}^{N} \sum_{k\in\varepsilon\mathbb{Z}^N\cap Q} l^y_{\varepsilon,i}(k) \right) dy \;\le\; \sqrt{N}\mathcal{H}^{N-1}(J),$$

so that for any $\delta > 0$, there exists a set A of positive measure in $(0,1)^N$, such that

$$y \in A \;\Rightarrow\; \liminf_{\varepsilon\to 0} \varepsilon^{N-1} \sum_{i=1}^{N} \sum_{k\in\varepsilon\mathbb{Z}^N\cap Q} l^y_{\varepsilon,i}(k) \;\le\; \sqrt{N}\mathcal{H}^{N-1}(J) + \delta. \tag{3.3}$$

Step 2. Let now $\Delta(t) := \max\{1 - |t|, 0\}$ ($t \in \mathbb{R}$) and $\Delta_N(\xi) = \prod_{i=1}^N \Delta(\xi_i)$ for all $\xi \in \mathbb{R}^N$ (which is known in finite elements approximation as the "Q1" interpolation function). If we let

$$v^y_\varepsilon(x) := \sum_{k\in\varepsilon\mathbb{Z}^N\cap Q} u(\varepsilon y + k)\Delta_N\left(\frac{x-k}{\varepsilon} - y\right),$$

it is well known that for a.e. $y \in (0,1)^N$, $v^y_\varepsilon \to u$ in $L^1(Q)$ (see for instance [7]).

Step 3. The slicing properties of BV functions (see [5]) also ensure that for all i and $\mathcal{H}^{N-1}$-a.e. $z \in \{x \in Q \,:\, x_i = 0\}$, the function $(0,1) \ni t \mapsto u(z + te_i)$ is in $SBV(0,1)$, with finite jump set given by $\{t \,:\, z + te_i \in J\}$, and whose derivative is given by $t \mapsto \partial_i u(z + te_i)$, which by assumption is bounded by L. We deduce that for a.e. $y \in (0,1)^N$, the discrete function v^y_ε satisfies $|v^y_\varepsilon(k + \varepsilon e_i) - v^y_\varepsilon(k)| \le L\varepsilon$ for any $i = 1, \dots, N$ and $k \in \varepsilon\mathbb{Z}^N \cap Q$ such that $k + \varepsilon e_i \in Q$ and $J \cap [\varepsilon y + k, \varepsilon y + k + \varepsilon e_i] = \emptyset$, which is equivalent to $l^y_{\varepsilon,i}(k) = 0$.

Step 4. From steps 1, 2 and 3, there exists $y \in A$ such that:

$$\liminf_{\varepsilon\to 0} \varepsilon^{N-1} \sum_{i=1}^{N} \sum_{k\in\varepsilon\mathbb{Z}^N\cap Q} l^y_{\varepsilon,i}(k) \;\le\; \sqrt{N}\mathcal{H}^{N-1}(J) + \delta\,, \tag{3.4}$$

$v^y_\varepsilon \to u$ in $L^1(Q)$, and $|v^y_\varepsilon(k+\varepsilon e_i) - v^y_\varepsilon(k)| \le L\varepsilon$ for any $i = 1, \dots, N$ and $k \in \varepsilon\mathbb{Z}^N \cap Q$ such that $k + \varepsilon e_i \in Q$ and $l^y_{\varepsilon,i}(k) = 0$. We choose a sequence $(\varepsilon_j)_{j\ge 1}$ such that the lim inf in (3.4) is in fact a limit, and let $v_j := v^y_{\varepsilon_j}$, and $l_{j,i} := l^y_{\varepsilon_j,i}$.

From now on, since we refer only to the grids $\{\varepsilon_j y + \varepsilon_j k \,:\, k \in \mathbb{Z}^N\}$ which we use to interpolate u, we can assume (up to translation) that $y = 0$, so that they coincide with the grids $\{\varepsilon_j k \,:\, k \in \mathbb{Z}^N\}$.

In a small cube $k + (0, \varepsilon_j)^N$ in Q ($k \in \varepsilon_j\mathbb{Z}^N$), as soon as J does not intersect any edge of the cube, one has $|\partial_i v_j| \le L$ for all $i = 1, \dots, N$ so that $|\nabla v_j| \le \sqrt{N}L$ inside the cube. Given an edge $[k, k + \varepsilon_j e_i]$, if $l_{j,i}(k) = 1$, then J intersects the edge. In this case, we cannot control $|\nabla v_j|$ in all the cubes in Q that share this edge, whose total number is at most 2^{N-1}. We let K_j be the union of all such cubes: by (3.4) we have the estimate

$$|K_j| \;\le\; 2^{N-1}\varepsilon_j^N \sum_{i=1}^{N} \sum_{k\in\varepsilon_j\mathbb{Z}^N\cap Q} l_{j,i}(k) \;\le\; c\varepsilon_j\,. \tag{3.5}$$

On the other hand we have

$$\mathcal{H}^{N-1}(\partial K_j) \;\le\; (N+1)2^{N-1}\varepsilon_j^{N-1} \sum_{i=1}^{N} \sum_{k\in\varepsilon_j\mathbb{Z}^N\cap Q} l_{j,i}(k)\,,$$

so that (using (3.4), with the "$\liminf_{\varepsilon\to 0}$" replaced with "$\lim_{j\to\infty}$")

$$\limsup_{j\to\infty} \mathcal{H}^{N-1}(\partial K_j) \le C(N)(N+1)2^{N-1}(\sqrt{N}\mathcal{H}^{N-1}(J) + \delta). \tag{3.6}$$

Let $v'_j = v_j \mathbf{1}_{Q\setminus K_j}$. By (3.5), we still have $v'_j \to u$ in $L^1(Q)$ as $j \to \infty$. The previous discussion shows that in any $Q' \subset\subset Q$, for j large enough, $v'_j \in SBV(Q')$ with $\|\nabla v'_j\|_\infty \le \sqrt{N}L$, v'_j is piecewise smooth and $S(v'_j) \subset \mathcal{H}^{N-1}(\partial K_j)$ is a subset of a finite number of facets of hypercubes.

By Ambrosio's theorem [5, Theorem 4.36], we know that $\nabla v'_j \rightharpoonup \nabla u$ in $L^p(Q')$ (for any $p < +\infty$). Hence $\operatorname{curl} \nabla v'_j \stackrel{*}{\rightharpoonup} \operatorname{curl} \nabla u$ as $j \to \infty$, in the distributional sense. On the other hand, since

$$Dv'_j \;=\; \nabla v'_j(x)\, dx \;+\; v'_j\, \nu_{K_j} \mathcal{H}^{N-1} \llcorner \partial K_j,$$

(where ν_{K_j} is the exterior normal to K_j and v'_j stands here for the non-zero trace of v'_j on the exterior surface of K_j), and since $\operatorname{curl} Dv'_j = 0$, one has

$$\operatorname{curl} \nabla v'_j \;=\; -\operatorname{curl}(v'_j\, \nu_{K_j} \mathcal{H}^{N-1} \llcorner \partial K_j)\,,$$

which can be shown to be equal to

$$-\,(\nabla_\tau v'_j) \wedge \nu_{K_j} \mathcal{H}^{N-1} \llcorner \partial K_j$$

where $a \wedge b$ denotes the antisymmetric tensor product $a \otimes b - b \otimes a$. Hence its total variation, as a measure, is bounded by $\sqrt{2N}L\mathcal{H}^{N-1}(\partial K_j)$. If $\varphi \in C^\infty_c(Q; M^{N\times N})$ is fixed, one has therefore (choosing Q' such that $\mathrm{supp}\varphi \subset\subset Q'$),

$$\langle \mathrm{curl}\,\nabla v_j', \varphi \rangle \;\leq\; \sqrt{2N}L\,\|\varphi\|_\infty \mathcal{H}^{N-1}(\partial K_j)\,.$$

Passing to the limit and recalling (3.6), we get

$$\langle \mathrm{curl}\,\nabla u, \varphi \rangle \;\leq\; \sqrt{2N}L\,\|\varphi\|_\infty (N+1)2^{N-1}(\sqrt{N}\mathcal{H}^{N-1}(J)+\delta)\,.$$

Sending δ to zero and recalling $J = J_u \cap Q$ and $L = \|\nabla u\|_\infty$, we conclude that (3.1) holds with a constant $c \leq \sqrt{2}N(N+1)2^{N-1}$. This shows the thesis of the Theorem. □

Remark 3.3. The set J_u is rectifiable: for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$, if $\rho > 0$ is small enough, $J_u \cap B(x,\rho)$ is a C^1 hypersurface that cuts the ball $B = B(x,\rho)$ into two disjoint Lipschitz sets, up to a set of $\mathcal{H}^{N-1}$ measure $o(\rho^{N-1})$. Moreover, up to a change of basis, we have $\nu \simeq e_1$ (and $|\nu_1| \simeq 1$, $|\nu_i| << 1$ for $i \geq 2$) in $J_u \cap B$. A similar study (see again [7, 8]) will show that in such a ball B, $|\mathrm{curl}\,\nabla u|(B) \lesssim 2\sqrt{2N}\|\nabla u\|_\infty \mathcal{H}^{N-1}(J_u \cap B)$. Passing to the limit $\rho \to 0$, we improve the constant c in the Theorem: $c \leq 2\sqrt{2N}$. We expect, however, that a different approximation technique, possibly not based on a discretization, would help remove the $\sqrt{N}$ in that constant.

Remark 3.4. Notice that the assumption $u \in SBV_\infty(\Omega)$ is essential in order to obtain that $\mathrm{curl}\,\nabla u$ is a measure absolutely continuous with respect to $\mathcal{H}^{N-1} \llcorner J_u$. In general, $\mathrm{curl}\,\nabla u$ is not even a measure in Ω for $u \in SBV(\Omega)$. In fact it suffices to consider $f \in L^1(\Omega)$ such that $\mathrm{curl}\, f$ is a distribution of order one in Ω, and the function $u \in SBV(\Omega)$ given by Alberti's result [1] such that $\nabla u = f$. More explicit counterexamples can be constructed as follows. We consider functions defined on $\Omega \subseteq \mathbb{R}^2$, so that we can identify $\mathrm{curl}\,\nabla u$ with the distribution

$$\langle \mathrm{curl}\,\nabla u, \varphi \rangle := \int_\Omega (\partial_2 u \partial_1 \varphi - \partial_1 u \partial_2 \varphi)\, dx,$$

where $\varphi \in C_c^\infty(\Omega)$.

(a) If we drop the assumption $\nabla u \in L^\infty(\Omega, \mathbb{R}^N)$, we can consider Ω as the square $Q_1 =]-1,1[^2$ of $\mathbb{R}^2$ and $u \in SBV(Q_1)$ defined as

$$u(x,y) := \begin{cases} \ln(x^2+y^2) & \text{if } y > 0 \\ 0 & \text{if } y < 0. \end{cases}$$

It can be easily checked that $\mathrm{curl}\,\nabla u$ is a distribution of order one.

(b) If we drop the assumption $\mathcal{H}^{N-1}(J_u) < +\infty$, we can reason as follows. Let ϑ be the 2-periodic function on $\mathbb{R}$ such that $\vartheta(x) = 1 - |x|$ for $x \in [-1,1]$, and let $\vartheta_k(x) := \frac{1}{k}\vartheta(kx)$. Let $Q_1 =]-1,1[^2$, and let for $n \geq 1$

$$S_n := \left\{ (x,y) \in Q_1 \,:\, \frac{1}{n+1} < y < \frac{1}{n} \right\}.$$

We can find $k_n \in \mathbb{N}$ in such a way that $k_n \nearrow +\infty$ and

$$u(x,y) := \begin{cases} \vartheta_{2^{k_n}}(x) & \text{if } (x,y) \in S_n \\ 0 & \text{if } y < 0 \end{cases}$$

belongs to $SBV(Q_1)$. Moreover, $|\nabla u(x,y)| = 1$ a.e. on Q_1, so that $\nabla u \in L^\infty(Q_1, \mathbb{R}^N)$. Clearly $\mathrm{curl}\,\nabla u$ is a Radon measure on every open set $A_n := \{(x,y) \in Q_1 \,:\, -1/2 < x < 1/2, \frac{1}{n+1} < y < \frac{3}{4}\}$ (which is compactly contained in Q_1), but $|\mathrm{curl}\,\nabla u|(A_n) = n-1$. As a consequence $\mathrm{curl}\,\nabla u$ cannot be a measure on Q_1.

4. Some estimates for vector fields in a cube

Let us first show the following estimate, valid for smooth vector fields.

Proposition 4.1 (Helmholtz's type estimate). *Let $Q = (0,1)^N$ be the unit cube of $\mathbb{R}^N$, let $f \in L^1(Q)$ and let $\varphi \in C^\infty(\overline{Q};\mathbb{R}^N)$ be a vector field such that*

$$\begin{cases} \operatorname{div}\varphi = 0 & \text{in } Q,\\ \operatorname{curl}\varphi = f & \text{in } Q,\\ \varphi\cdot\nu = 0 & \text{on } \partial Q. \end{cases}$$

Then for every $1 \le p < \frac{N}{N-1}$ there exists a constant C depending only on N and p such that

$$\|\varphi\|_{L^p(Q)} \le C\|f\|_{L^1(Q)}. \tag{4.1}$$

Proof. Let us consider $\eta = (\eta_1,\dots,\eta_N) \in C_c^\infty(Q;\mathbb{R}^N)$, and let $g = (g_1,\dots,g_N) \in H^1(Q;\mathbb{R}^N)$ be a solution of the equation

$$\begin{cases} \Delta g = \eta & \text{in } Q,\\ g_i = 0 & \text{on } \partial_{e_i^\perp} Q\\ \frac{\partial g_i}{\partial\nu} = 0 & \text{on } \partial_{e_i^\parallel} Q\,, \end{cases} \tag{4.2}$$

where still, $\{e_i : i = 1,\dots,N\}$ is the canonical basis of $\mathbb{R}^N$, and $\partial_{e_i^\perp}Q$ and $\partial_{e_i^\parallel}Q$, denote the faces of ∂Q orthogonal and parallel to e_i respectively. (Observe that (4.2) corresponds to finding g that minimizes the energy $\int_Q |\nabla g|^2 + 2\eta\cdot g$, with boundary condition $g\cdot\nu = 0$ on ∂Q.)

It is quite standard that such a g is smooth, and we will show later on that for every $1 < q < +\infty$, we have the estimate

$$\|g\|_{W^{2,q}(Q)} \le C\|\eta\|_{L^q(Q)}, \tag{4.3}$$

where C depends only on N and q. If in particular $q > N$, by Sobolev's embedding theorem, (4.3) yields

$$\|\nabla g\|_{L^\infty(Q)} \le C\|\eta\|_{L^q(Q)}. \tag{4.4}$$

Let $\varphi = (\varphi_1,\dots,\varphi_N)$. We observe (also g being smooth) that

$$\begin{aligned} \int_Q \operatorname{curl}\varphi\cdot\nabla g\,dx &= \sum_{i,j}\int_Q (\partial_i\varphi_j - \partial_j\varphi_i)\partial_j g_i\,dx \\ &= \sum_{i,j}\int_{\partial Q}(\varphi_j\partial_j g_i\nu_i - \varphi_i\partial_j g_i\nu_j)\,d\mathcal{H}^{N-1} + \sum_{i,j}\int_Q(-\varphi_j\partial^2_{i,j}g_i + \varphi_i\partial^2_{j,j}g_i)\,dx\,. \end{aligned} \tag{4.5}$$

We claim that

$$\sum_{i,j}\int_{\partial Q}(\varphi_j\partial_j g_i\nu_i - \varphi_i\partial_j g_i\nu_j)\,d\mathcal{H}^{N-1} = 0. \tag{4.6}$$

Indeed, for $i \ne j$, in view of the boundary conditions in (4.2), we have that $\partial_j g_i = 0$ on $\partial_{e_i^\perp}Q$ and on $\partial_{e_j^\perp}Q$, so that (since by definition, $\nu = \pm e_i$ on $\partial_{e_i^\perp}Q$)

$$\int_{\partial Q}\varphi_j\partial_j g_i\nu_i\,d\mathcal{H}^{N-1} \;=\; \pm\int_{\partial_{e_i^\perp}Q}\varphi_j\partial_j g_i\,d\mathcal{H}^{N-1} = 0$$

and

$$\int_{\partial Q}\varphi_i\partial_j g_i\nu_j\,d\mathcal{H}^{N-1} \;=\; \pm\int_{\partial_{e_j^\perp}Q}\varphi_i\partial_j g_i\,d\mathcal{H}^{N-1} = 0.$$

By (4.5) and (4.6) we get

$$\int_Q \operatorname{curl}\varphi\cdot\nabla g\,dx=\sum_{i,j}\int_Q(-\varphi_j\partial^2_{i,j}g_i+\varphi_i\partial^2_{j,j}g_i)\,dx=-\int_Q\varphi\cdot\nabla(\operatorname{div}g)\,dx+\int_Q\varphi\cdot\Delta g\,dx.$$

Since $\operatorname{div}\varphi=0$ in Q and $\varphi\cdot\nu=0$ on ∂Q, we have

$$\int_Q\varphi\cdot\nabla(\operatorname{div}g)\,dx=0,$$

so that

$$\int_Q \operatorname{curl}\varphi\cdot\nabla g\,dx=\int_Q\varphi\cdot\Delta g\,dx=\int_Q\varphi\cdot\eta\,dx.$$

By (4.4) we conclude that

$$\int_Q\varphi\cdot\eta\,dx\;\le\;\|\nabla g\|_{L^\infty(Q)}\|\operatorname{curl}\varphi\|_{L^1(Q)}\;\le\;C\|\operatorname{curl}\varphi\|_{L^1(Q)}\|\eta\|_{L^q(Q)}\,,$$

hence (taking the supremum on $\eta\in C^\infty_c(\Omega;\mathbb{R}^N)$ with $\|\eta\|_{L^q(Q)}\le 1$),

$$\|\varphi\|_{L^{q'}(Q)}\le C\|\operatorname{curl}\varphi\|_{L^1(Q)}$$

where $q'=q/(q-1)$. As q varies on $(N,+\infty)$, we get that $p=q'$ ranges over $(1,\frac{N}{N-1})$, so that (4.1) holds.

In order to complete the proof, we need to justify the estimate (4.3). Consider the component g_1 of g (the proof for any other component is identical): we have

$$\begin{cases}\Delta g_1=\eta_1 & \text{on } Q\\ g_1=0 & \text{on } \partial_{e_1^\perp}Q\\ \frac{\partial g_1}{\partial\nu}=0 & \text{on } \partial_{e_1^\parallel}Q\,.\end{cases}\tag{4.7}$$

We proceed extending g_1 and η_1 to $\hat g_1$ and $\hat\eta_1$ defined on $\mathbb{R}^N$ and two-periodic in each variable, in such a way that

$$\Delta\hat g_1=\hat\eta_1\qquad\text{on }\mathbb{R}^N.\tag{4.8}$$

First of all, we extend g_1 and η_1 on the cube $[0,2]^N$. For $1\le x_1\le 2$ and $0\le x_i\le 1$ with $i=2,\dots,N$, we set

$$\begin{cases}\hat g_1(x):=-g_1(2-x_1,x_2,\dots,x_N)\\ \hat\eta_1(x):=-\eta_1(2-x_1,x_2,\dots,x_N).\end{cases}$$

Then for $1\le x_2\le 2$ and $0\le x_i\le 1$ with $i=3,\dots,N$ we set

$$\begin{cases}\hat g_1(x):=\hat g_1(x_1,2-x_2,x_3,\dots,x_N)\\ \hat\eta_1(x):=\hat\eta_1(x_1,2-x_2,x_3,\dots,x_N)\end{cases}$$

and we proceed in the same way for the coordinates $x_3,x_4,\dots x_N$.

We can then extend $\hat g_1$ and $\hat\eta_1$ by periodicity to the entire $\mathbb{R}^N$. We get immediately that equation (4.8) is satisfied, so that in particular $\hat g_1$ is smooth. Moreover we have that $\hat g_1=0$ on the hyperplanes orthogonal to e_1 and passing through the points of the form $(k,0,0,\dots,0)$ with $k\in\mathbb{Z}$.

By L^p-regularity estimates [15, Theorem 9.11] we get that there exists a constant C depending only on N and q such that

$$\|\hat g_1\|_{W^{2,q}(Q)}\le C\left(\|\hat g_1\|_{L^q(\tilde Q)}+\|\hat\eta_1\|_{L^q(\tilde Q)}\right),$$

where $\tilde{Q}$ is the cube centered at the origin with side of length 4. In view of the periodicity of $\hat{g}_1$ and $\hat{\eta}_1$, we obtain that

$$\|g_1\|_{W^{2,q}(Q)} \le C\left(\|g_1\|_{L^q(Q)} + \|\eta_1\|_{L^q(Q)}\right). \tag{4.9}$$

Let us assume by contradiction that claim (4.3) is false. Then there exist $\hat{g}_1^n$ and $\hat{\eta}_1^n$ periodic on $\mathbb{R}^N$ such that

$$\|\hat{g}_1^n\|_{W^{2,q}(Q)} \ge n\|\hat{\eta}_1^n\|_{L^q(Q)}. \tag{4.10}$$

We can assume that $\|\hat{g}_1^n\|_{L^q(Q)} = 1$. Then by (4.9) we obtain

$$\|\hat{g}_1^n\|_{W^{2,q}(Q)} \le C\left(1 + \frac{\|\hat{g}_1^n\|_{W^{2,q}(Q)}}{n}\right).$$

so that

$$\|\hat{g}_1^n\|_{W^{2,q}(Q)} \le \tilde{C}. \tag{4.11}$$

In particular $\hat{g}_1^n$ is compact in $W^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for all $\alpha < \frac{Nq}{N-q}$ (or in a suitable Hölder space). Then there exists $\hat{g}$ periodic in $\mathbb{R}^N$ such that

$$\hat{g}_1^n \to \hat{g} \qquad \text{strongly in } W^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N).$$

In particular we get $\|\hat{g}\|_{L^q(Q)} = 1$. By (4.7), (4.10) and (4.11) we get

$$\|\Delta \hat{g}_1^n\|_{L^q(Q)} \le \frac{\tilde{C}}{n} \to 0,$$

so that $\hat{g}$ is harmonic in $\mathbb{R}^N$. Since $\hat{g}$ is periodic, we conclude that it is constant. As $\hat{g}_1^n = 0$ on $\partial_{e_1^\perp} Q$, we finally deduce that $\hat{g} = 0$. But this is against $\|\hat{g}\|_{L^q(Q)} = 1$, so that claim (4.3) is proved. □

The following corollary will be used in the proof of Theorem 1.2.

Corollary 4.2 (Helmholtz's type estimate with a curl measure). *Let $Q = (0,1)^N$ be the unit cube in $\mathbb{R}^N$. Let $\mu \in \mathcal{M}(\mathbb{R}^N; M^{N\times N})$ and $\varphi \in L^1(Q, \mathbb{R}^N)$ be respectively a Radon measure on Q and a vector field such that*

$$\begin{cases} \operatorname{div}\varphi = 0 & \text{in } Q,\\ \operatorname{curl}\varphi = \mu & \text{in } Q,\\ \varphi\cdot\nu = 0 & \text{on } \partial Q, \end{cases}$$

Then for every $1 \le p < \frac{N}{N-1}$ we have that

$$\|\varphi\|_{L^p(Q,\mathbb{R}^N)} \le C|\mu|(Q),$$

where C depends only on N and p, and $|\cdot|$ denotes the total variation.

Proof. Let $\{\rho_\varepsilon\}_{\varepsilon>0}$ be smooth radial symmetric kernels. We claim that we can extend φ and μ to $\hat{\varphi} \in L^1_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ and $\hat{\mu} \in \mathcal{M}(\mathbb{R}^N; M^{N\times N})$ in such a way that

$$|\hat{\mu}|(\partial Q) = 0, \tag{4.12}$$

and such that

$$\varphi_\varepsilon := \hat{\varphi} * \rho_\varepsilon, \qquad \mu_\varepsilon := \hat{\mu} * \rho_\varepsilon \tag{4.13}$$

satisfy

$$
\begin{cases} \operatorname{div} \varphi_\varepsilon = 0 & \text{in } Q \\ \varphi_\varepsilon \cdot \nu = 0 & \text{on } \partial Q \\ \operatorname{curl} \varphi_\varepsilon = \mu_\varepsilon & \text{in } Q. \end{cases} \tag{4.14}
$$

If this is true, by (4.1) we will obtain that for all $p < \frac{N}{N-1}$

$$
\|\hat{\varphi}_\varepsilon\|_{L^p(Q)} \leq C \|\hat{\mu}_\varepsilon\|_{L^1(Q)},
$$

where C depends only on N and p. Letting $\varepsilon \to 0$, in view of (4.12) we will deduce

$$
\|\varphi\|_{L^p(Q;\mathbb{R}^N)} \leq C|\mu|(Q) = C|\operatorname{curl} \varphi|(Q).
$$

As in the proof of Proposition 4.1, we now build the extension $(\hat{\varphi}, \hat{\mu})$ satisfying (4.12) and (4.14). First of all, we extend φ and μ to the cube $Q(0,2)$. For $1 \leq x_1 \leq 2$ and $0 \leq x_i \leq 1$ with $i = 2, \dots, N$, let us set

$$
\hat{\varphi} := \begin{pmatrix} -\varphi_1(2 - x_1, x_2, \dots, x_N) \\ \varphi_2(2 - x_1, x_2, \dots, x_N) \\ \vdots \\ \varphi_N(2 - x_1, x_2, \dots, x_N) \end{pmatrix}
$$

For $i, j \neq 1$, and E Borel set contained in $Q + e_1$, let us set

$$
\hat{\mu}_{1j}(E) := -\mu_{1j}(T_1^{-1}(E)), \quad \hat{\mu}_{j1}(E) := -\mu_{j1}(T_1^{-1}(E)), \quad \hat{\mu}_{ij}(E) := \mu_{ij}(T_1^{-1}(E)),
$$

where $T_1(x) := (2 - x_1, x_2, \dots, x_N)$.

Then we proceed in the same way for the components $x_2, x_3, \dots, x_N$. We obtain

$$
\hat{\varphi} \in L^1(Q(0,2); \mathbb{R}^N) \quad \text{and} \quad \hat{\mu} \in \mathcal{M}_b(Q(0,2); M^{N \times N}).
$$

We extend now $\hat{\varphi}$ and $\hat{\mu}$ to the entire $\mathbb{R}^N$ by periodicity. We obtain

$$
\hat{\varphi} \in L^1_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N) \quad \text{and} \quad \hat{\mu} \in \mathcal{M}_b(\mathbb{R}^N; M^{N \times N})
$$

with $\hat{\mu}$ satisfying (4.12). By construction we have

$$
\begin{cases} \operatorname{div} \hat{\varphi} = 0 \\ \operatorname{curl} \hat{\varphi} = \hat{\mu}\,. \end{cases} \tag{4.15}
$$

The first identity in (4.15) is easily checked by appropriate integration against test functions. For the second, we need to show that for any $\psi \in C_c^\infty(\mathbb{R}^N; M^{N \times N})$, one has

$$
\sum_{i,j=1}^N \int_{\mathbb{R}^N} \hat{\varphi}_i \partial_j (\psi_{i,j} - \psi_{j,i})\, dx \;=\; \sum_{i,j=1}^N \int_{\mathbb{R}^N} \psi_{i,j}\, d\hat{\mu}_{i,j}\,. \tag{4.16}
$$

By construction, this clearly holds if ψ has compact support in $\bigcup_{k \in \mathbb{Z}^N} (k + Q)$. We thus need to show that any other test function ψ can be approximated by functions with such a support, without perturbing too much both terms of the equality in (4.16).

We observe that not only (4.12) holds, but also,

$$
|\hat{\mu}| \left(\bigcup_{k \in \mathbb{Z}^N} (k + \partial Q) \right) \;=\; 0.
$$

Hence, if η_ε is a family of cut-off functions that go locally uniformly to zero in $\mathbb{R}^N \setminus \bigcup_{k\in\mathbb{Z}^N}(k+\partial Q)$, one has

$$\sum_{i,j=1}^{N} \int_{\mathbb{R}^N} \psi_{i,j}\, d\hat{\mu}_{i,j} \;=\; \lim_{\varepsilon\to 0} \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} (1-\eta_\varepsilon)\psi_{i,j}\, d\hat{\mu}_{i,j} \tag{4.17}$$

for any smooth test function ψ.

Let us choose a smooth, even $\eta \in C_c^\infty(-1,1)$ with $\eta \equiv 1$ in a neigborhood of 0, and $0 \le \eta \le 1$, and let $\eta_\varepsilon(x) := \eta(x_1/\varepsilon)$ for any $x \in \mathbb{R}^N$. We have

$$\begin{aligned}\sum_{i,j=1}^{N} \int_{\mathbb{R}^N} &\hat{\varphi}_i \partial_j((1-\eta_\varepsilon)\psi_{i,j} - (1-\eta_\varepsilon)\psi_{j,i})\, dx \\ &= \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} (1-\eta_\varepsilon)\hat{\varphi}_i \partial_j(\psi_{i,j} - \psi_{j,i})\, dx \;-\; \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{\eta'(\frac{x_1}{\varepsilon})}{\varepsilon} \hat{\varphi}_i(\psi_{i,1} - \psi_{1,i})\, dx\,.\end{aligned}$$

Hence, if we can show that the last sum goes to 0 as $\varepsilon \to 0$, together with (4.17), this will show that ψ in (4.16) may be replaced with a function whose support avoids $\{x_1 = 0\}$. In an obvious way, it will be identical to show that ψ may be replaced with a function whose support avoids $\{x_i = k\}$ for any $i = 1,\ldots,N$ and $k \in \mathbb{Z}$. This will show that ψ may be replaced with a function with compact support in $\bigcup_{k\in\mathbb{Z}^N}(k+Q)$, in which case, as observed, (4.16) clearly holds. It therefore remains to show that

$$\lim_{\varepsilon\to 0} \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{\eta'(\frac{x_1}{\varepsilon})}{\varepsilon} \hat{\varphi}_i(x)(\psi_{i,1}(x) - \psi_{1,i}(x))\, dx \;=\; 0\,. \tag{4.18}$$

First of all, the first term in the sum is clearly zero. Then, since $\hat{\varphi}_i$ is even with respect to x_1 for any $i \ge 2$, and since η' is odd, one has for $i \ge 2$ (letting $x = (x_1, x')$ for any $x \in \mathbb{R}^N$)

$$\int_{\mathbb{R}^N} \frac{\eta'(\frac{x_1}{\varepsilon})}{\varepsilon} \hat{\varphi}_i(x)(\psi_{i,1}(x) - \psi_{1,i}(x))\, dx \;=\; \int_0^\varepsilon \int_{\mathbb{R}^{N-1}} \eta'(\frac{x_1}{\varepsilon})\hat{\varphi}_i(x) \frac{\tilde{\psi}_{i,1}(x_1,x') - \tilde{\psi}_{i,1}(-x_1,x')}{\varepsilon}\, dx'\, dx_1$$

where $\tilde{\psi}_{i,1} := \psi_{i,1} - \psi_{1,i}$. The function $x \mapsto \eta'(\frac{x_1}{\varepsilon})(\tilde{\psi}_{i,1}(x_1,x') - \tilde{\psi}_{i,1}(-x_1,x'))/\varepsilon$ is clearly bounded (by $c = 2\|\eta'\|_\infty \|\partial_1 \tilde{\psi}_{i,1}\|_\infty$) so that this integral is less than $c\int_0^\varepsilon \int_{\mathbb{R}^{N-1}} |\hat{\varphi}_i|\, dx$ which goes to 0 as $\varepsilon \to 0$. Summing from $i = 2$ to N shows (4.18).

Let us now consider the convolutions (4.13). Clearly we have, from (4.15),

$$\begin{cases} \operatorname{div} \varphi_\varepsilon = 0 \\ \operatorname{curl} \varphi_\varepsilon = \mu_\varepsilon. \end{cases}$$

Moreover, since we have extended the i-th component oddly in the direction e_i, it is readily checked that

$$\varphi_\varepsilon \cdot \nu = 0 \quad \text{on } \partial Q.$$

Since the claims (4.12) and (4.14) are proved, the proof is concluded. □

5. Proof of Theorem 1.2

Let us first deduce from the results in the two previous section the following rigidity estimate, which is valid for any compact $\mathcal{K}$ such that estimate (1.1) holds.

Proposition 5.1 (The rigidity estimate). *Let $Q = (0,1)^N$ be the unit cube in $\mathbb{R}^N$ and let $1 \le p < N/(N-1)$. Let $u \in SBV(Q;\mathbb{R}^N)$ be such that $\nabla u(x) \in \mathcal{K}$ for a.e. $x \in Q$. Then $\mu_u := \operatorname{curl} \nabla u$ is a measure concentrated on J_u and there exists $K \in \mathcal{K}$ such that*

$$\|\nabla u - K\|_{L^p(Q)} \le C|\mu_u|(Q), \tag{5.1}$$

where C depends only on N and p.

Proof. By Theorem 3.1 we have that $\mu_u := \operatorname{curl} \nabla u$ is a measure concentrated on J_u such that

$$|\mu_u| \leq c\mathcal{H}^{N-1} \llcorner J_u,$$

where c is a constant depending only on $\|\nabla u\|_\infty$.

Let us consider $w \in H^1(Q; \mathbb{R}^N)$ solution of the minimization problem

$$\min \left\{ \|\nabla v - \nabla u\|^2 \,:\, v \in H^1(Q; \mathbb{R}^N), \int_Q v(x)\, dx = 0 \right\}.$$

Let $\varphi := \nabla u - \nabla w$. We have that $\varphi \in L^2(Q; M^{N\times N})$, and by minimality, that $\int_Q \varphi : \nabla v \, dx = 0$ for any $v \in H^1(Q; \mathbb{R}^N)$, hence:

$$\begin{cases} \operatorname{div} \varphi = 0 & \text{in } Q \\ \varphi \cdot \nu = 0 & \text{on } \partial Q. \end{cases}$$

Moreover we have that

$$\operatorname{curl} \varphi = \operatorname{curl} \nabla u - \operatorname{curl} \nabla w = \mu_u,$$

i.e., $\operatorname{curl} \varphi \in \mathcal{M}(Q; M^{N\times N})$.

By corollary 4.2 (applied to each component of φ), there exists a constant C depending only on p and N such that

$$\|\varphi\|_{L^p(Q)} \leq C|\mu_u|(Q)$$

so that

$$\|\nabla u - \nabla w\|_{L^p(Q)} \leq C|\mu_u|(Q). \tag{5.2}$$

Moreover, by the rigidity estimate (1.1) we have that there exists $K \in \mathcal{K}$ such that

$$\|\nabla w - K\|_{L^p(Q)} \leq C\|\operatorname{dist}(\nabla w, \mathcal{K})\|_{L^p(Q)} \tag{5.3}$$

(possibly changing C, which still depends only on p and N). In view of (5.2) and (5.3), and since $\nabla u(x) \in \mathcal{K}$ for a.e. $x \in Q$, we deduce that

$$\begin{aligned} \|\nabla u - K\|_{L^p(Q)} &\leq \|\nabla w - K\|_{L^p(Q)} + \|\nabla u - \nabla w\|_{L^p(Q)} \\ &\leq C\|\operatorname{dist}(\nabla w, \mathcal{K})\|_{L^p(Q)} + \|\nabla u - \nabla w\|_{L^p(Q)} \\ &\leq C\|\operatorname{dist}(\nabla u, \mathcal{K})\|_{L^p(Q)} + (1 + C)\|\nabla u - \nabla w\|_{L^p(Q)} \\ &\leq (1 + C)C|\mu_u|(Q) \end{aligned}$$

so that (5.1) holds. ☐

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Since $\nabla u(x) \in \mathcal{K}$ for a.e. $x \in \Omega$, by Theorem 3.1 we have that $\mu_u := \operatorname{curl} \nabla u$ is a measure concentrated on J_u and such that

$$|\mu_u| \leq c\mathcal{H}^{N-1} \llcorner J_u, \tag{5.4}$$

where $c = c(\|\nabla u\|_\infty)$. Let us cover $\mathbb{R}^N$ by means of disjoint cubes of side h, and let $\{Q(a_i, h)\}_{i\in I}$ be the family of these cubes contained in Ω. We carry out the proof in several steps.

Step 1: Piecewise constant approximation of ∇u. By Proposition 5.1, using a rescaling argument, we have that for every $i \in I$ there exists $K_i^h \in \mathcal{K}$ such that

$$\|\nabla u - K_i^h\|_{L^p(Q(a_i,h))} \leq C\frac{h^{N/p}}{h^{N-1}}|\mu_u|(Q(a_i, h)), \tag{5.5}$$

where C depends only on p and N.

Let us consider the piecewise constant function ψ_h defined on Ω such that

$$\psi_h(x) := \begin{cases} K_i^h & \text{if } x \in Q(a_i, h) \\ 0 & \text{if } x \notin \bigcup_{i\in I} Q(a_i, h) \end{cases} \tag{5.6}$$

Step 2: Estimate for $|D\psi_h|$. Let us estimate the total variation $|D\psi_h|$ of ψ_h. We consider two neighbouring cubes $Q(a_i, h)$ and $Q(a_j, h)$. By applying estimate (5.1) to the rectangle $R^h_{i,j} = \mathrm{int}(Q(a_i, h) \cup Q(a_j, h))$ (of size $2h$ in one direction and h in the $N-1$ other: the proof of Corollary 4.2 in that case is identical to the proof in the case of a cube, or, alternatively, can be easily deduced by an appropriate transformation of the cube), we have that there exists $K \in \mathcal{K}$ such that

$$\|\nabla u - K\|_{L^p(R^h_{i,j})} \le \tilde{C} \frac{h^{N/p}}{h^{N-1}} |\mu_u|(R^h_{i,j}) \tag{5.7}$$

where $\tilde{C}$ depends only on N and p. Then, in view of (5.5) we get that

$$\begin{aligned}
|K_i^h - K_j^h| &\le |K_i^h - K| + |K - K_j^h| \le 2^{1-1/p}\left(|K_i^h - K|^p + |K - K_j^h|^p\right)^{1/p} \\
&= 2^{1-1/p} h^{-N/p} \|K - (K_i^h \mathbf{1}_{Q(a_i,h)} + K_j^h \mathbf{1}_{Q(a_j,h)})\|_{L^p(R^h_{i,j})} \\
&\le 2^{1-1/p} h^{-N/p}\left(\|K - \nabla u\|_{L^p(R^h_{i,j})} + \|\nabla u - (K_i^h \mathbf{1}_{Q(a_i,h)} + K_j^h \mathbf{1}_{Q(a_j,h)})\|_{L^p(R^h_{i,j})}\right) \\
&\le 2^{1-1/p} h^{-N/p}\left(\|K - \nabla u\|_{L^p(R^h_{i,j})} + \|\nabla u - K_i^h\|_{L^p(Q(a_i,h))} + \|\nabla u - K_j^h\|_{L^p(Q(a_j,h))}\right) \\
&\le 2^{1-1/p} \frac{\tilde{C} + C}{h^{N-1}} |\mu_u|(R^h_{i,j})
\end{aligned}$$

so that

$$h^{N-1} |K_i^h - K_j^h| \le C |\mu_u|(R^h_{i,j}) \tag{5.8}$$

for some C depending only on N and p. We conclude that the variation of $D\psi_h$ accross the interface $\partial Q(a_i, h) \cap \partial Q(a_j, h)$ is estimated with the variation of the measure μ_u in the union of the two cubes $Q(a_i, h)$ and $Q(a_j, h)$ and their common interface.

Let now A, B be open and such that $\overline{B} \subseteq A \subseteq \overline{A} \subseteq \Omega$. By (5.8) we get that for h large enough

$$|D\psi_h|(B) \le C|\mu_u|(A) \tag{5.9}$$

for some C depending only on N and p.

Step 3: ∇u is piecewise constant. Since $\mathcal{K} \subseteq M^{N\times N}$ is compact, we have that ψ_h is uniformly bounded in $L^\infty(\Omega; M^{N\times N})$. In view of (5.9), and since $|\mu_u| \le \mathcal{H}^{N-1} \llcorner J_u$, we can use the compactness in BV (see [5, Theorem 3.23]) obtaining $\psi \in BV(\Omega)$ such that

$$\psi_h \to \psi \qquad \text{strongly in } L^1(\Omega, M^{N\times N})$$

and

$$|D\psi|(A) \le C\mathcal{H}^{N-1}(J_u \cap A) \tag{5.10}$$

for every open set $A \subseteq \Omega$.

Let us check that $\psi = \nabla u$. Since ∇u and ψ_h are uniformly bounded in $L^\infty(\Omega; M^{N\times N})$, and since $p < \frac{N}{N-1}$, by (5.5) we have that

$$\limsup_{h\to+\infty} \|\nabla u - \psi_h\|_{L^p(\Omega)} \le \limsup_{h\to+\infty} \sum_{i\in I} \|\nabla u - \psi_h\|_{L^p(Q(a_i,h))}$$
$$\le \limsup_{h\to+\infty} \sum_{i\in I} C\frac{h^{N/p}}{h^{N-1}}|\mu_u|(Q(a_i,h)) \le \limsup_{h\to+\infty} C\frac{h^{N/p}}{h^{N-1}}|\mu_u|(\Omega) = 0$$

so that $\psi_h \to \nabla u$ strongly in $L^p(\Omega; M^{N\times N})$, and $\psi = \nabla u$.

By (5.10) we get that $\nabla u \in SBV(\Omega; M^{N\times N})$, and that $D(\nabla u)$ is concentrated on J_u. Since $\mathcal{H}^{N-1}(J_u) < +\infty$, by [5, Theorem 4.23] we deduce that ∇u is piecewise constant, i.e. there exists a Caccioppoli partition $\{D_j\}_{j\in\mathbb{N}}$ and matrices $K_j \in \mathcal{K}$ such that

$$\partial^* D_j \subseteq J_u, \qquad \sum_{j\in\mathbb{N}} \mathcal{H}^{N-1}(\partial^* D_j) = 2\mathcal{H}^{N-1}(S(\nabla u)) \le 2\mathcal{H}^{N-1}(J_u) \tag{5.11}$$

and

$$\nabla u = \sum_{j\in\mathbb{N}} K_j \mathbf{1}_{D_j}. \tag{5.12}$$

Step 4: Conclusion. Let us consider the map $w \in SBV(\Omega)$ defined by

$$w(x) := \sum_{j\in\mathbb{N}} (K_j \cdot x)\mathbf{1}_{D_j}(x).$$

Since $\nabla w = \nabla u$, and $J_w \subseteq J_u$ in view of (5.11), we deduce that $D(u-w)$ is supported by J_u. By [5, Theorem 4.23], we conclude that there exists a Caccioppoli partition $\{F_k\}_{k\in\mathbb{N}}$ of Ω, and $b_k \in \mathbb{R}^N$, such that

$$\partial^* F_k \cap \Omega \subseteq J_u, \qquad \sum_{k\in\mathbb{N}} \mathcal{H}^{N-1}(\partial^* F_k \cap \Omega) = 2\mathcal{H}^{N-1}(J_u)$$

and

$$u - w = \sum_{k\in\mathbb{N}} b_k \mathbf{1}_{F_k}. \tag{5.13}$$

Considering the Caccioppoli partition $\{E_i\}_{i\in\mathbb{N}}$ determined by the intersection of the families $\{D_j\}_{j\in\mathbb{N}}$ and $\{F_k\}_{k\in\mathbb{N}}$, we deduce that there exist $K_i \in \mathcal{K}$ and $b_i \in \mathbb{R}^N$ such that

$$u = \sum_{i\in\mathbb{N}} (K_i \cdot x + b_i)\mathbf{1}_{E_i}(x)$$

and the proof is concluded. □

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